# Dynamics of Glassy Systems :

## an overview


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# Journal of Statistical Mechanics: Theory and Experiment

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# Experiments

## Equilibrium

# The compounds

**Dilute magnetic impurities in some non-magnetic host**

**Metallic spin glasses**

— $Cu_{1-x}Mn_x$ the archetypal one

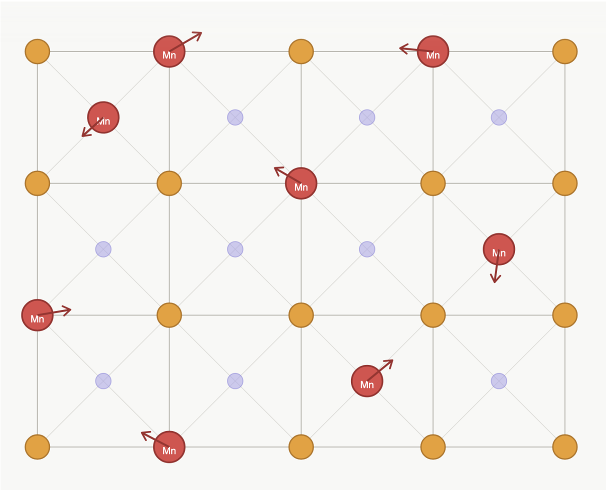
— $Au_{1-x}Fe_x$ extensively studied

— $Ag_{1-x}Mn_x$ similar to CuMn

**Insulating spin glasses**

— $Eu_xSr_{1-x}$ a semiconductor

— $Fe_{1-x}Mn_xTiO_3$ a frustrated insulator




How much spin-glass material is there in physics labs ? Grams.

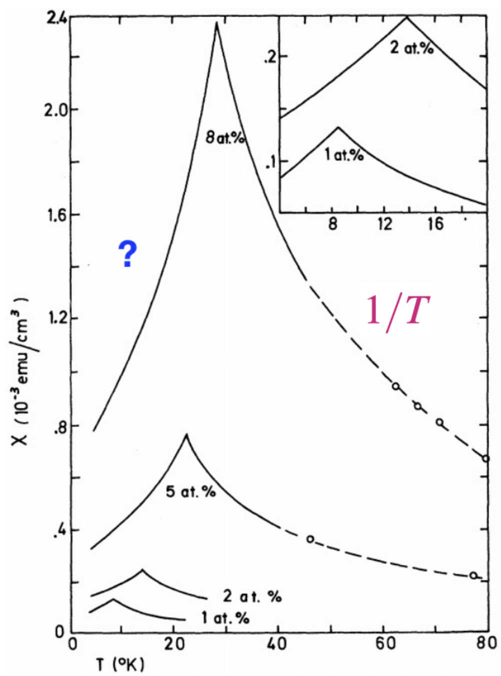
Are they useful materials ? **No**

Have they been inspirational ? **So much so - Fundamental research**

# The anomaly

## The experiment & the phase transition - 70s

Magnetic susceptibility




Temperature

**Canella & Mydosh 1972**

The cusp indicates a phase transition towards **which phase ?**

Hypothesis : **canonical equilibrium**

for simplicity, Ising spins $s_i = \pm 1$ or $\uparrow, \downarrow$

partition function $\mathcal{Z} = \sum_{\text{conf}} e^{-\beta \mathcal{H}}$

linear susceptibility $\chi = \frac{\partial m_h}{\partial h}\Big|_{h=0}$

magnetization density $m_h = N^{-1} \sum_i \langle s_i \rangle_h$

average with $\mathcal{Z}_h$ and $\mathcal{H}_h = \mathcal{H} - h \sum_i s_i$

$\langle s_i \rangle_h = \sum_{\text{conf}} s_i \, e^{-\beta \mathcal{H}_h} / \mathcal{Z}_h$

Taking the derivative explicitly & $h \to 0$ :

Fluctuation-dissipation theorem :

$$\chi = \beta(N^{-1} \sum_i \langle s_i^2 \rangle - (N^{-1} \sum_i \langle s_i \rangle)^2)$$

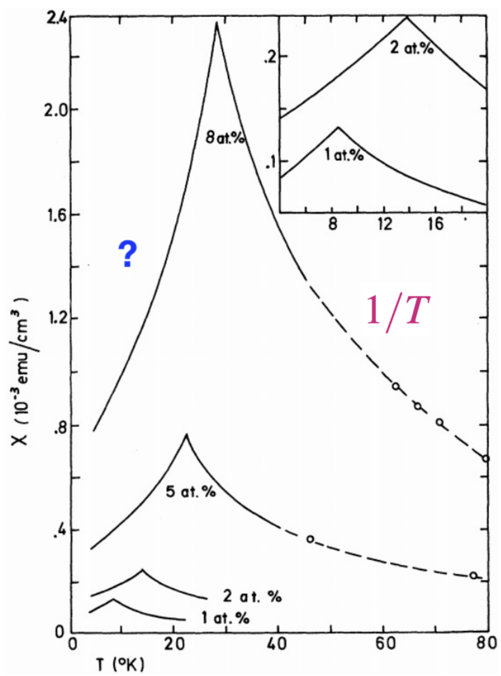
Thus, if $\langle s_i \rangle = 0$ and $s_i = \pm 1$

$$\boxed{\chi = \beta = 1/T}$$ **Curie law**

# The anomaly

## The experiment & the phase transition - 70s

Magnetic susceptibility




Temperature

**Canella & Mydosh 1972**

The cusp indicates a phase transition towards **which phase ?**

Hypothesis : **canonical equilibrium**

**FDT** $\chi = \beta(1 - N^{-1} \sum_i \langle s_i \rangle^2)$

The low $T$ phase cannot be ferro since

$m = N^{-1} \sum_{i=1}^N \langle s_i \rangle = 0$ measured

but $\langle s_i \rangle$ could be $\neq 0$ for each $i$

decrease in $\chi$ implies

$$q_{\text{EA}} \equiv \frac{1}{N} \sum_i \langle s_i \rangle^2 \begin{cases} = 0 & T \geq T_c \\ \neq 0 & T < T_c \end{cases}$$

**EA order parameter**

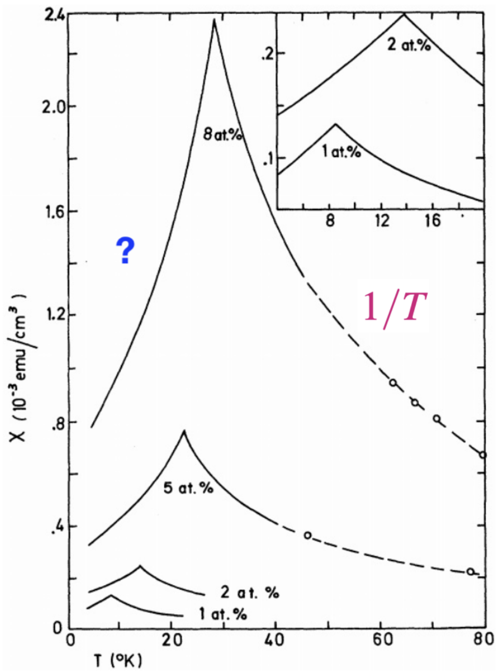
# Theory

## Models & analysis

# Edwards-Anderson Model

## The experiment, the paper & the authors - 70s

Magnetic susceptibility




Temperature

**Canella & Mydosh 1972**

The cusp indicates a phase transition towards **which phase ?**

J. Phys. F: Metal Phys.. Vol. 5, May 1975. Printed in Great Britain. © 1975.

**Theory of spin glasses**

S F Edwards† and P W Anderson‡
Cavendish Laboratory, Cambridge, UK

Received 14 October 1974, in final form 13 February 1975

# Edwards-Anderson Model

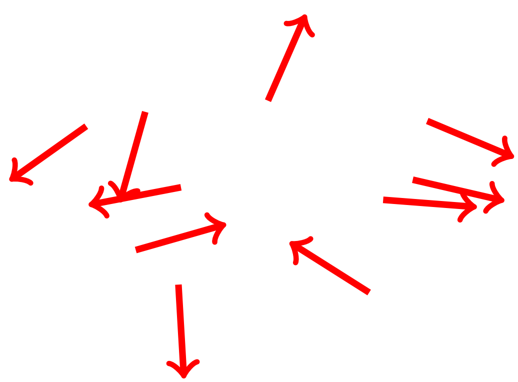
## Disordered low temperature equilibrium

**Theory of spin glasses**

S F Edwards† and P W Anderson‡
Cavendish Laboratory, Cambridge, UK

Received 14 October 1974, in final form 13 February 1975

**Abstract.** A new theory of the class of dilute magnetic alloys, called the spin glasses, is proposed which offers a simple explanation of the cusp found experimentally in the susceptibility. The argument is that because the interaction between the spins dissolved in the matrix oscillates in sign according to distance, there will be no mean ferro- or antiferromagnetism, but there will be a ground state with the spins aligned in definite directions, even if these directions appear to be at random. At the critical temperature, the existence of these preferred directions affects the orientation of the spins, leading to a cusp in the susceptibility. This cusp is smoothed by an external field. If the potential



$$\mathcal{H} = -\frac{1}{2}\sum_{i\neq j} J_{ij}\,\vec{s}_i\cdot\vec{s}_j$$

with

$$J_{ij} = \frac{F(2k_F r_{ij})}{r_{ij}^3}$$ RKKY

# Edwards-Anderson Model

## A minimal model

$$\mathcal{H} = -\frac{1}{2} \sum_{\langle ij \rangle} J_{ij} s_i s_j$$

$s_i = \pm 1$ Ising spins - the variables

from magnetic moments to bimodal variables

$\langle ij \rangle$ nearest-neighbours on a cubic lattice

from random positions to the vertices of a regular lattice

$J_{ij}$ quenched random, taken from a probability distribution - the couplings

$P[\{J_{ij}\}]$ bimodal, Gaussian, or other with no fat tails, with

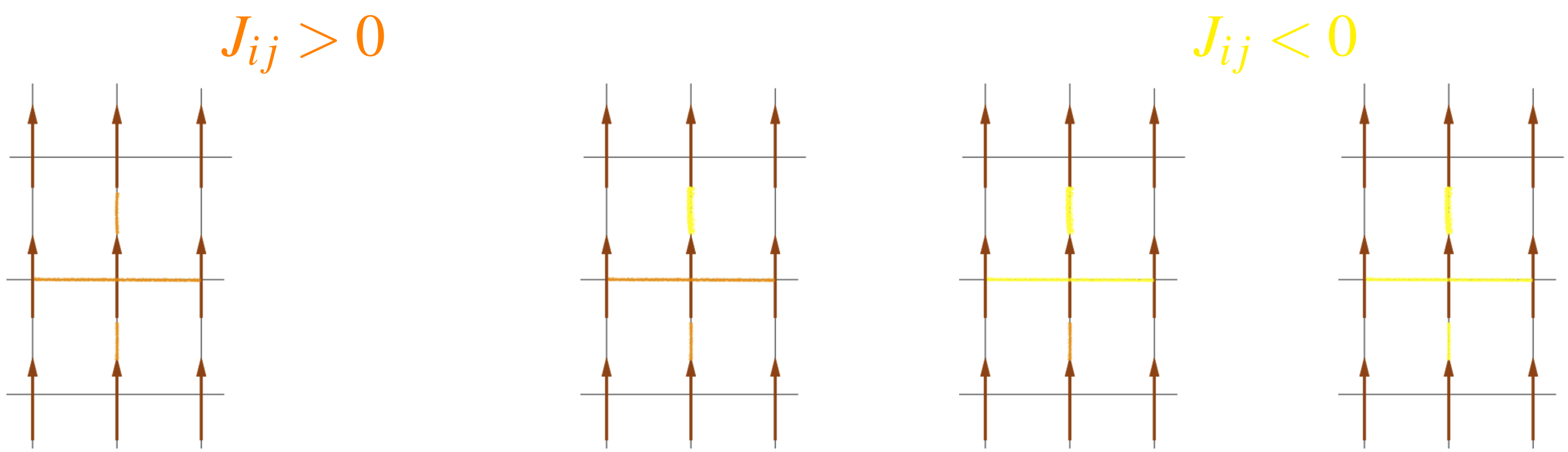
$[J_{ij}] = 0$ and $[J_{ij}^2] = J^2$

from RKKY to first neighbour random interactions

# Properties

## Heterogeneity

Each variable, spin or other, feels a different local field, $h_i = \sum_{j=1}^{z} J_{ij} s_j$, contrary to what happens in a ferromagnetic sample, for instance.




**Homogeneous** $\qquad$ **Heterogeneous**

$h_i = 4J \ \forall\ i \qquad h_j = 2J \qquad h_k = -2J \qquad h_l = -4J$.

Each sample is *a priori* different but,

**do they all have a different thermodynamic and dynamic behavior ?**

# Low temperature phases

## Phenomenology : homogeneity vs inhomogeneity

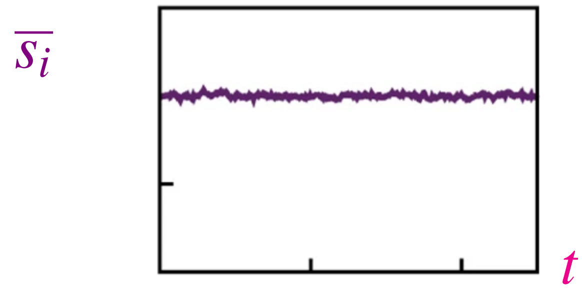
In a **ferromagnet in equilibrium** at temperature $T < T_c$, $\langle s_i \rangle = m(T)\ \forall i$ or $\langle s_i \rangle = -m(T)\ \forall i$ in the two homogeneous, symmetric and degenerate equilibrium states

# Low temperature phases

## Phenomenology : homogeneity vs inhomogeneity

In a **ferromagnet in equilibrium** at temperature $T < T_c$, $\langle s_i \rangle = m(T)\ \forall i$ or $\langle s_i \rangle = -m(T)\ \forall i$ in the two homogeneous, symmetric and degenerate equilibrium states

If one were to follow the time evolution of each spin in one of the two equilibrium states at $T < T_c$, one would see $\overline{s_i}(t) = m(T) + \delta_i(t)$ with $\delta_i(t)$ a small time-dependent fluctuation and the overline for a running time average $\overline{s_i}(t) = \tau^{-1} \int_t^{t+\tau} dt'\, s_i(t')$



# Low temperature phases

## Phenomenology : homogeneity vs inhomogeneity

In a **spin-glass in equilibrium** at temperature $T < T_c$, one expects $\langle s_i \rangle = m_i(T)$, with a different value for each $i$, in each inhomogeneous and degenerate equilibrium state.

There may be many different ensembles $\{m_i(T)\}$ that are equilibrium states (degeneracy, similar to what we saw in the frustrated magnets for the ground states but here in the full low $T$ phase)

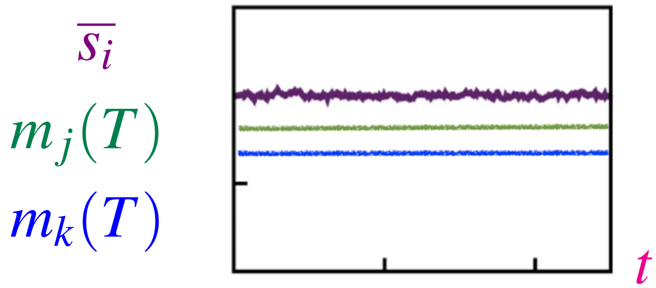
There is also the up-down symmetry $\{m_i(T)\} \mapsto \{-m_i(T)\}$

# Low temperature phases

## Phenomenology : homogeneity vs inhomogeneity

In a **spin-glass in equilibrium** at temperature $T < T_c$, one expects $\langle s_i \rangle = m_i(T)$, with a different value for each $i$, in each inhomogeneous and degenerate equilibrium state.

If one were to follow the time evolution of each spin in one of the possibly many equilibrium states at $T < T_c$, one would see $\overline{s_i}(t) = m_i(T) + \delta_i(t)$ with $\delta_i(t)$ a small time-dependent fluctuation and the overline, a running time average $\overline{s_i}(t) = \tau^{-1} \int_t^{t+\tau} dt' \, s_i(t')$



# Edwards-Anderson Model

## Dynamic order parameter

**Theory of spin glasses**

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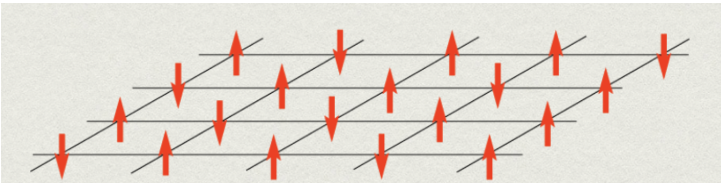
Received 14 October 1974, in final form 13 February 1975

viewed later will be the *same* random coil. Thus what we must argue is that if on one observation a particular spin is $s_i^{(1)}$ then if it is studied again a long time later, there is a nonvanishing probability that $s_i^{(2)}$ will point in the same direction, ie

$$q = \langle s_i^{(1)} . s_i^{(2)} \rangle \neq 0.$$

$$q_{\mathrm{EA}} = \lim_{t \gg t'} \lim_{N \to \infty} \langle s_i(t) s_i(t') \rangle \neq 0$$



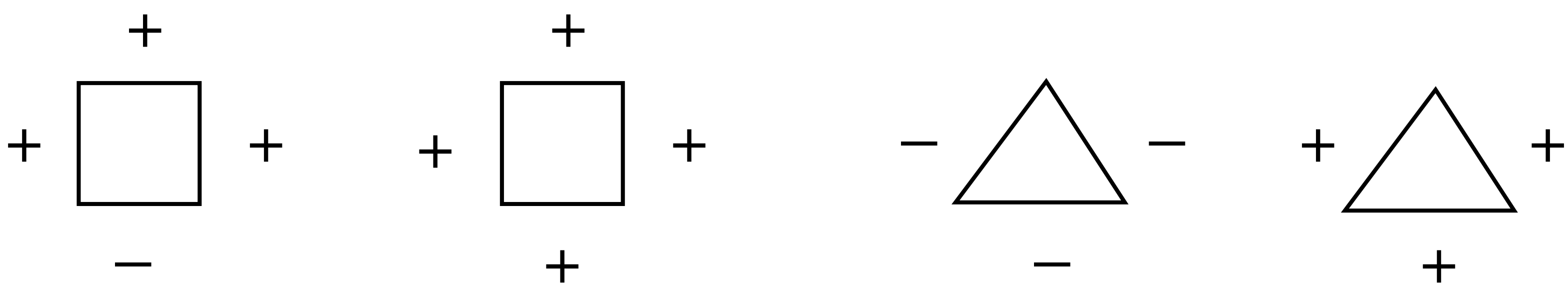
Spatially disordered but dynamically frozen, on average

$$\chi = \beta(1 - q_{\mathrm{EA}})$$

# Properties

## Frustration

$$H_J[\{s\}] = -\sum_{\langle ij \rangle} J_{ij}\, s_i s_j$$ **Ising model**



**Disordered** **Geometric**

$$E_{\rm GS}^{\rm frust} > E_{\rm GS}^{\rm FM} \quad \text{and} \quad S_{\rm GS}^{\rm frust} > S_{\rm GS}^{\rm FM}$$

Frustration enhances the **ground-state** energy and degeneracy (entropy)

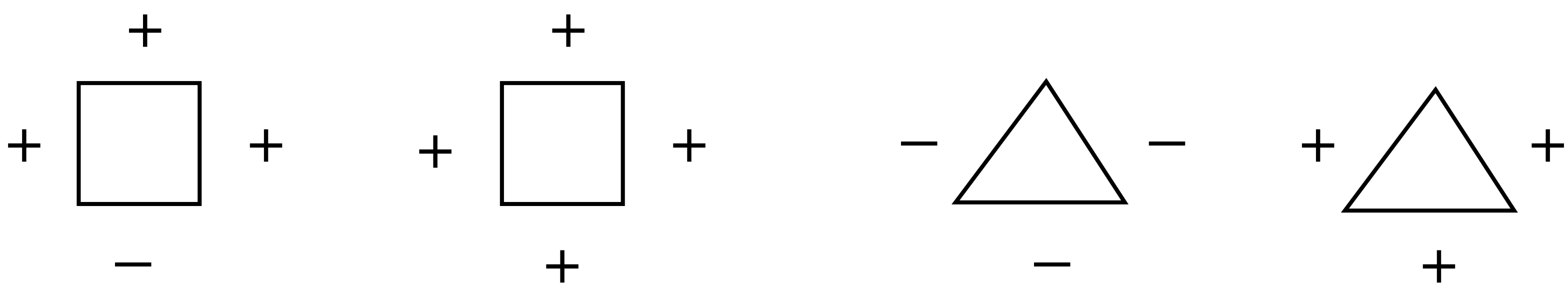
One can expect to have many **metastable states** too

One cannot satisfy all couplings simultaneously if $\prod_{\rm loop} J_{ij} < 0$

# Properties

## Frustration

$H_J[\{s\}] = -\sum_{\langle ij \rangle} J_{ij}\, s_i s_j$ **Ising model**



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Frustration enhances the **ground-state** energy and **degeneracy**

One can expect to have many **metastable states** too

One cannot satisfy all couplings simultaneously if $\prod_{\rm loop} J_{ij} < 0$

# Properties

## Self-averageness

Given a quantity $A_J$, which depends on the quenched randomness $J_{ij}$, it is distributed according to

$$P(A) = \int \prod_{ij} dJ_{ij}\, P(\{J_{ij}\})\, \delta(A - A_J)$$

For "additive" quantities, $P(A)$ expected to be more peaked as $N \to \infty$

Therefore, one will observe $A = A_{\mathrm{typ}}$ such that $P(A_{\mathrm{typ}}) = \max_A P(A)$

However, it is difficult to calculate $A_{\mathrm{typ}}$

What about calculating $[A] = \int \prod_{ij} dJ_{ij}\, P(\{J_{ij}\})\; A$ ?

**Self-averageness** $\boxed{[A] = A_{\mathrm{typ}}}$

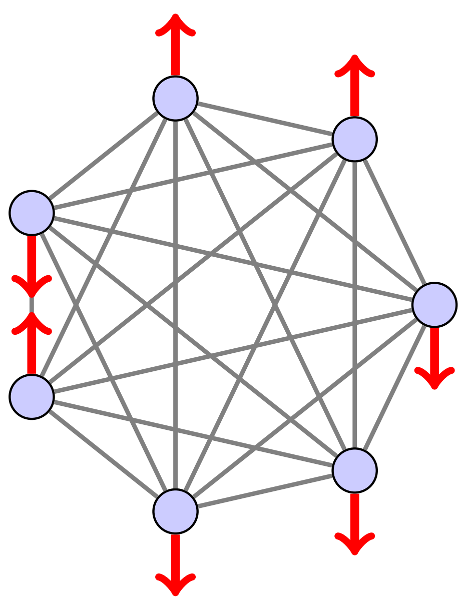
# Sherrington-Kirkpatrick Model

## The mean-field extension - all to all couplings - 70s

### Solvable Model of a Spin-Glass

David Sherrington* and Scott Kirkpatrick
*IBM Thomas J. Watson Research Center, Yorktown Heights, New York 10598*
(Received 16 October 1975)

We consider an Ising model in which the spins are coupled by infinite-ranged random interactions independently distributed with a Gaussian probability density. Both "spin-glass" and ferromagnetic phases occur. The competition between the phases and the type of order present in each are studied.



$$\mathcal{H} = -\frac{1}{2}\sum_{i\neq j} J_{ij}\, s_i s_j$$

$$[J_{ij}] = 0 \qquad [J_{ij}^2] = \frac{J^2}{N}$$

# Sherrington-Kirkpatrick Model

## Self-averageness & the Replica Method

Take a fully-connected ising spin $\{s_i\}$, $i = 1, \ldots, N$ model with quenched random interactions $J_{ij}$ drawn from a probability distribution $P(\{J_{ij}\})$

$$\boxed{\mathcal{H} = -\sum_{i \neq j} J_{ij} s_i s_j}$$

In the $N \to \infty$ limit, disorder averaged & typical free-energy densities, coincide

$$\boxed{f \underset{N\to\infty}{=} [f] = -\frac{k_B T}{N}[\ln Z]}$$

**self-averageness**

The disorder average can be evaluated with the help of the **replica trick** which uses the identify $x^n = \exp(n \ln x)$ Taylor expanded around $n = 0$

$$x^n \underset{n\to 0}{=} 1 + n \ln x + O(n^2) \quad \Rightarrow \quad \boxed{[\ln Z] \underset{n\to 0}{=} \frac{[Z^n] - 1}{n}}$$

# Sherrington-Kirkpatrick Model

## The replica method - a sketch

$$-\beta[f] = \lim_{N\to\infty} \frac{[\ln Z]}{N} = \lim_{N\to\infty}\lim_{n\to 0} \frac{[Z^n]-1}{Nn}$$

$Z^n$ is the partition function of $n$ independent copies of the system : the **replicas**

Gaussian $P(\{J_{ij}\})$ average over disorder $\Rightarrow$ $a, b$ **replica coupling**

$$\sum_a \sum_{i\neq j} J_{ij} s_i^a s_j^a \;\Rightarrow\; \sum_{i\neq j}\sum_{ab} s_i^a s_j^a s_i b s_j^b$$

Quadratic decoupling with the Hubbard-Stratonovich (Gaussian) trick

$$Q_{ab}\sum_i s_i^a s_i^b + \frac{1}{2} Q_{ab}^2$$

The elements of $Q_{ab}$ can be evaluated by saddle-point if one exchanges

$$\lim_{N\to\infty}\lim_{n\to 0}\ldots \;\mapsto\; \lim_{n\to 0}\lim_{N\to\infty}\ldots$$

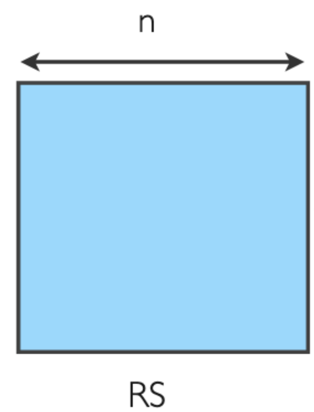
$Q^*_{ab}$ is a $0\times 0$ matrix but it admits an interpretation in terms of **overlaps**

$$NQ^*_{ab} = \sum_i [\langle s_i^a s_i^b\rangle]$$

# Sherrington-Kirkpatrick Model

## The structure of the matrix $Q^*_{ab} = [\langle s_a s_b \rangle]$

### Replica Symmetry




**Sherrington & Kirkpatrick**, *Solvable model of a spin-glass*, PRL 35, 1792 (1975)

but $S(T=0) < 0$ and the saddle-point $Q^*_{ab} = q$ is not stable

**de Almeida & Thouless** , *Stability of the Sherrington-Kirkpatrick solution of a spin glass model*, J. Phys. A : Math. Gen. **11**, 983 (1978)

Fig. from **Morone, Caltagirone, Harrison & Parisi**, *Replica Theory and Spin Glasses,* Les Houches 2013

# Sherrington-Kirkpatrick Model

## Parisi's solution - Replica Symmetry Breaking    Nobel 2021

J. Phys. A: Math. Gen. **13** (1980) L115–L121. Printed in Great Britain

**LETTER TO THE EDITOR**

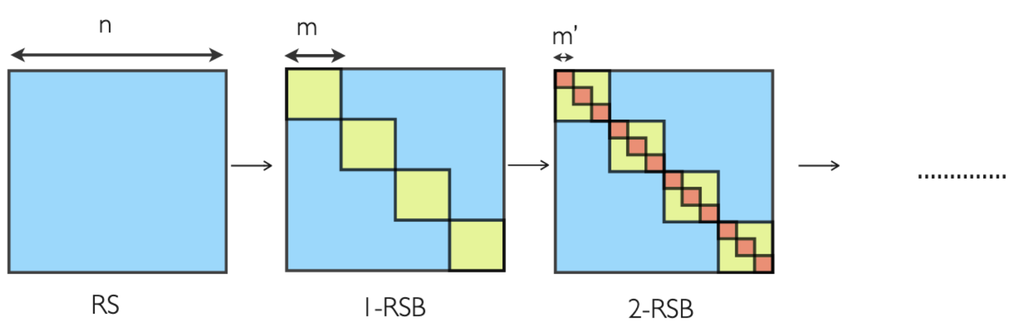
**A sequence of approximated solutions to the S–K model for spin glasses**

G Parisi

Istituto Nazionale de Fisica Nucleare, Laboratori Nazionali di Frascati, Casella Postale 13, 0004 Frascati, Roma, Italy

Received 4 January 1980

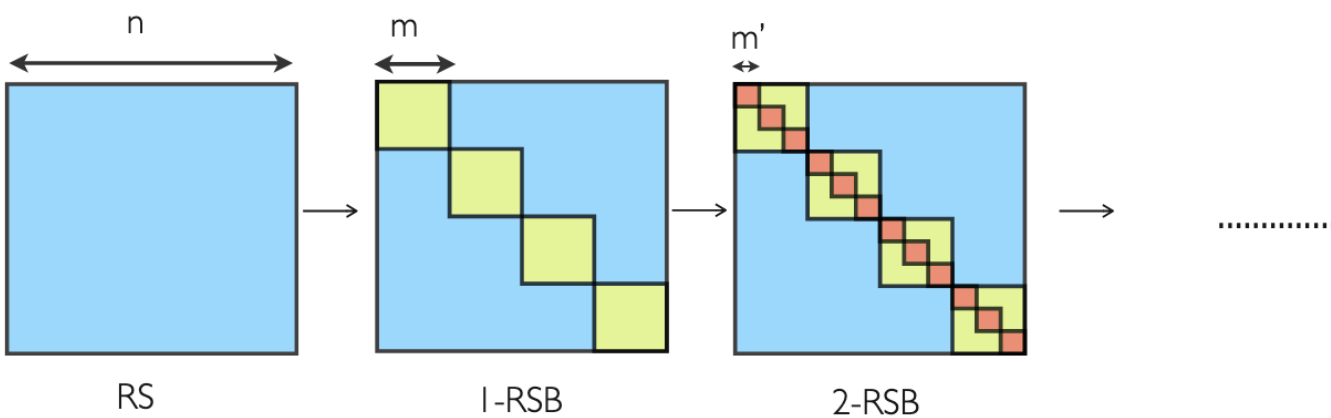
**Abstract.** In the framework of the new version of the replica theory, we compute a sequence of approximated solutions to the Sherrington–Kirkpatrick model of spin glasses.



# Sherrington-Kirkpatrick Model

**The structure of the overlap matrix $Q^*_{ab} = [\langle s_a s_b \rangle]$**

**Replica Symmetry Breaking**




Now $S(T=0)=0$ and the saddle-point $Q_{ab}$ is stable

Fig. from **Morone, Caltagirone, Harrison & Parisi**, *Replica Theory and Spin Glasses,* Les Houches 2013

# Sherrington-Kirkpatrick Model

## Multiplicity of equilibrium states

Sample the equilibrium states

at fixed realization of $J_{ij}$

Get the configurations $\{s_i^a\}$

calculate the **overlap/correlation**

$q = N^{-1} \sum_i \langle s_i^{(a)} s_i^{(b)} \rangle$

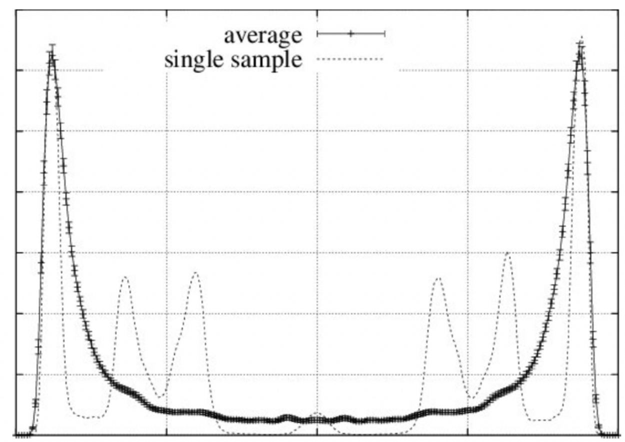
Result : $\pm q_{\mathrm{EA}}$ :

same state or reversed found

Result : $q \neq q_{\mathrm{EA}}$

two different states sampled

$P_J$




$q$

**Marinari, Martin & Zuliani, 2001**

**Real replicas**

Bold curve, $[P_J](q)$

# Sherrington-Kirkpatrick Model

## Multiplicity of equilibrium states & ultrametricity

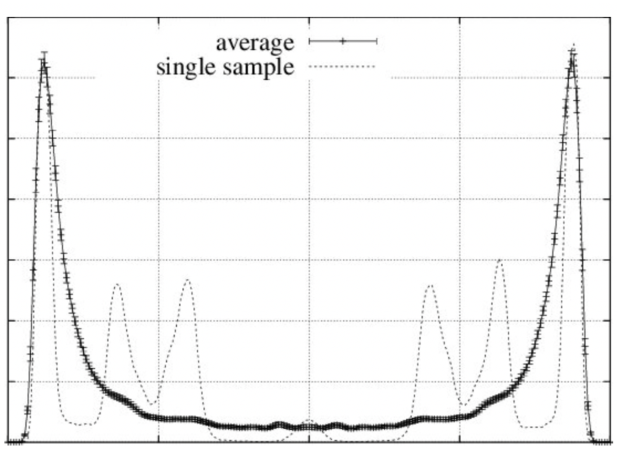
VOLUME 52, NUMBER 13 PHYSICAL REVIEW LETTERS 26 MARCH 1984

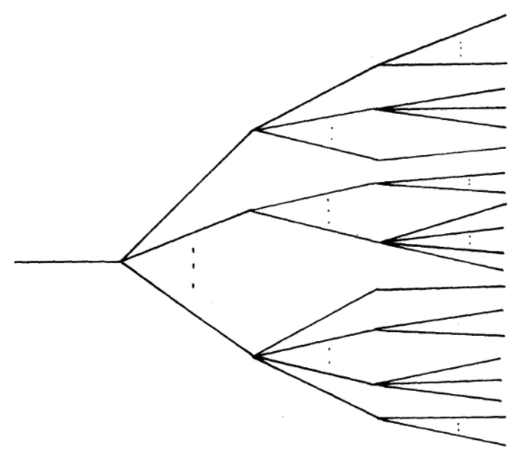
### Nature of the Spin-Glass Phase

M. Mézard, G. Parisi,[a] N. Sourlas, G. Toulouse,[b] and M. Virasoro[c]
*Laboratoire de Physique Théorique de l'Ecole Normale Supérieure, F-75231 Paris-Cedex, France*
(Received 12 December 1983)

A probability distribution has been proposed recently by one of us as an order parameter for spin-glasses. We show that this probability depends on the particular realization of the couplings even in the thermodynamic limit, and we study its distribution. We also show that the space of states has an ultrametric topology.

$P_J$



$$q = N^{-1} \sum_i \langle s_i^{(a)} s_i^{(b)} \rangle$$

Weird organization of the equilibrium states

# Sherrington-Kirkpatrick Model

## Mathematics entered the game

Francesco Guerra 1942 - 2025

$\downarrow t$

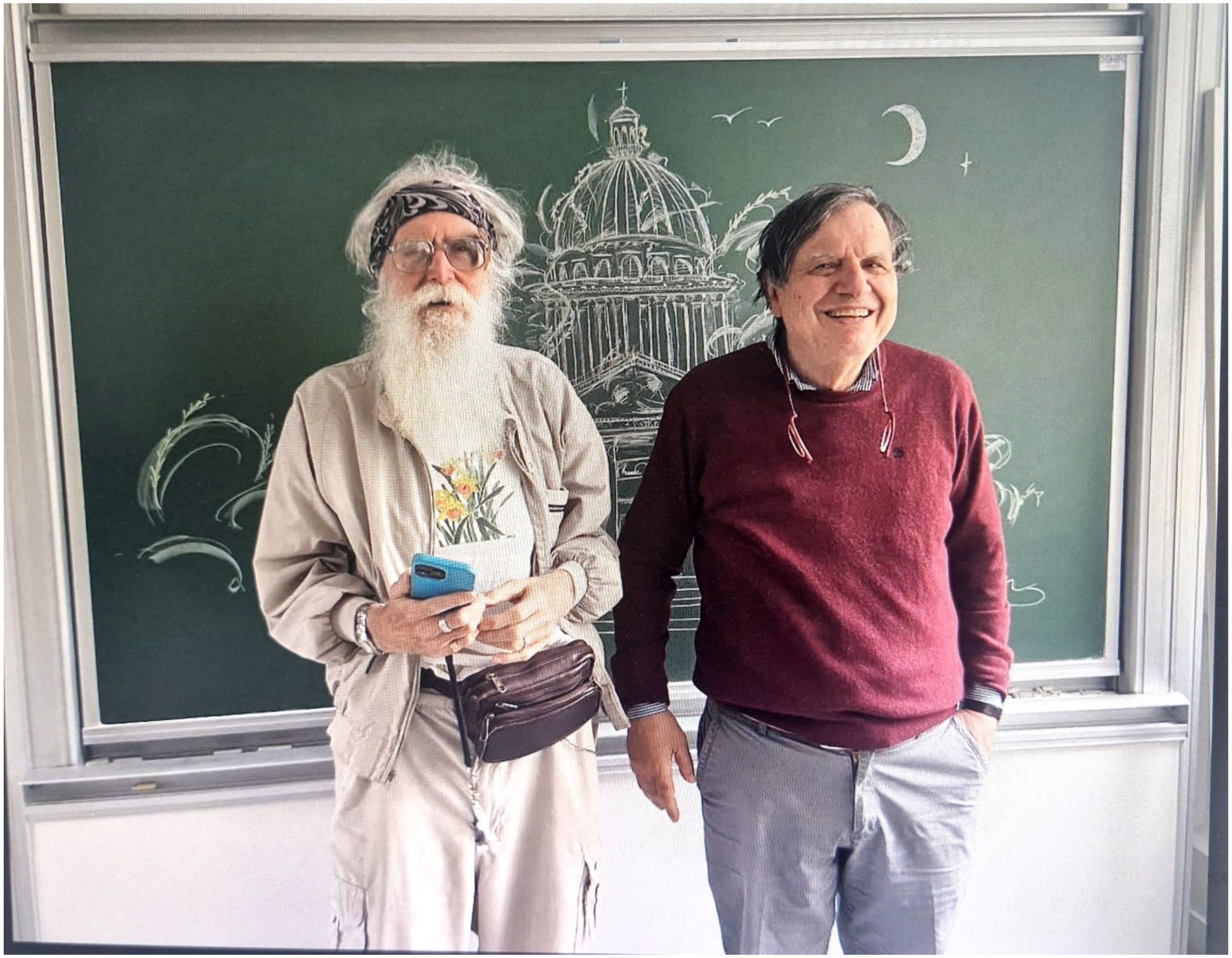
Michel Talagrand - Prix Abel 2024

$$\boxed{f \geq f_{\mathrm{FRSB}} \geq f \qquad \Rightarrow \qquad f = f_{\mathrm{FRSB}}}$$

Guerra 2003 Talagrand 2006

# A recent picture in Paris

**Talagrand & Parisi**



# Free-energy landscapes

## Thouless, Anderson & Palmer - working at fixed disorder

PHILOSOPHICAL MAGAZINE, 1977, VOL. 35, NO. 3, 593–601

### Solution of ‘Solvable model of a spin glass’

By D. J. THOULESS
Department of Mathematical Physics, University of Birmingham, Birmingham, England

and P. W. ANDERSON†‡ and R. G. PALMER
Department of Physics, Princeton University, Princeton, New Jersey 08540, U.S.A.‡

[Received 12 October 1976]

ABSTRACT

The Sherrington–Kirkpatrick model of a spin glass is solved by a mean field technique which is probably exact in the limit of infinite range interactions. At and above $T_c$ the solution is identical to that obtained by Sherrington and Kirkpatrick (1975) using the $n \to 0$ replica method, but below $T_c$ the new result exhibits several differences and remains physical down to $T = 0$.

$$\langle s_i \rangle = m_i = \tanh\{\beta \sum_{j(\neq i)} [J_{ij} m_j - \beta J_{ij}^2 (1 - m_j^2) m_i]\}$$

# TAP equations

## Orders of magnitude

The Thouless-Anderson-Palmer (TAP) equations read

$$m_i = \tanh\left\{ \sum_{j(\neq i)} [\beta J_{ij} m_j - \beta^2 m_i J_{ij}^2 (1 - m_j^2)] \right\}$$

Recall that $m_i = \langle s_i \rangle$

The first term in the rhs $\sum_{j(\neq i)} J_{ij} m_j \simeq \frac{1}{\sqrt{N}} \sqrt{N} = O(1)$ because of the central limit theorem.

The second term $\sum_{j(\neq i)} J_{ij}^2 (1 - m_j^2) \simeq \frac{1}{N} N = O(1)$ because all terms in the sum are positive definite ($m_j \leq 1 \; \forall j$)

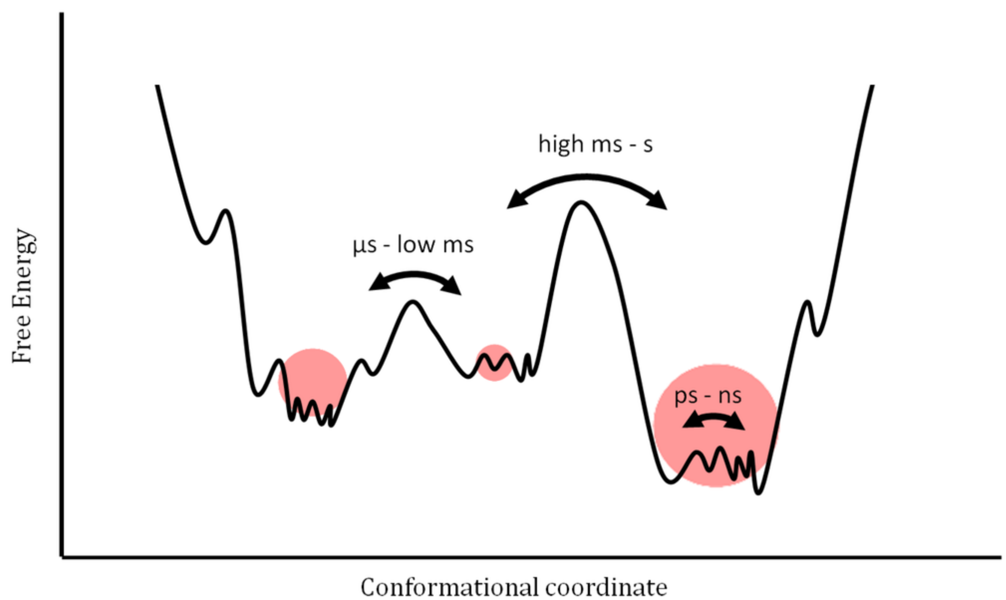
**Onsager reaction term** eliminates back reaction

# Free-energy landscapes

## Free-energy density at fixed randomness

The TAP equations are the extremization conditions on the *TAP free-energy*

$$F_J^{\text{tap}}(\{m_i\}) = -\frac{1}{2}\sum_{i\neq j} J_{ij} m_i m_j - \frac{\beta}{4}\sum_{i\neq j} J_{ij}^2 (1-m_i^2)(1-m_j^2) + T\sum_{i=1}^{N}\left[\frac{1+m_i}{2}\ln\frac{1+m_i}{2} + \frac{1-m_i}{2}\ln\frac{1-m_i}{2}\right]$$




**At low temperatures** $\{m_i\}$

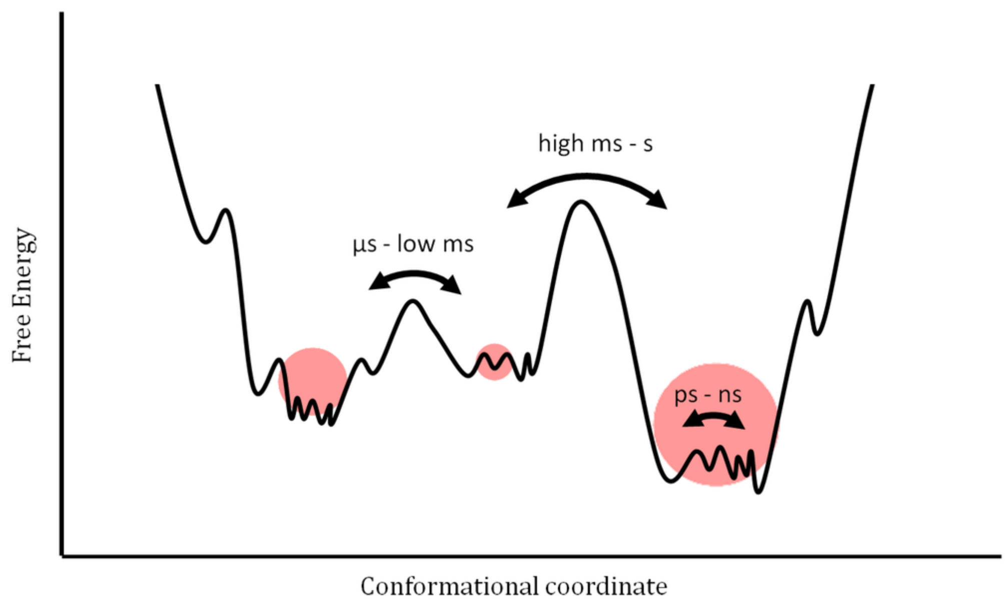
# Free-energy landscapes

## Free-energy density at fixed randomness

The TAP equations are the extremization conditions on the *TAP free-energy*

$$\frac{\delta F_J^{\rm tap}(\{m_i\})}{\delta m_j} = 0$$

The stability of the solutions is determined by the Hessian $\frac{\delta^2 F_J^{\rm tap}(\{m_i\})}{\delta m_j \delta m_k}$




**At low temperatures** $\{m_i\}$

# Free-energy landscapes

## Free-energy density at fixed randomness

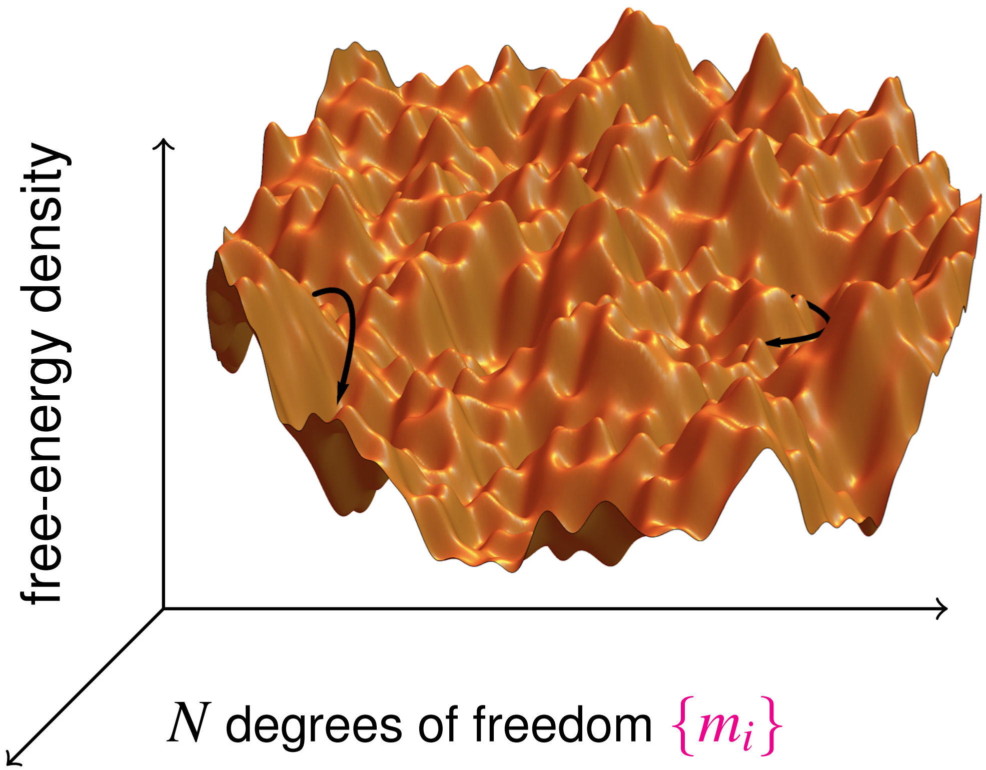



Figure adapted from a picture by **C. Cammarota**

For each $\{J_{ij}\}$ realization the landscape will be different

but the **statistical properties** will be the same

**Bray & Moore**, **Cavagna, Giardina & Parisi**, **Fyodorov & Ros**, **Kent-Dobias**, etc.

# Free-energy landscapes

## Free-energy density at fixed randomness

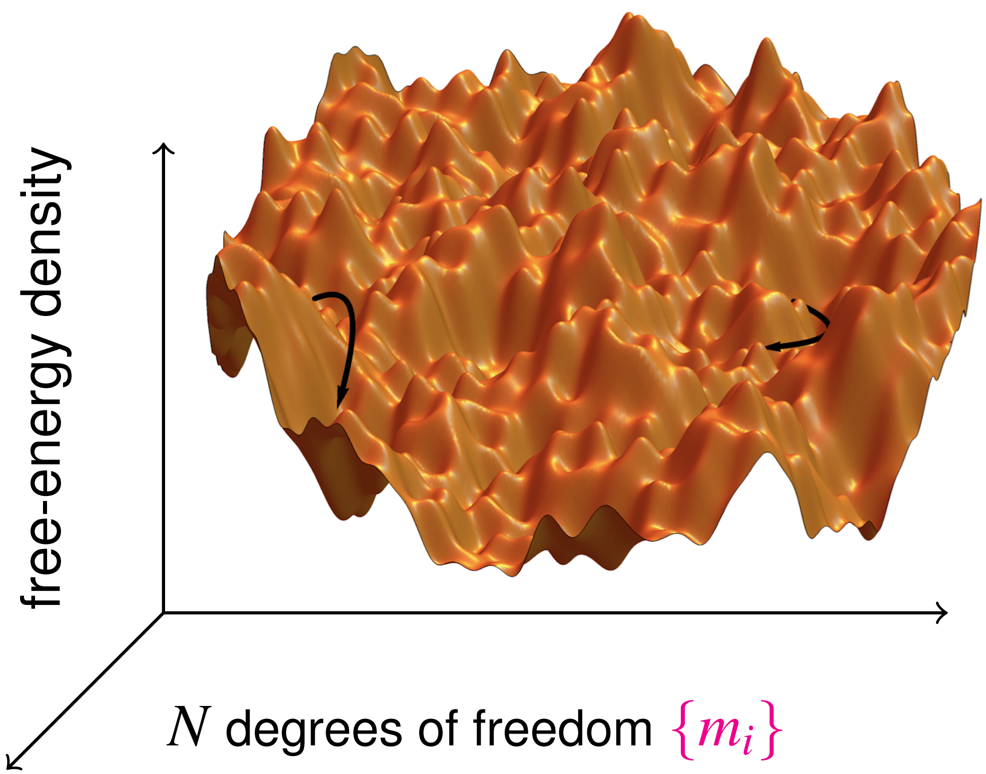



Figure adapted from a picture by **C. Cammarota**

**Topography of the landscape on the $N$-dimensional substrate made by the $N$ order parameters ?**

# Mean-field classes

## According to equilibrium and metastability

$$\mathcal{H} = - \sum_{i_1 \neq \ldots \neq i_p} J_{i_1 \ldots i_p} s_{i_1} \ldots s_{i_p}$$

$p$ integer

quenched random couplings $J_{i_1 \ldots i_p}$ drawn from a Gaussian $P[\{J_{i_1 \ldots i_p}\}]$

with $[J_{i_1 \ldots i_p}] = 0$ and $[J^2_{i_1 \ldots i_p}] = J^2/N^{p-1}$

Choice of spin variables

Ising $s_i = \pm 1$

Globally spherical $\sum_{i=1}^{N} s_i^2 = N$ and $\mathcal{H} \to \mathcal{H} + \frac{z}{2}(\sum_{i=1}^{N} s_i^2 - N)$

$p = 2$ Ising Sherrington-Kirkpatrick **spin-glass** Model **FRSB**

$p = 2$ sherical : ferromagnetic **coarsening** **Berlin-Kac 52** **RS**

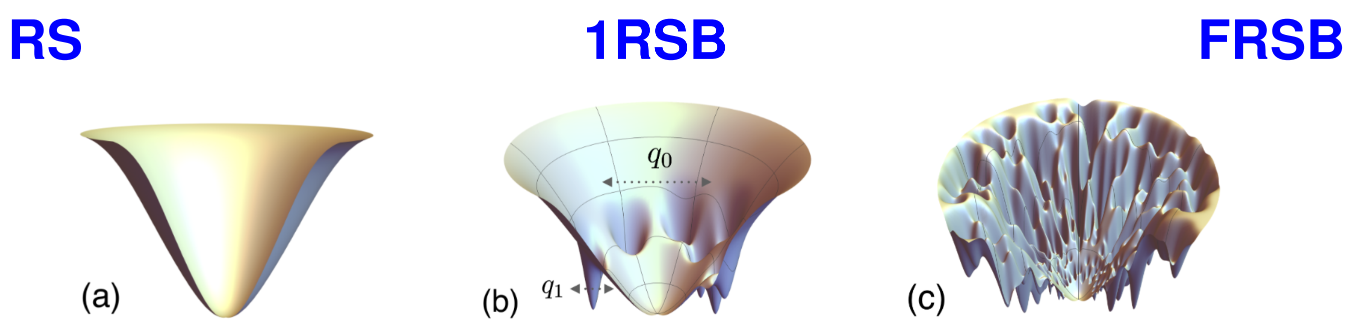
$p \geq 3$ Ising or spherical RFOT modelling of **glasses** **1RSB**

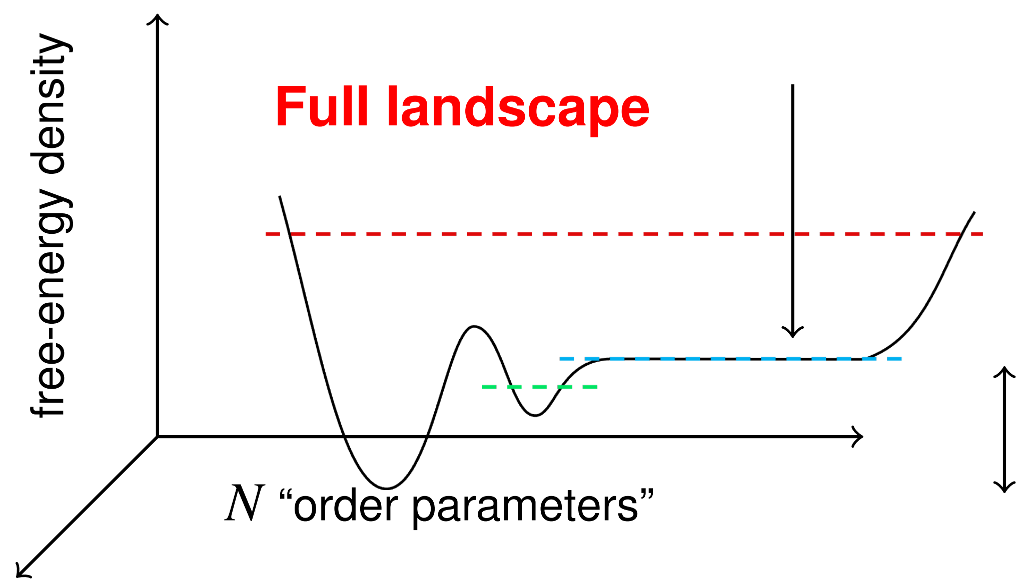
**Kirkpatrick, Thirumalai & Wolynes 87-89**

# Mean-field classes

## According to Replica Theory and TAP analysis

### Equilibrium states



# Finite dimensions

## Scaling Theory

**Two ground states** related by spin reversal $\Longrightarrow$ no replica symmetry breaking.

Compact clusters (droplets) in excitation above the ground state.

The surface of the droplets is fractal, with dimension $d_f > d - 1$.

A droplet of size $L$ costs free-energy proportional to $L^\theta$,

where $\theta$ is the stiffness exponent.

Unstable in any magnetic field (no AT-line in finite dimensions).

*The spin glass is like a magnet with a rough energy landscape but a single ordered phase – excitations cost more and more energy as they grow, unlike the RSB picture where infinitely many competing states exist at no cost.*

The alternative view of **Bray & Moore, Fisher & Huse** late 80s

**Issue not settled**

# More history

## Texts & Interviews

“For posterity and future study, we are conducting a series of interviews with the original scientific participants and deposit them at the Centre d’Archives de Philosophie, d’Histoire et d’Édition des Sciences (CAPHÉS) of École normale supérieure de Paris (ENS)

`https://caphes.ens.fr/history-of-replica-symmetry-breaking-in-physics/`

The interviews are also assembled on a dedicated site

`https://historyrsb.nakala.fr/en`

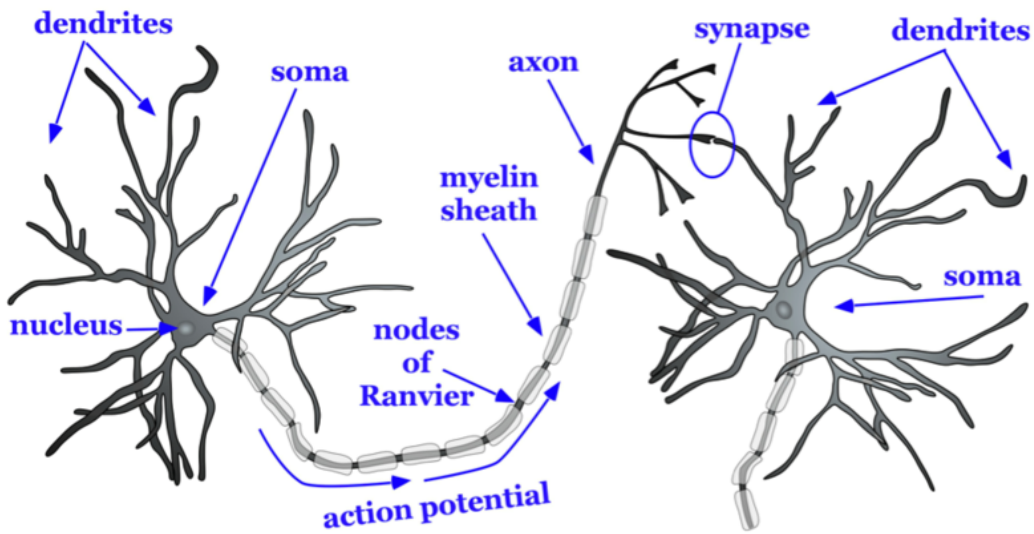
Interestingly, two recent Nobel prize winners fall within the scope of this project : Giorgio Parisi (Physics, 2021) and John Hopfield (Physics, 2024).”

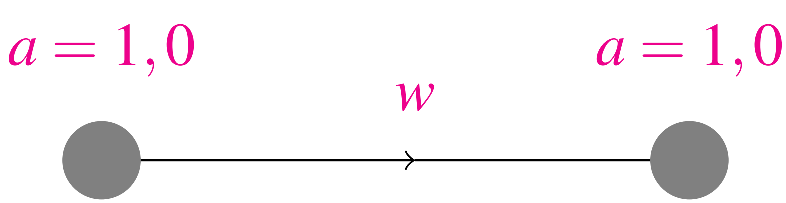
# Theory

## Beyond : neural networks

# Neural networks

## From biology to physics : simplest possible modelling



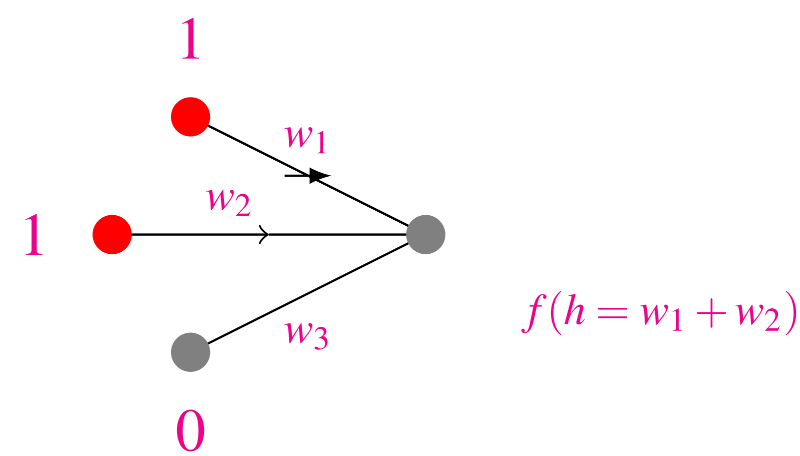




**Ramón y Cajal, ca. 1890 Nobel 1906 W. S. McCulloch & W. Pitts 43**

(father of neuroscience working in Madrid, neurophysiologist, autodidact math & logic at Chicago)

# Neural networks

## Random graphs with weighted connections




At each instant, each neuron calculates the sum of the entries sent but its neighbors

weighted by the strength of the synapses, $h_i = \sum_j w_{ij} a_j$, and it applies a function, $f$.

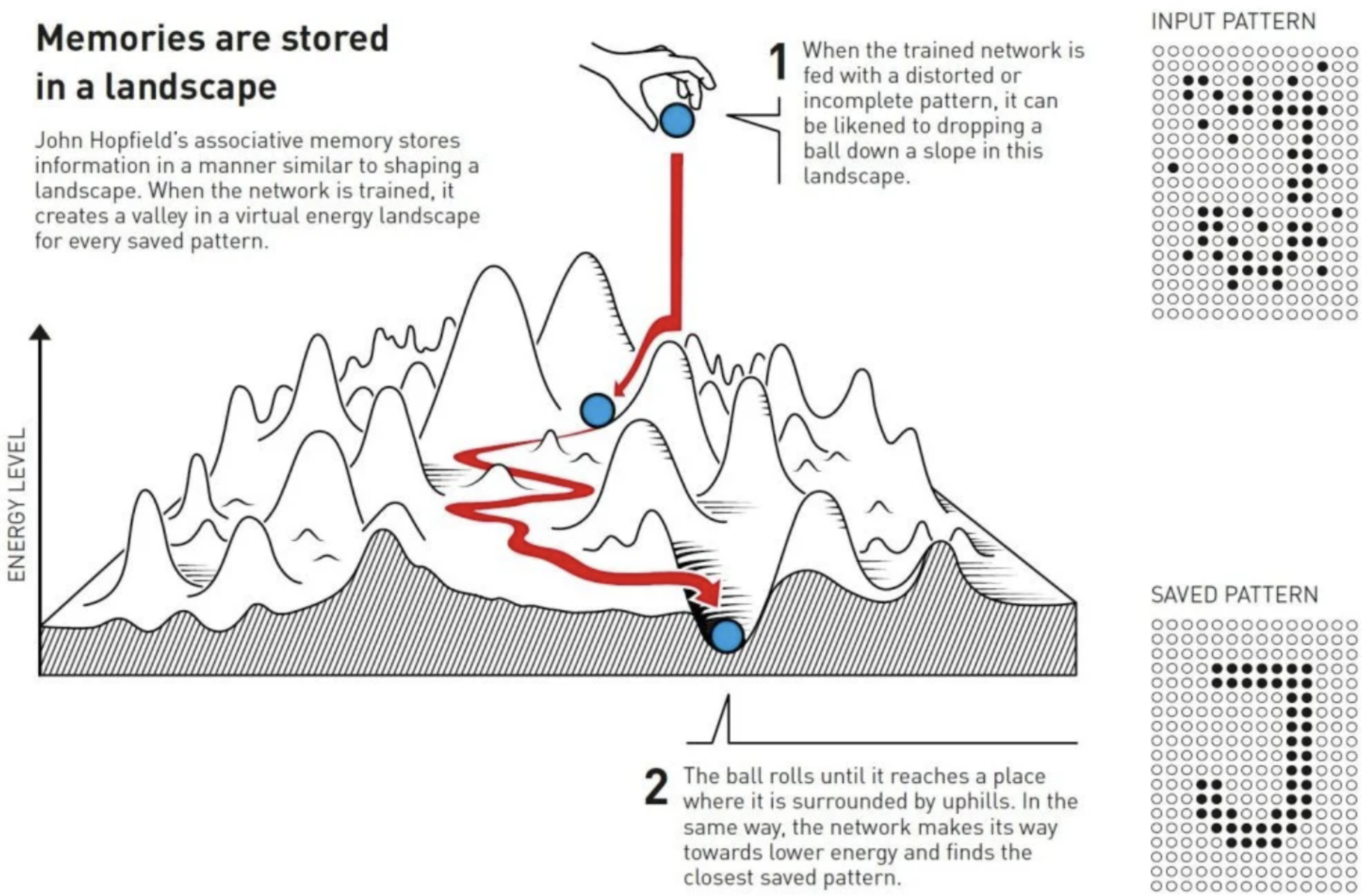
if the result overcomes a threshold, $f(h_i) > \theta$, the receptor neuron spikes, $a_i = 1$.

So on and so forth on the whole network

Details to make precise, e.g. parallel or random sequential dynamics

# Hopfield Model

**Associative memories : one chooses the synaptic weights $w_{ij}$**




**Hopfield 82 Nobel 2024** Image `www.nobelprize.org`

# Hopfield Model

## Properties

- The **synapses** $\{w_{ij}\}$ shape the landscape & encode the memories **Hebb 49**
- **Basins of attraction** : input objects must sufficiently resemble the network's learned templates for successful recognition.
- The network has a limiting **capacity** : beyond a threshold

$$\left(\frac{\text{number of learnt objets}}{\text{number of neurons}}\right)_c$$

it fails **Phase Transition : order (recall possible) - disorder (failure)**

- The network builds **illegitimate valleys** which do not correspond to learnt objects

**Amit, Gutfreund & Sompolinsky 85** maximal capacity of Hopfiled-like networks

**Elizabeth Gardner 88** Space of interactions in neural network models : all possible connection weights that could successfully store a given set of memories

with replica methods as developed by Parisi **Nobel 2021**

# More history

## Personal views

A field with early contributions from many Argentine physicists, in part thanks to **Miguel Virasoro's** (Roma Sapienza, ICTP & Buenos Aires) influence.

**Néstor Parga** (Bariloche $\mapsto$ Madrid)

**Roberto Perazzo** (Buenos Aires)

**Edgardo Ferrán** (Buenos Aires $\mapsto$ bioinformatics at Sanofi & others, France)

**Walter Theumann** (Porto Alegre)

**Tato Moscato** (La Plata $\mapsto$ Newcastle, Australia)

**After the early 90s : Neural networks & AI winter**

Some people stayed **Geoffrey Hinton** **Nobel 2024**, **Yann LeCun**, etc.

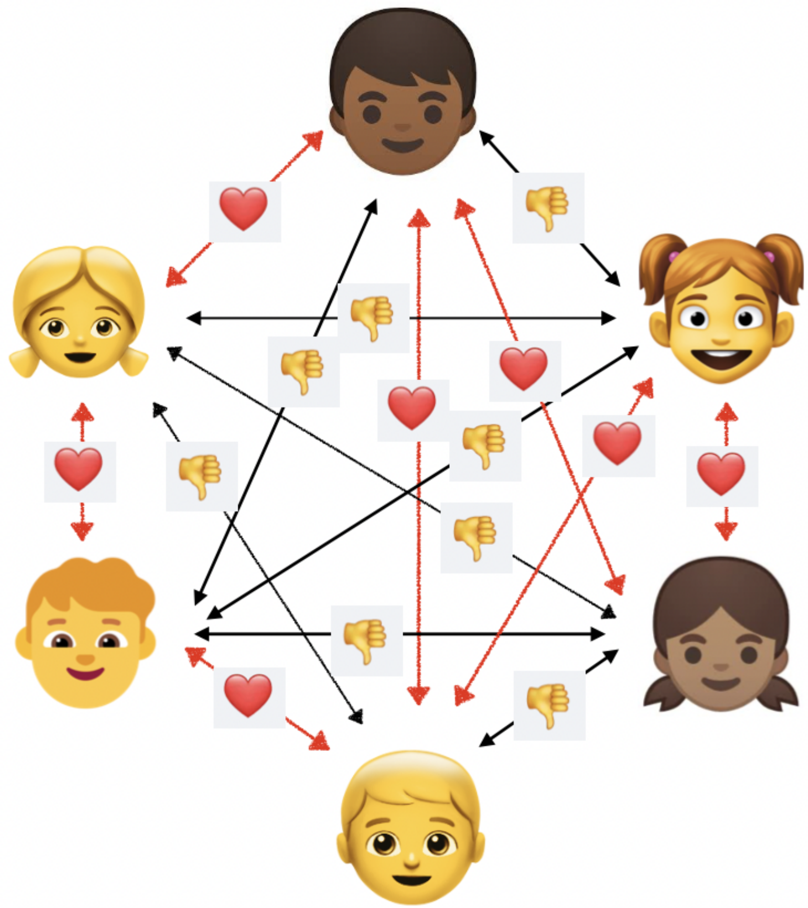
Huge amount of very important work by **Sara Solla** (Chicago, USA)

# Theory

## Beyond : random optimizoptimizationation

# Random optimization

**How do we split the group in two equal sized parts**



**and minimize frustration ?**

# Random optimization

## Mapping to a statistical physics model

In the **graph partitioning** - **group splitting** example

- $i, j = 1, \ldots, N$ label the persons.
- Predetermined $J_{ij} = -1$ for ❤️ love or $J_{ij} = 1$ for 👎 hate feelings and $J_{ij} = 0$ for strangers
- $s_i = 1$ if $i$ is in group $A$ or $s_i = -1$ if $i$ is in group $B$

**find the assignment** of all the $s_i$ so that they **add up to zero** ($\sum_{i=1}^{N} s_i = 0$) & the **cost function**

$\mathcal{H} =$ sum over all pairs of the love/hate values in the same group

is **minimized**

# Random optimization

## Mapping to a statistical physics model

In the **graph partitioning** - **group splitting** example

- $i, j = 1, \ldots, N$ label the persons.
- Predetermined $J_{ij} = -1$ for ❤️ love or $J_{ij} = 1$ for 👎 hate feelings
- $s_i = 1$ if $i$ is in group $A$ or $s_i = -1$ if $i$ is in group $B$

**find the assignment** of all the $s_i$ so that they **add up to zero** ($\sum_{i=1}^{N} s_i = 0$) & the

**Cost function should be minimised** like finding a ground state

$$\mathcal{H} = \underbrace{\sum_{i \neq j}}_{\text{sum over all pairs}} \underbrace{J_{ij}}_{\substack{\text{quenched} \\ \text{love/hate}}} \underbrace{\left( \frac{1 + s_i s_j}{2} \right)}_{\substack{\text{selects pairs in same group} \\ \text{vanishes if } i,\, j \text{ in different groups}}}$$

# Random optimization

## Mapping to a statistical physics model

Some successes of the statistical physics approach

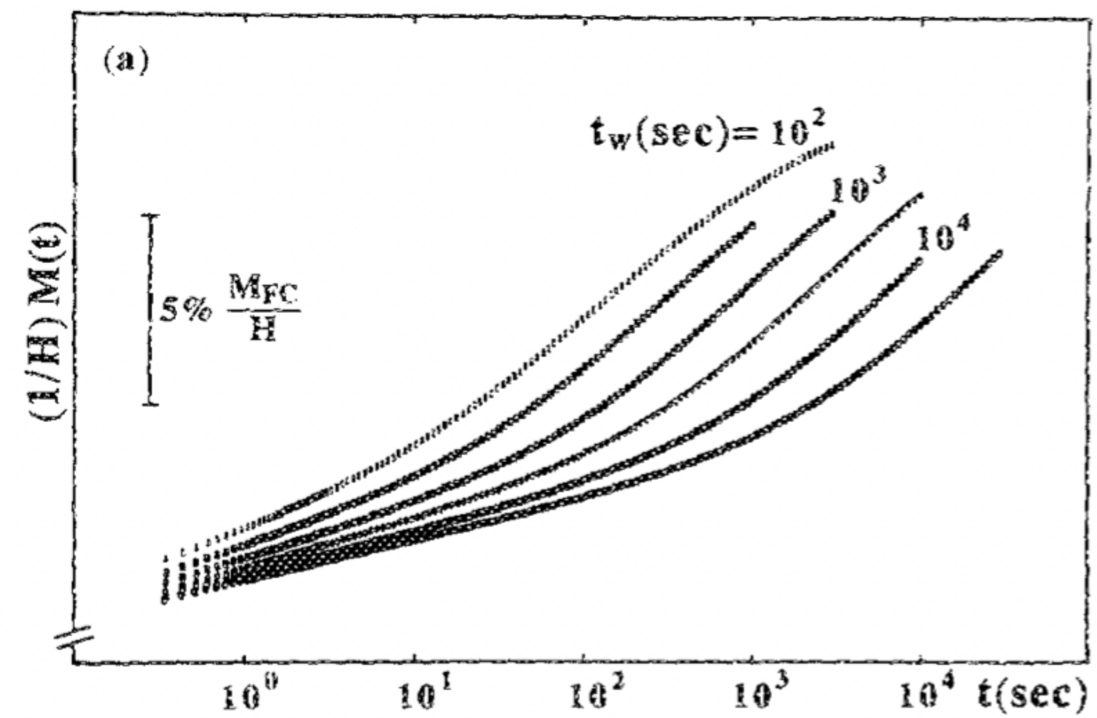
- **S. Kirkpatrick, C. D. Gelatt Jr and M. P. Vecchi**, “Optimization by **Simulated Annealing**". Science 220, 671 (1983). **Dynamics**
- **R. Monasson, R. Zecchina, S. Kirkpatrick, B. Selman and L. Troyansky**, “Determining Computational Complexity from Characteristic **Phase Transitions**” Nature 400, 133 (1999)
- **J. S. Yedidia, W. T. Freeman and Y. Weiss**, “Understanding **Belief Propagation** and its Generalizations” Exploring artificial intelligence in the new millennium 8 (236-239), 0018-9448 (2003) **TAP**
- **M. Mézard, G. Parisi and R. Zecchina**, Science 297, 812 (2002) “Analytic and Algorithmic Solution of Random Satisfiability Problems” **TAP++**
  **Onsager Prize, APS, 2016**

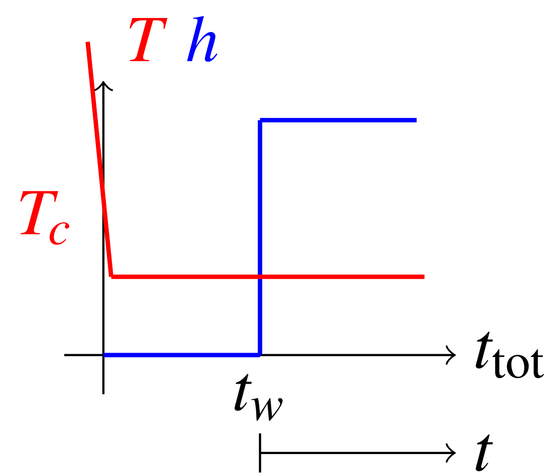
# Back to SG experiments

## Dynamics

# Quenches

**Zero field cooled magnetization - 80s**



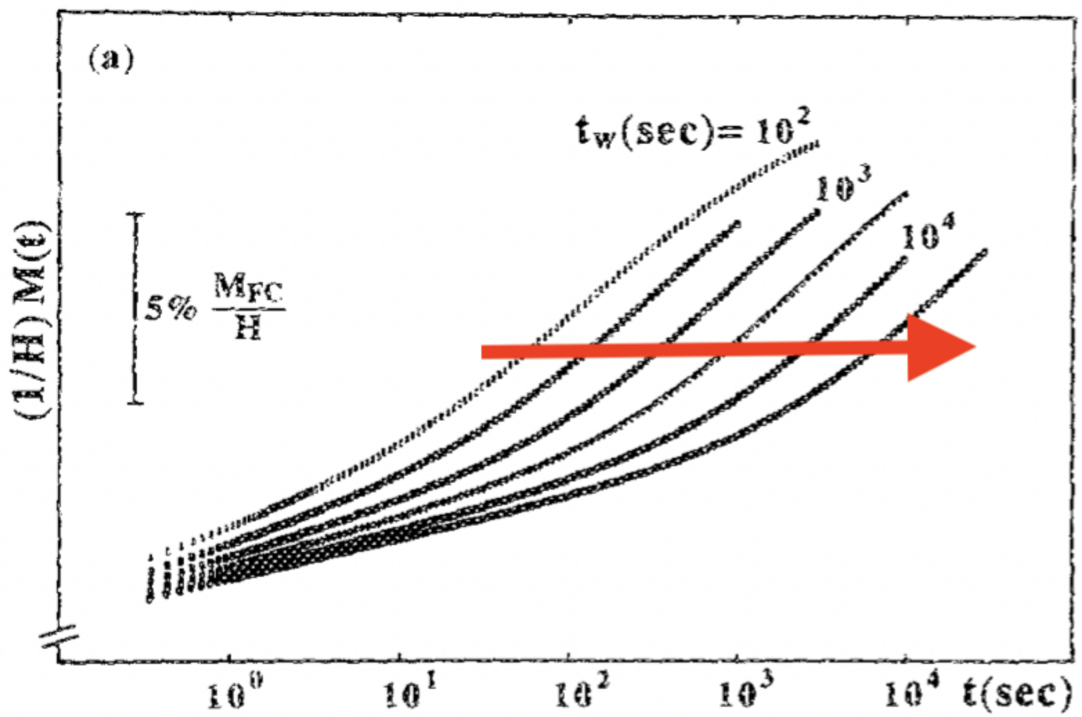

$t_w$ are the waiting times, before the magnetic field is applied.

**Sandlund, Svedlindh, Granberg, Nordblad & Lundgren 88**

# Aging

**Breakdown of stationarity : out of equilbrium**

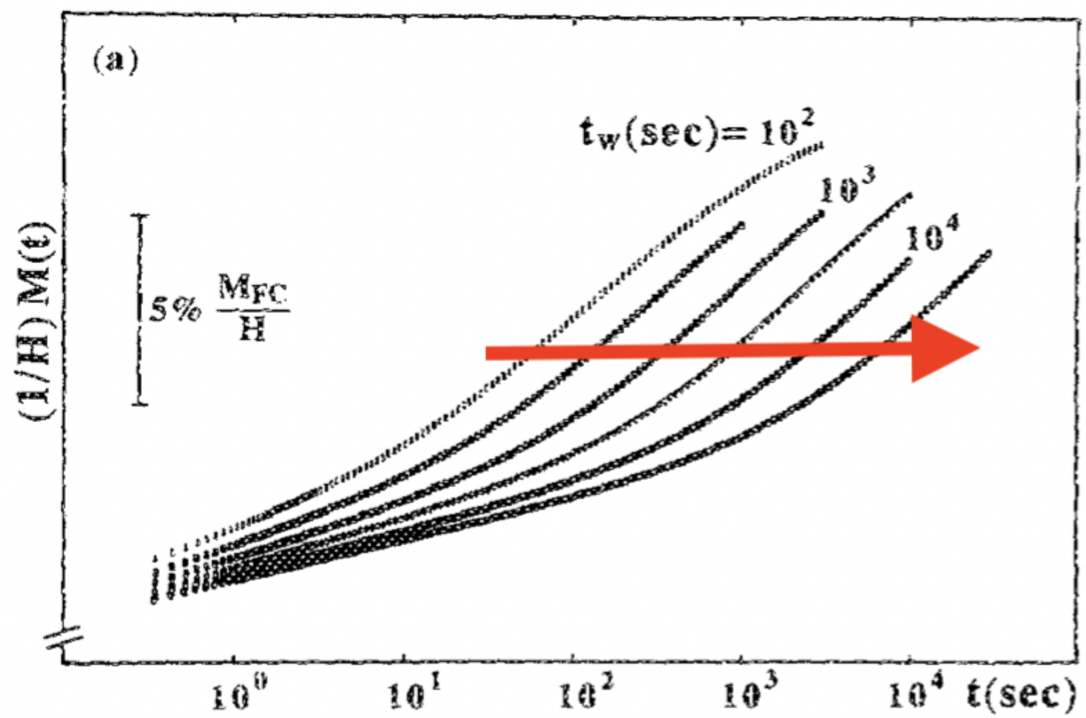
$$m(t, t_w) \neq m(t - t_w)$$




Older systems (longer $t_w$) relax more slowly than younger ones

# Aging

**Breakdown of stationarity : out of equilbrium**



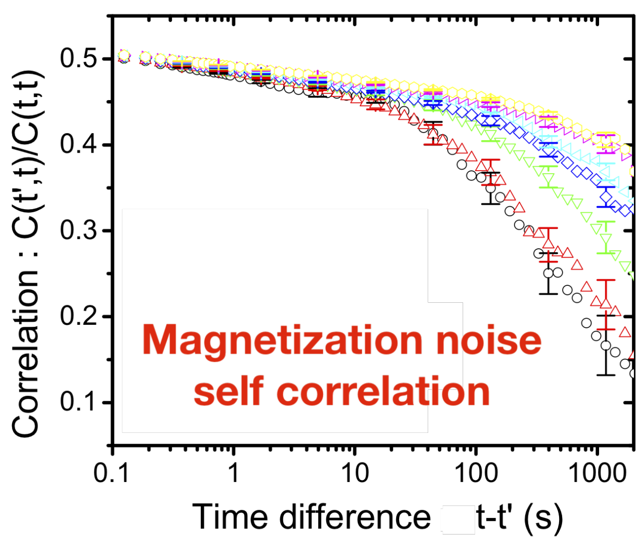

Older systems (longer $t_w$) relax more slowly than younger ones

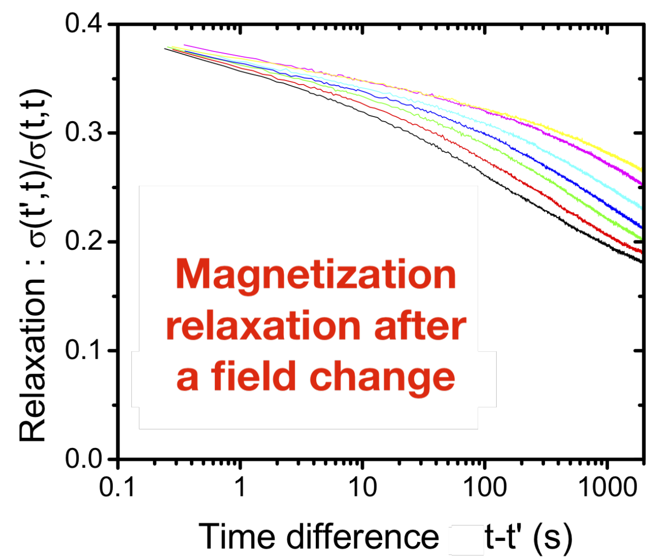
Similar to previous measurements in all kinds of **polymer glasses**

**Kovacs, Struik, etc. 60s**

# Induced & spontaneous

**slow relaxation & loss of stationarity (aging)**




**Correlation** **Linear response**

$0$ $t'$ $t'$ $t'$ $t'$ time

Different curves are measured after log-spaced reference times $t'$ after the quench : **breakdown of stationarity** $\Longrightarrow$ far from equilibrium

No identifiable growing length $\mathcal{R}(t)$ : **glassy microscopic mechanisms ?**

**Spin-glass experiments** **Hérisson & Ocio 02-04**

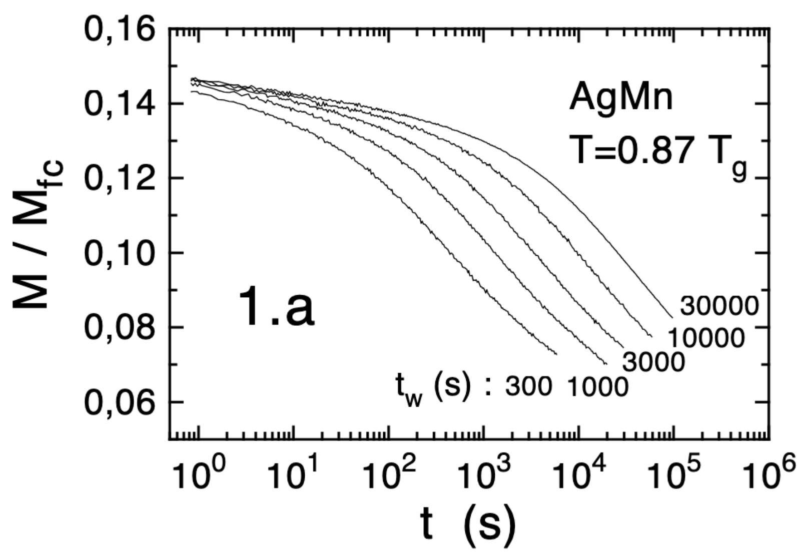
# Induced & spontaneous

## thermo-remanent magnetization decay & self correlation

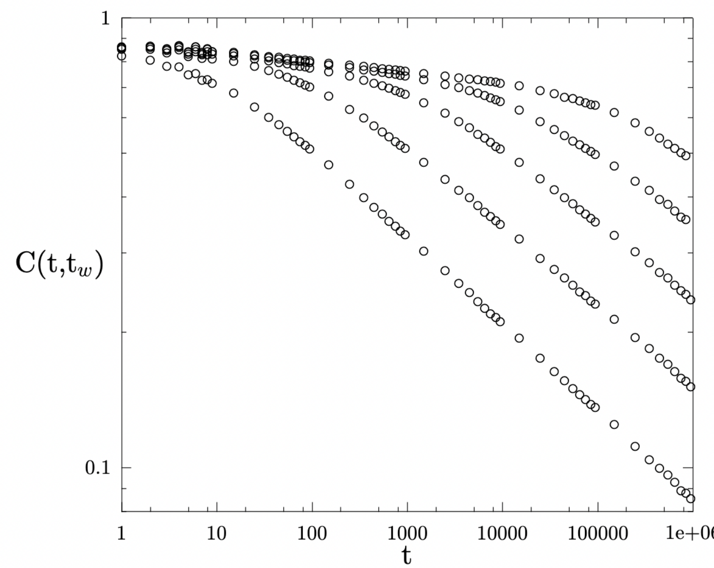
$m(t_{tot}, t_w)$

$C(t_{tot}, t_w) = N^{-1} \sum_i s_i(t_{tot}) s_i(t_w)$

**Experiments**




**Monte Carlo 3D EA**




In these plots, $t_w$ are the waiting times when the magnetic field is switched off or the reference spin configuration is stored ; $t$ is the time-difference

**Hammann, Ocio & Vincent 1990**

**Rieger 1993**

The equilibration time of **macroscopic** **spin-glasses** below $T_c$

goes beyond the experimentally accessible time window $t_{\rm exp}$

$$\lim_{N \gg 1} t_{\rm eq}(N,T) \gg t_{\rm exp} \qquad\qquad T < T_c$$

$t_{\rm eq}$ grows with the system size $N$ at low temperatures

They show **aging** both in magnetization (induced) and in correlation (spontaneous) measurements

Above $T_c$, in the paramagnetic phase, they equilibrate,

$$\lim_{N \gg 1} t_{\rm eq}(N,T) < t_{\rm exp} \qquad\qquad T > T_c$$

The equilibration time of **macroscopic coarsening & glassy systems** in a wide range of parameters,

e.g. low temperatures,

goes beyond the experimental window $t_{\rm exp}$

$$\boxed{\lim_{N\gg 1} t_{\rm eq}(N, \text{parameters}) \gg t_{\rm exp}}$$

$t_{\rm eq}$ grows with the system size $N$ at low temperatures

and they also show **aging** though with some differences, to be discussed

# Theory

## Explanations

# Kinetic Ising Model

## Archetypical example for classical magnetic systems

$$H = -J \sum_{\langle ij \rangle} s_i s_j$$

$s_i = \pm 1$ Ising spins.

$\langle ij \rangle$ sum over nearest-neighbours on the lattice.

$J > 0$ ferromagnetic coupling constant.

critical temperature $T_c > 0$ for $d > 1$.

Evolution, coupling to bath

Monte Carlo rule $s_i \to -s_i$ accepted with

$p = 1$ if $\Delta E < 0$

$p = e^{-\beta \Delta E}$ if $\Delta E > 0$

$p = 1/2$ if $\Delta \mathcal{E} = 0$

Non-conserved order parameter dynamics [ $\uparrow\downarrow$ towards $\uparrow\uparrow$ ] etc. allowed.

[$m = 0$ to $m = 2$]

# Scalar Field Theory

## Summary : discrete *vs.* continuous

| Ising spin models | Scalar field theories |
|---|---|
| $H = -J \sum_{\langle ij \rangle} s_i s_j$ | $\mathcal{F}[\phi] = \int d^d r \left[ \frac{1}{2} (\nabla \phi)^2 - \frac{\mu}{2} \phi^2 + \frac{g}{4} \phi^4 ) \right]$ |
| NCOP [ $\uparrow\downarrow \mapsto \uparrow\uparrow$ ] | $\partial_t \phi(r,t) = -\delta_{\phi(r,t)} \mathcal{F}[\phi] + \xi(r,t)$ |
| COP [ $\uparrow\downarrow \mapsto \downarrow\uparrow$ ] | $\partial_t \phi(r,t) \ = \ -\nabla^2 \delta_{\phi(r,t)} \mathcal{F}[\phi] \ + \ \eta(r,t)$ |

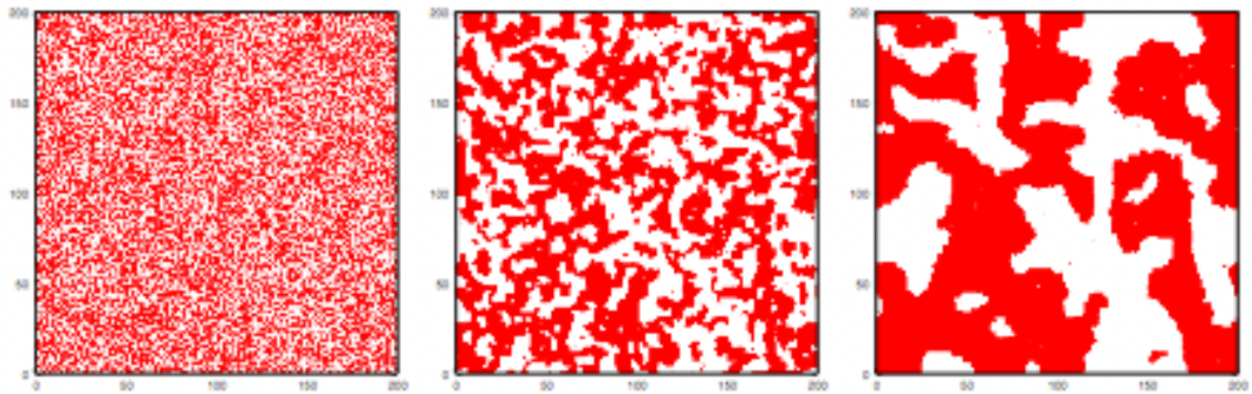
Overdamped limit is fine ; rescaling of time to eliminate $\gamma_0$

In the COP case $\langle \eta(x,t) \eta(y,t') \rangle = 2 k_B T \nabla^2 \delta(x-y) \delta(t-t')$

Generalisations for vector cases. **Quenched disorder** can be introduced by taking the $J_{ij}$ or the parameters in the field theory, *e.g.* $\mu$, from a pdf.

# Aging

## One possible explanation : domain growth



Growth of the two degenerate ground states slowed down by quenched randomness

### Dynamic scaling

Thermal activation arguments for a growing length $\mathcal{R} \sim (\ln t)^{1/\psi}$

**Fisher & Huse, Bray & Moore, late 80s**, for spin-glasses, general review **Bray 94**

# Dynamic Scaling

Growth of ‘red and white’ patches separated by

interfaces which surround such geometric domains

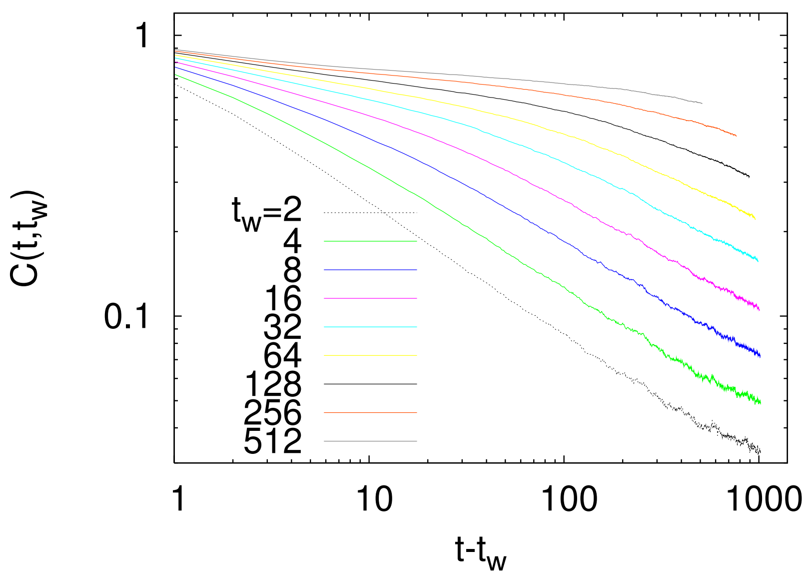
Spatial regions of local equilibrium (with non-vanishing order parameter, at $T < T_c$) grow in time

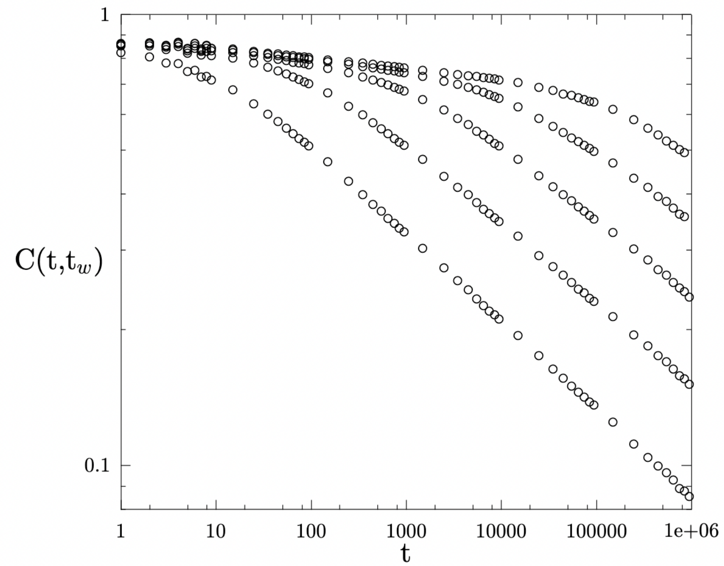
A single growing length $\mathcal{R}(t,T)$ can be identified and will be at the heart of dynamic scaling.

Here and in the following we measure $T$ in units of $J$

# Ferromagnet vs spin-glass

**Not so different as long as correlations are concerned**




**2D FM Ising model** - spin-spin

**Sicilia *et al.* 07**

**3D Edwards Anderson Model**

**Rieger 93**

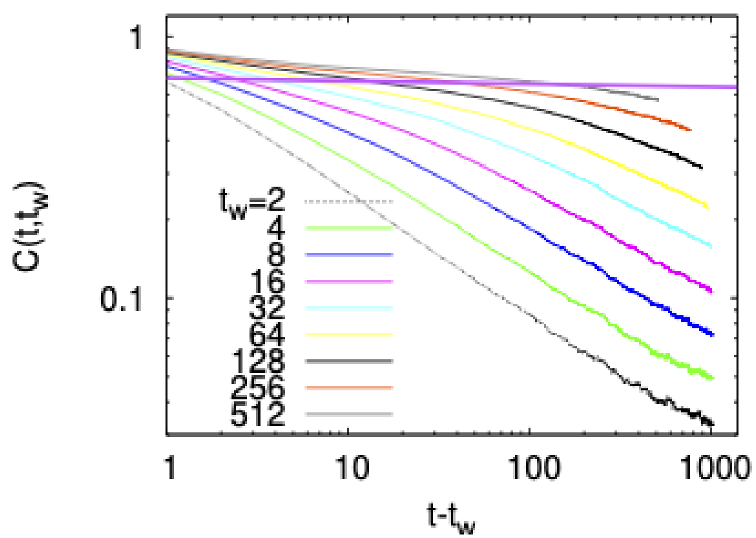
**One correlation can exhibit stationary and non stationary relaxation**

**in different two-time regimes**

# Two-time self-correlation

**e.g., MC simulation of the $2d$IM at $T < T_c$**

$$C(t,t_w) = N^{-1} \sum_{i=1}^{N} \langle s_i(t) s_i(t_w) \rangle$$




Stationary relaxation

$m_{eq}^2$

Aging decay

Separation of time-scales : stationary – aging

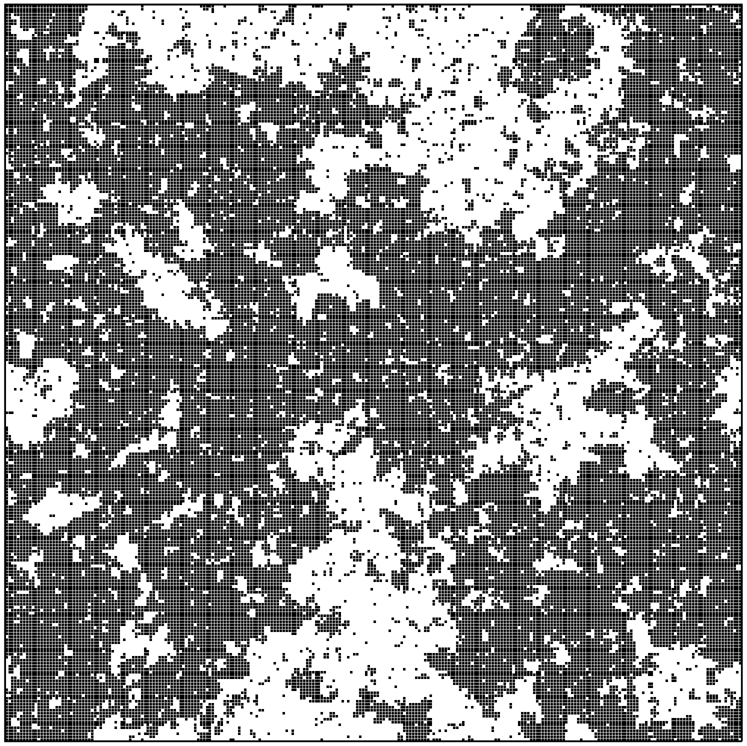
$$C(t,t_w) = C_{st}(t-t_w) + m_{eq}^2 \, f\left(\frac{\mathcal{R}(t,T)}{\mathcal{R}(t_w,T)}\right)$$

$$C_{st}(0) = 1 - m_{eq}^2, \quad \lim_{x\to\infty} C_{st}(x) = 0, \quad f(1) = 1, \quad \lim_{x\to\infty} f(x) = 0$$

# Interpretation

**Thermal fluctuations within domains & domain-wall motion**

$C_{\mathrm{st}}(t-t_w)$ $\quad\quad$ $m_{\mathrm{eq}}^2\, f(\mathcal{R}(t,T)/\mathcal{R}(t_w,T))$

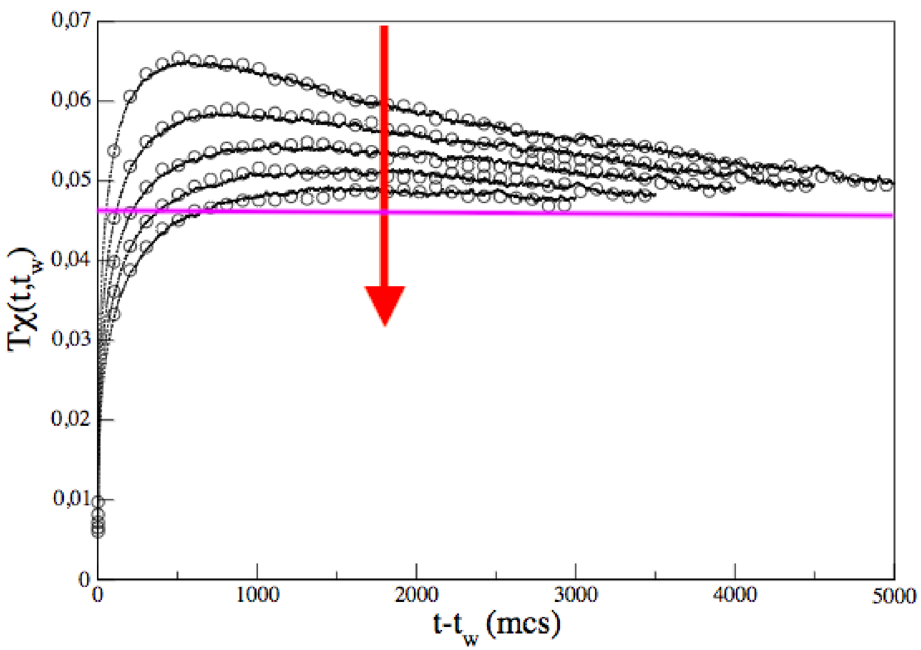


# Thermo-remanent magnetisation

**Critical and sub-critical coarsening**

$$m(t,t_w) = m_{eq}(t-t_w) + [\mathcal{R}(t_w,T)]^{-a_\chi}\, g\left(\frac{\mathcal{R}(t,T)}{\mathcal{R}(t_w,T)}\right)$$

$$m_{eq}(t-t_w) = -(k_B T)^{-1} dC_{eq}(t-t_w)/d(t-t_w)$$




$$1 - m_{eq}^2$$

Lippiello, Corberi & Zannetti 05

The equilibration time of a **coarsening** system scales as

$$\mathcal{R}(t_{\rm eq}) \sim L$$

$$\boxed{\lim_{N\gg 1} t_{\rm eq}(N, \text{parameters}) \gg t_{\rm exp}}$$

$t_{\rm eq}$ grows with the system size $N \propto L^D$

Correlations **age** similarly to the ones in spin-glasses

Integrated responses, e.g. thermo-remanent magnetisation, do not

# Characterisation of the Collective Relaxation

when there is no “visible” length $\mathcal{R}$

## Global Observables

two-time correlations and linear responses

# Two-time dependencies

## Self displacement & correlation – integrated linear response

$$\Delta^2(t,t') \equiv \frac{1}{N}\sum_i [\langle (x_i(t)-x_i(t'))^2\rangle] \qquad \text{Displacement}$$

$$C(t,t') \equiv \frac{1}{N}\sum_i [\langle x_i(t)x_i(t')\rangle] \qquad \text{Correlation}$$

} Unperturbed

$$\chi(t,t') \equiv \frac{1}{N}\sum_i \int_{t'}^{t} dt''\, R_i(t,t'') = \frac{1}{N}\sum_i \int_{t'}^{t} dt'' \left[ \left.\frac{\delta \langle x_i(t)\rangle_h}{\delta h_i(t'')}\right|_{h=0} \right]$$

**Extend the notion of order parameter**

They <u>are not</u> related by FDT out of equilibrium

The averages are thermal (and over initial conditions) $\langle \dots \rangle$

and over quenched randomness $[\dots]$ (if present)

$t'$ "waiting-time" and $t$ "measuring-time" after preparation

# Mean-Field Modelling

## Usual Curie-Weiss for PM-FM
## Unusual for Glasses

# Mean-Field Modelling

## Classical $p$-spin Spherical Models

**Potential energy**

$$\mathcal{V} = - \sum_{i_1 \neq \ldots \neq i_p} J_{i_1 \ldots i_p} s_{i_1} \ldots s_{i_p} \qquad p \text{ integer}$$

quenched random couplings $J_{i_1 \ldots i_p}$ drawn from a Gaussian $P[\{J_{i_1 \ldots i_p}\}]$

(over-damped) **Langevin dynamics** for continuous spins $s_i \in \mathbb{R}$

coupled to a white bath $\langle \xi_i(t) \rangle = 0$ and $\langle \xi_i(t) \xi_j(t') \rangle = 2\gamma k_B T \, \delta_{ij} \, \delta(t - t')$

$$\gamma \frac{ds_i}{dt} = - \frac{\delta \mathcal{V}}{\delta s_i} + z_t s_i + \xi_i$$

$z_t$ is a Lagrange multiplier that fixes the spherical constraint $\sum_{i=1}^{N} s_i^2 = N$

$p = 2$ mean-field **coarsening**

$p \geq 3$ **RFOT** modelling of **glasses** **Kirkpatrick, Thirumalai & Wolynes 87-89**

# Dynamic equations

## Integro-differential eqs. on the correlation and linear response

In the $N \to \infty$ limit exact and closed causal Schwinger-Dyson equations

(Average over randomness, random initial conditions and thermal noise)

$$(\gamma\partial_t - z_t)C(t,t') = \int dt'' \left[\Sigma(t,t'')C(t'',t') + D(t,t'')R(t',t'')\right]$$
$$+2\gamma k_B T R(t',t)$$
$$(\gamma\partial_t - z_t)R(t,t') = \int dt'' \, \Sigma(t,t'')R(t'',t') + \delta(t-t')$$

where $\Sigma$ and $D$ are the self-energy and vertex, which for $p$ spin models read

$$D(t,t') = \frac{p}{2} C^{p-1}(t,t') \qquad \Sigma(t,t') = \frac{p(p-1)}{2} C^{p-2}(t,t') R(t,t')$$

$z_t$ is fixed by $C(t,t) = 1$ **Sompolinsky & Zippelius 82, LFC & Kurchan 93**

Similar to **Mode-Coupling Theory** for liquids **Götze et al 80s** or **DMFT** for quantum systems **Georges & Kotliar 90s**, but not necessarily in equilibrium

## How to solve these equations?

**Input from numerical solutions $\Longrightarrow$**

**Asymptotic Ansatz**

# Weak ergodicity breaking

$$\lim_{t-t'\to\infty} \lim_{t'\to\infty} C(t,t') = q_{\rm EA} \qquad\qquad \lim_{t\gg t'} C(t,t') = 0$$

**Bouchaud 92**

# Weak long-term memory

$$\lim_{t-t'\to\infty} \lim_{t'\to\infty} R(t,t') \simeq 0$$

## but

$$\sigma(t,t') = \int_0^{t'} dt''\, R(t,t'') \longrightarrow f_\sigma(C(t,t')) = \text{finite}$$

**LFC & Kurchan 93**

**allow us to solve the integro-differential eqs. asymptotically**

## Weak ergodicity breaking

$$\lim_{t-t'\to\infty}\ \lim_{t'\to\infty} C(t,t') = q_{\mathrm{EA}} \qquad\qquad \lim_{t\gg t'} C(t,t') = 0$$

**Bouchaud 92**

## Weak long-term memory

$$\lim_{t-t'\to\infty}\ \lim_{t'\to\infty} R(t,t') \simeq 0$$

**but**

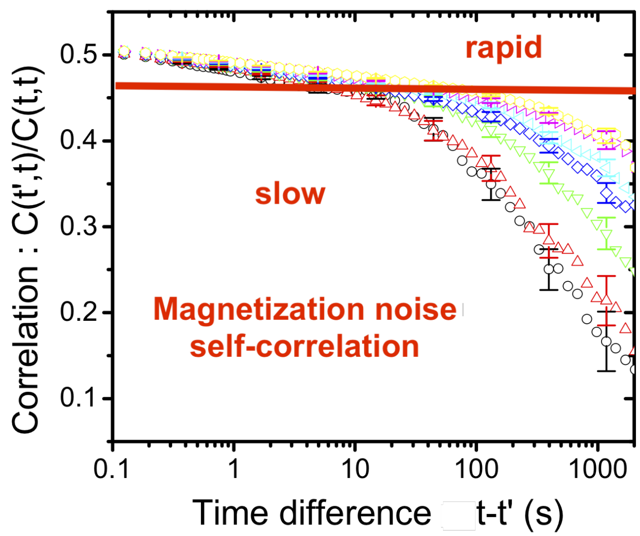
$$\sigma(t,t') = \int_0^{t'} dt''\, R(t,t'') \longrightarrow f_\chi(C(t,t')) = \text{finite}$$

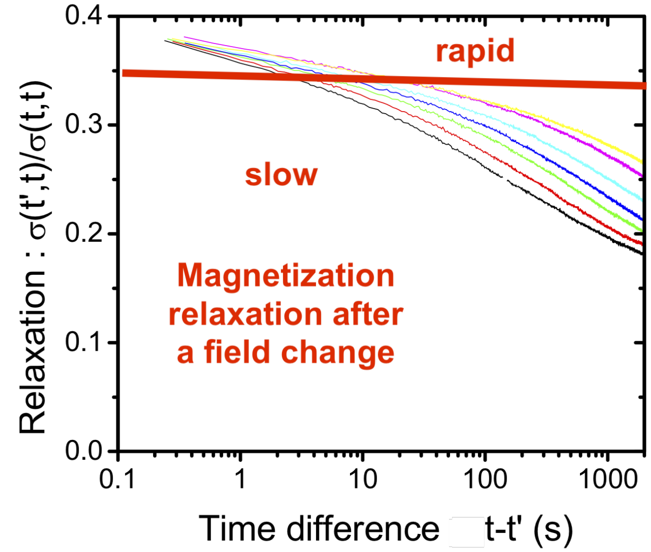
**LFC & Kurchan 93**

**and capture aging**

# Slow Relaxation & Aging

**Separation of time scales & loss of stationarity**




**Correlation** **Integrated linear response**

$0$ $t'$ $t'$ $t'$ $t'$ time

Different curves are measured after log-spaced times $t'$ after the quench

**breakdown of stationarity** $\Longrightarrow$ **aging**, far from equilibrium

**microscopic mechanisms ?**

**Spin-glass experiments** **Hérisson & Ocio 02-04**

# Physical Aging

## in words & scaling

**Older systems** (more time elapsed after the quench, longer $t'$)

relax **more slowly** than younger ones

Breakdown of stationarity of correlation & integrated response

$$C(t,t') \neq C(t-t') \qquad \sigma(t,t') \neq \sigma(t-t')$$

In each regime – $\underbrace{\text{rapid } (r)}_{\text{above plateau}}$ and $\underbrace{\text{slow } (s)}_{\text{below plateau}}$ – there is scaling*

$$C(t,t') = C_{r,s}\left(\frac{\mathcal{R}_{r,s}(t)}{\mathcal{R}_{r,s}(t')}\right) \qquad \sigma(t,t') = \sigma_{r,s}\left(\frac{\mathcal{R}_{r,s}(t)}{\mathcal{R}_{r,s}(t')}\right)$$

* proven from general properties of temporal correlation functions

**LFC & Kurchan 94**

**but no “visual” interpretation of $\mathcal{R}(t)$ in glassy systems**

# Physical Aging & Memory

**the rate of relaxation**

**Older systems** (more time elapsed after the quench, longer $t'$)

relax **more slowly** than younger ones

Time variation - rate of change

$$\partial_t C_{r,s}\left(\frac{\mathcal{R}_{r,s}(t)}{\mathcal{R}_{r,s}(t')}\right) = C'_{r,s}\left(\frac{\mathcal{R}_{r,s}(t)}{\mathcal{R}_{r,s}(t')}\right)\frac{\dot{\mathcal{R}}_{r,s}(t)}{\mathcal{R}_{r,s}(t')} = \overline{C}_{r,s}\left(C_{r,s}(t,t')\right)\frac{\dot{\mathcal{R}}_{r,s}(t)}{\mathcal{R}_{r,s}(t)}$$

difference between **rapid** and **slow**, depending on $\mathcal{R}(t)$

$$\underbrace{\partial_t C_r(t,t') \simeq C_r(t,t')}_{\text{e.g. } \mathcal{R} \sim e^{t/\tau} \text{ exponential}} \qquad \underbrace{\partial_t C_s(t,t') \ll C_s(t,t')}_{\text{e.g. } \mathcal{R} \sim t \text{ algebraic}}$$

# Interpretation

## Complex Landscapes

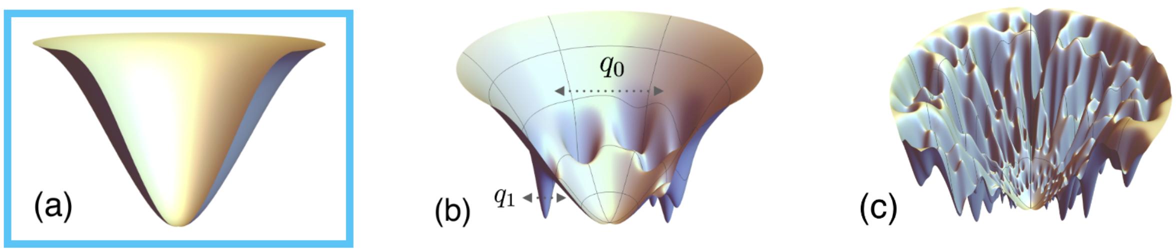
## Beyond Ginzburg-Landau

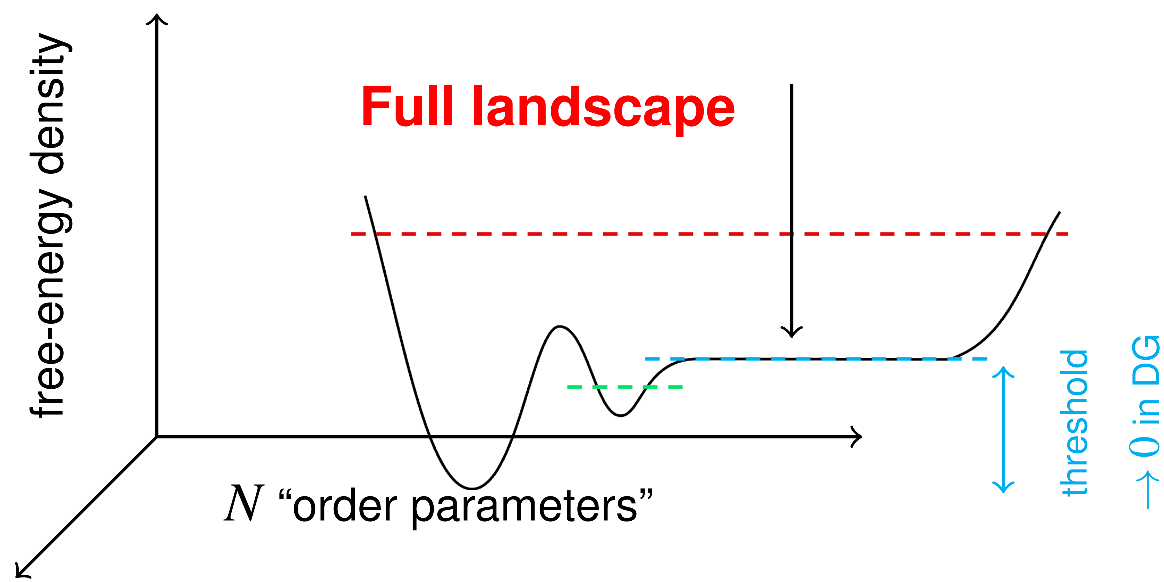
# Mean-field classes

## Domain growth

## Equilibrium states

**RS** **1RSB** **FRSB**



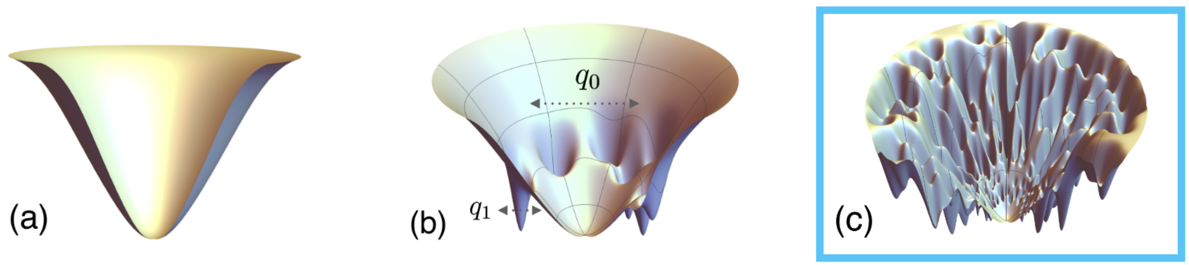



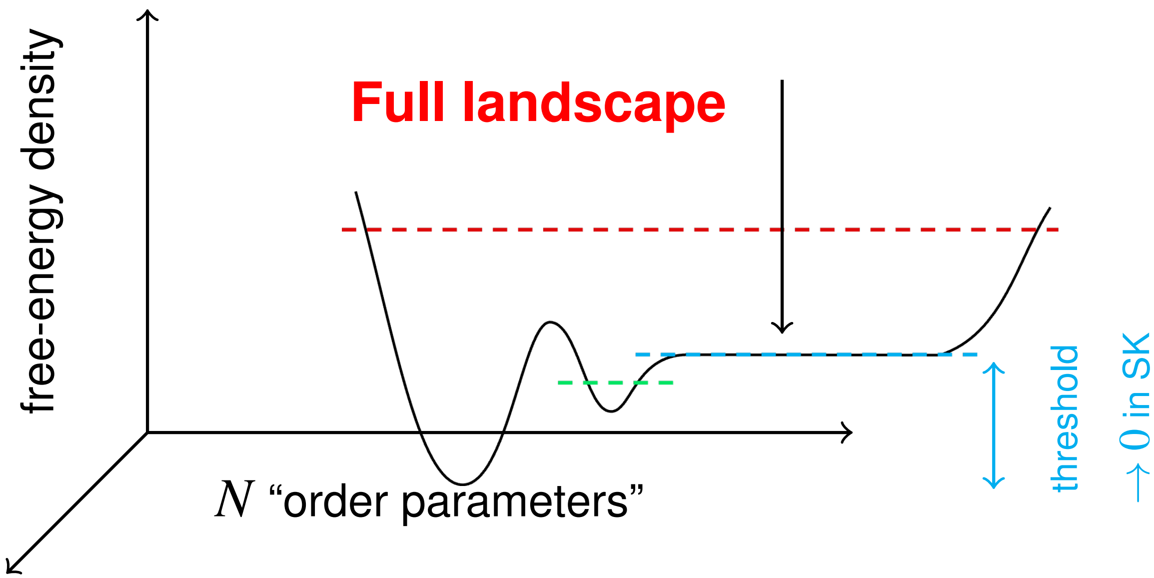
# Mean-field classes

## Sherrington-Kirkpatrick

### Equilibrium states

**RS** **1RSB** **FRSB**



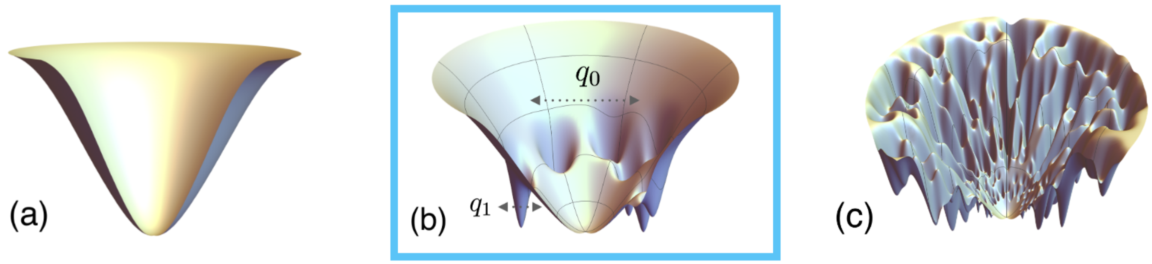



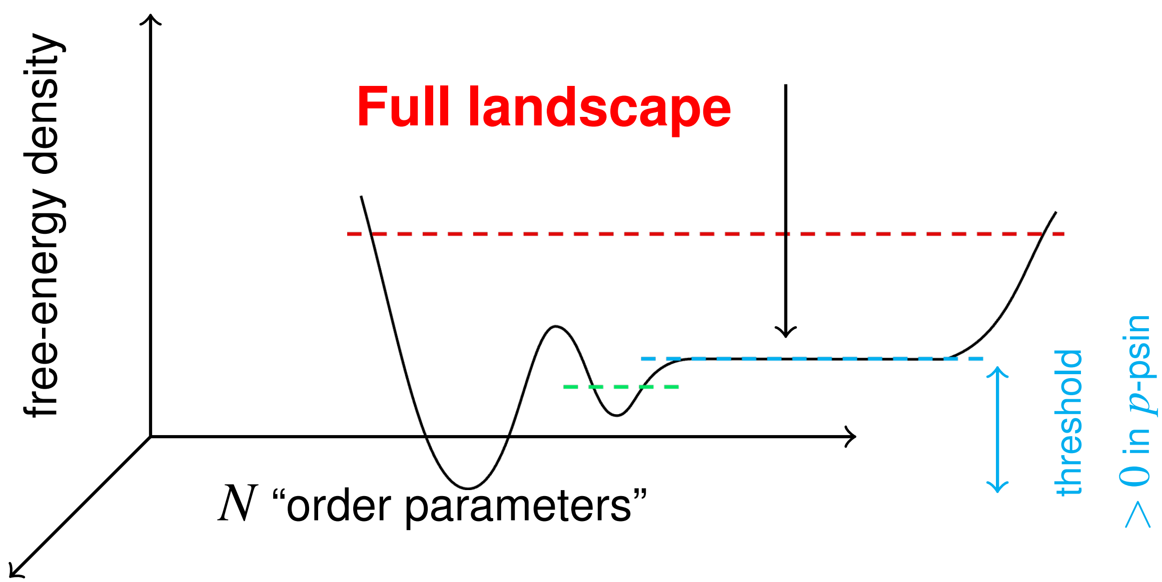
# Mean-field classes

$p \geq 3$ **spherical and Ising models**

## Equilibrium states

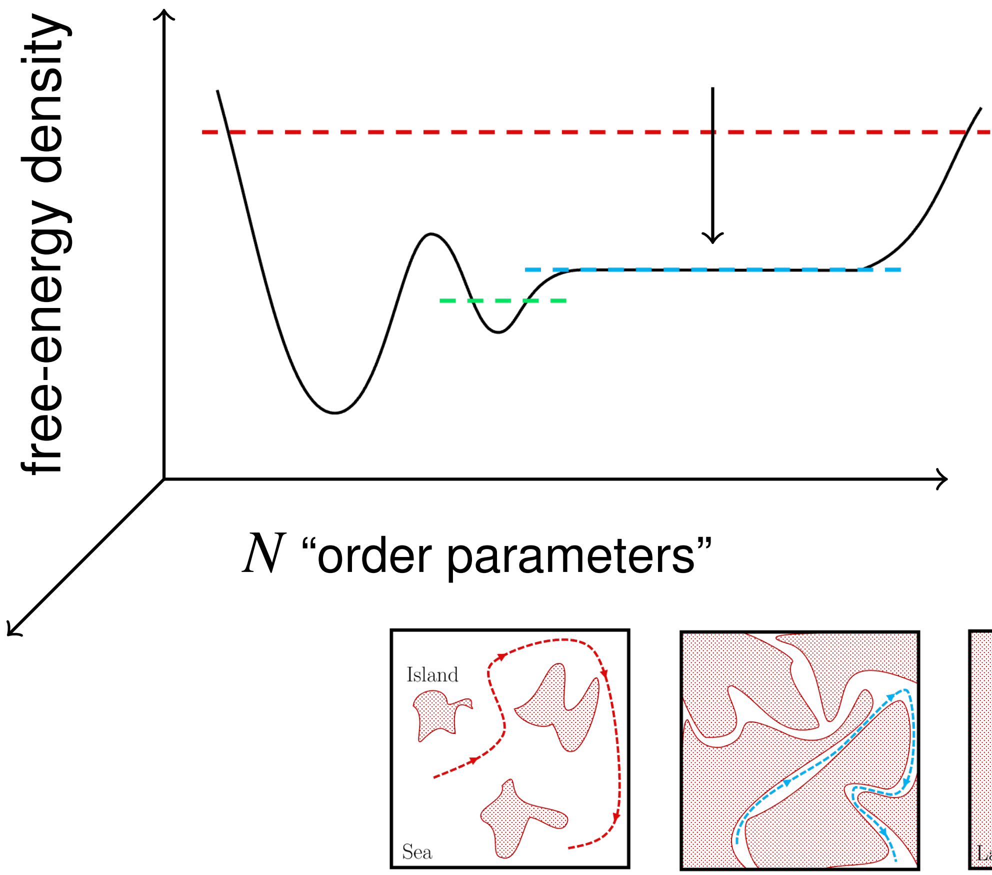
**RS** **1RSB** **FRSB**



# $p \geq 3$ spin models

## Glassy class




mostly maxima

threshold approached from quench

$$\lim_{t\to\infty} e(t) = e_{\rm th}$$

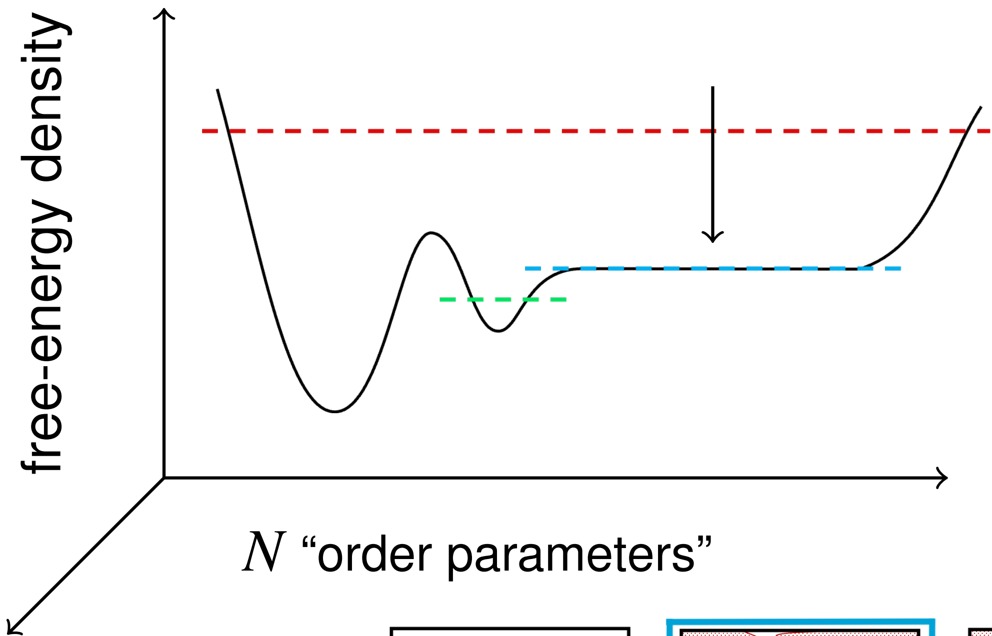
mostly minima

**The dynamics is linked to the topography of the landscape**

**LFC & Kurchan 93**

# $p \geq 3$ spin models

## Glassy class

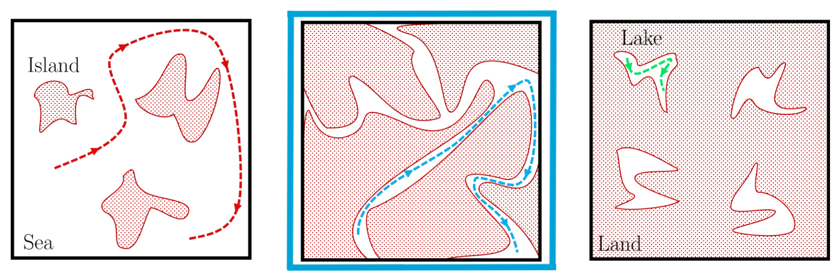



mostly maxima

threshold approached from quench

$$\lim_{t\to\infty} e(t) = e_{\text{th}}$$

mostly minima




**Flat threshold as an attractor for the $p$-spin relaxation**

Both for **physical** and **algorithmic** dynamic rules

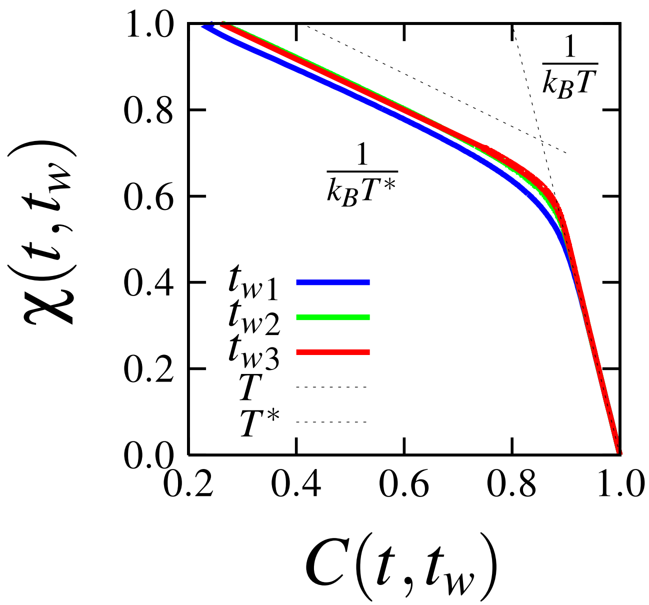
# Some surprising predictions

## with physical consequences

# Fluctuation-dissipation

## Induced vs. spontaneous fluctuations in glasses

A quench from $T_0 \to \infty$ to $T < T_c$




parametric construction

$t_w$ fixed

$t_{w_1} < t_{w_2} < t_{w_3}$

$t - t_w$ : $0 \to \infty$

used as a parameter

$T^* > T$

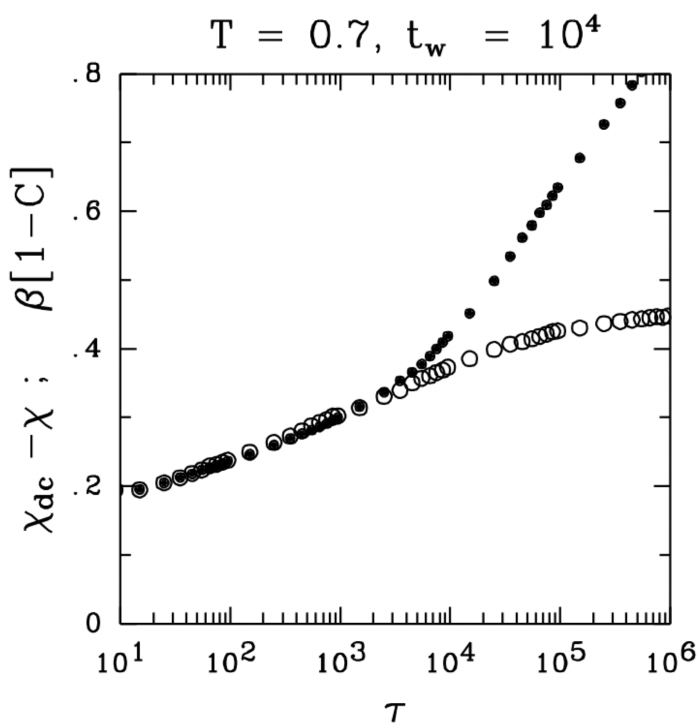
Breakdown of the equilibrium FDT $k_B T \chi = C$

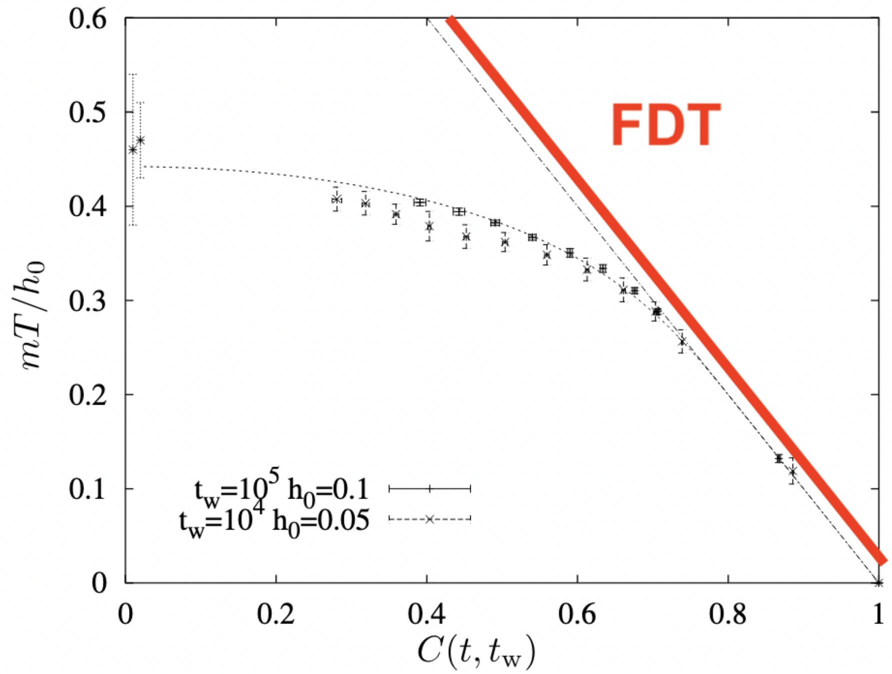
Convergence to $k_B T \chi(C)$, two linear relations for $C \lessgtr q_{\rm EA}$

Mean-field **LFC & Kurchan 93** & effective temperature interpretation **LFC, Kurchan & Peliti 97**

# Fluctuation-dissipation

## Induced vs. spontaneous fluctuations in 3D EA

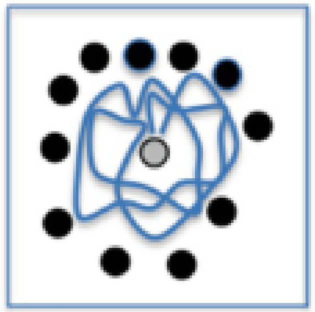
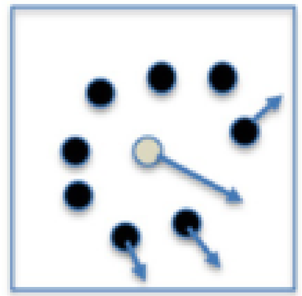



**Franz & Rieger 95** **Marinari, Parisi, Ricci-Tersenghi & Ruiz Lorenzo 98**

# Fluctuation-dissipation

## Interpretation

**Short-scale** re-arrangements ruled by the **equilibrium external bath** & **local properties of landscape**

The fluctuation-dissipation relation holds with the bath temperature $T$

**Large-scale** re-arrangements follow the systems' **internal dynamics** & **large scale props of landscape**

The fluctuation-disspation relation holds with another temperature $T^*$

After **cooling** from equilibrium at $T_0 > T_d$, hotter $T^* > T$

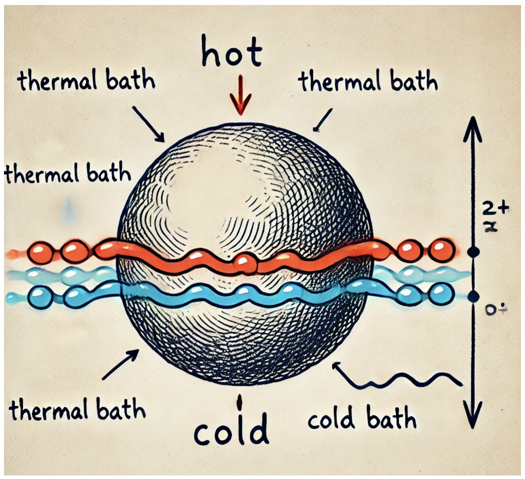
After **heating** from equilibrium at $T_0 < T_d$, colder $T^* < T$

**Support for this interpretation :**

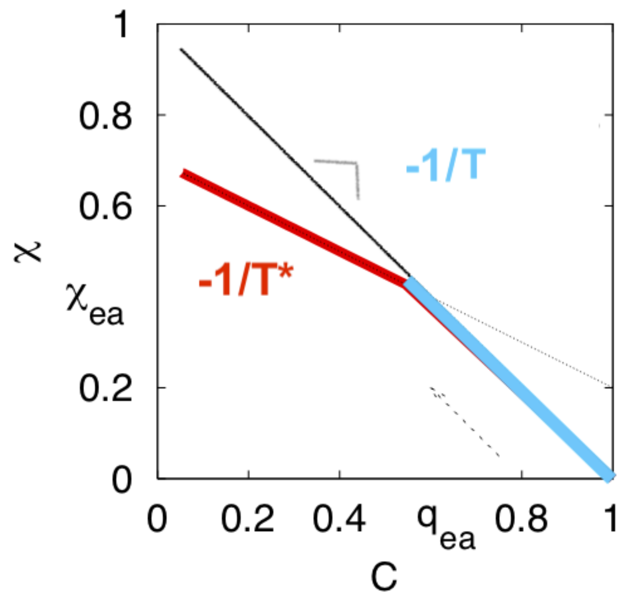
# Effective temperatures

**Induced by two (or more) baths**

**A tracer in contact with a complex bath**




Sketch created by ChatGPT




$\Gamma = \Gamma_{\rm cold}$ + $\Gamma_{\rm hot}$

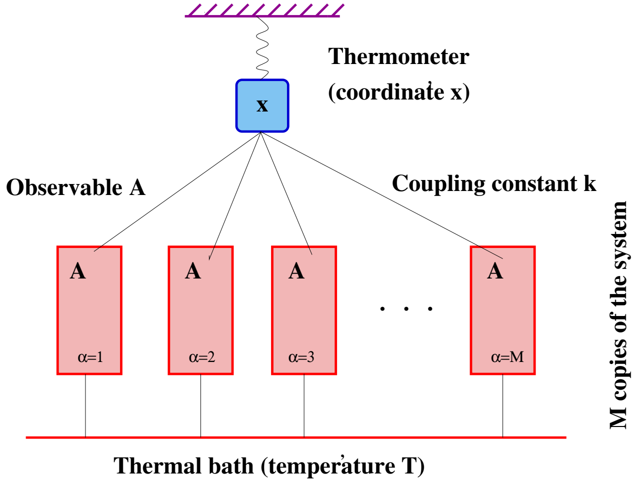
$\Gamma_{\rm cold}(t-t') = 2\gamma\delta(t-t')$ and $T$ $\qquad$ $\Gamma_{\rm hot}(t-t') = \gamma_{\rm hot}\, e^{-(t-t')/\tau}$ and $T^*$

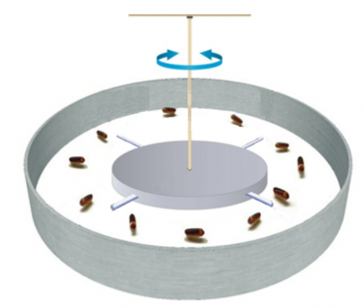
**LFC & Kurchan 00, Ilg & Barrat 07, etc.,** cfr. tracer in pasive & active bath

# Effective temperatures

## Measurement with thermometers

LFC, Kurchan & Peliti 97



**Grigera & Israeloff 99** - glassy

**D'Anna, Mayor, Barrat, Loreto & Nori 03** - granular

**Boudet, Jagielka, Guerin, Barois, Pistolesi & Kellay 24**

artificial active matter - robots

- Short internal time scale fast dynamics is tested and $T$ is recorded.
- Long internal time scale slow dynamics is tested and $T^*$ is recorded.

Related to the phenomenological *fictive temperatures* of **Tool 46, Gardon & Narayanaswamy 70, Moynihan et al 76, etc.** but measurable & with a thermodynamic interpretation

Also appearing in stochastic thermodynamic relations **Zamponi et al 05**

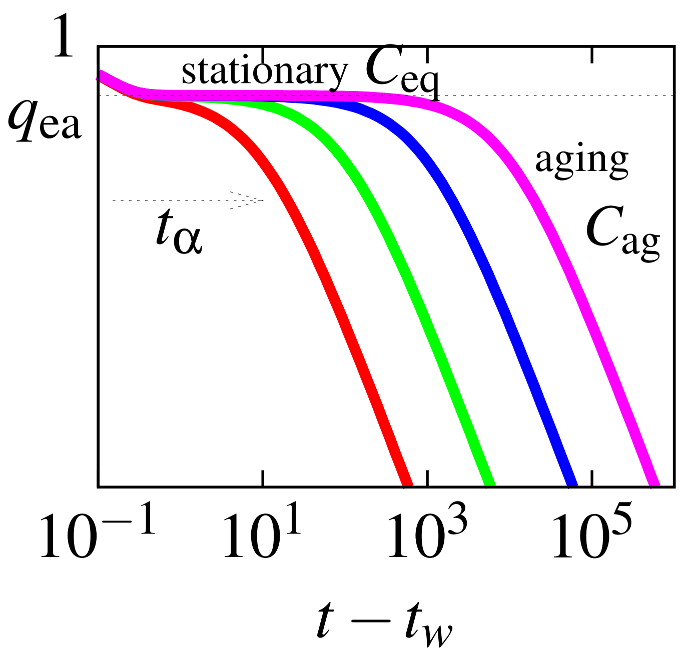
# Time reparametrization invariance

## and fluctuations

# Time reparametrization invariance

**In the long $t_w$ limit**

**Fast** $t - t_w \ll t_w$




The aging part is **slow**

$$\mathcal{R}(t)/\mathcal{R}(t_w) = \mathcal{O}(1)$$

$$C_{ag}(t,t_w) \sim f_{ag}\left(\frac{\mathcal{R}(t)}{\mathcal{R}(t_w)}\right)$$

$$\partial_t C_{ag}(t,t_w) \propto \frac{\dot{\mathcal{R}}(t)}{\mathcal{R}(t)} \xrightarrow[t\to\infty]{} 0$$

$$\boxed{\partial_t C_{ag}(t,t_w) \ll C_{ag}(t,t_w)}$$

Eqs. for the slow relaxation $C_{ag} < q_{EA}$ are invariant under

$$\boxed{t \to h(t) \quad C(t,t_w) \to C(h(t),h(t_w)) \quad R(t,t_w) \to \dot{h}(t')/h(t_w)\, R(h(t),h(t_w))}$$

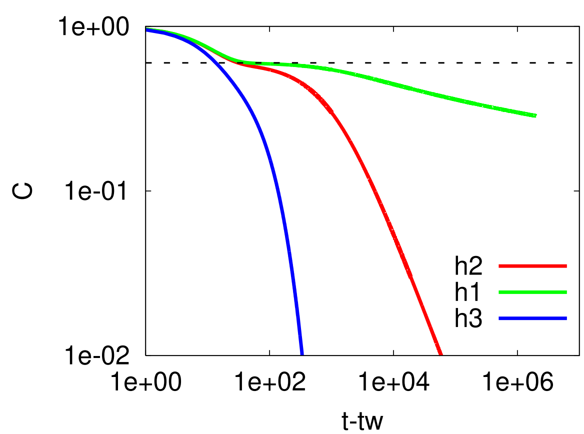
# Leading fluctuations

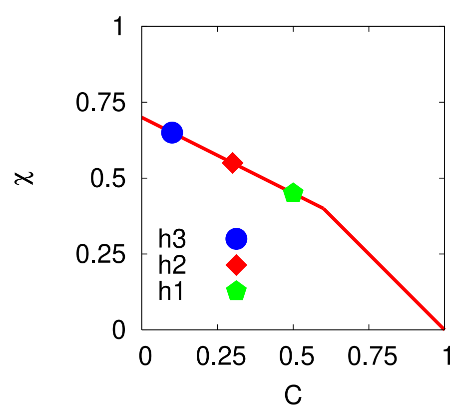
## Global to local correlations & linear responses

$$C_{\text{ag}}(t,t_w) \approx f_{\text{ag}}\left(\frac{\mathcal{R}(t)}{\mathcal{R}(t_w)}\right)$$ **global correlation**

**Global time-reparametrization invariance** $\Rightarrow \; C^{\text{ag}}_{\vec{r}}(t,t_w) \sim f_{\text{ag}}\left(\frac{h_{\vec{r}}(t)}{h_{\vec{r}}(t_w)}\right)$

Ex. $h_{\vec{r}_1} = \frac{t}{t_0}$, $h_{\vec{r}_2} = \ln\left(\frac{t}{t_0}\right)$, $h_{\vec{r}_3} = e^{\ln^{a>1}\left(\frac{t}{t_0}\right)}$ in different spatial regions




**Castillo, Chamon, LFC, Iguain & Kennett 02, 03**

**Chamon, Charbonneau, LFC, Reichman & Sellitto 04**

**Jaubert, Chamon, LFC & Picco 07**

**Avila, Castillo & Parsaeian 12**

**Dyre 22-25**

More recent perspective : time-reparametrization invariance in SYK models

**Kitaev 15, Maldacena & Stanford 16**

# Triangular relations

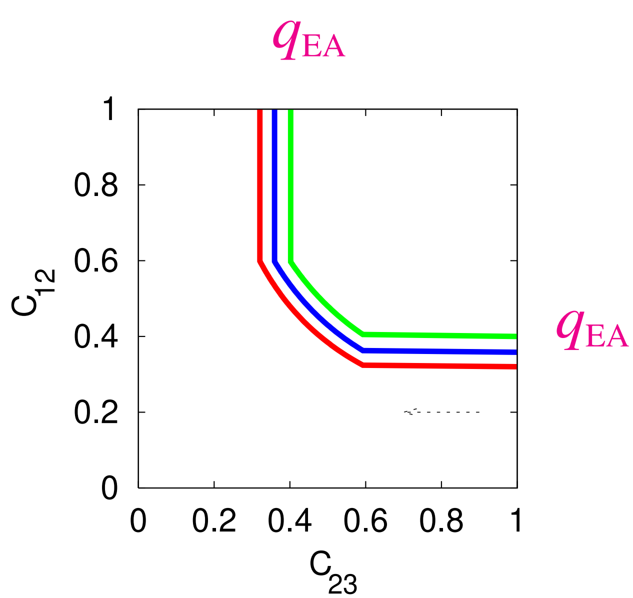
## Scaling of the aging global correlation

Take three times $t_1 \geq t_2 \geq t_3$ and compute the three global correlations

$$C(t_1,t_2),\ C(t_2,t_3),\ C(t_1,t_3)$$

If, in the aging regime $C_{\rm ag}^{ij} \equiv C_{\rm ag}(t_i,t_j) = f_{\rm ag}\left(\frac{\mathcal{R}(t_i)}{\mathcal{R}(t_j)}\right)$ with $t_i \geq t_j \Rightarrow$

$$C_{\rm ag}^{12} = f_{\rm ag}\left(\frac{\mathcal{R}(t_1)}{\mathcal{R}(t_3)}\frac{\mathcal{R}(t_3)}{\mathcal{R}(t_2)}\right) = f_{\rm ag}\left(\frac{f_{\rm ag}{}^{-1}(C_{\rm ag}^{13})}{f_{\rm ag}{}^{-1}(C_{\rm ag}^{23})}\right)$$




choose $t_3$ and $t_1$ so that $C^{13} = 0.3$

the arrow shows the $t_2$ 'flow' from $t_3$ to $t_1$

*e.g.* $C_{\rm ag}^{12} = q_{\rm EA}\, C_{\rm ag}^{13}/C_{\rm ag}^{23}$

in the spherical $p$ spin model

# Triangular relations

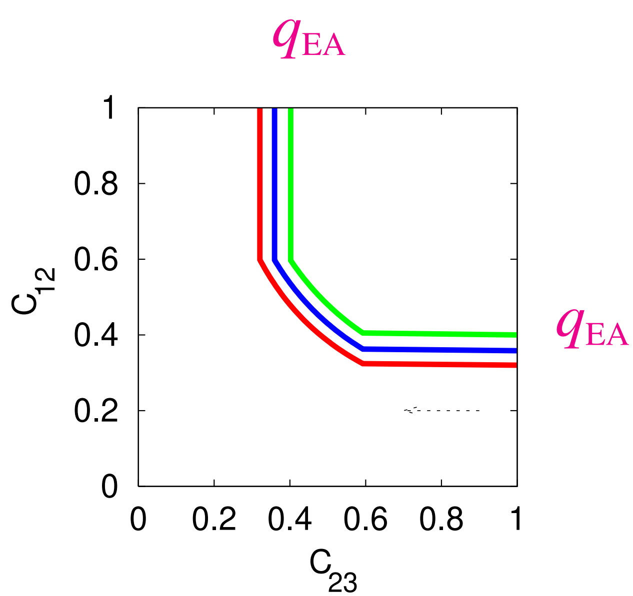
## Scaling of the slow part of the global correlation

Take three times $t_1 \geq t_2 \geq t_3$ and compute the three local correlations

$$C_{\vec{r}}(t_1, t_2),\ C_{\vec{r}}(t_2, t_3),\ C_{\vec{r}}(t_1, t_3)$$

If, in the aging regime $C_{\vec{r}}^{ij} \equiv C_{\vec{r}}(t_i, t_j) = f_{\rm ag}\left(\frac{h_{\vec{r}}(t_i)}{h_{\vec{r}}(t_j)}\right)$ with $t_i \geq t_j \Rightarrow$

$$C_{\vec{r}}^{12} = f_{\rm ag}\left(\frac{f_{\rm ag}{}^{-1}(C_{\vec{r}}^{13})}{f_{\rm ag}{}^{-1}(C_{\vec{r}}^{23})}\right)$$




choose $t_3$ and $t_1$ so that $C^{13} = 0.3$

the arrow shows the $t_2$ ‘flow’ from $t_3$ to $t_1$

*e.g.* $C_{\vec{r}}^{12} = q_{\rm EA}\, C_{\vec{r}}^{13} / C_{\vec{r}}^{23}$

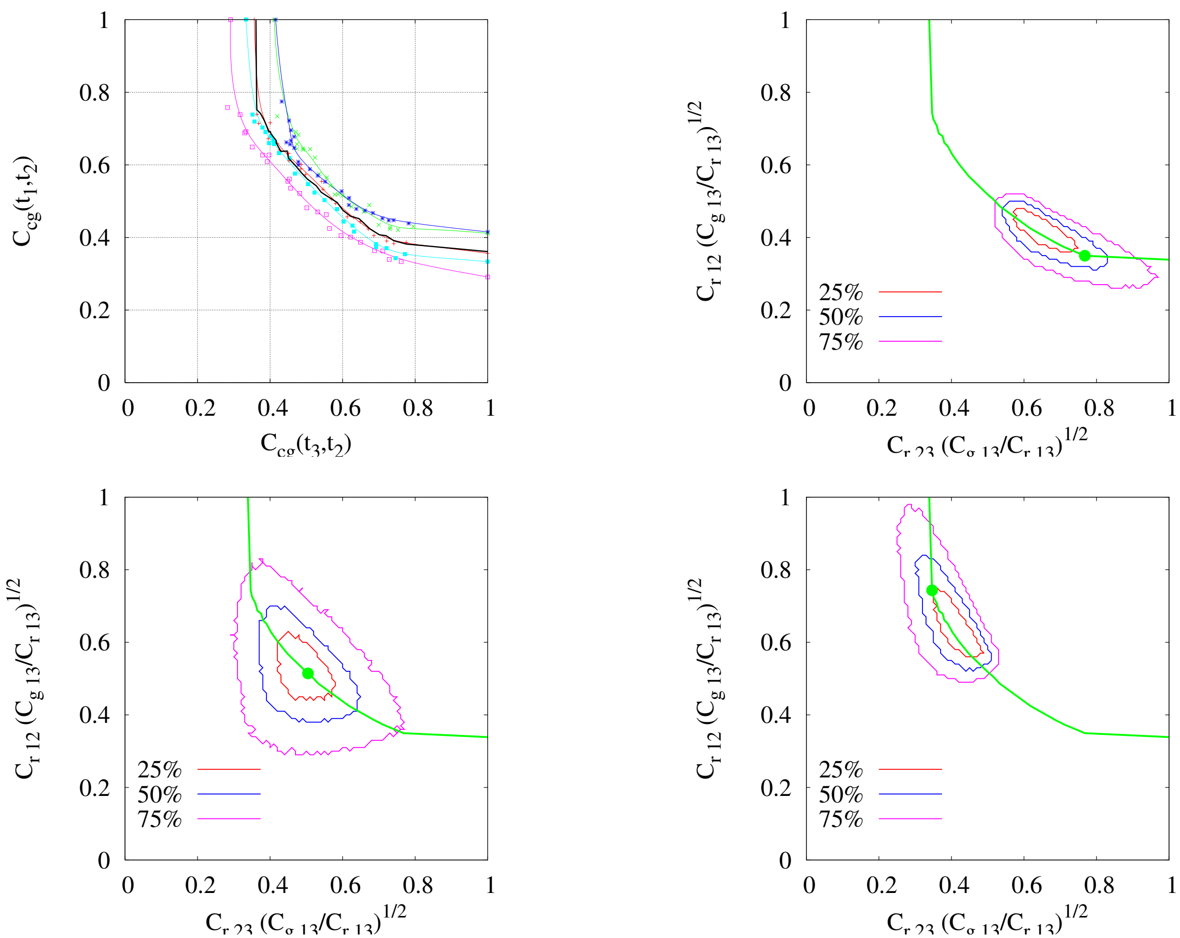
in the spherical $p$ spin model

# Triangular relations

## 3D Edwards-Anderson model

Black : global, other : five regions with size $\ell$

Green : global ; other : 2d projections of pdf

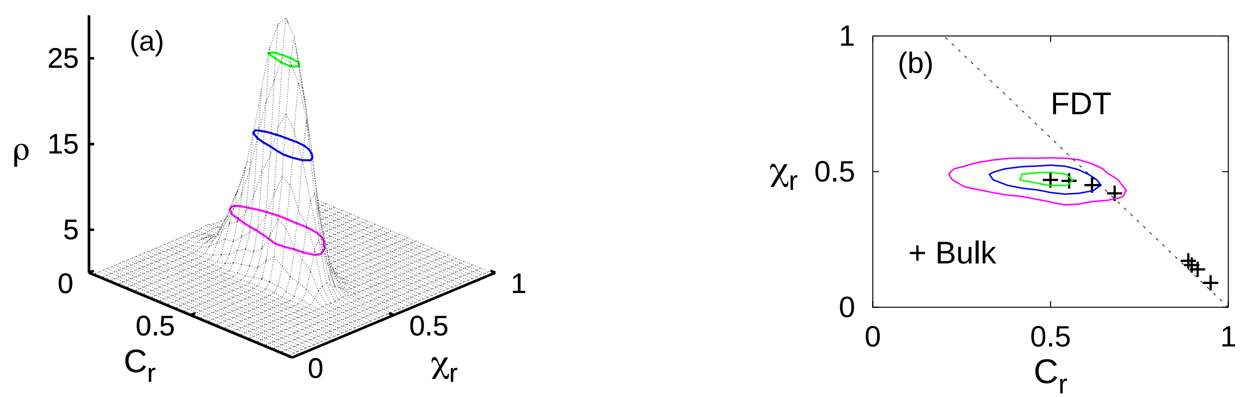



**Jaubert, Chamon, LFC & Picco 07**

# Local FDTs

## 3D Edwards-Anderson

$$C_{\vec{r}}(t,t_w) \equiv \frac{1}{V_{\vec{r}}} \sum_{i\in V_{\vec{r}}} s_i(t)s_i(t_w) \qquad \chi_{\vec{r}}(t,t_w) \equiv \frac{1}{V_{\vec{r}}} \sum_{i\in V_{\vec{r}}} \int_{t_w}^{t} dt' \left. \frac{\delta s_i(t)}{\delta h_i(t')} \right|_{h=0}$$




+ Bulk : Parametric plot $\chi(t,t_w)$ vs $C(t,t_w)$ for $t_w$ fixed and 7 $t\ (> t_w)$

$\rho$ corresponds to the maximum $t$ yielding the smallest global $C$ (left-most +)

**Castillo, Chamon, LFC, Iguain, Kennett 02**

**Kinetically constrained models + Charbonneau, Reichman & Sellitto 04**

# Short Overview of Glassy Dynamics

## Statics

## & a bit of Relaxation Dynamics