Theory of Disordered Systems

Statics and Dynamics

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Plan

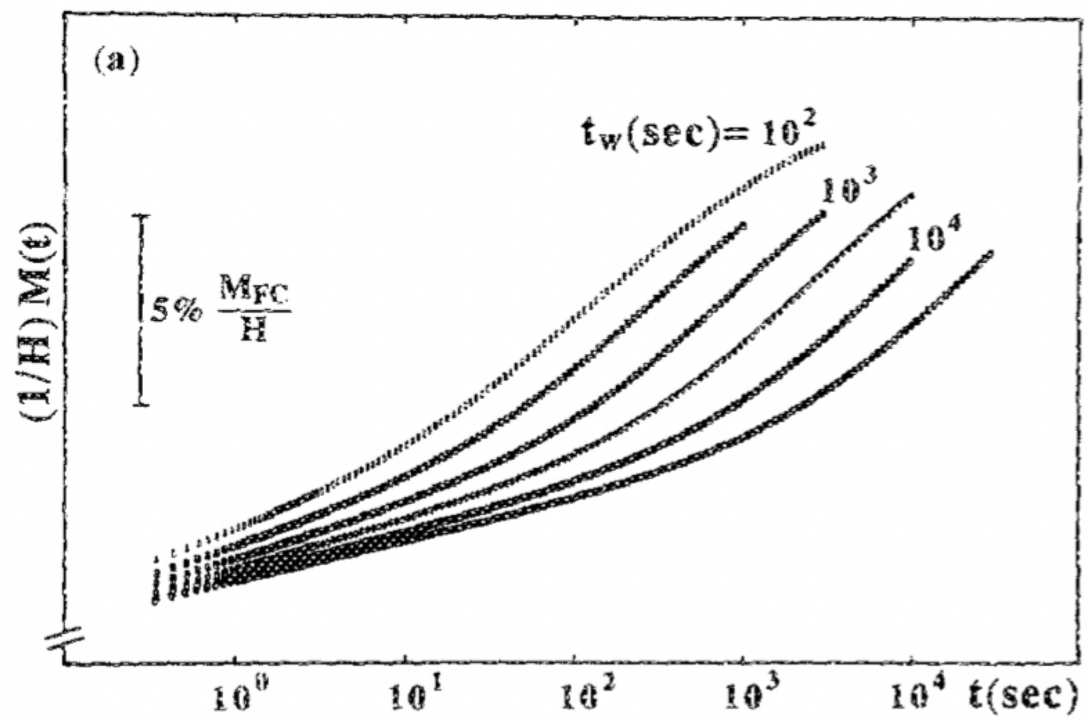
B. Dynamics

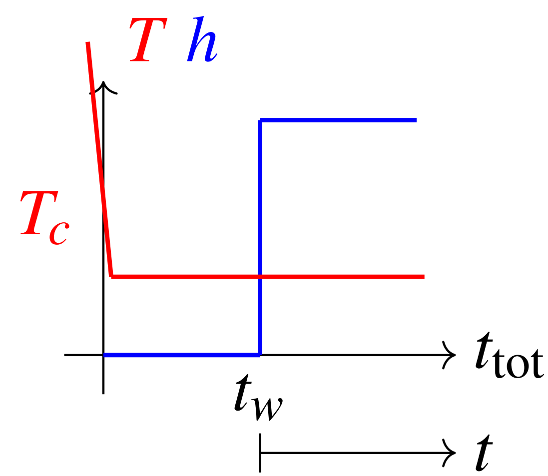
1. Spin-Glass Experiments
2. Theory
   (a) One dimensional energy landscapes
   (b) Coarsening
      i. Discrete & continuous models
      ii. Phenomenology
      iii. Dynamic scaling
   (c) Glassy dynamics
      i. Dynamic mean-field theory
      ii. Free-energy landscape interpretation
      iii. FDT & effective temperatures
      iv. Time reparametrization invariance & fluctuations

1. Back to SG experiments

Quenches

Zero field cooled magnetization - 80s





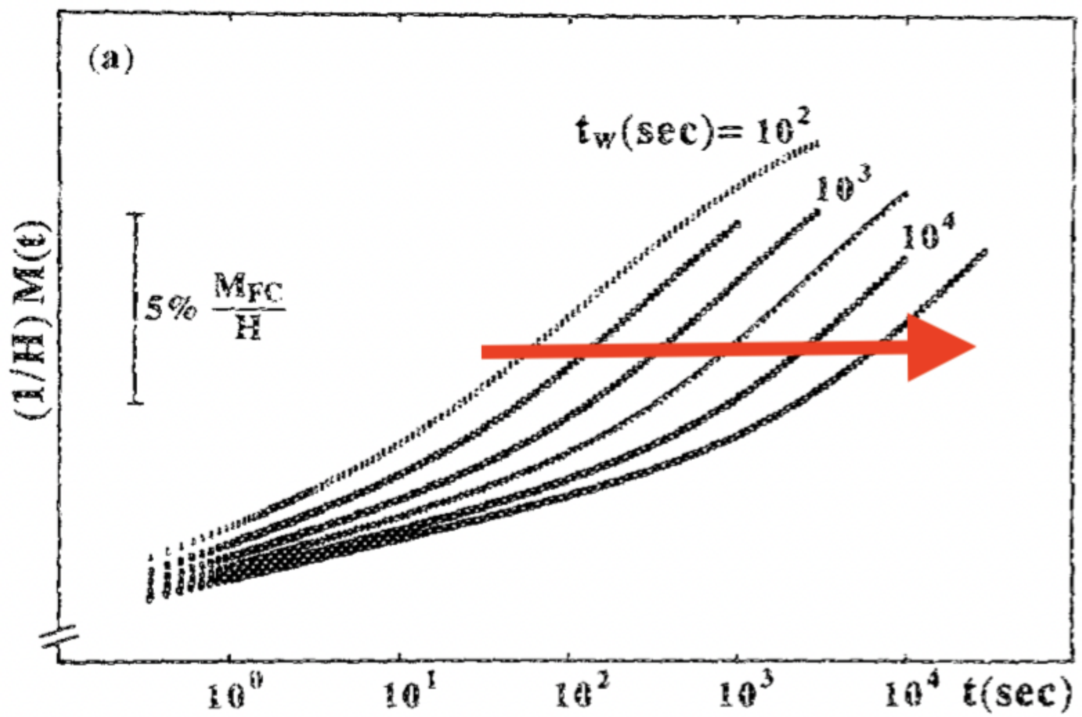
t_w are the waiting times, before the magnetic field is applied.

Sandlund, Svedlindh, Granberg, Nordblad & Lundgren 88

Aging

Breakdown of stationarity : out of equilbrium

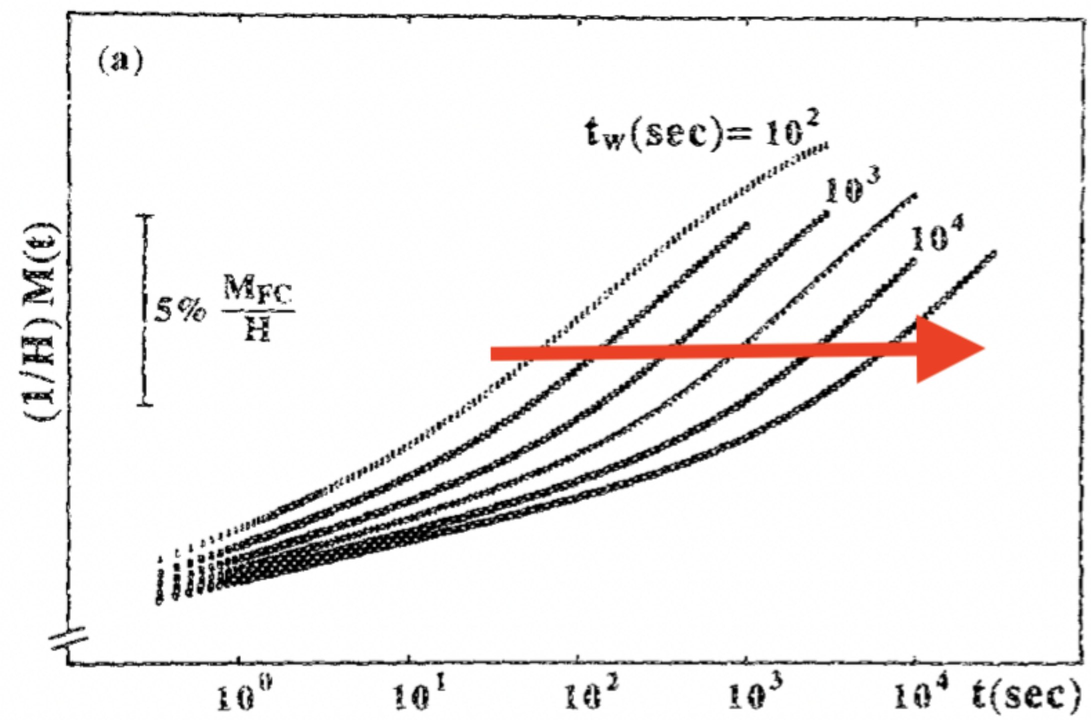
$$\boxed{m(t,t_w) \neq m(t-t_w)}$$



Older systems (longer t_w) relax more slowly than younger ones

Aging

Breakdown of stationarity : out of equilbrium



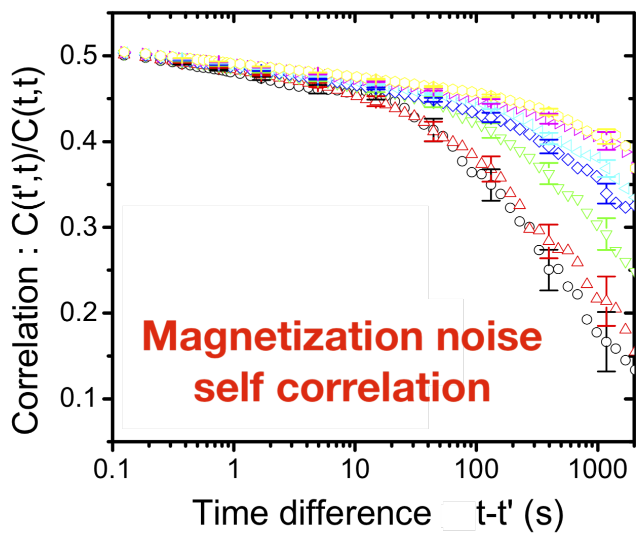
Older systems (longer t_w) relax more slowly than younger ones

Similar to previous measurements in all kinds of **polymer glasses**

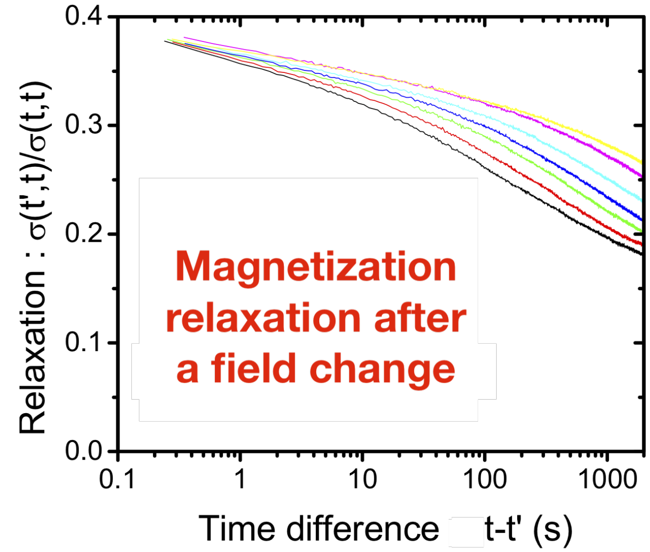
Kovacs, Struik, etc. 60s

Induced & spontaneous

slow relaxation & loss of stationarity (aging)



Correlation



Linear response

$0 \quad t' \quad t' \quad t' \quad t' \quad$ time

Different curves are measured after log-spaced reference times t' after the quench : **breakdown of stationarity** $\Longrightarrow$ far from equilibrium

No identifiable growing length $\mathcal{R}(t)$: **glassy microscopic mechanisms ?**

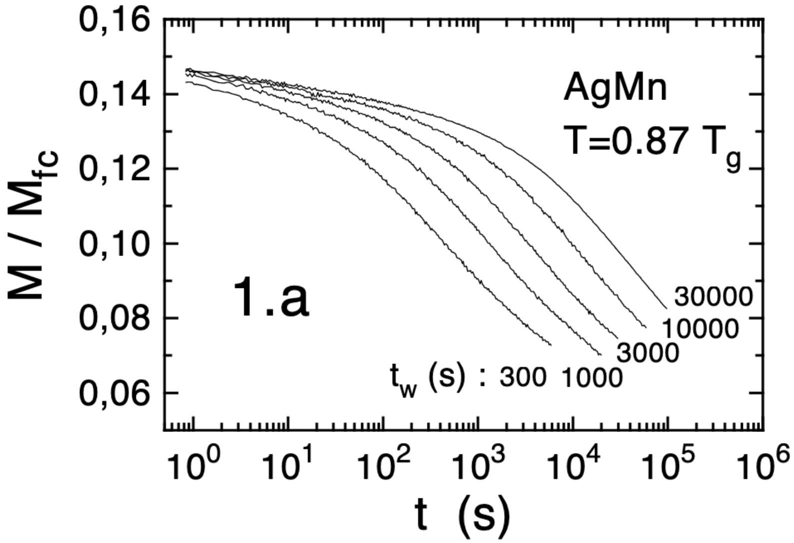
Spin-glass experiments **Hérisson & Ocio 02-04**

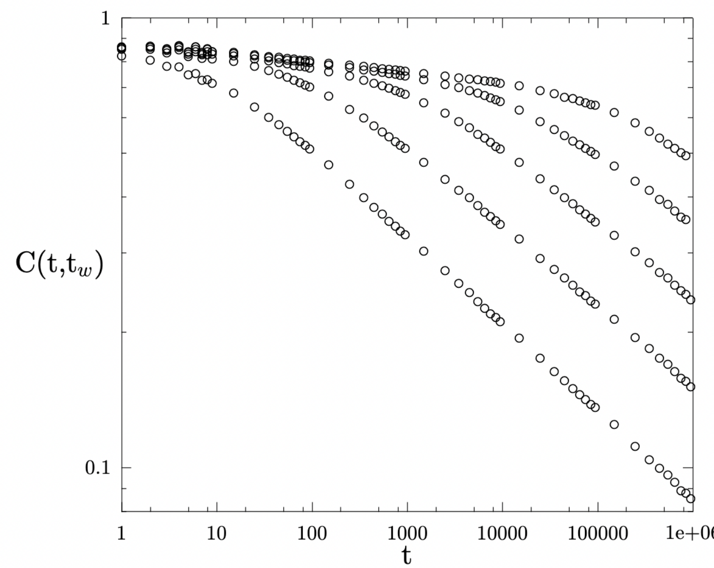
Induced & spontaneous

thermo-remanent magnetization decay & self correlation

$m(t_{tot}, t_w)$ $C(t_{tot}, t_w) = N^{-1} \sum_i s_i(t_{tot}) s_i(t_w)$

Experiments **Monte Carlo 3D EA**





In these plots, t_w are the waiting times when the magnetic field is switched off or the reference spin configuration is stored ; $t = t_{tot} - t_w$ is the time-difference

Hammann, Ocio & Vincent 90 **Rieger 93**

The equilibration time of **macroscopic** **spin-glasses** below T_c

goes beyond the experimentally accessible time window $t_{\rm exp}$

$$\boxed{\lim_{N\gg 1} t_{\rm eq}(N,T) \gg t_{\rm exp} \qquad\qquad T < T_c}$$

$t_{\rm eq}$ grows with the system size N at low temperatures

They show **aging** both in magnetization (induced) and in correlation (spontaneous) measurements

Above T_c, in the paramagnetic phase, they equilibrate, no aging

$$\boxed{\lim_{N\gg 1} t_{\rm eq}(N,T) < t_{\rm exp} \qquad\qquad T > T_c}$$

The equilibration time of **macroscopic coarsening & glassy systems** in a wide range of parameters,

e.g. low temperatures,

goes beyond the experimental window $t_{\rm exp}$

$$\boxed{\lim_{N\gg 1} t_{\rm eq}(N, \text{parameters}) \gg t_{\rm exp}}$$

$t_{\rm eq}$ grows with the system size N at low temperatures

and they also show **aging** though with some differences, to be discussed

2. Theory

Explanations

Dynamics

How does a (large) system reach equilibrium or

what does it do while it evolves out of equilibrium

- **Dynamics in one dimensional energy landscapes**
- **" in the simplest free-energy landscapes**
- **" in complex free-energy landscapes**

In all cases, discussion of time scales, thermalisation, fluctuations

Dynamics

How does a (large) system reach equilibrium or

what does it do while it evolves out of equilibrium

- **Dynamics in one dimensional energy landscapes**

  Conservative dynamics : Newton

  Dissipative dynamics : Langevin

  - Equilibrium vs. out of equilibrium
  - White (Markov) vs. coloured (memory) noises
  - Exponential vs. non-exponential relaxation
  - Relaxation, diffusion & activation

- **Dynamics in the simplest free-energy landscapes**

- **Dynamics in complex free-energy landscapes**

In all cases, discussion of time scales, thermalisation, fluctuations

Newton equation

Motion of a particle in one dimension

A particle with mass M and time-dependent position $r(t)$ evolves according to **Newton's eq.**

$$\underbrace{M\ddot{r}(t)}_{\text{inertia}} = \underbrace{F\left(r(t)\right)}_{\text{force}}$$

Inertia : mass times acceleration

Force :

gradient descent $F\left(r(t)\right) = -\left.\frac{dV(r)}{dr}\right|_{r=r(t)}$ in $d = 1$

but other options in higher dimensions

or generic $F(t)$

Newton equation

Conservative dynamics in a one dimensional harmonic potential

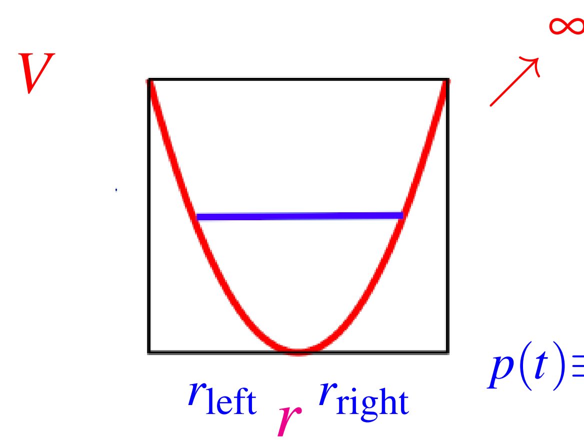
A particle with mass M and time-dependent position $r(t)$ evolves according to **Newton's eq.**

$$\boxed{\underbrace{M\ddot{r}(t)}_{\text{inertia}} = \underbrace{F(r(t))}_{\text{force}}}$$

Force $F(r(t)) = -\left.\dfrac{dV(r)}{dr}\right|_{r(t)} \Rightarrow$

Constant energy $E = \frac{(p(t))^2}{2M} + V(r(t))$

and confinement $r_{\text{left}} < r < r_{\text{right}}$



$p(t) \equiv M\dot{r}(t)$

Langevin equation

Effects of an environment : dissipation and noise

A particle with mass M and time-dependent position $r(t)$ coupled to a **memory-less equilibrium environment** evolves according to

$$\boxed{\underbrace{M\ddot{r}(t)}_{\text{inertia}} + \underbrace{\gamma\dot{r}(t)}_{\text{friction}} = \underbrace{F(r(t))}_{\text{force}} + \underbrace{\xi(t)}_{\text{noise}}}$$

Gaussian white noise,

zero mean $\langle \xi(t) \rangle = 0$ and correlation $\langle \xi(t)\xi(t') \rangle = 2\gamma k_B T \delta(t - t')$

γ is the friction coefficient, T is the temperature of the equilibrium bath and k_B the Boltzmann constant. $\beta = (k_B T)^{-1}$

The noise is delta-correlated, factor 2γ ensures equilibration of the bath

Langevin equation

Relaxation dynamics in a one dimensional harmonic potential

A particle with mass M and time-dependent position $r(t)$ coupled to a **memory-less equilibrium environment** evolves according to

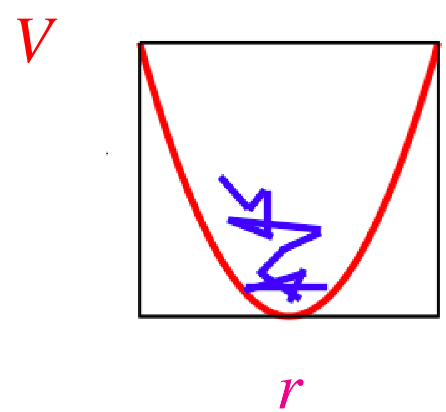
$$\underbrace{M\ddot{r}(t)}_{\text{inertia}} + \underbrace{\gamma\dot{r}(t)}_{\text{friction}} = \underbrace{-M\omega_0^2 r(t)}_{\text{potential force}} + \underbrace{\xi(t)}_{\text{noise}}$$

Friction/Dissipation $F_d(t) = -\gamma\dot{r}(t) \quad \Rightarrow$

Energy decay $\dfrac{d\langle \mathcal{H}(t)\rangle}{dt} = -\gamma\langle \dot{r}^2(t)\rangle$

Relaxation $\langle r(t)\rangle \to r_{\min} + c\,\boxed{e^{-t/\tau}}\,f(\omega t)$

Time-scale $\tau_{\rm over} = \frac{\gamma}{M\omega_0^2}$ and $\tau_{\rm under} = \frac{M}{\gamma}$



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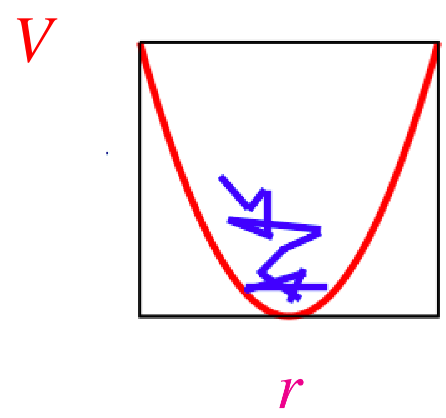
Bath+Potential $\Rightarrow$ Equilibration

$$\boxed{P(p,r,t) \to P_{\mathrm{GB}}(p,r) = \frac{e^{-\beta\mathcal{H}(p,r)}}{\mathcal{Z}}}$$

$$\mathcal{H}(p,r) = \frac{p^2}{2M} + \frac{M\omega_0^2\,(r-r_{\min})^2}{2}$$



Newton equation

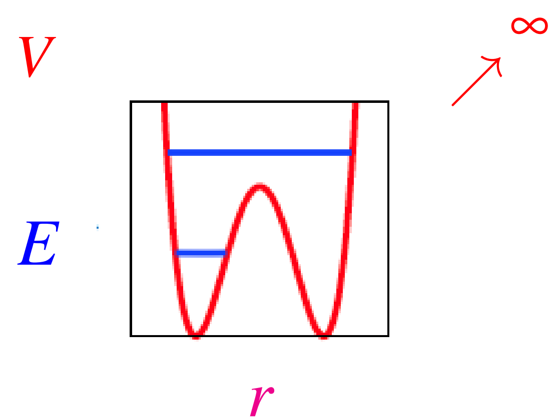
Conservative dynamics in a one dimensional double-well potential

A particle with mass M and time-dependent position $r(t)$ evolves according to **Newton's eq.**

$$\underbrace{M\ddot{r}(t)}_{\text{inertia}} = \underbrace{F(r(t))}_{\text{potential force}}$$

Force $F(r(t)) = -\left.\dfrac{dV(r)}{dr}\right|_{r(t)} \Rightarrow$

Constant energy $E = \frac{M}{2}\dot{r}^2(t) + V(r(t))$



Langevin equation

Thermal activation over finite barriers

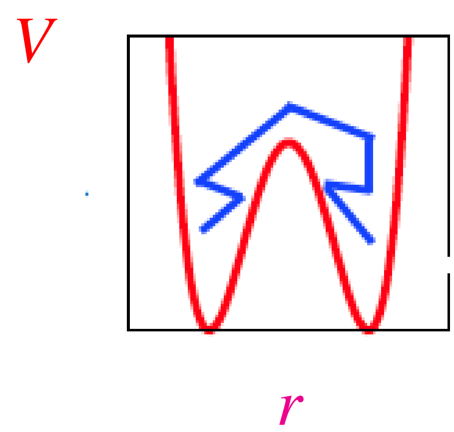
A particle with mass M and time-dependent position $r(t)$ coupled to a **memory-less equilibrium environment** evolves according to

$$\boxed{\underbrace{M\ddot{r}(t)}_{\text{inertia}} + \underbrace{\gamma\dot{r}(t)}_{\text{friction}} = \underbrace{\mu r(t) - g r^3(t)}_{\text{double well force}} + \underbrace{\xi(t)}_{\text{noise}}}$$

Friction/Dissipation $F_d(t) = -\gamma\dot{r}(t) \quad \Rightarrow$

$$\frac{d\langle\mathcal{H}(t)\rangle}{dt} = -\gamma\langle\dot{r}^2(t)\rangle$$

Activation over the barrier $\boxed{t_A = \tau_0\, e^{\beta\Delta V}}$



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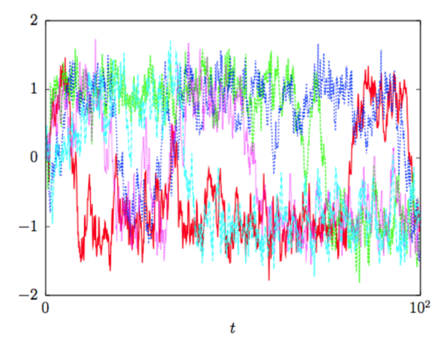
Five independent runs $r(t)$

Bath+Potential $\Rightarrow$ Equilibration

$$\boxed{P(p,r,t) \;\to\; P_{\mathrm{GB}}(p,r) = \frac{e^{-\beta\mathcal{H}(p,r)}}{\mathcal{Z}}}$$

r



t

Dynamics in equilibrium

Two properties

- One-time quantities reach their equilibrium values :

$$\boxed{\langle A(\{\vec{r}\}_\xi)(t)\rangle \;\to\; \langle A(\{\vec{r}\})\rangle_{\rm GB}}$$

- All time-dependent correlations are stationary **(steady state)**

$$\langle A_1(\{\vec{r}\}_\xi)(t_1)A_2(\{\vec{r}\}_\xi)(t_2)\cdots A_n(\{\vec{r}\}_\xi)(t_n)\rangle =$$
$$\langle A_1(\{\vec{r}\}_\xi)(t_1+\Delta)A_2(\{\vec{r}\}_\xi)(t_2+\Delta)\cdots A_n(\{\vec{r}\}_\xi)(t_n+\Delta)\rangle$$

  for any n and Δ. In particular, $\boxed{C(t,t_w)=C(t-t_w)}$

- The fluctuation-dissipation theorem (FDT), a model independent relation between linear responses (susceptibilities) and correlation functions, holds.

  In particular, $\boxed{\chi(t,t_w)=(k_BT)^{-1}[C(t,t)-C(t,t_w)]}$ **(thermal)**

Langevin equation

Out of equilibrium diffusion due to flatness of potential

A particle with mass M and time-dependent position $r(t)$ coupled to a **memory-less equilibrium environment** evolves according to

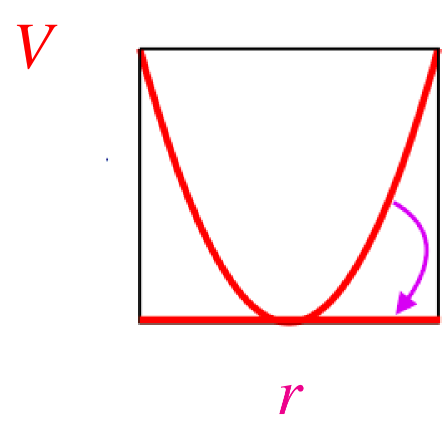
$$\underbrace{M\ddot{r}(t)}_{\text{inertia}} + \underbrace{\gamma\dot{r}(t)}_{\text{friction}} = \underbrace{\cancelto{0}{F(r(t))}}_{\text{flat potential}} + \underbrace{\xi(t)}_{\text{noise}}$$

Friction/Dissipation $F_d(t) = -\gamma\dot{r}(t) \quad \Rightarrow$

$$\langle \mathcal{H}(t)\rangle \to \langle p^2\rangle_{\mathrm{GB}}/(2M) \qquad \tau_{\mathrm{under}} = \frac{M}{\gamma}$$

Normal diffusion

$$\Delta_r(t,t') \equiv \langle (r(t)-r(t'))^2\rangle \to 2\mathcal{D}|t-t'|$$



Langevin equation

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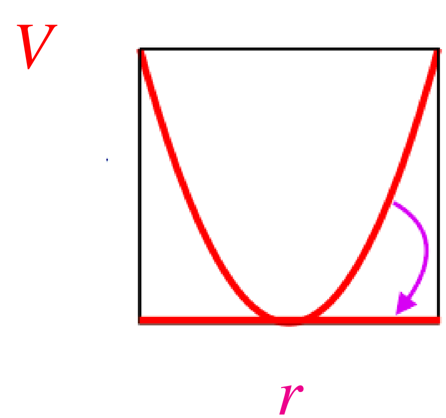
$$\langle \mathcal{H}(t)\rangle \to \langle p^2\rangle_{\mathrm{GB}}/(2M) \qquad \tau_{\mathrm{under}} = \frac{M}{\gamma}$$

Self-correlation

$$\boxed{C_r(t,t') \equiv \langle r(t)r(t')\rangle \to \min(t,t')}$$



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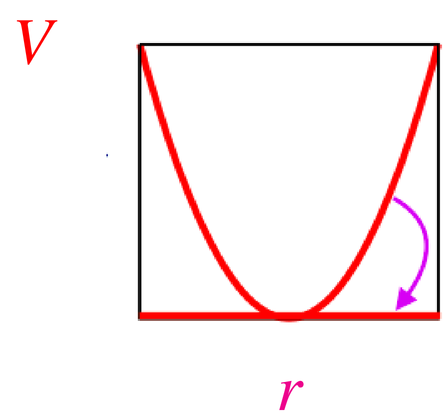
Bath+flat potential

$$P(p,r,t) \to \frac{e^{-\beta\frac{p^2}{2M}}}{Z_p} \times \boxed{\text{unbounded}}$$

momenta equilibrate but coordinates do not



Statements

One dimensional Markov (white noise) Langevin dynamics

Under an environment in equilibrium,

a confining potential $\lim_{r\to\pm\infty} V(r) = \infty$

and finite barriers $\Delta V(r) < \infty$

equilibration of both p and r.

exponential approach to equilibrium $e^{-t/\tau}$ (different τ)

Arrhenius activation over finite barriers $t_A = \tau_0 \, e^{\beta \Delta V}$

Flat potential (or unconfining one)

equilibration of p and out of equilibrium dynamics of r

neither stationarity nor FDT of r observables

General Langevin equation

Non-Markovian dynamics

A particle with mass M and time-dependent position $r(t)$ coupled to an **equilibrium environment with memory** evolves according to

$$\boxed{\underbrace{M\ddot{r}(t)}_{\text{inertia}} + \underbrace{\int_{t_0}^{t} dt'\, \Sigma_B(t-t')\dot{r}(t')}_{\text{friction}} = \underbrace{F(t)}_{\text{force}} + \underbrace{\xi(t)}_{\text{noise}}}$$

Coloured noise with correlation $\boxed{\langle \xi(t)\xi(t') \rangle = k_B T\, \Sigma_B(t-t')}$ and zero mean.

Equilibrium environment : the same friction kernel and noise correlation $\Sigma_B(t-t')$.

Derivations : see, *e.g.*, **Weiss 99, LFC Les Houches 02**

Coloured noise

Power-law correlations

Generic : Most of the exact **fluctuation-dissipation relations** in and out of equilibrium remain unaltered for generic Σ_B, *e.g.* the fluctuation-dissipation theorem, fluctuation theorems, *etc.*

Aron, Biroli & LFC 10

Particular : The **functional form** of the observables depends on the characteristics of the noise, *i.e.* on Σ_B.

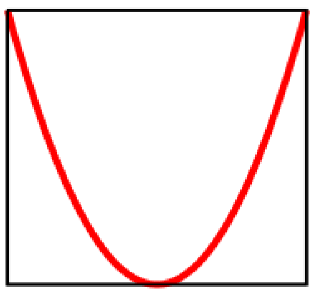
Some interesting cases are

$$\boxed{\Sigma_B(t-t') = \frac{g}{\Gamma_E(1-\alpha)}\, |t-t'|^{-\alpha} \quad \text{with} \quad \alpha > 0}$$

g the 'friction coefficient' and Γ_E the Euler-function.

Example

a particle in a harmonic potential



$$V(r) = \frac{1}{2} M\omega_0^2 r^2$$

After a relatively short transient,

independently of the initial condition

equilibrium dynamics

$$\boxed{C_r(t,t') \equiv \langle r(t) r(t') \rangle \to \frac{1}{M\omega_0^2} E_{\alpha,1}\left(-\frac{M\omega_0^2 |t-t'|^\alpha}{\gamma}\right)}$$

with $E_{\alpha,1} = \sum_{k=0} \frac{z^k}{\Gamma_E(\alpha k+1)}$ the Mittag-Leffler function.

Only for an Ohmic bath $\alpha = 1$ the relaxation is exponential $E_{1,1}(z) = e^z$

non-Ohmic bath $\alpha \neq 1$ $E_{\alpha,1}(z) \to z^{-1}$ for $z \to -\infty$ **power-law relaxation.**

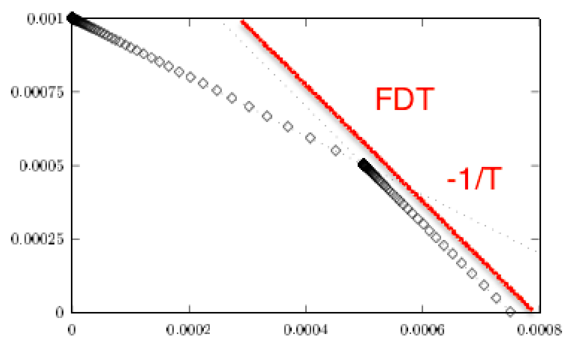
Example

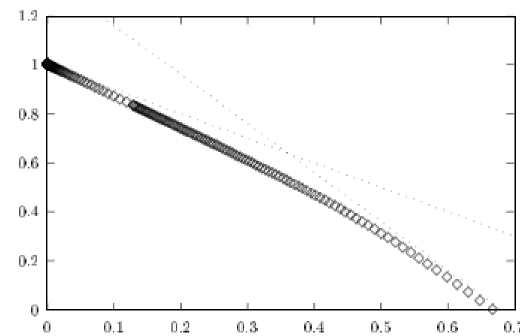
Time-scales & FDT violations in the simple oscillator

Coupling to one bath at temperature T *vs.*

Coupling to several baths with different time-scales and temperatures

$$\chi(t,t_w) = \chi(t-t_w)$$





$$C(t,t_w) = C(t-t_w)$$

Two scales — Not well-separated scales

LFC & Kurchan 99

Statements

One dimensional Markov Langevin dynamics

Even the long-term dynamics may depend on the microscopic dynamics, in this case as induced by the bath.

The approach to equilibration can be non-exponential
in this example, it is dictated by the noise correlation

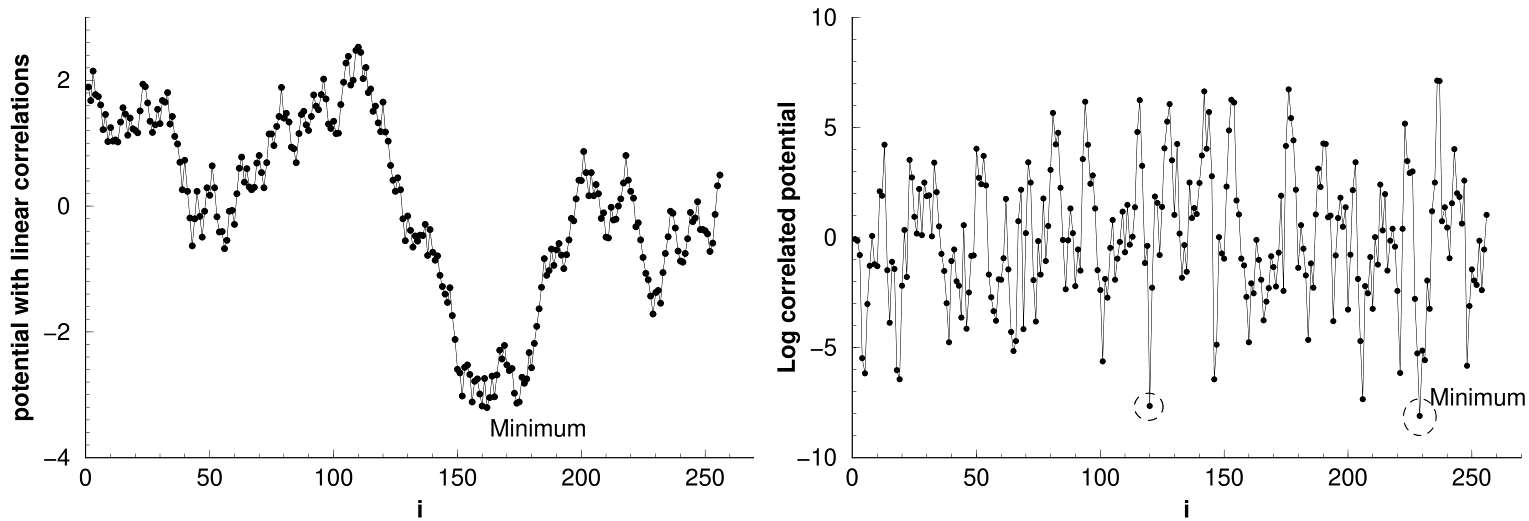
It is hard to conclude on thermal equilibration from the functional form of the observables or stationary correlation functions

The comparison between correlation functions and linear response functions gives more detailed information in this respect

Langevin equation

Diffusion in random quenched potentials

V



Figs. from **Carpentier & Le Doussal 00**

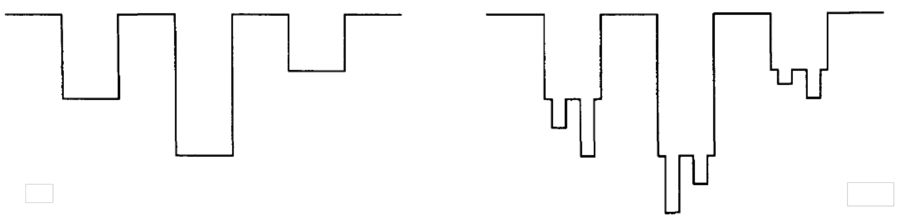
MSD $$\boxed{\Delta_r(t,t') \equiv \langle (r(t) - r(t'))^2 \rangle \to 2\mathcal{D}|t-t'|^a}$$

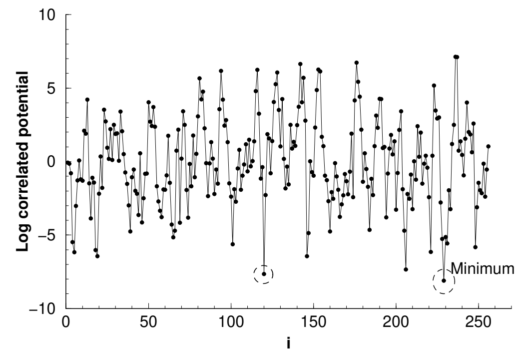
Interest in anomalous diffusion a, renormalisation of diffusion constant $\mathcal{D}$, etc.

Reviews **Bouchaud & Georges 90, Metzler & Klafter 00, Havlin & ben-Avraham 02**

Trap models

For dynamics in real but also phase space





Bouchaud 92 Bouchaud & Dean 95 **Carpentier & Le Doussal 00**

Variations on the connectivity of the traps, the trapping time distributions, the nesting of traps within traps, etc.

Many studies including

Ben Arous, Bovier, Gayrard, Jerny 02-08, Baity-Jesi, Biroli, Cammarota 17-18

More on the relation with disordered systems' dynamics later

Dynamics

How does a (large) system reach equilibrium or

what does it do while it evolves out of equilibrium

- **One dimensional energy landscapes**
- **The simplest free-energy landscape : many variables - large dimensions**

  Stochastic dynamics of the Ising model

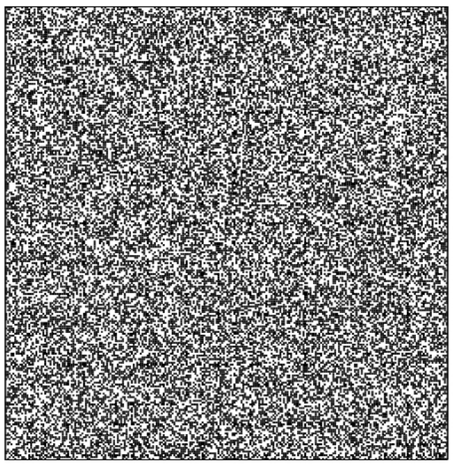
  Time-dependent Ginzburg-Landau equation

  - Collective effects
  - Critical relaxation
  - Dynamic scaling
- **Complex free-energy landscapes**
- **Time scales, thermalisation**

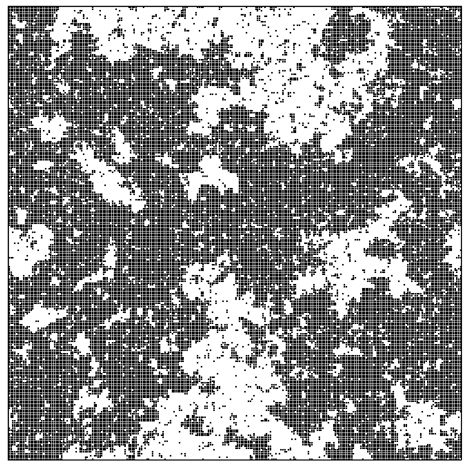
Phenomenology

Ferromagnetism

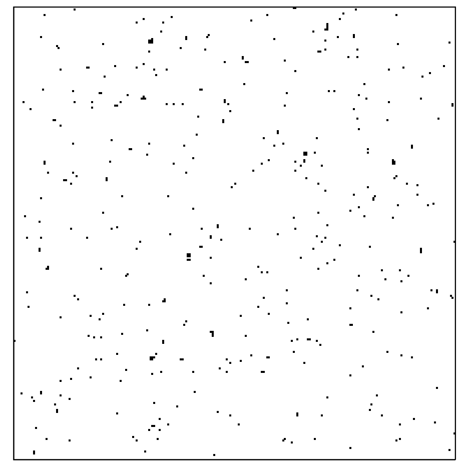
Equilibrium configurations - up & down spins in a $2d$ Ising model



$T \to \infty$



$T = T_c$

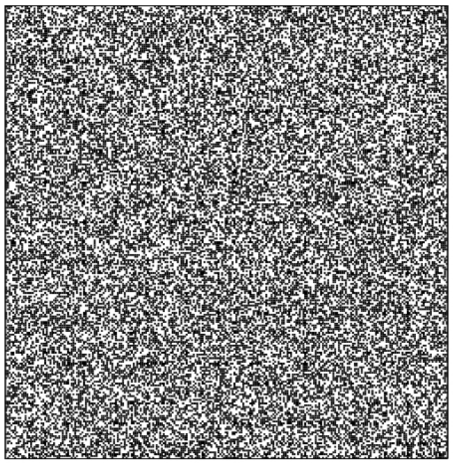


$T < T_c$

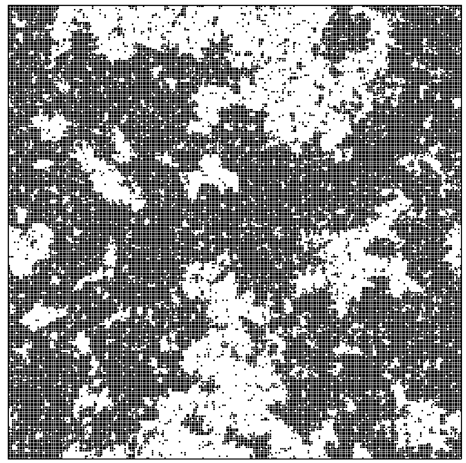
Simplest physical problem with a phase transition : Disordered high temperature phase, 2nd order phase transition, spontaneous symmetry breaking below T_c, two ordered low temperature phases related by symmetry (all up or all down decorated with thermal fluctuations).

Ferromagnetism

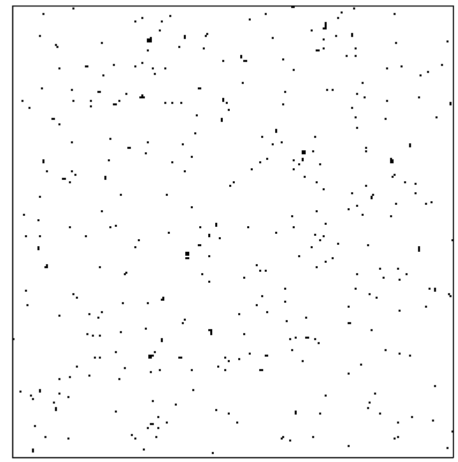
Equilibrium configurations - up & down spins in a $2d$ Ising model



$T \to \infty$



$T = T_c$



$T < T_c$

Static questions: coarse-grained local order parameter, effective “weight” functional *i.e.* free-energy functional $\Rightarrow$ Ginzburg-Landau approach

Models

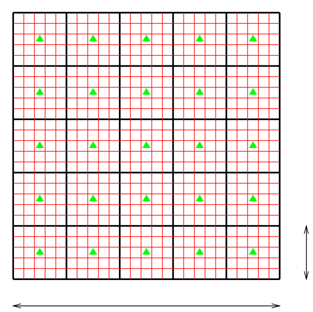
Discrete *vs.* continuous

Ising spin models

$$\mathcal{H} = -\sum_{\langle ij \rangle} J_{ij} s_i s_j$$

Field theory for the order parameter

$$F[\phi] = \int d^d r \left[\frac{1}{2} (\nabla \phi)^2 + \frac{\mu}{2} \phi^2 + \frac{g}{4} \phi^4 \right]$$



$$\phi(\vec{r}) = \mathcal{V}_{\vec{r}}^{-1} \sum_{i \in V_{\vec{r}}} s_i$$

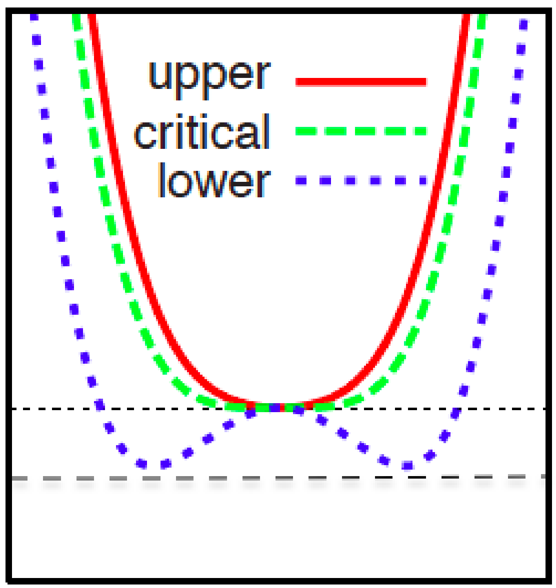
$F[\phi] \propto L^d$ while $\mathcal{V}_{\vec{r}} = \delta^d \ll \ell^d \ll L^d$

Generalisations to vector models. **Quenched disorder** can be introduced by taking the J_{ij} or the parameters in the field theory to be random distributed.

Ferromagnetism

Bi-valued equilibrium states related by symmetry

$$v[\phi] = \frac{\mu}{2}\phi^2(\vec{r}) + \frac{g}{4}\phi^4(\vec{r})$$



Potential density

$$\Delta v[\phi] = O(1)$$

$$\phi(\vec{r})$$

Ginzburg-Landau free-energy

Scalar order parameter

Disordered phase (high T, $\mu > 0$)

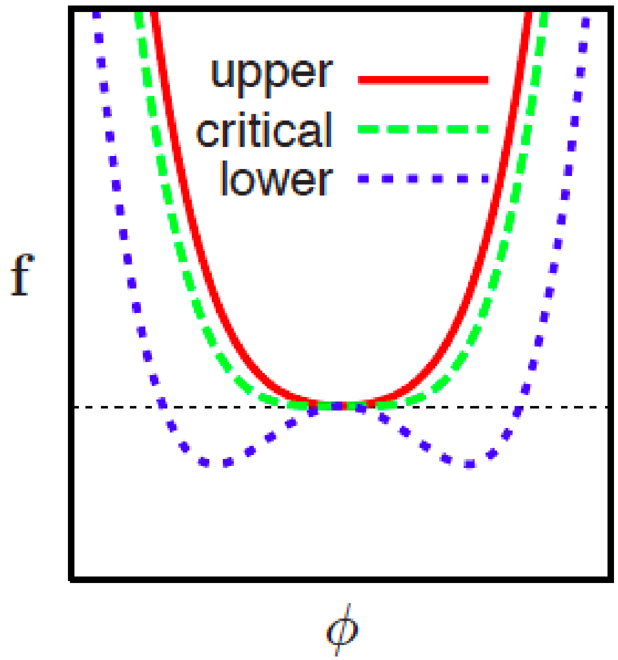
Ordered phases (low T, $\mu < 0$)

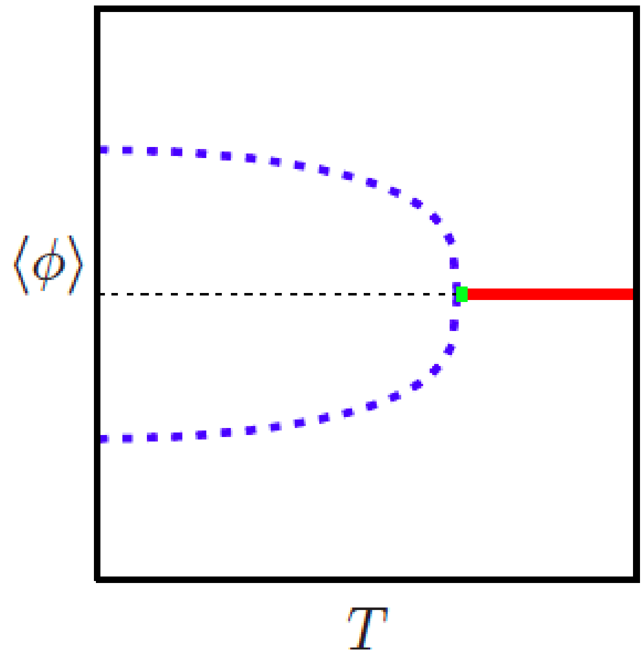
Critical point (T_c, $\mu = 0$)

Ferromagnetism

Free-energy density for the global order parameter

$\int d^d r\, (\nabla\phi)^2 \Rightarrow \phi(\vec{r}) = \phi$ and $f = L^{-d} \int d^d r\, v[\phi(\vec{r})] = v(\phi)$



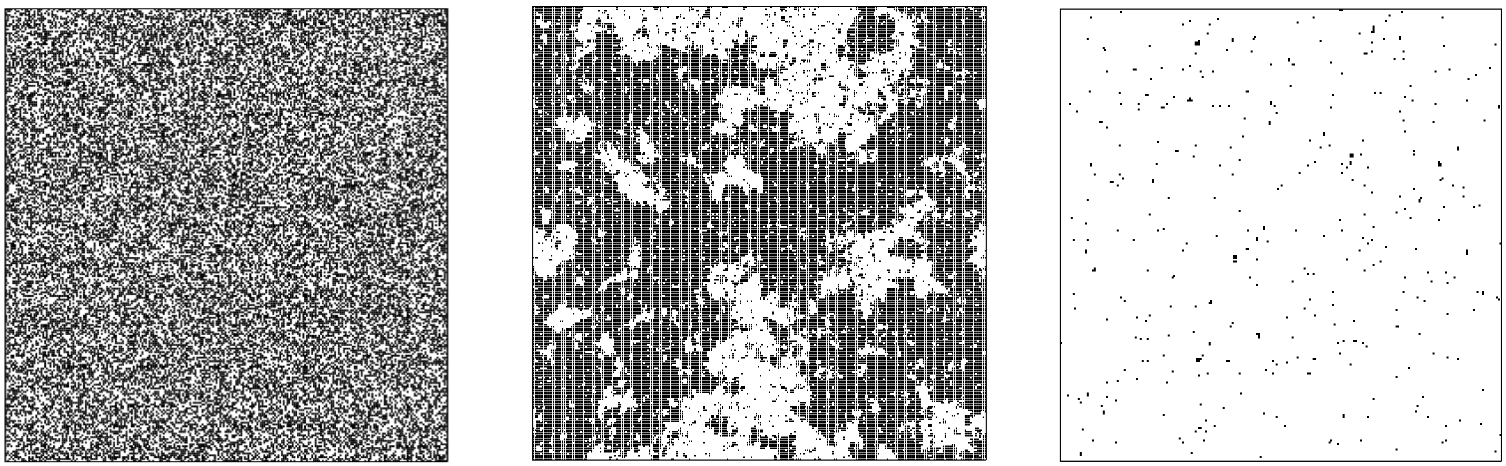


$\langle\phi\rangle = \phi^* = \phi_{\min} \equiv \phi$ s. t. $v'[\phi] = 0$ & $v''[\phi] > 0$

The barrier is $\Delta f = O(1)$ but $\boxed{\Delta F \sim L^{d-1}}$ diverges with L **Ex.**

Ferromagnetism

Equilibrium configurations - up & down spins in a $2d$ Ising model



$T \to \infty$ $\quad\quad$ $T = T_c$ $\quad\quad$ $T < T_c$

Dynamic question : starting from equilibrium at $T_0 \to \infty$ how is equilibrium at $T = T_c$ or $T < T_c$ approached and attained ?

Models

Dynamics for discrete and continuous

Ising spin models

$$\mathcal{H} = -\sum_{\langle ij \rangle} J_{ij} s_i s_j$$

NCOP $[\, \uparrow\downarrow \mapsto \uparrow\uparrow \,]$

COP $[\, \uparrow\downarrow \mapsto \downarrow\uparrow \,]$

Field theory for the order parameter

$$F[\phi] = \int d^d r \left[\frac{1}{2} (\nabla\phi)^2 - \frac{\mu}{2}\phi^2 + \frac{g}{4}\phi^4) \right]$$

$$\gamma \partial_t \phi(\vec{r},t) = -\delta_{\phi(\vec{r},t)} F[\phi] + \xi(\vec{r},t)$$

$$\gamma \partial_t \phi(\vec{r},t) \;=\; -\nabla^2 \delta_{\phi(\vec{r},t)} F[\phi] \;+\; \xi(\vec{r},t)$$

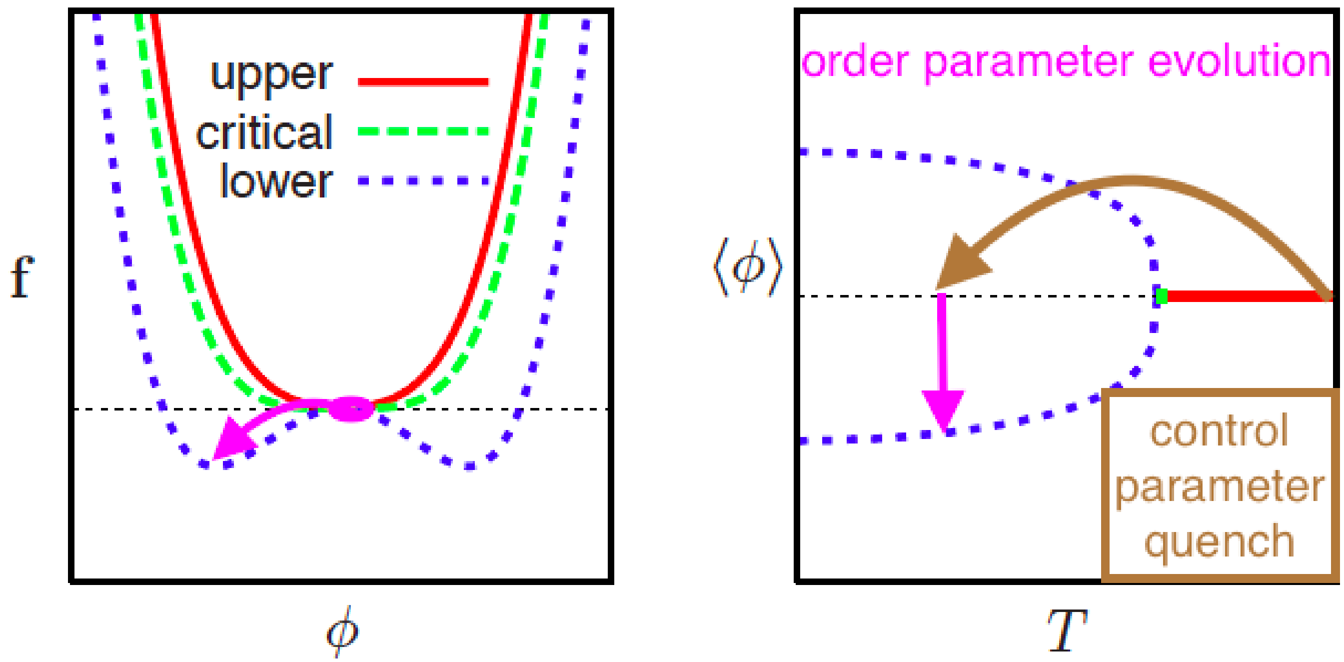
Overdamped limit and white noise commonly accepted

In the COP case $\langle \xi(\vec{x},t) \xi(\vec{y},t') \rangle = 2\gamma k_B T \nabla^2 \delta(\vec{x}-\vec{y}) \delta(t-t')$

Generalisations for vector cases. **Quenched disorder** can be introduced by taking the J_{ij} or the parameters in the field theory to be random distributed.

Ferromagnetism

Naively expected spontaneous symmetry breaking



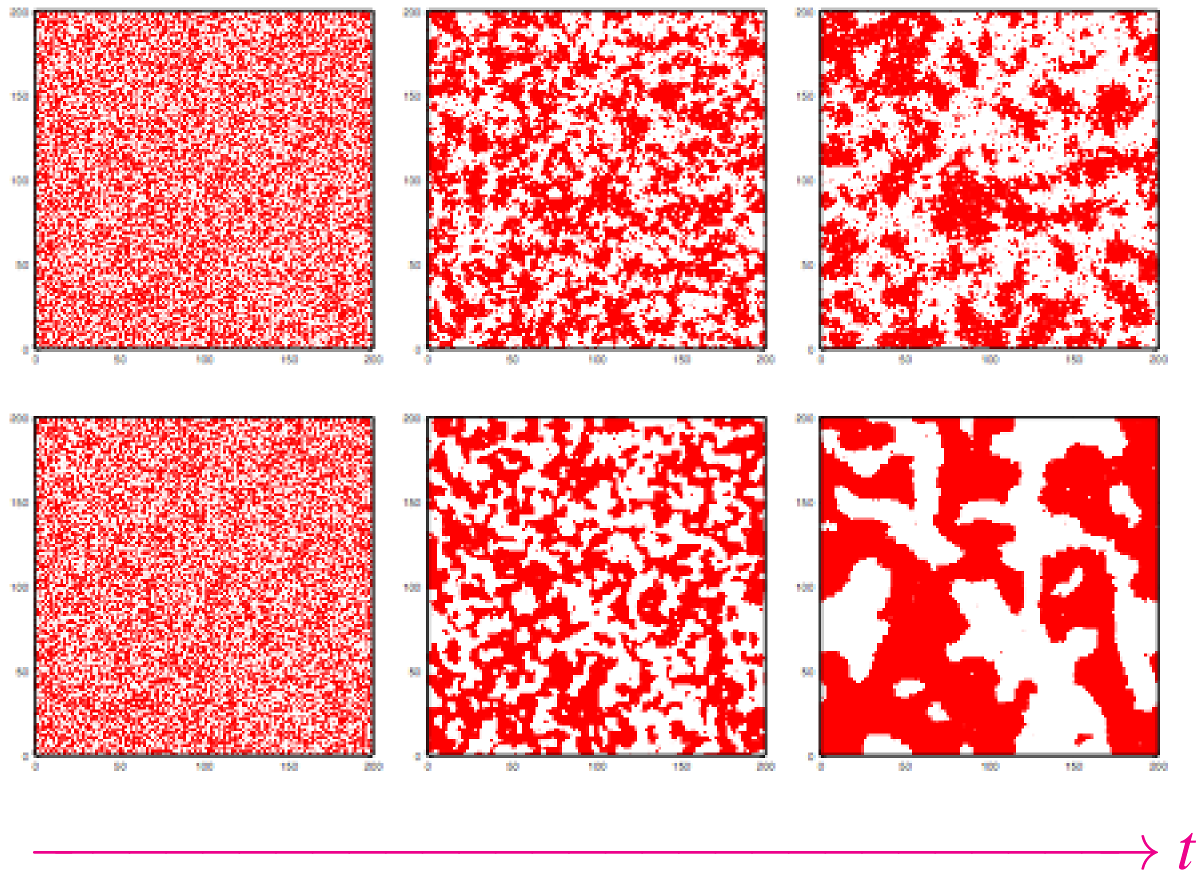
The magenta arrows are naively expected to be fast

but diverging times with the system size $t = O(L^{z_d})$ are needed to reach this regime

and even longer times $t_A = O(e^{\Delta F/(k_B T)})$ are needed to restore the symmetry

Coarsening

Snapshots after an instantaneous quench from $T_0 \to \infty$ to T at $t = 0$



$T = T_c$

$T < T_c$

At $T = T_c$ **critical dynamics**

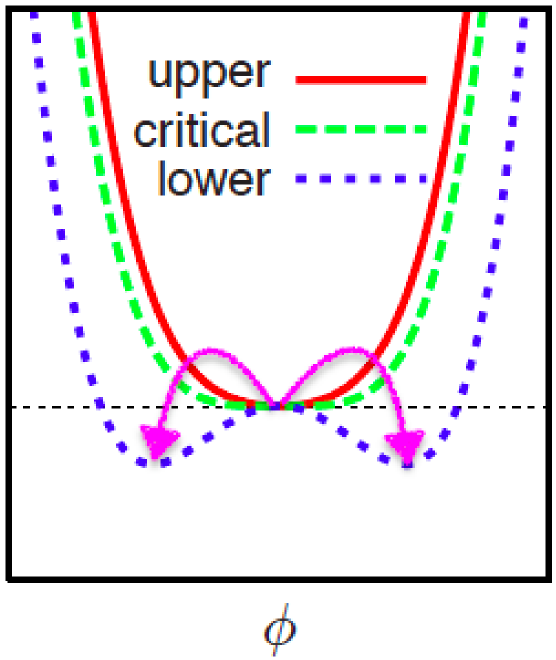
At $T < T_c$ **domain growth**

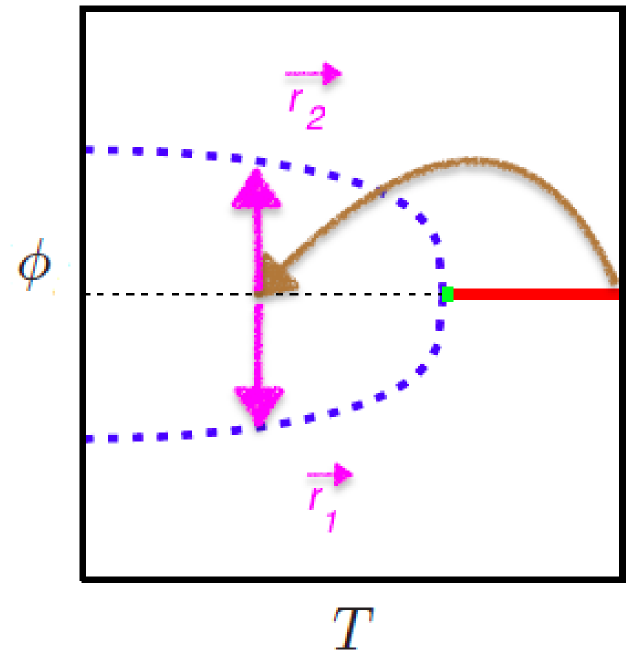
A certain number of interfaces or domain walls in the last snapshots.

Coarsening

Domain growth - dynamic scaling

$v[\phi(\vec{r})]$

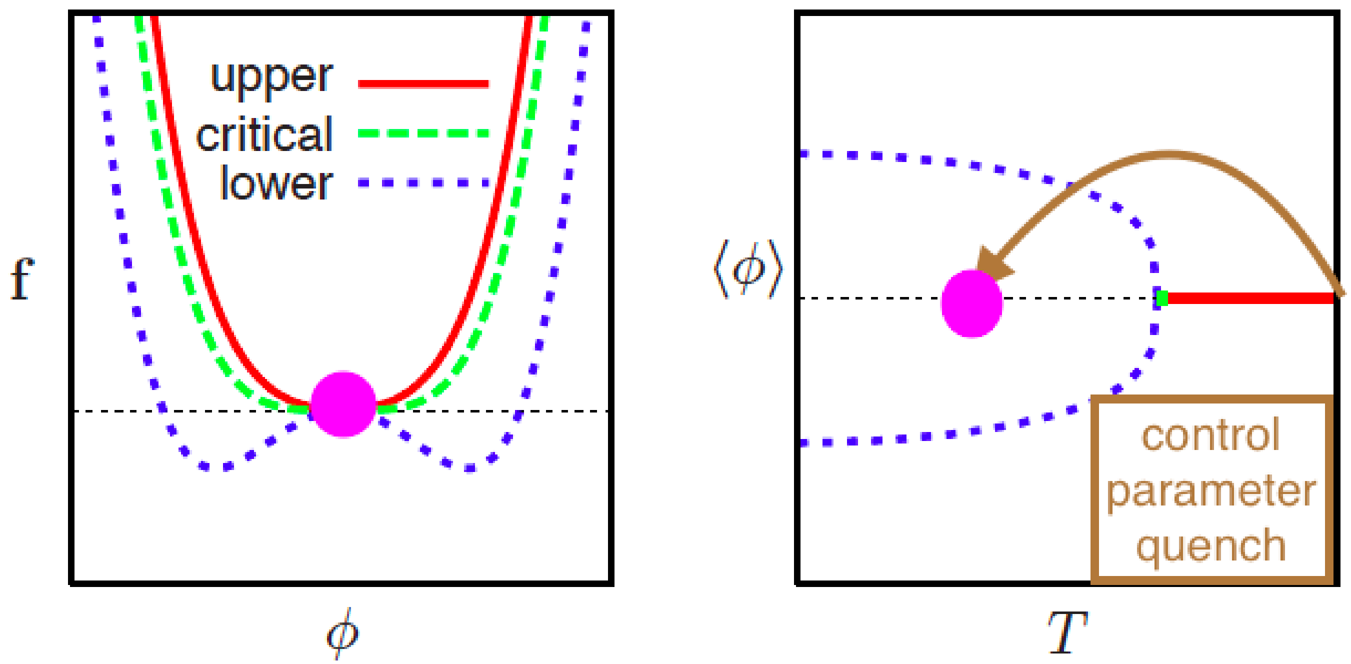




Locally the system chooses one or the other equilibrium state order with equal probability

Coarsening

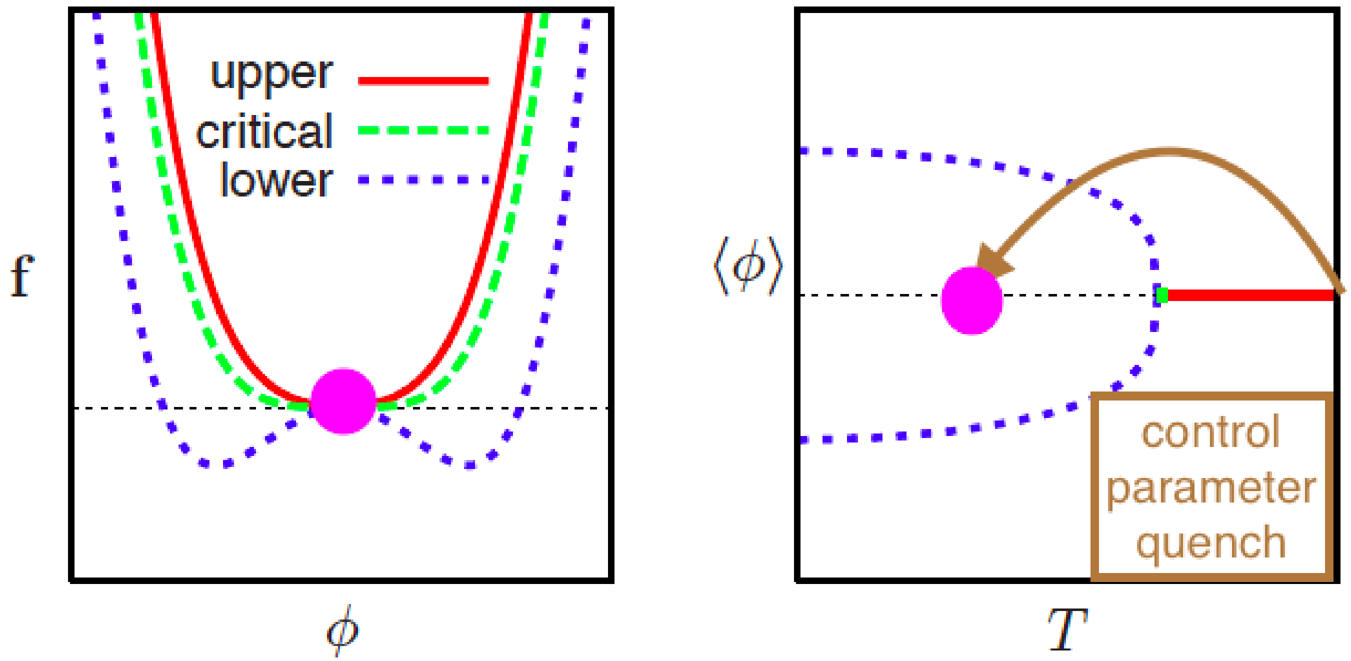
Domain growth - dynamic scaling



In the $L \to \infty$ limit, for finite times with respect to L, the global or averaged order parameter vanishes, $\langle \phi \rangle = 0$, and the representative point remains at the top of the barrier

Coarsening

Domain growth - dynamic scaling



Note that in this representation there is only one independent variable, the global ϕ. The remaining $L^d - 1$ "longitudinal" (flat) directions orthogonal to this one are not shown

Dynamic scaling

Dynamic Scaling

Growth of ‘red and white’ patches separated by

interfaces which surround such geometric domains

Spatial regions of local equilibrium (with non-vanishing order parameter, at $T < T_c$) grow in time

A single growing length $\mathcal{R}(t, T)$ can be identified and will be at the heart of dynamic scaling.

Here and in the following we measure T in units of J

Dynamic Scaling

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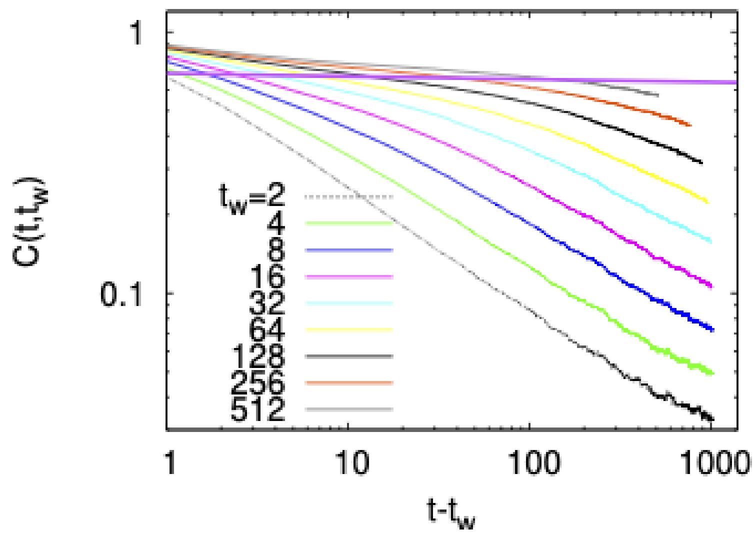
Here and in the following we measure T in units of J

Two-time self-correlation

e.g., MC simulation of the $2d$IM at $T < T_c$

$$\boxed{C(t,t_w) = N^{-1}\sum_{i=1}^{N}\langle s_i(t)s_i(t_w)\rangle}$$



Stationary relaxation

$m_{\rm eq}^2$

Aging decay

Separation of time-scales : stationary – aging

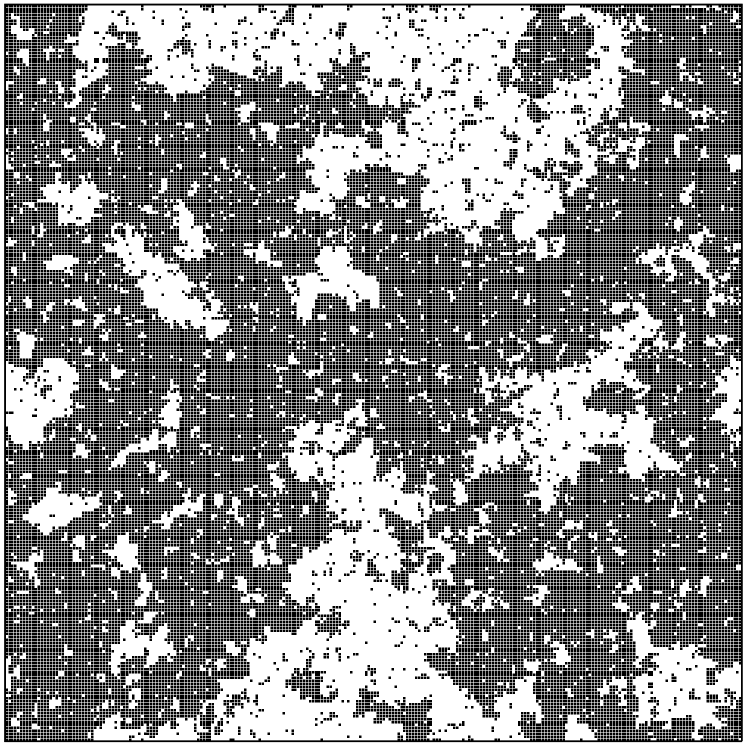
$$\boxed{C(t,t_w) = C_{st}(t-t_w) + m_{\rm eq}^2\, f\left(\frac{\mathcal{R}(t,T)}{\mathcal{R}(t_w,T)}\right)}$$

$$C_{st}(0) = 1 - m_{\rm eq}^2, \quad \lim_{x\to\infty} C_{st}(x) = 0, \quad f(1) = 1, \quad \lim_{x\to\infty} f(x) = 0$$

Interpretation

Thermal fluctuations within domains & domain-wall motion

$C_{\mathrm{st}}(t-t_w) \qquad\qquad m_{\mathrm{eq}}^2\, f(\mathcal{R}(t,T)/\mathcal{R}(t_w,T))$



Statements

Non-frustrated collective relaxation

Slow (non exponential) relaxation

Non-trivial collective effects and competition

From the point of view of landscapes :

importance of flat directions along hidden/longitudinal directions)

Effect of bath correlations (memory) on the relaxation

More on effective Langevin equations with memory and colou-red noise later

Scales

Times and system size

Thermodynamic limit taken first

$$\lim_{t\to\infty}\lim_{L\to\infty}$$

just coarsening is observed, dynamics far from equilibrium

Diverging times with system size, different regimes, depending on how one scales t and L :

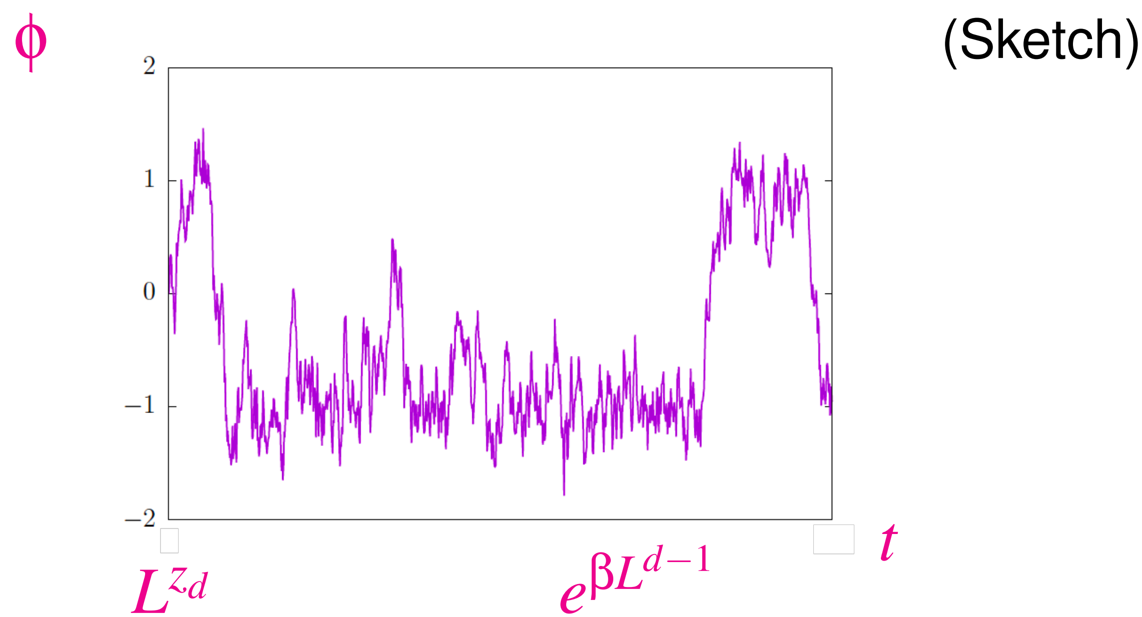
- $t = O(L^{z_d})$ approach to order, spontaneous symmetry breaking
- $t = O(e^{\beta\Delta F})$ jump over barrier global reversals

Scales

Times and system size

Diverging times with system size, different regimes, depending on how one scales t and L :

- $t = O(L^{z_d})$ approach to order, spontaneous symmetry breaking
- $t = O(e^{\beta L^{d-1}})$ jump over barrier global reversals **Ex.**



Scales

Times and system size

Diverging times with system size, different regimes, depending on how one scales t and L :

- $t = O(L^{z_d})$ approach to order, spontaneous symmetry breaking
- $t = O(e^{\beta L^{d-1}})$ jump over barrier global reversals

In this case, it is easy to beat the slow dynamics :

apply a pinning field that selects one equilibrium state

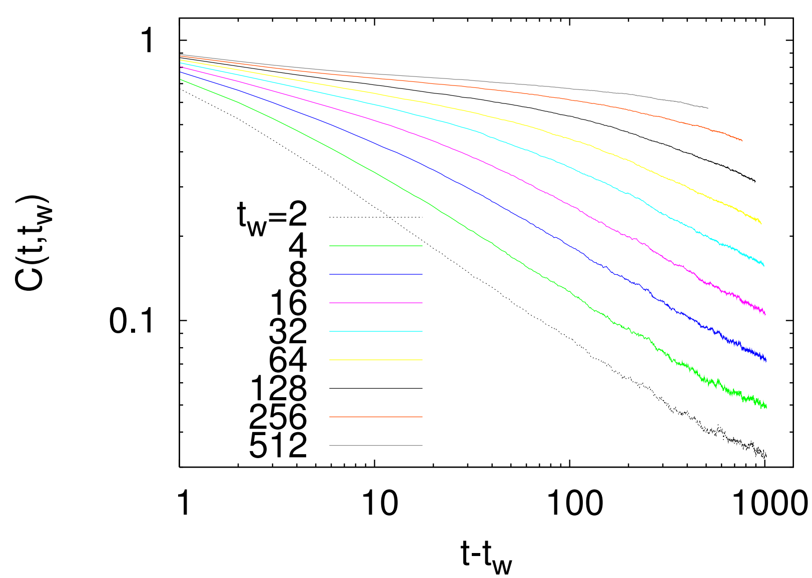
$\mathcal{H} \mapsto \mathcal{H} - h \sum_{i=1}^{N} s_i$ implying $F[\phi] \mapsto F[\phi] - h \int d^d r \, \phi(\vec{r})$

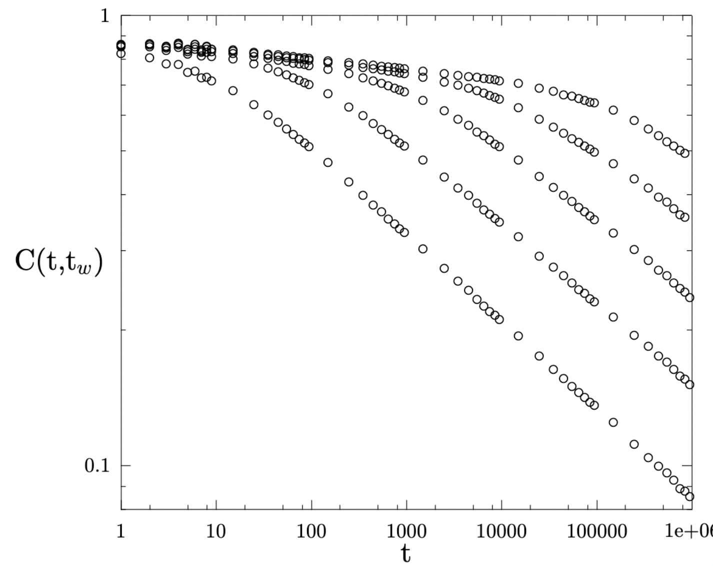
More on flat directions in this context **Kurchan & Laloux 96**

Spin glasses ?

Ferromagnet vs spin-glass

Not so different as long as correlations are concerned





$2d$ **FM Ising model** - spin-spin

Sicilia *et al.* 07

$3d$ **Edwards Anderson Model**

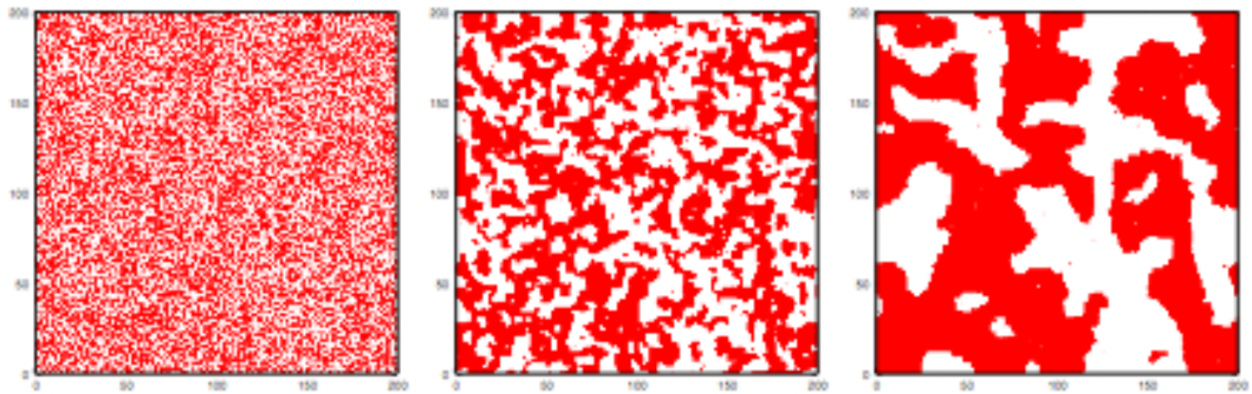
Rieger 93

One correlation can exhibit stationary and non stationary relaxation

in different two-time regimes

Aging in spin glasses

One possible explanation : domain growth



Growth of the two competing degenerate equilibrium states slowed down by quenched randomness

Dynamic scaling

Thermal activation arguments $t_A \sim e^{\beta \Delta F}$ & assumption $\Delta F \sim \mathcal{R}^{\psi}$ yield the unique growing length $\mathcal{R} \sim (k_B T \ln t)^{1/\psi}$

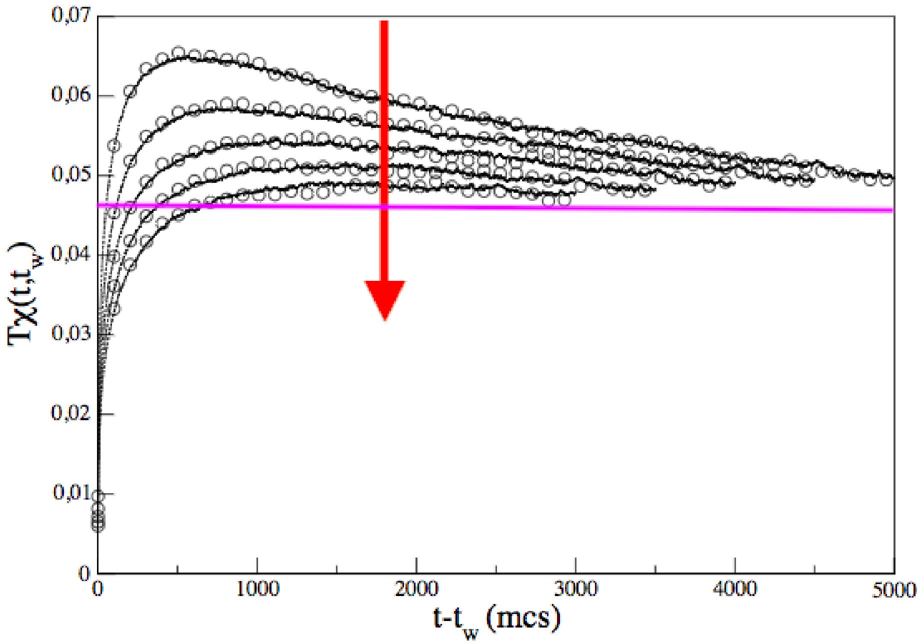
Fisher & Huse, Bray & Moore, late 80s, for spin-glasses, general review **Bray 94**

Thermo-remanent magnetisation

Critical and sub-critical coarsening

$$\boxed{m(t,t_w) = m_{\rm eq}(t-t_w) + [\mathcal{R}(t_w,T)]^{-a_\chi}\, g\left(\frac{\mathcal{R}(t,T)}{\mathcal{R}(t_w,T)}\right)}$$

$$m_{\rm eq}(t-t_w) = -(k_B T)^{-1} dC_{\rm eq}(t-t_w)/d(t-t_w)$$



$1 - m_{\rm eq}^2$

Lippiello, Corberi & Zannetti 05

The equilibration time of a **coarsening** system scales as

$$\mathcal{R}(t_{\rm eq}) \sim (k_B T \ln t_{\rm eq})^{\Psi} \sim L \quad \Rightarrow \quad t_{\rm eq} \sim \exp(\beta L^{1/\Psi})$$

$$\boxed{\lim_{N \gg 1} t_{\rm eq}(N, \text{parameters}) \gg t_{\rm exp}}$$

$t_{\rm eq}$ grows exponentially with a power of the system size L^d

Correlations **age** similarly to the ones in spin-glasses

Integrated responses, e.g. thermo-remanent magnetisation, **do not**

Dynamics

How does a (large) system reach equilibrium or

what does it do while it evolves out of equilibrium

- **One dimensional energy landscapes**
- **The simplest free-energy landscape : many variables - large dimensions**
- **Complex free-energy landscapes**

  Models

  Schwinger-Dyson equations & single variable equation

  Pure p-spin and SK-like models

  Separation of time scales, aging, FDT, effective temperatures

  Dynamics from the landscape point of view

  Non-potential forces

Characterisation of the Collective Relaxation

when there is no "visible" length $\mathcal{R}$

Global Observables

two-time correlations and linear responses

Two-time dependencies

Self displacement & correlation – integrated linear response

$$\left.\begin{aligned}\Delta^2(t,t') &\equiv \frac{1}{N}\sum_i [\langle (x_i(t)-x_i(t'))^2\rangle] && \text{Displacement}\\ C(t,t') &\equiv \frac{1}{N}\sum_i [\langle x_i(t)x_i(t')\rangle] && \text{Correlation}\end{aligned}\right\} \text{Unperturbed}$$

$$\chi(t,t') \equiv \frac{1}{N}\sum_i \int_{t'}^{t} dt''\, R_i(t,t'') = \frac{1}{N}\sum_i \int_{t'}^{t} dt'' \left[\left.\frac{\delta \langle x_i(t)\rangle_h}{\delta h_i(t'')}\right|_{h=0} \right]$$

Extend the notion of order parameter

They <u>are not</u> related by FDT out of equilibrium

The averages are thermal (and over initial conditions) $\langle \dots \rangle$

and over quenched randomness $[\dots]$ (if present)

t' "waiting-time" and t "measuring-time" after preparation

Mean-Field Modelling

Langevin dynamics

System coupled to a memory-less equilibrium bath

Overdamped limit $$\underbrace{\gamma \dot{s}_i(t)}_{\text{friction}} = \underbrace{\mathcal{F}_i[\{s_k(t)\}]}_{\text{force}} + \underbrace{\xi_i(t)}_{\text{white noise}}$$

Potential/conservative force $$\boxed{\mathcal{F}_i[\{s_k\}] = -\frac{\delta V[\{s_k\}]}{\delta s_i}}$$ with, *e.g.*

$\mathcal{V} = \sum_{i_1 \dots i_p} J_{i_1 \dots i_p} s_{i_1} \dots s_{i_p}$ and $J_{i_1 \dots i_p}$ fully symmetric w/r to $i_l \leftrightarrow i_m$

Non-potential force $$\boxed{\mathcal{F}_i[\{s_k\}] = -p \sum_{i_2 \dots i_p} J_{i i_2 \dots i_p} s_{i_2} \dots s_{i_p}}$$

with $J_{i_1 \dots i_p}$ fully symmetric w/r to $i_l \leftrightarrow i_m$ for $l, m = 2, \dots, p$ but asymmetric w/r to $i_1 \leftrightarrow i_m$ for $m = 2, \dots p$

Rheology, neural nets, etc.

Mean-Field Modelling

Classical p-spin Spherical Models

Potential energy

$$\mathcal{V} = - \sum_{i_1 \neq \ldots \neq i_p} J_{i_1 \ldots i_p} s_{i_1} \ldots s_{i_p} \qquad p \text{ integer}$$

quenched random couplings $J_{i_1 \ldots i_p}$ drawn from a Gaussian $P[\{J_{i_1 \ldots i_p}\}]$

(over-damped) **Langevin dynamics** for continuous spins $s_i \in \mathbb{R}$

coupled to a white bath $\langle \xi_i(t) \rangle = 0$ and $\langle \xi_i(t) \xi_j(t') \rangle = 2\gamma k_B T \, \delta_{ij} \, \delta(t - t')$

$$\gamma \frac{ds_i}{dt} = - \frac{\delta \mathcal{V}}{\delta s_i} + z_t s_i + \xi_i$$

z_t is a Lagrange multiplier that fixes the spherical constraint $\sum_{i=1}^{N} s_i^2 = N$

$p = 2$ mean-field **coarsening**

$p \geq 3$ **RFOT** modelling of **glasses** **Kirkpatrick, Thirumalai & Wolynes 87-89**

Langevin dynamics

System coupled to a memory-less equilibrium bath

Overdamped limit $$\underbrace{\gamma \dot{s}_i(t)}_{\text{friction}} = \underbrace{\mathcal{F}_i[\{s_k(t)\}]}_{\text{force}} + \underbrace{\xi_i(t)}_{\text{white noise}}$$

Initial conditions

- Rapid quench from very high temperature mimicked by random initial conditions $s_i(0)$, taken, e.g. from a Gaussian pdf $\Rightarrow$ uncorrelated with the quenched randomness $J_{i_1 \dots i_p}$.
- Correlated with $J^{\text{symm}}_{i_1 \dots i_p}$, e.g. $P_{\text{GB}}(\beta_{\text{in}})$ for some initial inverse temp. β_{in}

General : random potential correlation

$$\boxed{[V(\{s_k\})V(\{s'_l\})] = N f(N^{-1} \textstyle\sum_{k=1}^{N} s_k s'_k) = N f(C) \mapsto \frac{N}{2}\, C^p}$$

Dynamic equations

Integro-differential eqs. on the correlation and linear response

In the $N \to \infty$ limit exact causal Schwinger-Dyson equations

$$(\partial_t - z_t) C(t,t_w) = \int dt' \left[\Sigma(t,t') C(t',t_w) + D(t,t') R(t_w,t') \right] + 2T R(t_w,t) \; ,$$

$$(\partial_t - z_t) R(t,t_w) = \int dt' \, \Sigma(t,t') R(t',t_w) + \delta(t-t_w) \; ,$$

where the self-energy and vertex depend on C and R. For the potential p-spin model

$$\boxed{D(C) = f'(C) \qquad\qquad \Sigma(C,R) = D'(C) R = f''(C) R}$$

The Lagrange multiplier z_t is fixed by $C(t,t) = 1$. Random initial conditions.

LFC & Kurchan 93, Ben Arous, Dembo & Guionnet 06

Dynamic equations

Integro-differential eqs. on the correlation and linear response

In the $N \to \infty$ limit exact and closed causal Schwinger-Dyson equations

(Average over randomness, random initial conditions and thermal noise)

$$(\gamma\partial_t - z_t)C(t,t') = \int dt'' \left[\Sigma(t,t'')C(t'',t') + D(t,t'')R(t',t'')\right] + 2\gamma k_B T R(t',t)$$

$$(\gamma\partial_t - z_t)R(t,t') = \int dt''\, \Sigma(t,t'')R(t'',t') + \delta(t-t')$$

where Σ and D are the self-energy and vertex, which for p spin models read

$$D(t,t') = \frac{p}{2} C^{p-1}(t,t') \qquad \Sigma(t,t') = \frac{p(p-1)}{2} C^{p-2}(t,t')R(t,t')$$

z_t is fixed by $C(t,t)=1$ **Sompolinsky & Zippelius 82, LFC & Kurchan 93**

Similar to **Mode-Coupling Theory** for liquids **Götze et al 80s** or **DMFT** for quantum systems **Georges & Kotliar 90s**, but not necessarily in equilibrium

Dynamic equations

Equilibrium at $\beta_{\rm in}$ initial conditions

In the $N \to \infty$ limit exact causal Schwinger-Dyson equations

$$(\gamma\partial_t - z_t)R(t,t_w) = \int dt'\, \Sigma(t,t')R(t',t_w) + \delta(t-t_w)$$

$$(\gamma\partial_t - z_t)C(t,t_w) = \int dt' \left[\Sigma(t,t')C(t',t_w) + D(t,t')R(t_w,t')\right]$$

$$\boxed{+\beta_{\rm in}\sum_{a=1}^{n} D_a(t,0)C_a(t_w,0)} + 2\gamma k_B T R(t_w,t)$$

$$\boxed{(\gamma\partial_t - z_t)C_a(t,0) = \int dt'\,\Sigma(t,t')C_a(t',0) + \beta_{\rm in}\sum_{a=1}^{n} D_b(t,0)Q_{ab}}$$

$a = 1,\ldots,n \to 0$, replica method to deal with $e^{-\beta_{\rm in}H}$ and fix Q_{ab}

Houghton, Jain & Young 86, Franz & Parisi 95, Barrat, Burioni & Mézard 96

Dynamic equations

Single variable self-consistent over-damped Langevin equation

$$\underbrace{\gamma \dot{s}_i(t)}_{\text{friction}} + \underbrace{\int_0^t dt'\, \Sigma(t,t') s_i(t')}_{\text{self-generated friction}} - \underbrace{z(t) s_i(t)}_{\text{sph constraint}} =$$

$$= \underbrace{\xi_i(t)}_{\text{white noise}} + \underbrace{\eta_i(t)}_{\text{self-consist noise}}$$

$$\langle \eta_i(t) \eta_j(t') \rangle = \delta_{ij} D(t,t')$$

Σ and D are self-consistently determined in terms of correlations and linear responses of the original fields. The as in the SD eqs.

cfr. **Mode Coupling Th (Götze et al), Dynamic Mean-Field Th (Georges & Kotliar)**

Sompolinsky 81, LFC & Kurchan 00

How to solve these equations ?

Input from numerical solutions $\Longrightarrow$

Asymptotic Ansatz

Weak ergodicity breaking

$$\lim_{t-t'\to\infty} \lim_{t'\to\infty} C(t,t') = q_{\mathrm{EA}} \qquad \lim_{t\gg t'} C(t,t') = 0$$

Bouchaud 92

Weak long-term memory

$$\lim_{t-t'\to\infty} \lim_{t'\to\infty} R(t,t') \simeq 0$$

but

$$\sigma(t,t') = \int_0^{t'} dt''\, R(t,t'') \longrightarrow f_\sigma(C(t,t')) = \text{finite}$$

LFC & Kurchan 93

allow us to solve the integro-differential eqs. asymptotically

Weak ergodicity breaking

$$\lim_{t-t'\to\infty} \lim_{t'\to\infty} C(t,t') = q_{\mathrm{EA}} \qquad\qquad \lim_{t\gg t'} C(t,t') = 0$$

Bouchaud 92

Weak long-term memory

$$\lim_{t-t'\to\infty} \lim_{t'\to\infty} R(t,t') \simeq 0$$

but

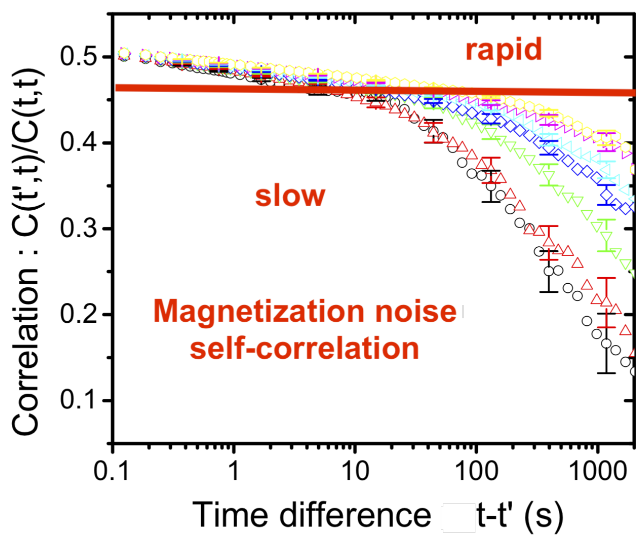
$$\sigma(t,t') = \int_0^{t'} dt''\, R(t,t'') \longrightarrow f_\chi(C(t,t')) = \text{finite}$$

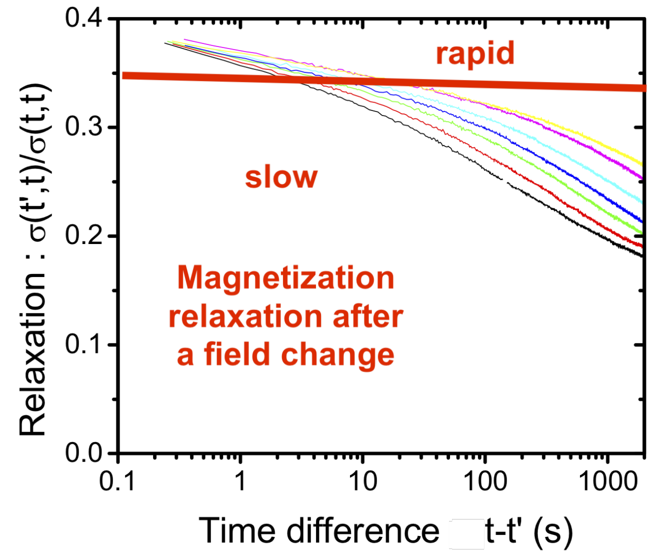
LFC & Kurchan 93

and capture aging

Slow Relaxation & Aging

Separation of time scales & loss of stationarity





Correlation **Integrated linear response**

0 t' t' t' t' time

Different curves are measured after log-spaced times t' after the quench

breakdown of stationarity $\Longrightarrow$ **aging**, far from equilibrium

$\boxed{\text{microscopic mechanisms ?}}$

Spin-glass experiments **Hérisson & Ocio 02-04**

Physical Aging

in words & scaling

Older systems (more time elapsed after the quench, longer t')

relax **more slowly** than younger ones

Breakdown of stationarity of correlation & integrated response

$$C(t,t') \neq C(t-t') \qquad \sigma(t,t') \neq \sigma(t-t')$$

In each regime – $\underbrace{\text{rapid } (r)}_{\text{above plateau}}$ and $\underbrace{\text{slow } (s)}_{\text{below plateau}}$ – there is scaling*

$$C(t,t') = C_{r,s}\left(\frac{\mathcal{R}_{r,s}(t)}{\mathcal{R}_{r,s}(t')}\right) \qquad \sigma(t,t') = \sigma_{r,s}\left(\frac{\mathcal{R}_{r,s}(t)}{\mathcal{R}_{r,s}(t')}\right)$$

* proven from general properties of temporal correlation functions

LFC & Kurchan 94

but no "visual" interpretation of $\mathcal{R}(t)$ in glassy systems

Physical Aging & Memory

the rate of relaxation

Older systems (more time elapsed after the quench, longer t')

relax **more slowly** than younger ones

Time variation - rate of change

$$\partial_t C_{r,s}\left(\frac{\mathcal{R}_{r,s}(t)}{\mathcal{R}_{r,s}(t')}\right) = C'_{r,s}\left(\frac{\mathcal{R}_{r,s}(t)}{\mathcal{R}_{r,s}(t')}\right)\frac{\dot{\mathcal{R}}_{r,s}(t)}{\mathcal{R}_{r,s}(t')} = \overline{C}_{r,s}\left(C_{r,s}(t,t')\right)\frac{\dot{\mathcal{R}}_{r,s}(t)}{\mathcal{R}_{r,s}(t)}$$

difference between **rapid** and **slow**, depending on $\mathcal{R}(t)$

$$\underbrace{\partial_t C_r(t,t') \simeq C_r(t,t')}_{\text{e.g. } \mathcal{R} \sim e^{t/\tau} \text{ exponential}} \qquad \underbrace{\partial_t C_s(t,t') \ll C_s(t,t')}_{\text{e.g. } \mathcal{R} \sim t \text{ algebraic}}$$

Interpretation

Complex Landscapes

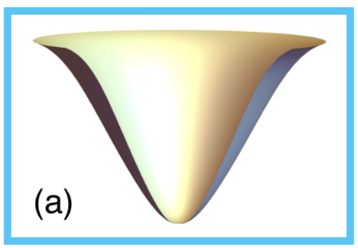
Beyond Ginzburg-Landau

Mean-field classes

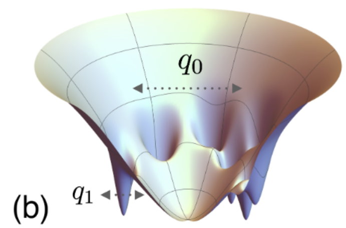
Domain growth

Equilibrium states

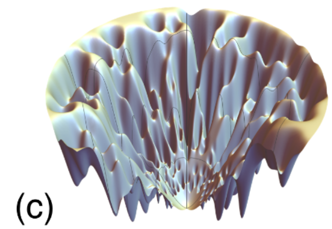
RS

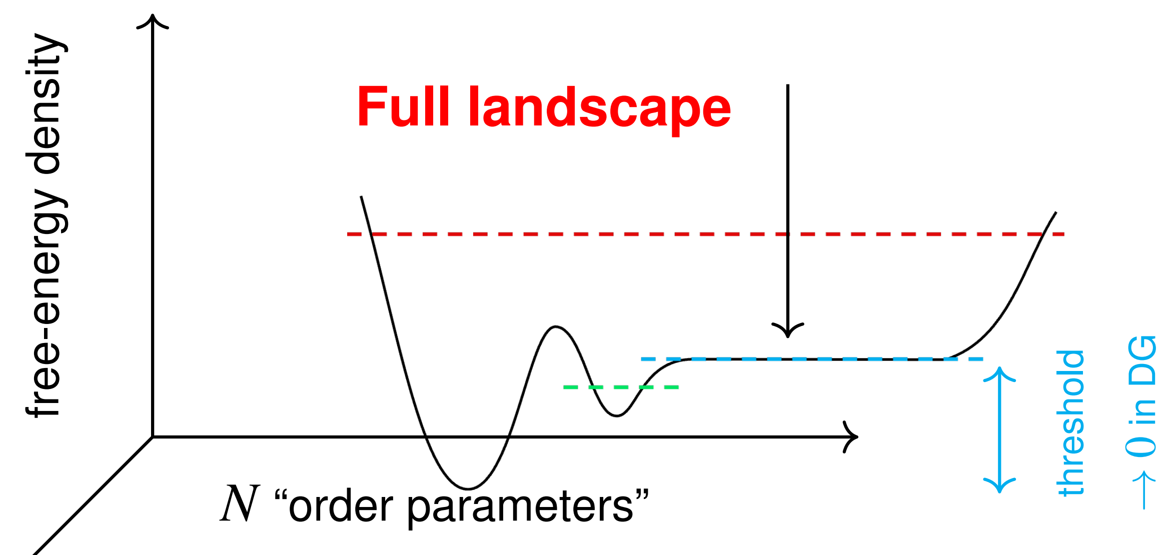


1RSB



FRSB





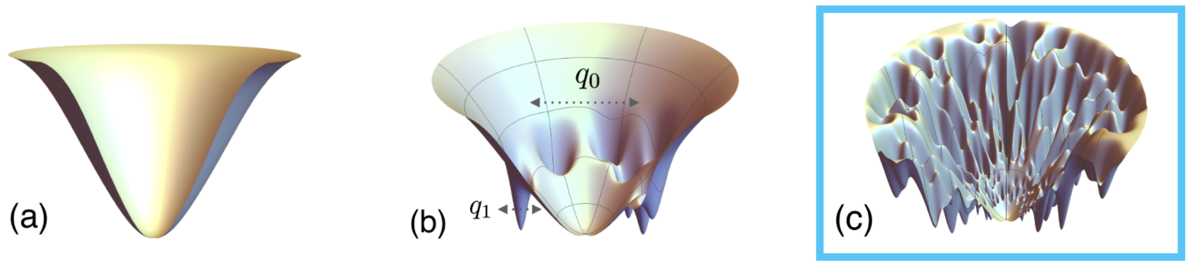
two minima - Ginzburg-Landau

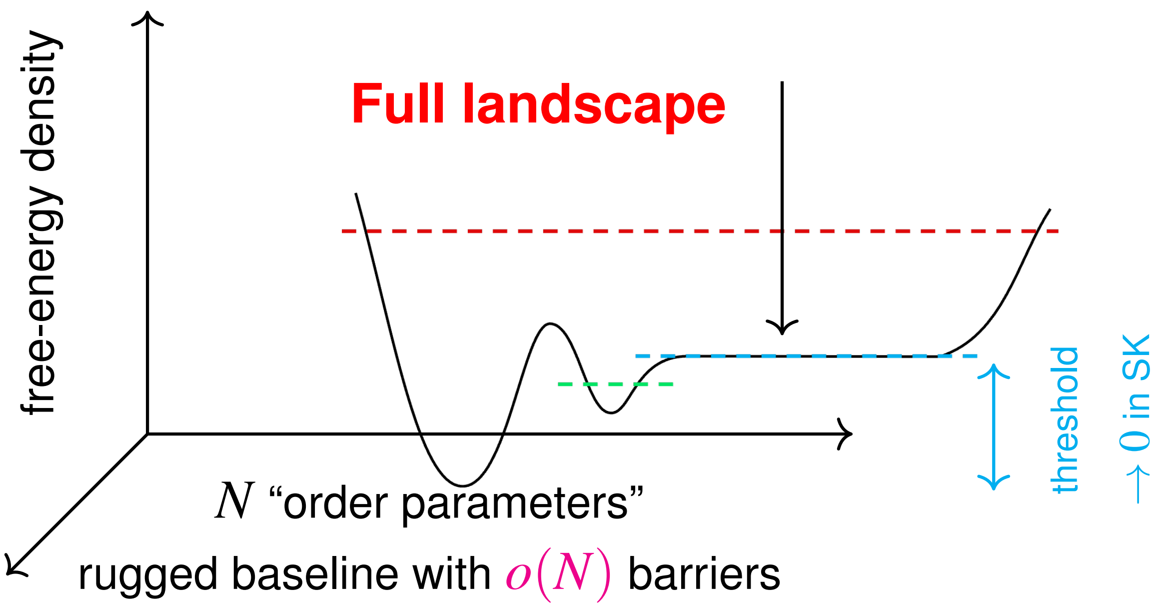
Mean-field classes

Sherrington-Kirkpatrick

Equilibrium states

RS **1RSB** **FRSB**



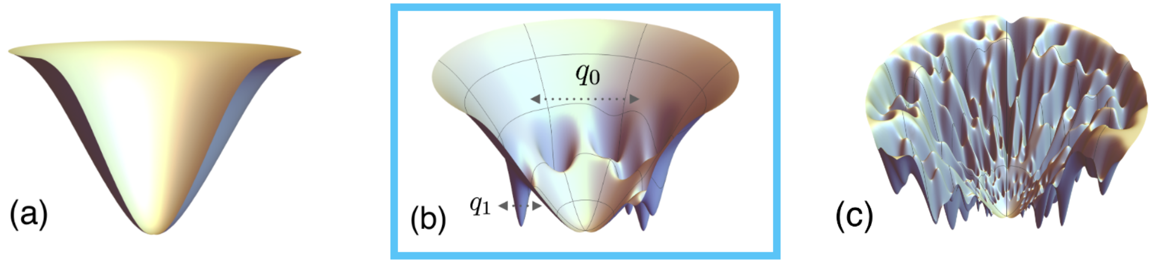


Mean-field classes

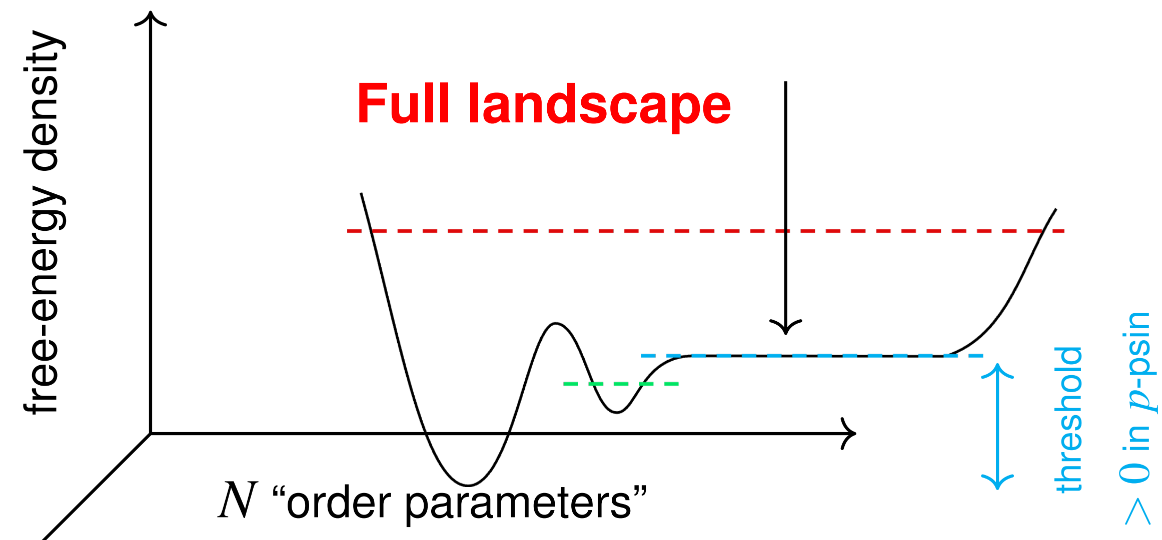
$p \geq 3$ spherical and Ising models

Equilibrium states

RS **1RSB** **FRSB**



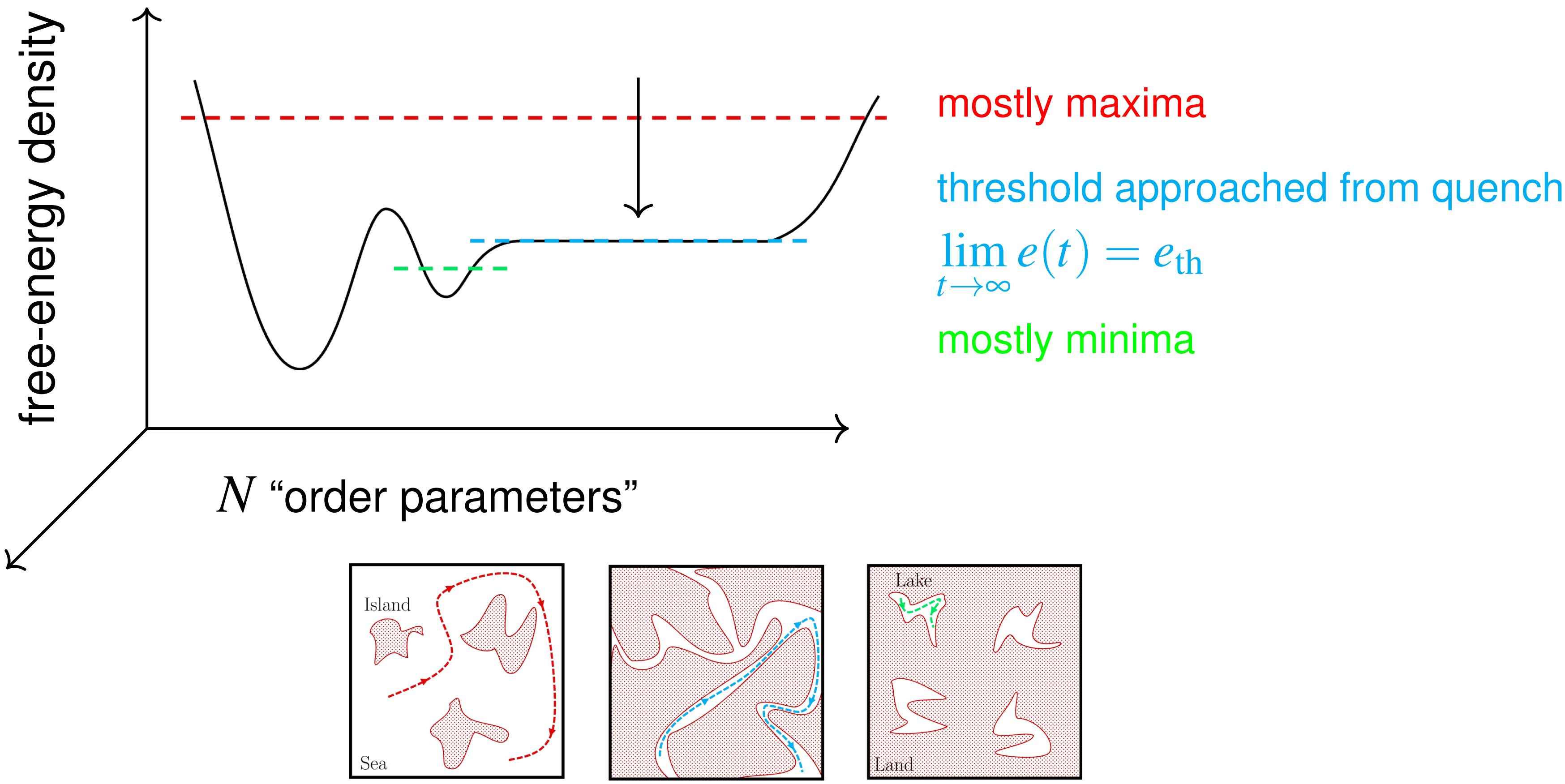
Full landscape



below threshold minima, $O(N)$ barriers

$p \geq 3$ spin models

Glassy class

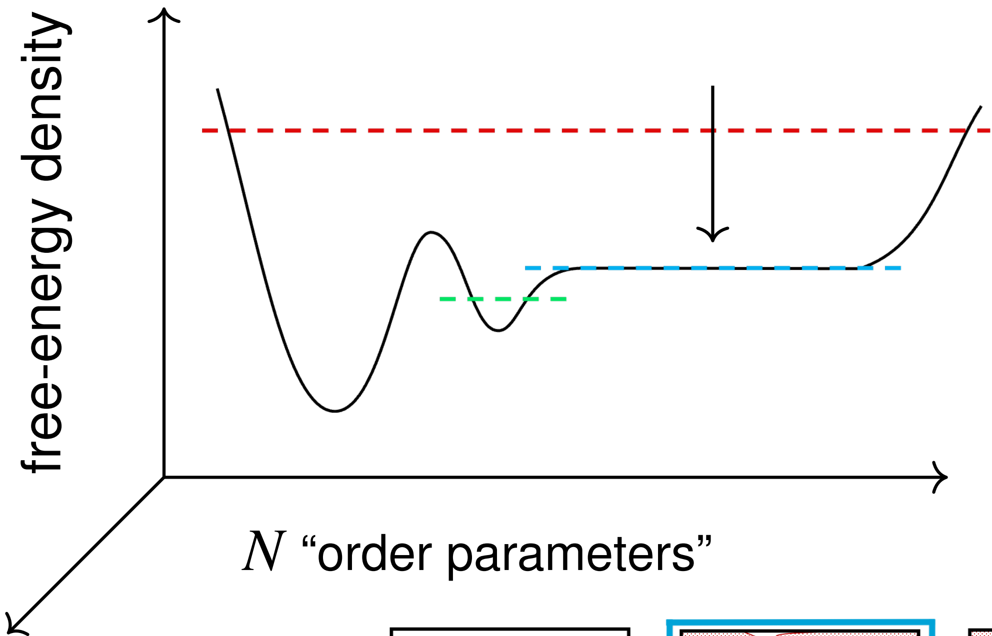


The dynamics are linked to the topography of the landscape

LFC & Kurchan 93

$p \geq 3$ spin models

Glassy class

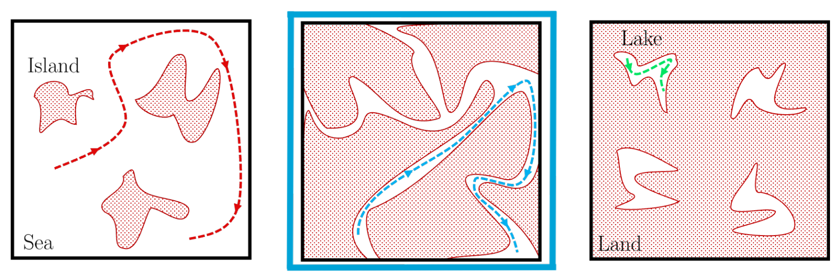


mostly maxima

threshold approached from quench

$$\lim_{t\to\infty} e(t) = e_{\text{th}}$$

mostly minima



Flat threshold as an attractor for the p-spin relaxation

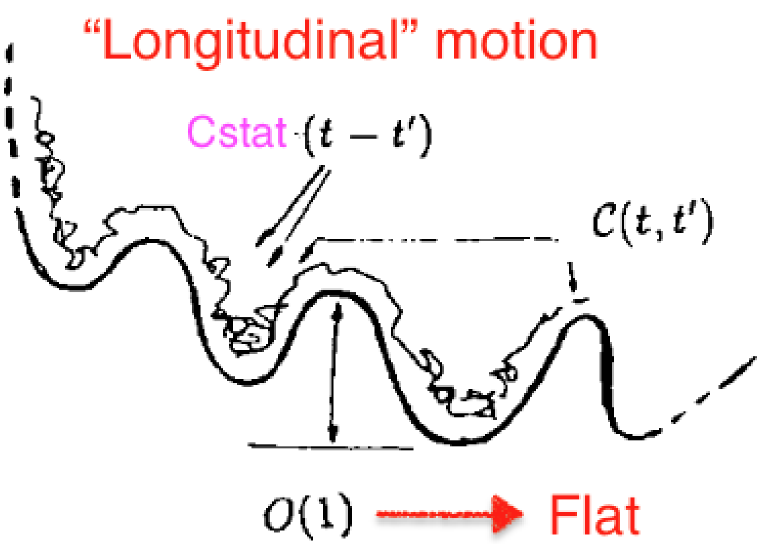
Both for **physical** and **algorithmic** dynamic rules

Quench from $T_0 > T_d$ to $T < T_d$

Long t' after $N \to \infty$: separation of time-scales

$$C(t,t') = C_{\rm ag}(t,t') + C_{\rm st}(t-t')$$

Slow motion



Fast motion

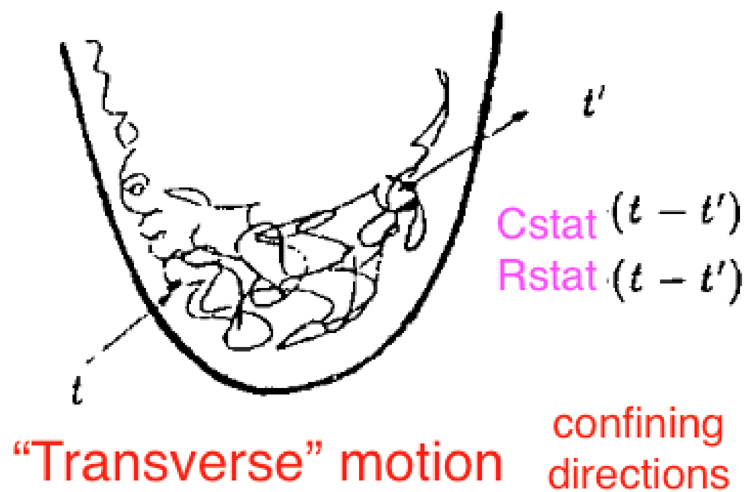
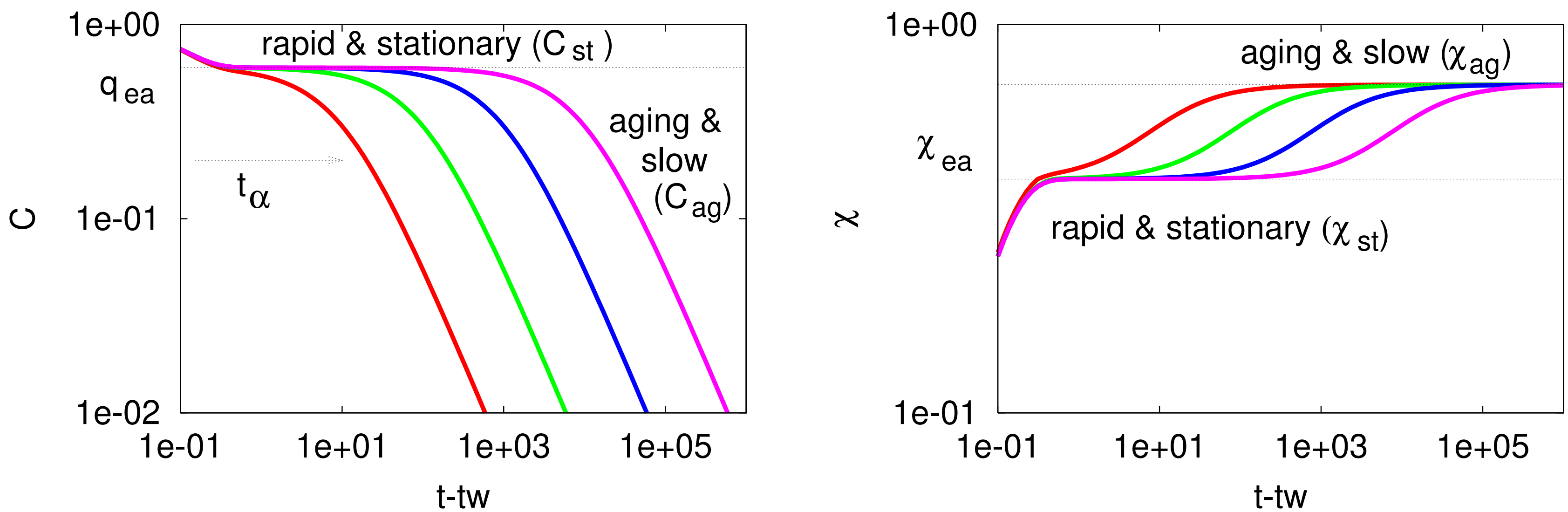


Image from **LFC & Kurchan 95**

Quench from $T_0 > T_d$ to $T < T_d$

Long t_w after $N \to \infty$: separation of time-scales



Sketch of the $\boxed{\text{separation of time-scales}}$ in the out of equilibrium relaxation

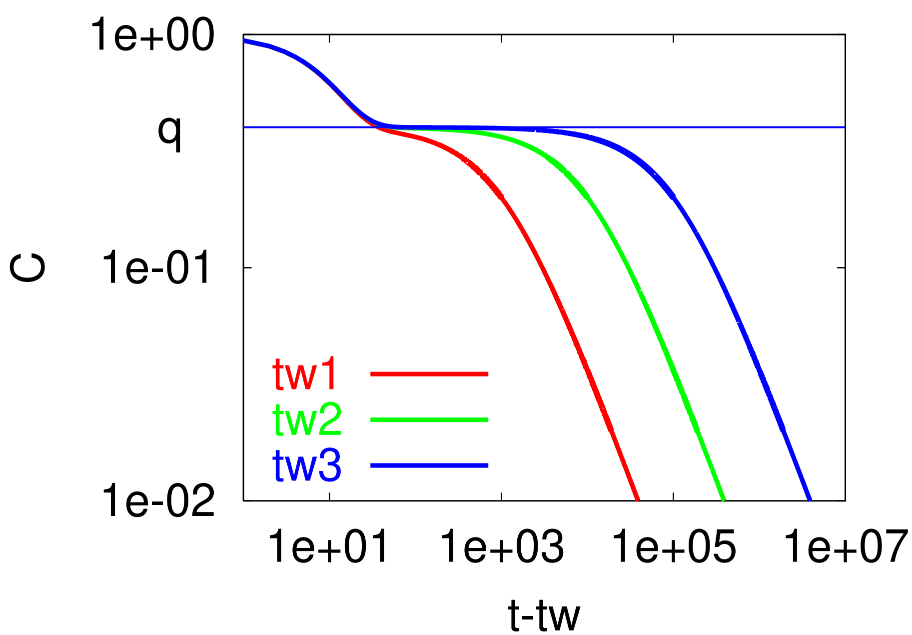
The actual form of the aging relaxation is different in the two families of models

Plugging such an *Ansatz* in the coupled integro-diff eqs. allows one to solve them

Quench from $T_0 > T_d$ to $T < T_d$

Long t_w after $N \to \infty$: separation of time-scales

Fast $\quad C(t,t_w) \approx f_{st}\left(\frac{L_{\rm st}(t)}{L_{\rm st}(t_w)}\right)$ with $L_{\rm st}(t) = e^{t/\tau}$



Slow

$C_{\rm ag}(t,t_w) \approx f_c\left(\frac{L(t)}{L(t_w)}\right)$

$\partial_t C_{\rm ag}(t,t_w) \ll C_{\rm ag}(t,t_w)$

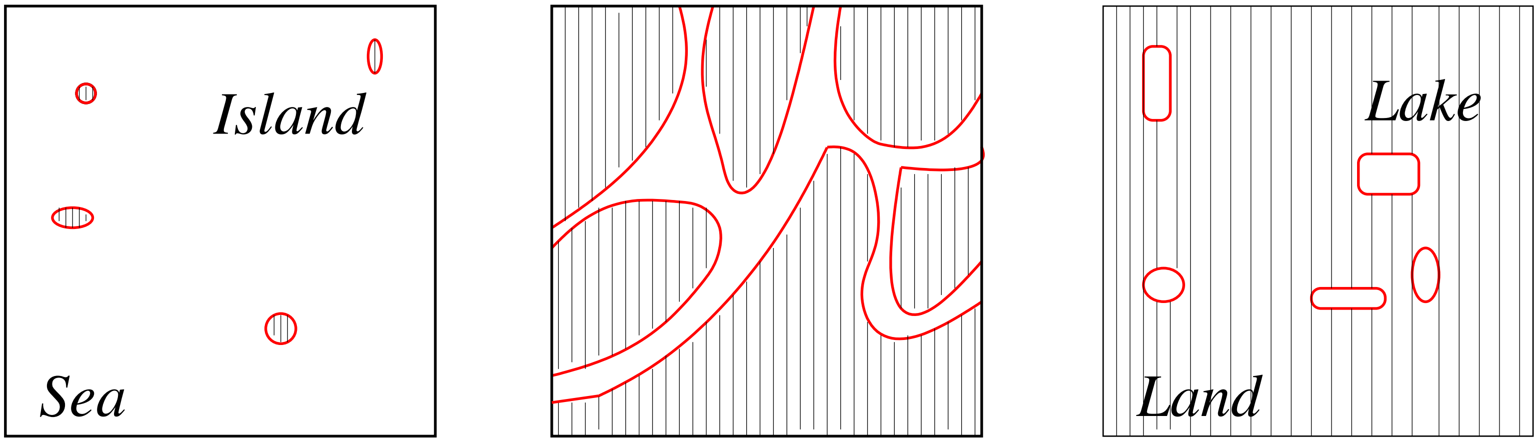
log-log scale !

Eqs. for the slow relaxation $C_{\rm ag} \equiv C < q$:

$\boxed{\text{Approx. asymptotic time-reparametization invariance}}$ $\qquad t \to h(t)$

View from above the landscape

Level determined by asymptotic (free-)energy density



Above threshold At threshold Below threshold

The system is like a ship :

freely navigating the sea above the threshold $T > T_d$

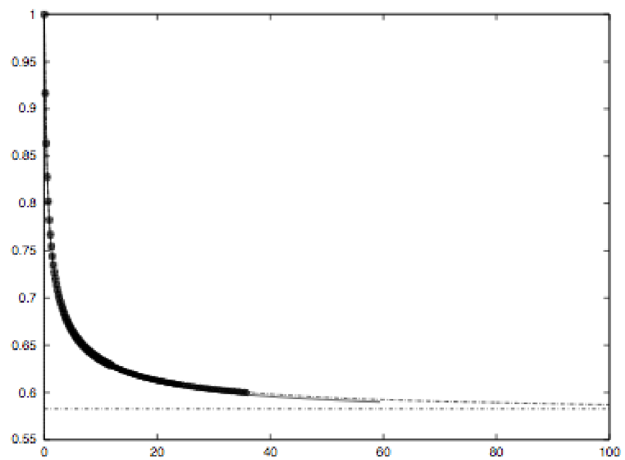
following "narrow" channels at the threshold $T < T_d$

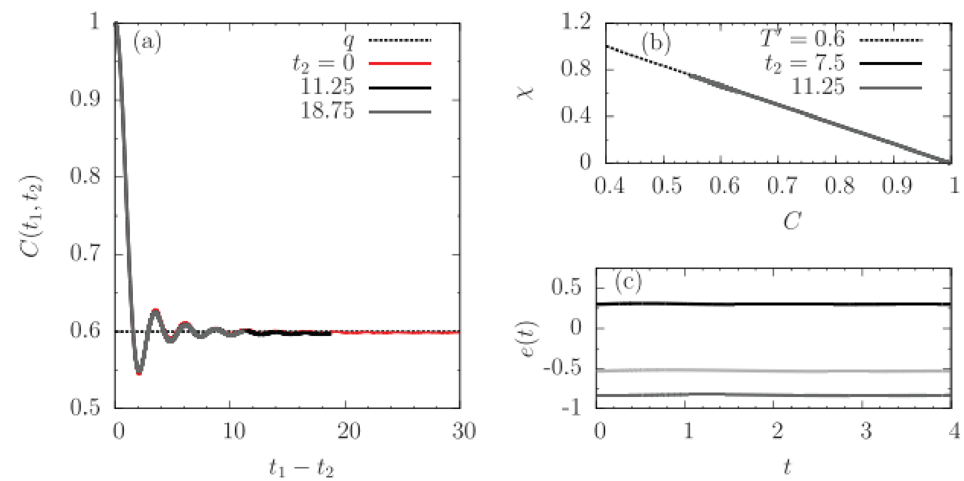
confined to lakes below the threshold $T_{\rm in} < T_d$ and $T < T_{\rm spinodal}$

Figs. from **LFC Les Houches 02**

Following TAP states

Long t_w after $N \to \infty$: trapping





$$C_0(t,0) \to c \neq 0$$

Barrat, Burioni & Mézard 96

Langevin dynamics

LFC, Lozano & Nessi 17

Newton dynamics isolated system

Statements

Pure p-spin model

The pure p spin model

Relaxation dynamics after a quench

separation of time-scales, stationary & aging

FDT and its violation

Partial thermal equilibration interpretation

Approach to the threshold flat level where marginal states are, $O(1)$ above the equilibrium energy density

Dynamic TAP equations

Following TAP states

No chaos in temperature, smooth

Some surprising predictions

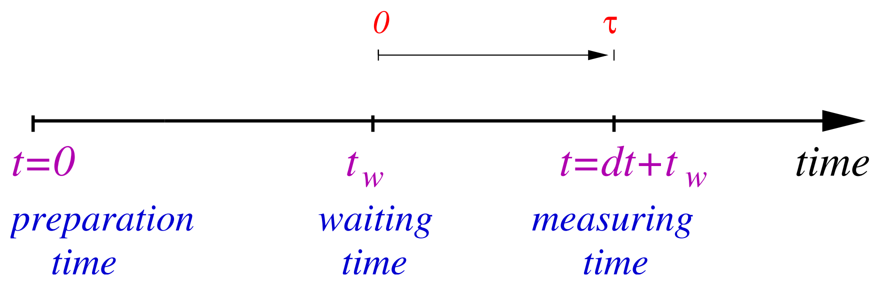
with physical consequences

In and out of equilibrium

Take a **mechanical point of view** and call $\{\vec{r}_i\}(t)$ the variables

e.g. the particles' coordinates $\{\vec{x}_i(t)\}$ and momenta $\{\vec{p}_i(t)\}$

Choose an initial condition $\{\vec{r}_i\}(0)$ and let the system evolve.



- For $t_w > t_{\mathrm{eq}}$: $\{\vec{r}_i\}(t)$ reach the equilibrium pdf and thermodynamics and statistical mechanics apply. **Temperature** is a well-defined concept.

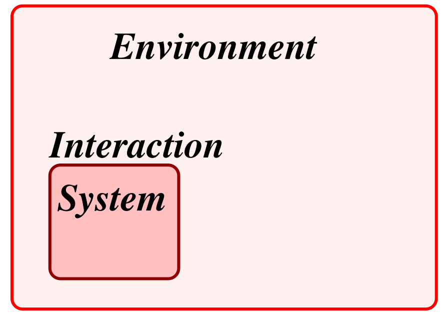
- For $t_w < t_{\mathrm{eq}}$: the system remains out of equilibrium and thermodynamics and (Boltzmann) statistical mechanics **do not** apply.

Is there a quantity to be associated with a temperature ?

Dynamics in equilibrium

Conditions

Take an open system coupled to an environment



Necessary :

— The bath should be in equilibrium

same origin of noise and friction.

— Deterministic force :

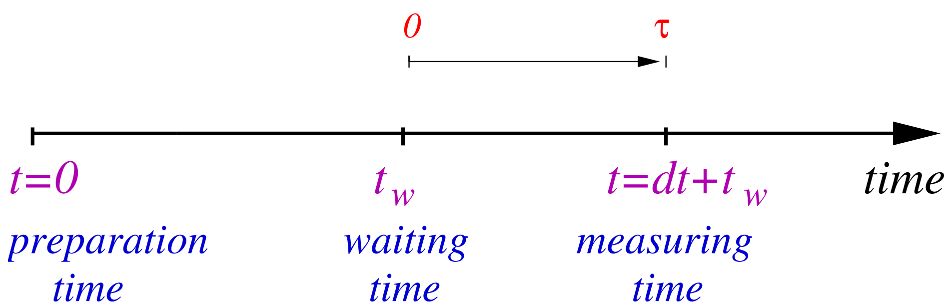
conservative forces only, $\vec{\mathcal{F}} = -\vec{\nabla}\mathcal{V}$.

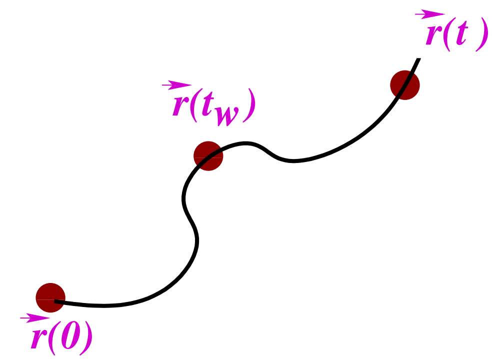
— Either the initial condition is taken from the equilibrium pdf, or the latter should be reached after an equilibration time $t_{\rm eq}$:

$$\boxed{P_{\rm eq}(v,x) \propto e^{-\beta(\frac{mv^2}{2}+\mathcal{V})}}$$

Two-time observables

Correlations





t_w not necessarily longer than $t_{\rm eq}$.

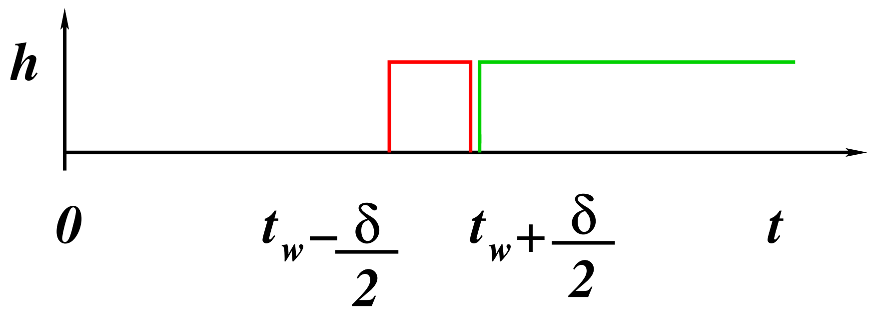
The two-time correlation between $A[\vec{x}(t)]$ and $B[\vec{x}(t_w)]$ is

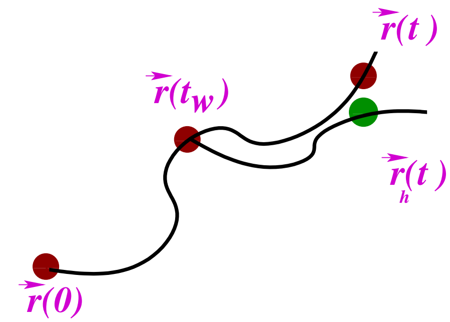
$$C_{AB}(t,t_w) \equiv \langle A[\vec{x}(t)]B[\vec{x}(t_w)]\rangle$$

average over realizations of the dynamics (initial conditions, random numbers in a MC simulation, thermal noise in Langevin dynamics, etc.)

Two-time observables

Linear response





The perturbation couples linearly to the observable $B[\vec{x}(t_w)]$

$$E \quad \to \quad E - hB[\vec{x}(t_w)]$$

The linear instantaneous response of another observable $A[\vec{x}(t)]$ is

$$R_{AB}(t,t_w) \equiv \left. \frac{\delta\langle A[\vec{x}(t)]\rangle_h}{\delta h(t_w)} \right|_{h=0}$$

The linear integrated response is $$\boxed{\chi_{AB}(t,t_w) \equiv \int_{t_w}^{t} dt' \, R_{AB}(t,t')}$$

Rue de Fossés St. Jacques et rue St. Jacques

Paris 5ème Arrondissement.

LFC

Fluctuation-dissipation

In equilibrium

$$\boxed{P(\vec{r}, t_w) = P_{\text{eq}}(\vec{r})}$$

- The dynamics are stationary

$$C_{AB} \to C_{AB}(t - t_w) \text{ and } R_{AB} \to R_{AB}(t - t_w)$$

- The fluctuation-dissipation theorem between spontaneous (C_{AB}) and induced (R_{AB}) fluctuations

$$R_{AB}(t - t_w) = -\frac{1}{k_B T} \frac{\partial C_{AB}(t - t_w)}{\partial t} \, \theta(t - t_w)$$

holds and implies

$$\boxed{\chi_{AB}(t - t_w) \equiv \int_{t_w}^{t} dt' \, R_{AB}(t, t') = \frac{1}{T}[C_{AB}(0) - C_{AB}(t - t_w)]}$$

Fluctuation-dissipation

Linear relation between χ and C

$$\boxed{P(\vec{r}, t_w) = P_{\text{eq}}(\vec{r})}$$

- The dynamics are stationary

$$C_{AB} \to C_{AB}(t - t_w) \text{ and } R_{AB} \to R_{AB}(t - t_w)$$

- The fluctuation-dissipation theorem between spontaneous (C_{AB}) and induced (R_{AB}) fluctuations

$$R_{AB}(t - t_w) = -\frac{1}{k_B T} \frac{\partial C_{AB}(t - t_w)}{\partial t} \theta(t - t_w)$$

holds and implies

$$\boxed{\chi_{AB}(t - t_w) \equiv \int_{t_w}^{t} dt' \, R_{AB}(t, t') = \frac{1}{k_B T} [C_{AB}(0) - C_{AB}(t - t_w)]}$$

Fluctuation-dissipation

A proof

The generic **Langevin equation** for a particle in $1d$ is

$$\boxed{m\ddot{x}(t) + M'[x(t)] \int_{-\mathcal{T}}^{t} dt'\, \Gamma(t-t') M'[x(t')]\dot{x}(t') = F(t) + \xi(t) M'[x(t)]}$$

with the coloured noise $\boxed{\langle \xi(t)\xi(t')\rangle = k_B T\, \Gamma(t-t')}$

The dynamic generating functional is a path-integral

$$\boxed{\mathcal{Z}_{dyn}[\eta] = \int dx_{-\mathcal{T}} d\dot{x}_{-\mathcal{T}} \int \mathcal{D}x \mathcal{D}i\hat{x}\; e^{-S[x,i\hat{x};\eta]}}$$

with $i\hat{x}(t)$ the ‘response’ variable.

$x_{-\mathcal{T}}$ and $\dot{x}_{-\mathcal{T}}$ are the initial conditions at time $-\mathcal{T}$.

Martin-Siggia-Rose-Jenssen-deDominicis formalism

Fluctuation-dissipation

A proof

The action has a deterministic part (Newton) that includes the initial condition and a dissipative part that depends upon Γ : $\boxed{S = S_{det} + S_{diss}}$

The transformation

$$\boxed{x(t) \to x(-t) \qquad\qquad i\hat{x}(t) \to i\hat{x}(-t) + \beta\dot{x}(-t)}$$

leaves S_{diss} and the path-integral measure invariant.

S_{det} is also invariant if $P(x_{-\mathcal{T}}, \dot{x}_{-\mathcal{T}}) = P_{\rm eq}(x_{-\mathcal{T}}, \dot{x}_{-\mathcal{T}})$, and $F = V'[x]$

The **FDT** valid for Newton or Langevin dynamics

$$\boxed{R_{AB}(t,t_w) + R_{AB}(t_w,t) = \beta\partial_{t_w} C_{AB}(t,t_w)}$$

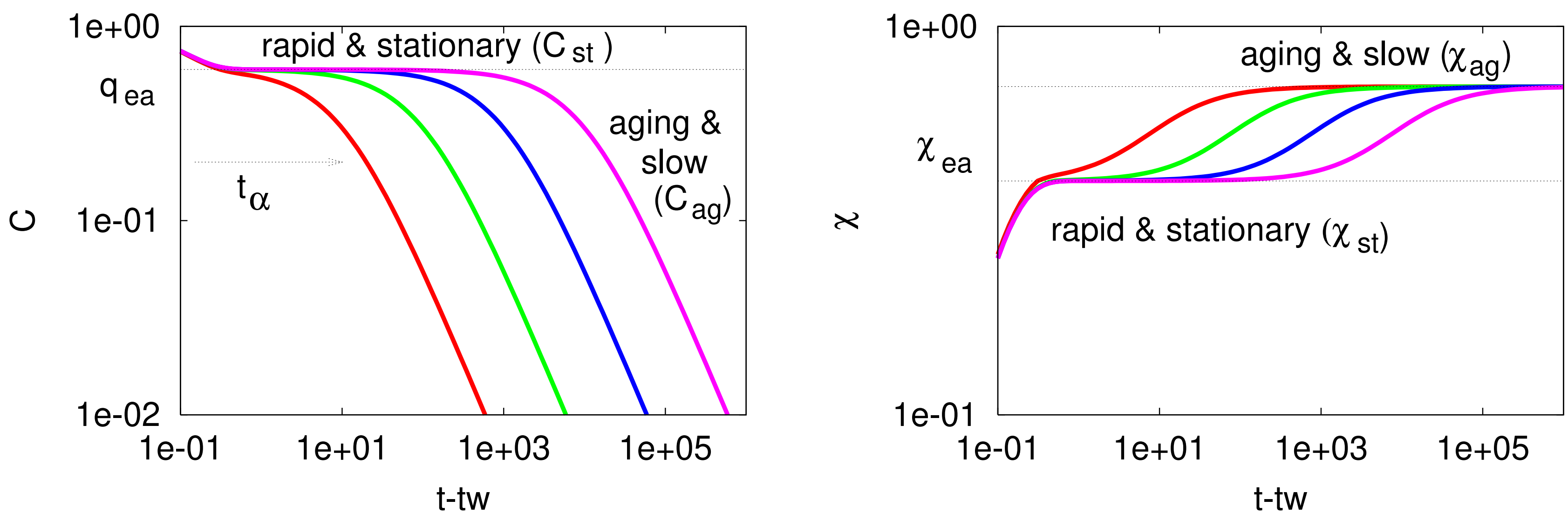
and higher-order extensions are Ward identities of this symmetry.

The fluctuation theorems can also be proven in this way.

Fluctuation-dissipation

Solvable glasses : p spin-models & mode-coupling theory

- Stochastic dynamics of a particle in an *infinite dimensional* space under the effect of a quenched random potential.
- A fully-connected (Curie approximation) spin model with as many ferromagnetic as antiferromagnetic couplings.

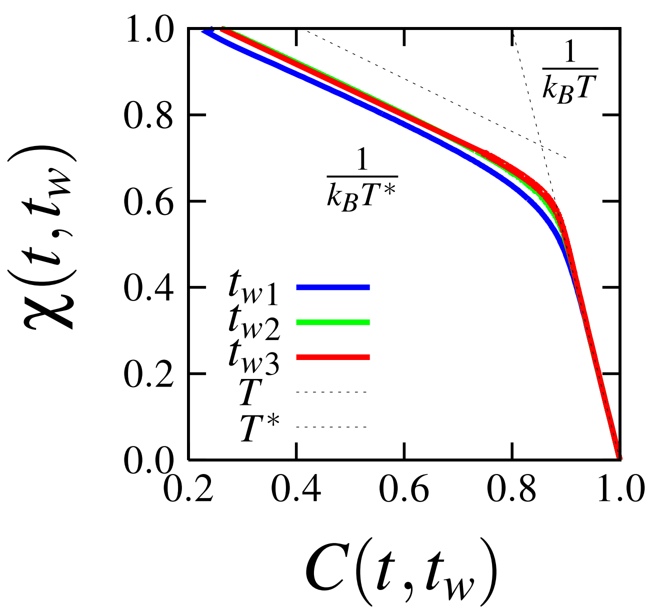


Sketch of the $\boxed{\text{separation of time-scales}}$ in the out of equilibrium relaxation

Fluctuation-dissipation

Induced vs. spontaneous fluctuations in glasses

A quench from $T_0 \to \infty$ to $T < T_c$



parametric construction

t_w fixed

$\boxed{t_{w_1} < t_{w_2} < t_{w_3}}$

$t - t_w$: $\quad 0 \to \infty$

used as a parameter

$\boxed{T^* > T}$

Breakdown of the equilibrium FDT $k_B T \chi = C$

Convergence to $k_B T \chi(C)$, two linear relations for $C \lessgtr q_{\rm EA}$

Mean-field **LFC & Kurchan 93** & effective temperature interpretation **LFC, Kurchan & Peliti 97**

Fluctuation-dissipation

Proposal

For non-equilibrium systems, relaxing slowly towards a situation such that **one-time quantities** [*e.g.* the energy-density $\mathcal{E}(t)$] **approach a finite value**

$$\boxed{\lim_{\substack{t_w \to \infty \\ C(t,t_w)=C}} \chi(t,t_w) = f_\chi(C)}$$

Also or weakly forced non-equilibrium systems in the limit of small work

The effective temperature is $$\boxed{-\frac{1}{k_B T_{\rm eff}(C)} \equiv \frac{\partial \chi(C)}{\partial C}}$$

LFC & Kurchan 94, Franz & Mézard 94

A short-time regime with FDT ?

A general property proven by a bound for Langevin dynamics

$$|\chi(t,t_w) - C(t,t) + C(t,t_w)| \leq K \left(-\frac{1}{N} \frac{dH(t_w)}{dt_w} \right)^{1/2}$$

with the “H-function”

$$H(t_w) = \int d\vec{x} d\vec{v}\, P(\vec{x},\vec{v},t_w)\, [k_B T \ln P(\vec{x},\vec{v},t_w) + \mathcal{H}(\vec{x},\vec{v})]$$

and its time variation $\frac{dH(t_w)}{dt_w} = -\langle \vec{f}(t_w) \cdot \vec{v}(t_w) \rangle - \sum_i g_i(t_w)$

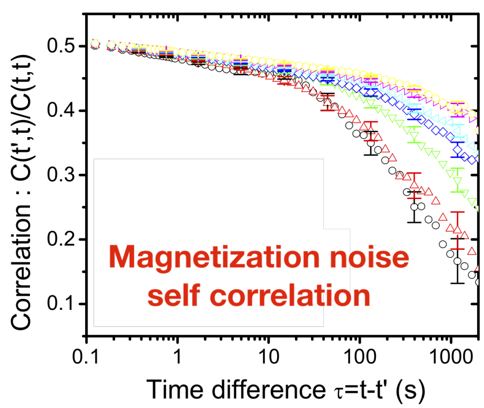
where the first term is the work done by eventual non-potential forces $\vec{f}$ and the second term is a sum of positive terms

$$g_i(t_w) = \gamma \int d\vec{x} d\vec{v}\, \frac{(m v_i P + T \partial_{v_i} P)^2}{m^2 P} \geq 0$$

LFC, Dean & Kurchan 97

Spin glass experiments

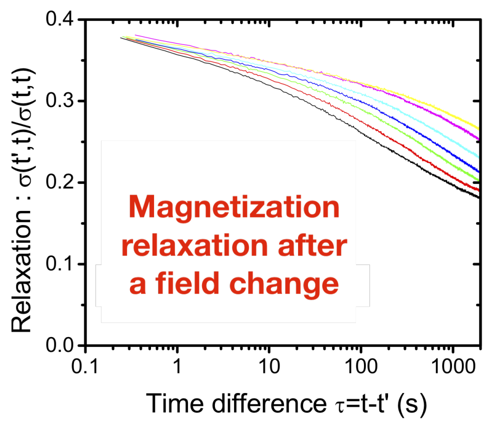
Aging, weak memory and fluctuation dissipation relations

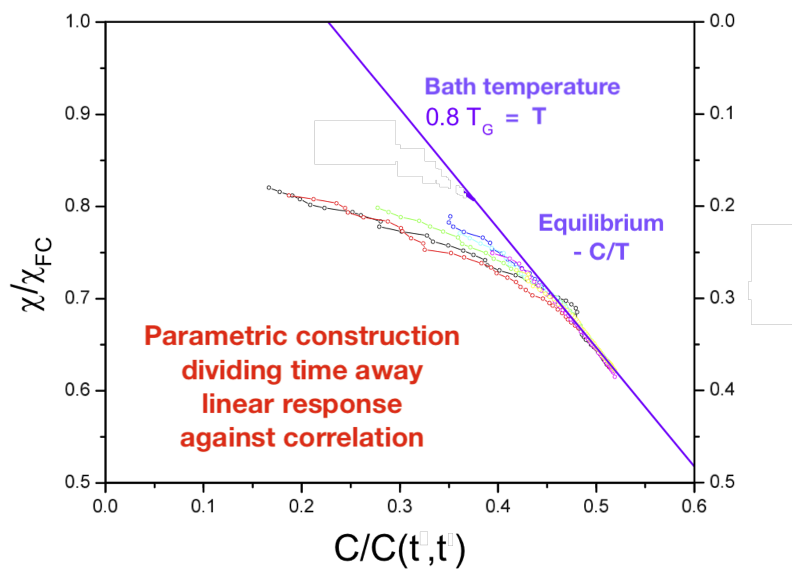


$$\chi(t,t') = \int_{t'}^{t} dt''\, \frac{1}{T_{\rm eff}(t,t'')}\, \frac{\partial C(t,t'')}{\partial t''} \qquad \text{General}$$

$$= \int_{t'}^{t} dt''\, \frac{1}{T_{\rm eff}(C(t,t''))}\, \frac{\partial C(t,t'')}{\partial t''} \qquad \text{Hypothesis}$$

$$= \int_{C(t,t')}^{C(t,t)} dC''\, \frac{1}{T_{\rm eff}(C'')} = \chi(C)$$

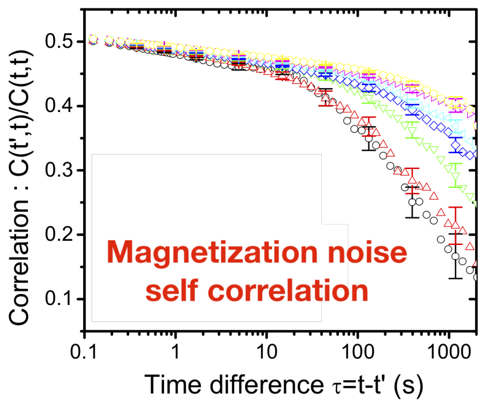


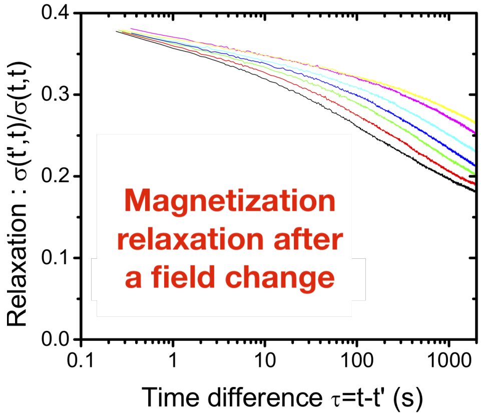


Insulating spin glass **Hérisson & Ocio 02-04**

Spin glass experiments

Putting two-time scales in evidence



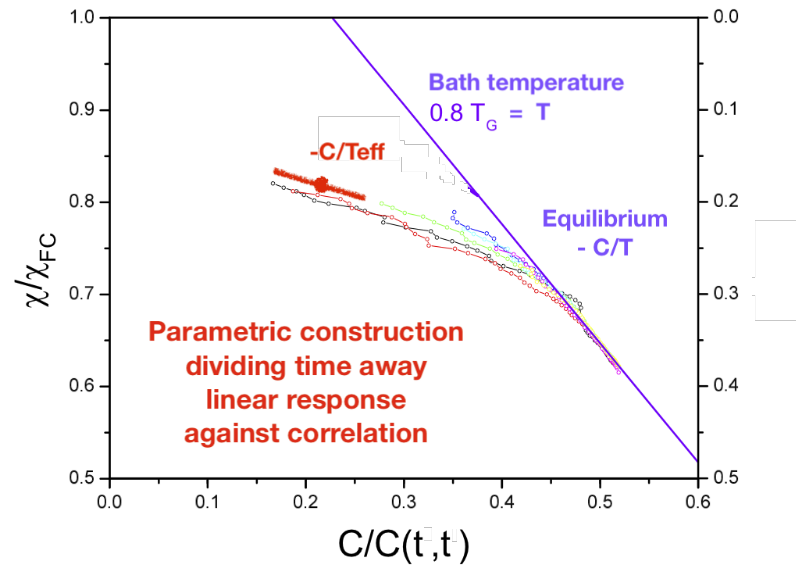


Global observables, χ and C

Separation of two-time scales

one with bath temperature T

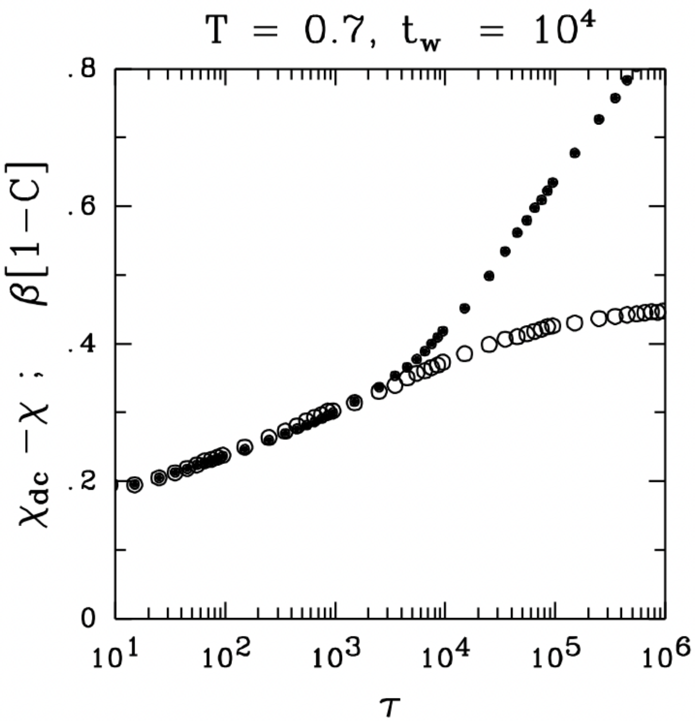
the rest with $T_{\rm eff}(C)$

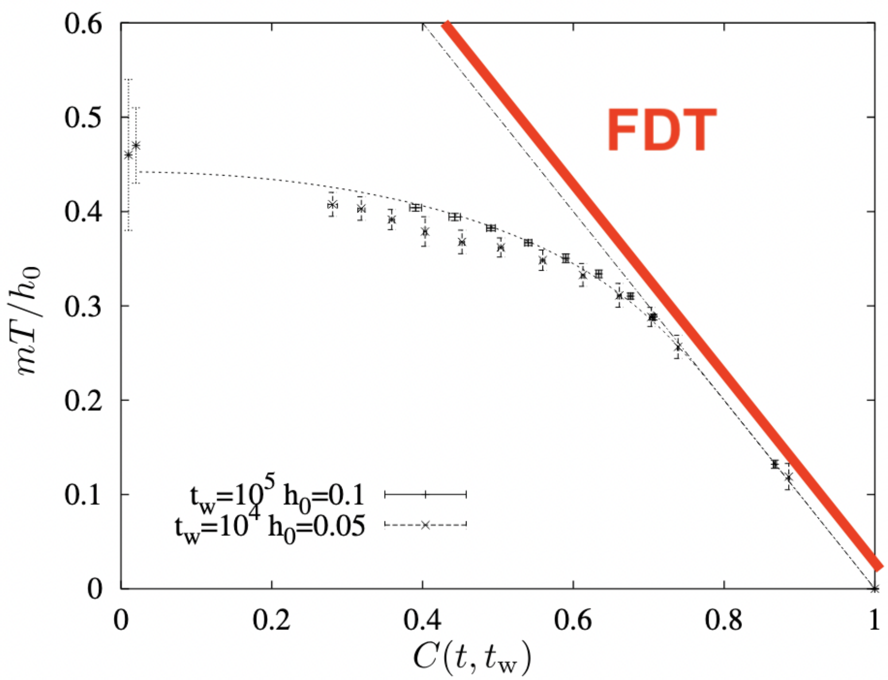


Insulating spin glass **Hérisson & Ocio 02-04**

Numerics

Induced vs. spontaneous fluctuations in 3D EA





Franz & Rieger 95 **Marinari, Parisi, Ricci-Tersenghi & Ruiz Lorenzo 98**

Dynamic FDT

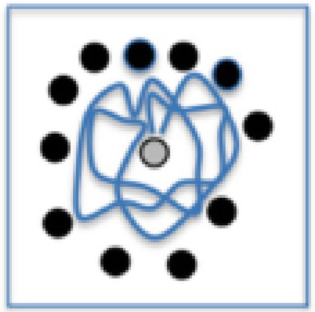
Effective Temperature

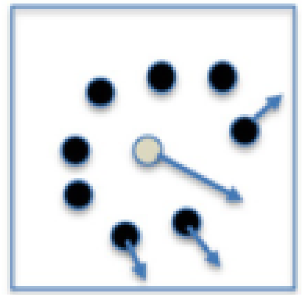
One can interpret the mismatch between the linear response and correlation function as evidence for an **effective temperature** whi-chdepends on the time-scale at which we look at the relaxation

$$\boxed{\frac{1}{k_B T_{\text{eff}}(C)} = -\frac{d\chi(C)}{dC}}$$

Fluctuation-dissipation

Interpretation





Short-scale re-arrangements ruled by the **equilibrium external bath** & **local properties of landscape**

The fluctuation-dissipation relation holds with the bath temperature T

Large-scale re-arrangements follow the systems' **internal dynamics** & **large scale props of landscape**

The fluctuation-disspation relation holds with another temperature T^*

After **cooling** from equilibrium at $T_0 > T_d$, hotter $T^* > T$

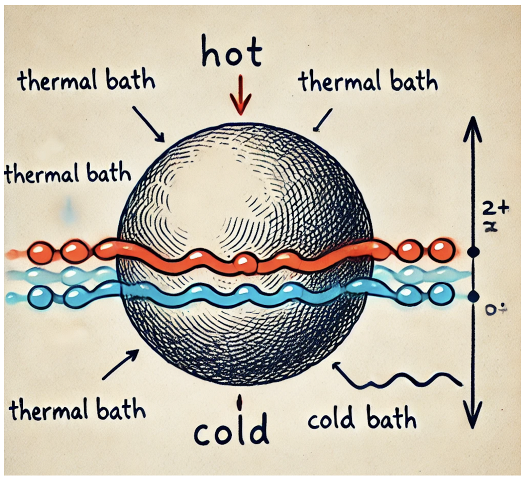
After **heating** from equilibrium at $T_0 < T_d$, colder $T^* < T$

Support for this interpretation :

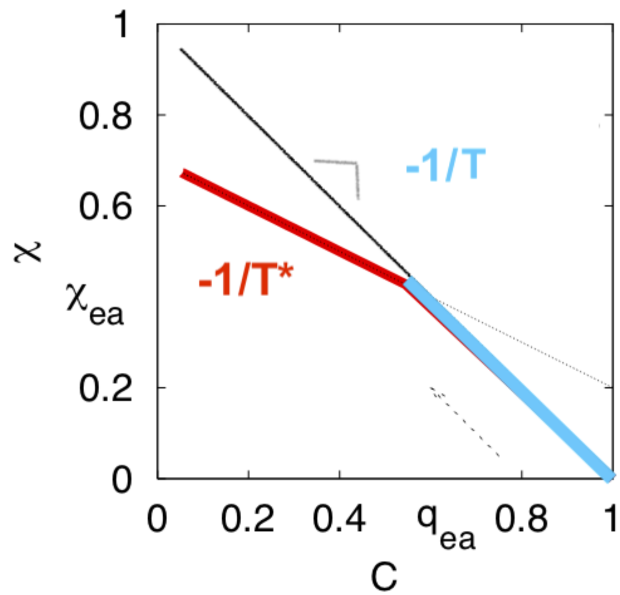
Effective temperatures

Induced by two (or more) baths

A tracer in contact with a complex bath



Sketch created by ChatGPT



$\Gamma = \Gamma_{\rm cold}$ + $\Gamma_{\rm hot}$

$\Gamma_{\rm cold}(t-t') = 2\gamma\delta(t-t')$ and T $\Gamma_{\rm hot}(t-t') = \gamma_{\rm hot}\, e^{-(t-t')/\tau}$ and T^*

LFC & Kurchan 00, Ilg & Barrat 07, etc., cfr. tracer in pasive & active bath

Dynamic equations

Single variable self-consistent over-damped Langevin equation

$$\underbrace{\gamma \dot{s}_i(t)}_{\text{friction}} + \underbrace{\int_0^t dt' \, \Sigma(t,t') s_i(t')}_{\text{self-generated friction}} - \underbrace{z(t) s_i(t)}_{\text{sph constraint}} =$$

$$= \underbrace{\xi_i(t)}_{\text{white noise}} + \underbrace{\eta_i(t)}_{\text{self-consist noise}}$$

$$\langle \eta_i(t) \eta_j(t') \rangle = \delta_{ij} D(t,t')$$

Σ and D are self-consistently determined in terms of correlations and linear responses of the original fields. The as in the SD eqs.

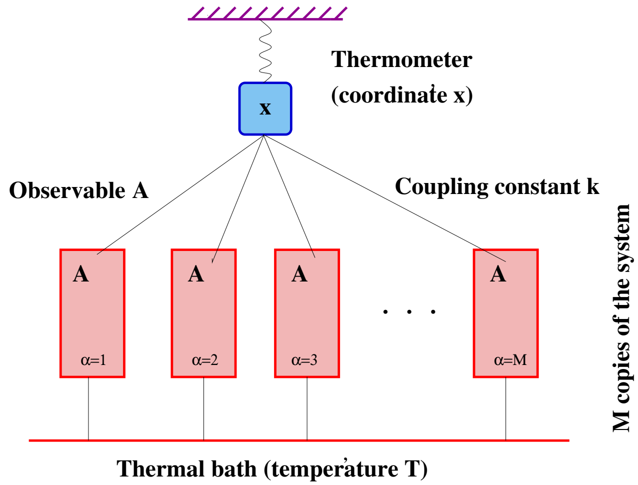
cfr. **Mode Coupling Th (Götze et al), Dynamic Mean-Field Th (Georges & Kotliar)**

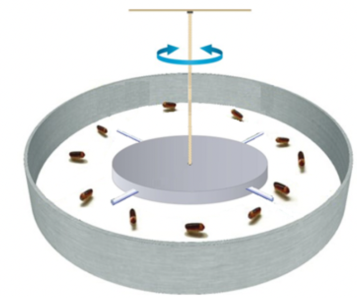
Sompolinsky 81, LFC & Kurchan 00

Effective temperatures

Measurement with thermometers

LFC, Kurchan & Peliti 97





Grigera & Israeloff 99 - glassy

D'Anna, Mayor, Barrat, Loreto & Nori 03 - granular

Boudet, Jagielka, Guerin, Barois, Pistolesi & Kellay 24
artificial active matter - robots

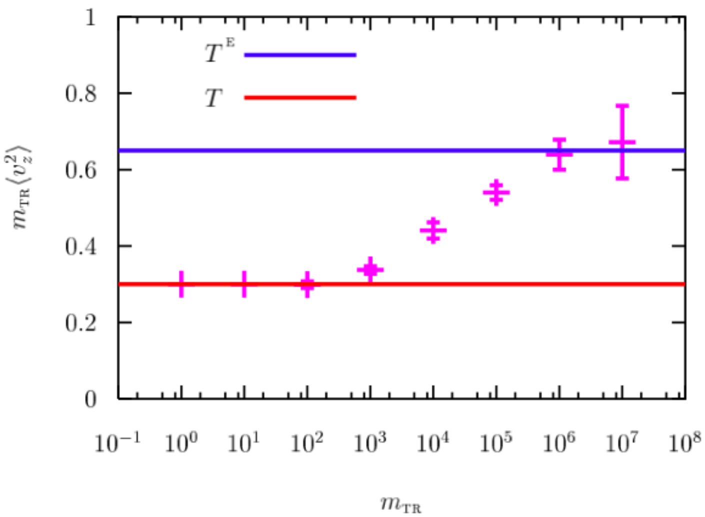
- Short internal time scale fast dynamics is tested and T is recorded.
- Long internal time scale slow dynamics is tested and T^* is recorded.

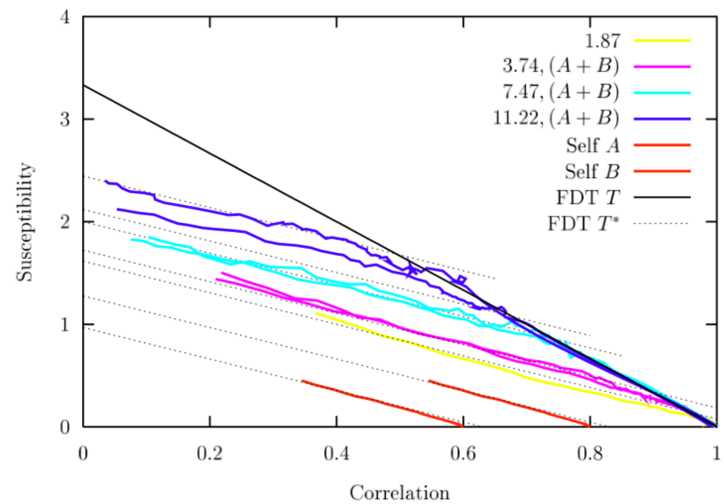
Related to the phenomenological *fictive temperatures* of **Tool 46, Gardon & Narayanaswamy 70, Moynihan et al 76, etc.** but measurable & with a thermodynamic interpretation

Also appearing in stochastic thermodynamic relations **Zamponi et al 05**

FDT & effective temperatures

Sheared binary Lennard-Jones mixture





Left : the kinetic energy of a tracer particle (the thermometer) as a function of its mass ($\tau_0 \propto \sqrt{m_{tr}}$)

$$\boxed{\frac{1}{2} m_{tr} \langle v_z^2 \rangle = \frac{1}{2} k_B T_{\text{eff}}}$$

Right : $\boxed{\chi_k(C_k)}$ plot for different wave-vectors k, partial equilibrations.

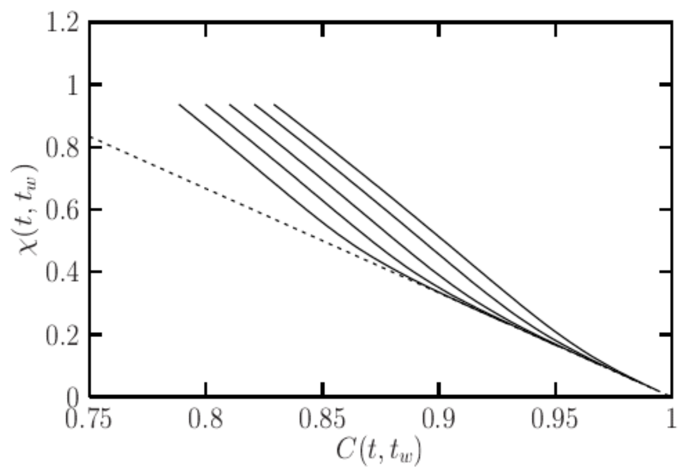
J-L Barrat & Berthier 00

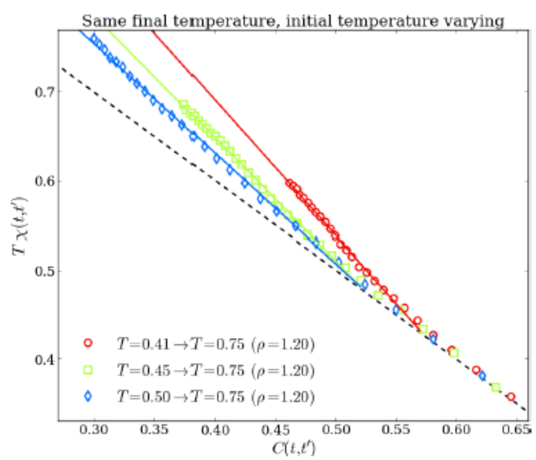
FDT & effective temperatures

Role of initial conditions

$T^* > T$ found for quenches from the disordered into the glassy phase

(Inverse) quench from an ordered initial state, $\boxed{T^* < T}$





2d XY model or O(2) field theory

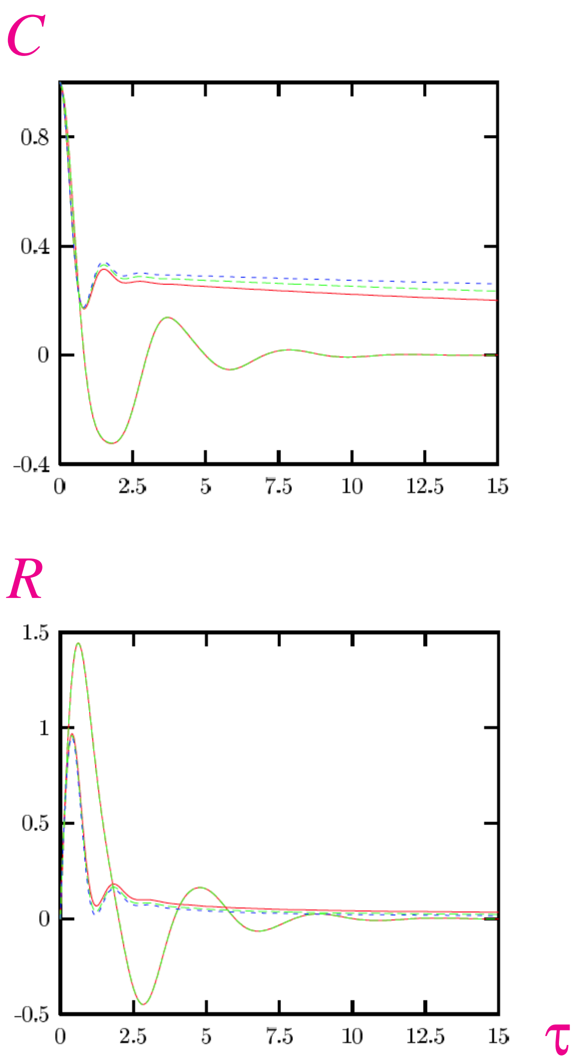
Berthier, Holdsworth & Sellitto 01

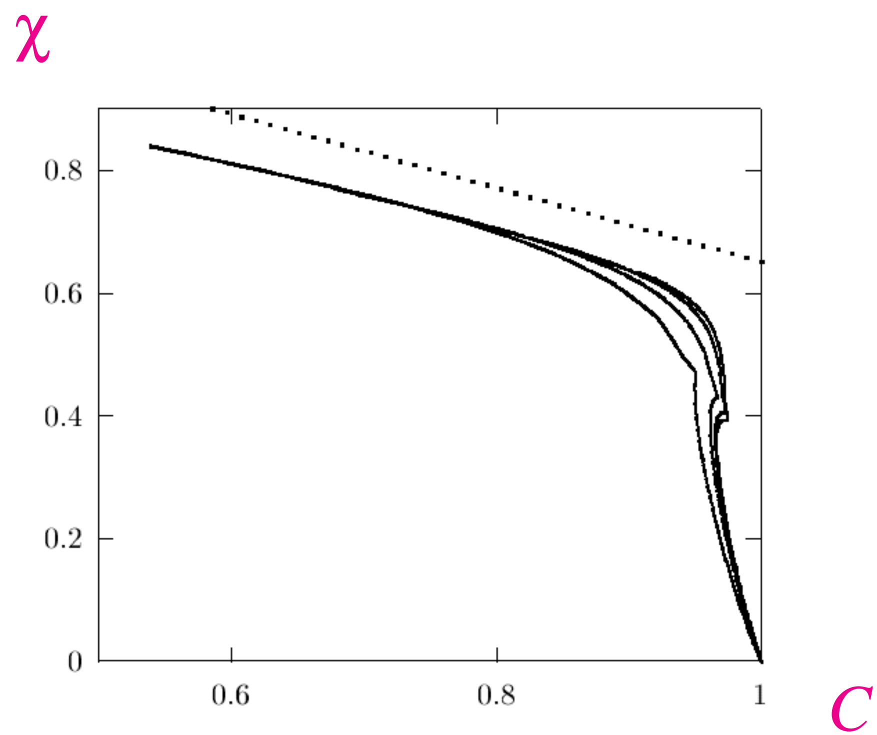
Binary Lennard-Jones mixture

Gnan, Maggi, Parisi & Sciortino 13

Dissipative quantum glasses

Quantum p-spin coupled to a bath of harmonic oscillators





Out of equilibrium decoherence

LFC & Lozano 98

Fluctuation-dissipation theorem

Classical dynamics in equilibrium

The classical FDT for a stationary system with $\tau \equiv t - t_w$ reads

$$\boxed{\chi(\tau) = \int_0^\tau dt'\, R(t') = -\beta[C(\tau) - C(0)] = \beta[1 - C(\tau)]}$$

choosing $C(0) = 1$.

Linear relation between χ and C

Quantum dynamics in equilibrium

The quantum FDT reads

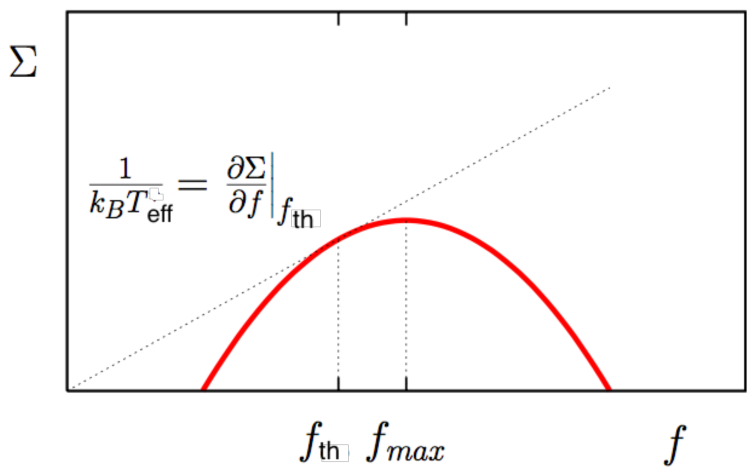
$$\boxed{\chi(\tau) = \int_0^\tau d\tau'\, R(\tau') = \int_0^\tau d\tau' \int_{-\infty}^{\infty} \frac{id\omega}{\pi\hbar}\, e^{-i\omega\tau'} \tanh\left(\frac{\beta\hbar\omega}{2}\right) C(\omega)}$$

Complicated relation between χ and C

Is $T_{\rm eff}$ related to an entropy ?

Yes, configurational entropy

$$\Sigma(f) = k_B \ln \mathcal{N}(f) \qquad \Rightarrow \qquad \boxed{\frac{1}{k_B T_{\rm eff}} = \left.\frac{\partial \Sigma(f)}{\partial f}\right|_{f_{\rm th}}}$$



NB $f_{max} \neq f_{\rm th} \Rightarrow$ failure of 'maximum entropy principles'.

Edwards & Oakshott 89, Monasson 95, Nieuwenhuizen 98

Sketchy view : many amorphous solid configurations ($\Sigma \Leftrightarrow T_{\rm eff}$) and vibrations around them ($f \Leftrightarrow T$).

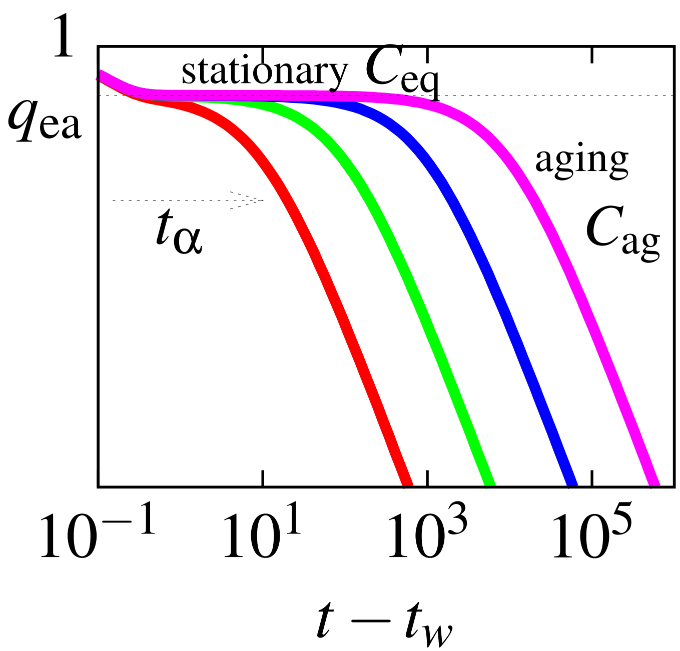
Time reparametrization invariance

and fluctuations

Time reparametrization invariance

In the long t_w limit

Fast $t - t_w \ll t_w$



The aging part is **slow**

$$\mathcal{R}(t)/\mathcal{R}(t_w) = O(1)$$

$$C_{ag}(t,t_w) \sim f_{ag}\left(\frac{\mathcal{R}(t)}{\mathcal{R}(t_w)}\right)$$

$$\partial_t C_{ag}(t,t_w) \propto \frac{\dot{\mathcal{R}}(t)}{\mathcal{R}(t)} \xrightarrow[t\to\infty]{} 0$$

$$\boxed{\partial_t C_{ag}(t,t_w) \ll C_{ag}(t,t_w)}$$

Eqs. for the slow relaxation $C_{ag} < q_{EA}$ are invariant under

$$\boxed{t \to h(t) \quad C(t,t_w) \to C(h(t),h(t_w)) \quad R(t,t_w) \to \dot{h}(t')/h(t_w)\,R(h(t),h(t_w))}$$

Time reparametrization

Example : the equation $(\partial_t - z_t)R(t,t_w) = \int dt'\, \Sigma(t,t')R(t',t_w)$

- Focus on times such that $z_t \to z_\infty$, $C \sim C_{\rm ag}$ and $R \sim R_{\rm ag}$

- Separation of time-scales (drop $\partial_t R$ and approximate the integral) :

$$-z_\infty R_{\rm ag}(t,t_w) \sim \int dt'\, D'[C_{\rm ag}(t,t')] R_{\rm ag}(t,t')\, R_{\rm ag}(t',t_w) \qquad (1)$$

- The transformation

$$t \to h_t \equiv h(t) \qquad \left\{ \begin{array}{l} C_{\rm ag}(t,t_w) \to C_{\rm ag}(h_t,h_{t_w}) \\ R_{\rm ag}(t,t_w) \to \frac{dh_{t_w}}{dt_w} R_{\rm ag}(h_t,h_{t_w}) \end{array} \right.$$

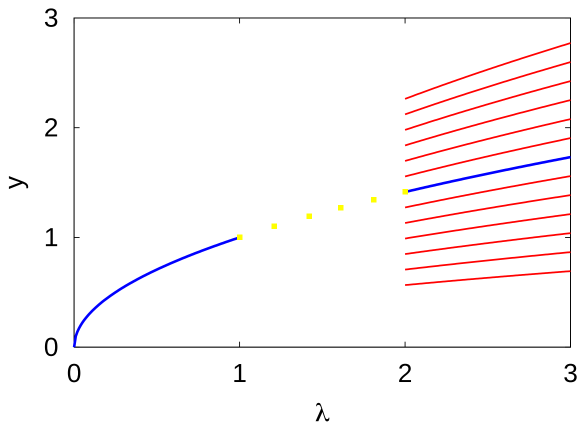
with h_t positive and monotonic leaves eq. (1) **invariant** :

$$-z_\infty R_{\rm ag}(h_t,h_{t_w}) \sim \int dh_{t'}\, D'[C_{\rm ag}(h_t,h_{t'})] R_{\rm ag}(h_t,h_{t'})\, R_{\rm ag}(h_{t'},h_{t_w})$$

Time reparametrization

A nuisance

Similar to the **matching problem** in non-linear diff. eqs.



$$\frac{dy}{d\lambda} = g[y(\lambda)]$$

Many asymptotic solutions if one sets $\frac{dy}{d\lambda} = 0$ for large λ

One is selected by the small λ behavior

A problem which is still open for the p-spin Schwinger-Dyson equations

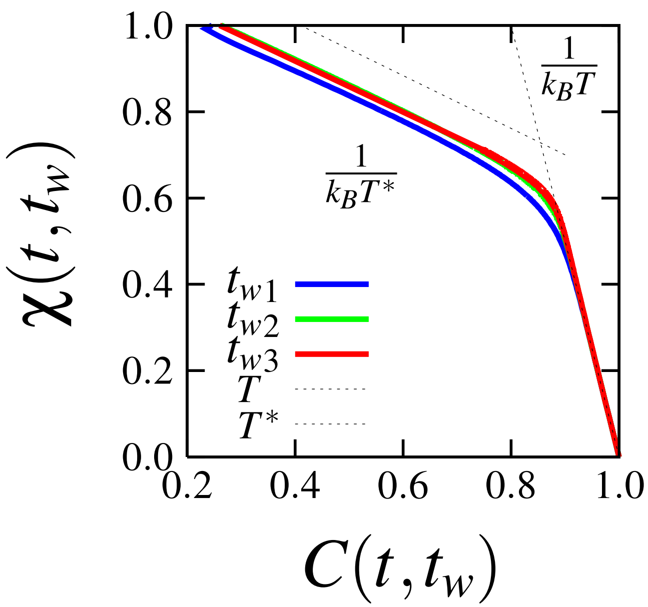
Time reparametrization

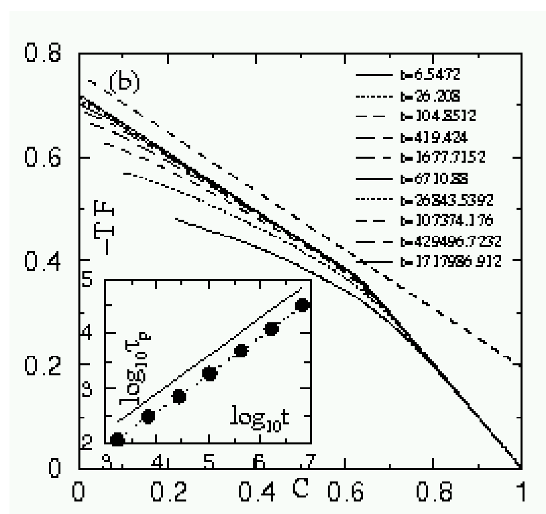
One can compute analytically $f_{\rm ag}$ and $\chi_{\rm ag}(C_{\rm ag})$

for times t and t_w such that $C_{\rm ag}(t,t_w) \sim f_{\rm ag}\left(\dfrac{\mathcal{R}(t)}{\mathcal{R}(t_w)}\right)$, $e.g.$

$$\chi_{\rm ag}(t,t_w) \sim \frac{1-q_{\rm EA}}{T} + \frac{1}{T^*}\left[q_{\rm EA} - C_{\rm ag}(t,t_w)\right]$$

but not the 'clock' $\mathcal{R}(t)$





Kim & Latz 00 very precise numerical solution

Remarks

Symmetry breaking terms $\partial_t C(t,t_w)$, etc.

vanish in the long $t_w \to \infty$ and $t - t_w \to \infty$ limits

Ultra soft mode

One can modify the actual $h(t)$ very easily by, *e.g.*,

– weak shearing $\Rightarrow$ stationary

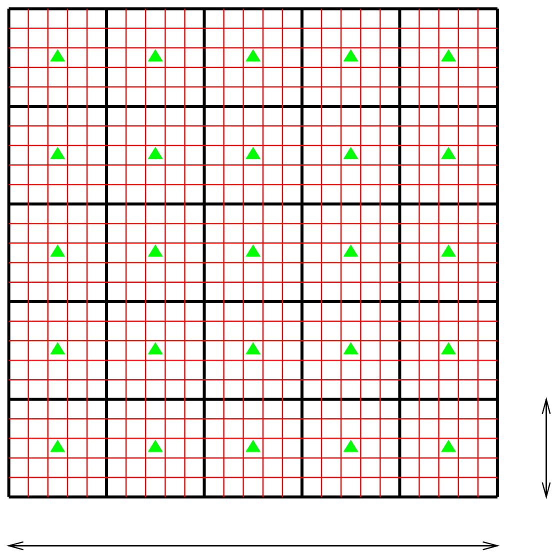
– weak periodic shaking $\Rightarrow$ periodic but stroboscopic aging

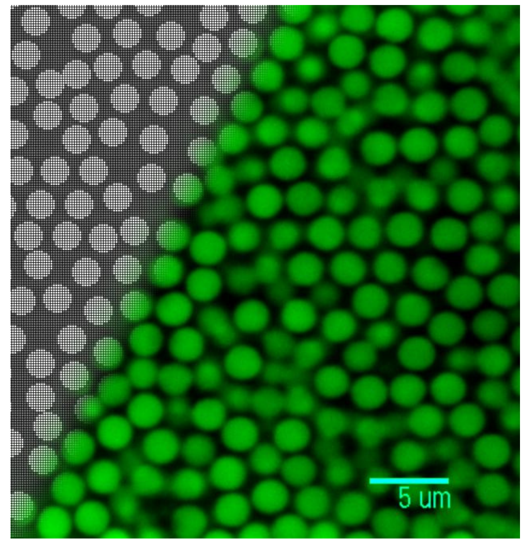
– coupling to various non-Markovian baths $\Rightarrow$

apply a thermal bath with a characteristic time-scale on one end and a different thermal bath with a different characteristic time-scale on the opposite end and see how a time-reparametrization flow establishes in the model

Turn it useful

Characterize the spatial fluctuations





$$C_{\vec{r}}(t,t_w) \equiv \frac{1}{V_{\vec{r}}} \sum_{i \in V_{\vec{r}}} s_i(t) s_i(t_w) \qquad \chi_{\vec{r}}(t,t_w) \equiv \frac{1}{V_{\vec{r}}} \sum_{i \in V_{\vec{r}}} \int_{t_w}^{t} dt' \left. \frac{\delta s_i^{(h)}(t)}{\delta h_i(t')} \right|_{h=0}$$

Review **Chamon & LFC 07**

Consequences

Characterize the spatial fluctuations

- There is an approximate dynamic symmetry :

  global time reparametrization invariance

- There is a **soft/massless** dynamic mode associated to it,

  with a **two-time diverging correlation length** $\xi(t,t_w)$

  Extract it from, *e.g*

$$C_4(r,t,t_w) = \frac{1}{N} \sum_{i,j/|\vec{r}_i-\vec{r}_j|=r} \langle s_i(t)s_i(t_w)s_j(t)s_j(t_w)\rangle_c$$

- Characterize **dynamic fluctuations - heterogeneities**

  $C_{\vec{r}}(t,t_w;\ell,\xi)$, $\rho(C_{\vec{r}},\chi_{\vec{r}};t,t_w;\ell,\xi)$, multi-time functions, *etc.*

- Disentangle simple **dynamic scaling** implications from **time reparametrization invariance** ones.

Consequences

Characterize the spatial fluctuations

In the **scaling limit**

lattice spacing $\ll$ coarse-graining length $\ll$ correlation length $\ll$ system size

$$a \ll \ell \ll \xi(t, t_w) \ll L$$

- The **'clock'** $h_{\vec{r}}(t)$ is **local** (analogy : angle - soft mode - in a Mexican hat potential)
- The **scaling functions** ($f_{\rm ag}$, $\chi_{\rm ag}(C_{\rm ag})$) **do not fluctuate** (modulus)

$\Rightarrow T_{\rm eff}$ **does not fluctuate**

In **practice (simulations, experiments)**

$$a \lesssim \ell \lesssim \xi(t, t_w) \ll L$$

- $\ell/\xi(t, t_w)$ is an additional scaling variable.

Implications on Fluctuations

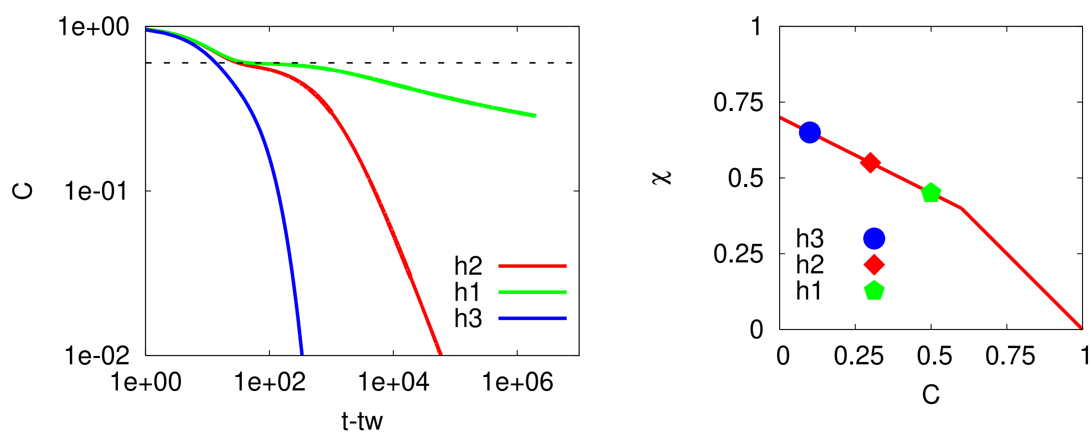
Leading fluctuations

Global to local correlations & linear responses

$$C_{\rm ag}(t,t_w) \approx f_{\rm ag}\left(\frac{\mathcal{R}(t)}{\mathcal{R}(t_w)}\right)$$ **global correlation**

Global time-reparametrization invariance $\Rightarrow \; C^{\rm ag}_{\vec r}(t,t_w) \sim f_{\rm ag}\left(\frac{h_{\vec r}(t)}{h_{\vec r}(t_w)}\right)$

Ex. $h_{\vec r_1} = \frac{t}{t_0}$, $h_{\vec r_2} = \ln\left(\frac{t}{t_0}\right)$, $h_{\vec r_3} = e^{\ln^{a>1}\left(\frac{t}{t_0}\right)}$ in different spatial regions



Castillo, Chamon, LFC, Iguain & Kennett 02, 03

Chamon, Charbonneau, LFC, Reichman & Sellitto 04

Jaubert, Chamon, LFC & Picco 07

Avila, Castillo & Parsaeian 12

Dyre 22-26

More recent perspective : time-reparametrization invariance in SYK models

Kitaev 15, Maldacena & Stanford 16

Triangular relations

Scaling of the aging global correlation

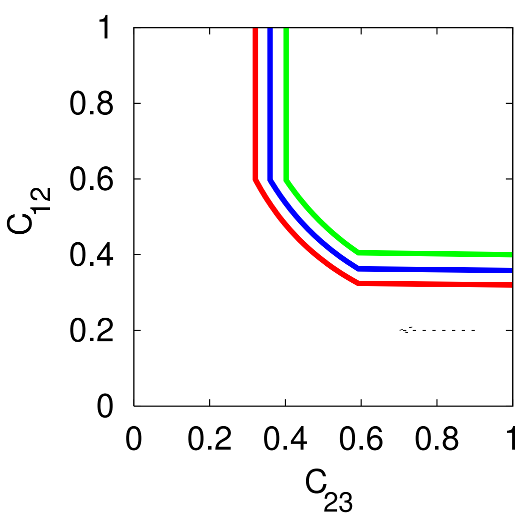
Take three times $t_1 \geq t_2 \geq t_3$ and compute the three global correlations

$$C(t_1, t_2),\ C(t_2, t_3),\ C(t_1, t_3)$$

If, in the aging regime $C^{ij}_{\rm ag} \equiv C_{\rm ag}(t_i, t_j) = f_{\rm ag}\left(\frac{\mathcal{R}(t_i)}{\mathcal{R}(t_j)}\right)$ with $t_i \geq t_j \Rightarrow$

$$C^{12}_{\rm ag} = f_{\rm ag}\left(\frac{\mathcal{R}(t_1)}{\mathcal{R}(t_3)}\frac{\mathcal{R}(t_3)}{\mathcal{R}(t_2)}\right) = f_{\rm ag}\left(\frac{f_{\rm ag}{}^{-1}(C^{13}_{\rm ag})}{f_{\rm ag}{}^{-1}(C^{23}_{\rm ag})}\right)$$

$q_{\rm EA}$



$q_{\rm EA}$

choose t_3 and t_1 so that $C^{13} = 0.3$

the arrow shows the t_2 ‘flow’ from t_3 to t_1

e.g. $C^{12}_{\rm ag} = q_{\rm EA}\, C^{13}_{\rm ag} / C^{23}_{\rm ag}$

in the spherical p spin model

Triangular relations

Scaling of the slow part of the global correlation

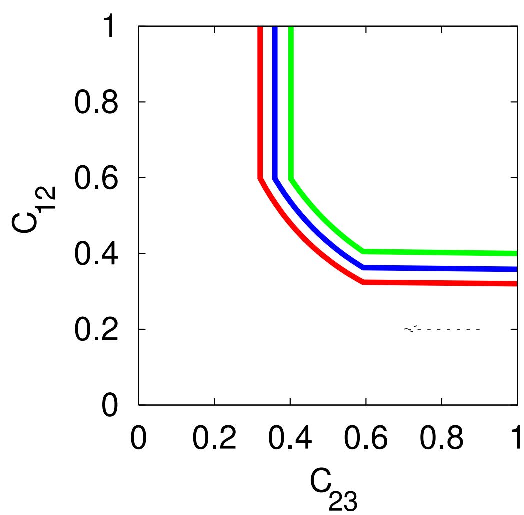
Take three times $t_1 \geq t_2 \geq t_3$ and compute the three local correlations

$$C_{\vec{r}}(t_1, t_2),\ C_{\vec{r}}(t_2, t_3),\ C_{\vec{r}}(t_1, t_3)$$

If, in the aging regime $C^{ij}_{\vec{r}} \equiv C_{\vec{r}}(t_i, t_j) = f_{\rm ag}\left(\frac{h_{\vec{r}}(t_i)}{h_{\vec{r}}(t_j)}\right)$ with $t_i \geq t_j \Rightarrow$

$$C^{12}_{\vec{r}} = f_{\rm ag}\left(\frac{f_{\rm ag}{}^{-1}(C^{13}_{\vec{r}})}{f_{\rm ag}{}^{-1}(C^{23}_{\vec{r}})}\right)$$

$q_{\rm EA}$



$q_{\rm EA}$

choose t_3 and t_1 so that $C^{13} = 0.3$

the arrow shows the t_2 'flow' from t_3 to t_1

e.g. $C^{12}_{\vec{r}} = q_{\rm EA}\, C^{13}_{\vec{r}} / C^{23}_{\vec{r}}$

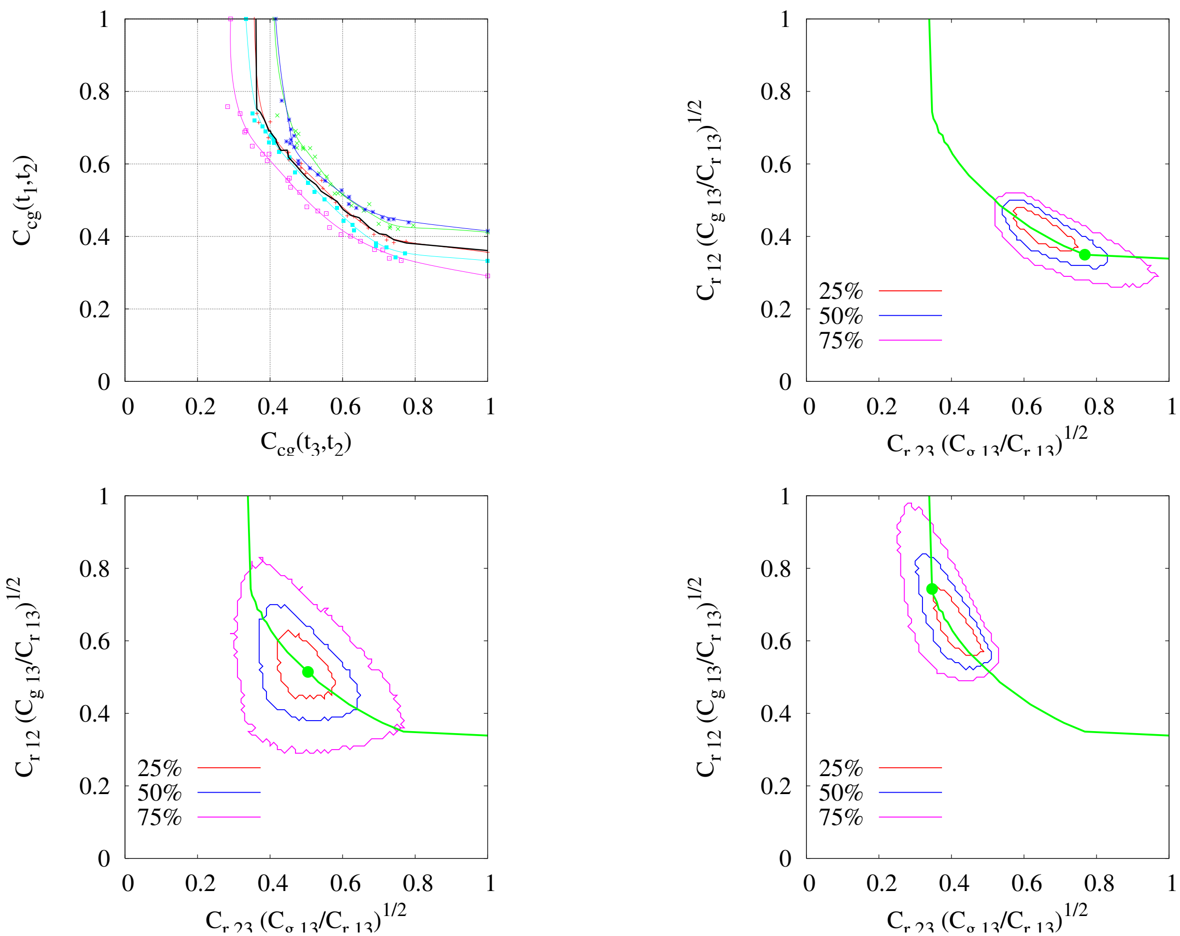
in the spherical p spin model

Triangular relations

3D Edwards-Anderson model

Black : global, other : five regions with size ℓ

Green : global ; other : 2d projections of pdf

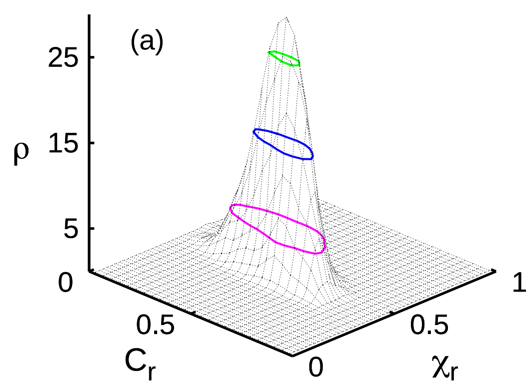


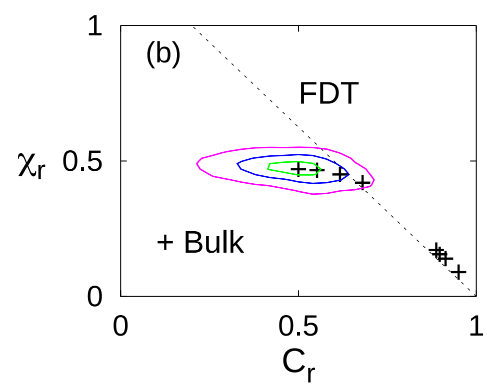
Jaubert, Chamon, LFC & Picco 07

Local FDTs

3D Edwards-Anderson

$$C_{\vec{r}}(t,t_w) \equiv \frac{1}{V_{\vec{r}}} \sum_{i \in V_{\vec{r}}} s_i(t) s_i(t_w) \qquad \chi_{\vec{r}}(t,t_w) \equiv \frac{1}{V_{\vec{r}}} \sum_{i \in V_{\vec{r}}} \int_{t_w}^{t} dt' \left. \frac{\delta s_i(t)}{\delta h_i(t')} \right|_{h=0}$$





+ Bulk : Parametric plot $\chi(t,t_w)$ vs $C(t,t_w)$ for t_w fixed and 7 $t\ (> t_w)$

ρ corresponds to the maximum t yielding the smallest global C (left-most +)

Castillo, Chamon, LFC, Iguain, Kennett 02

Kinetically constrained models + Charbonneau, Reichman & Sellitto 04

Conclusions on Fluctuations

Fluctuations

(Annoying) global time-reparametrization invariance $t \to h(t)$ in models in which

- $C_{\rm ag}(t,t_w) \gg \partial_t C_{\rm ag}(t,t_w)$ (slow dynamics)
- $\chi_{\rm ag}(t,t_w) \gg \partial_t \chi_{\rm ag}(t,t_w)$ (weak long-term memory)

and finite effective temperature $T_{\rm eff} < +\infty$ **Chamon, LFC & Yoshino 06**

Reason for the large dynamic fluctuations (heterogeneities) $h(\vec{r},t)$

Effective action for $\varphi(\vec{r},t)$ in $h(\vec{r},t) = e^{-\varphi(\vec{r},t)}$ **Chamon & LFC & Yoshino 07**

Quantum : the rapid equilibrium regime is modified but the slow aging one is classical controlled by a $T_{\rm eff} > 0 \Rightarrow$ the same applies

LFC & Lozano 98, 99 ; Kennett & Chamon 00, 01

Short Overview of Glassy Dynamics

Statics

& a bit of Relaxation Dynamics

Time ultrametricity

Proposal

For non-equilibrium systems, relaxing slowly towards a situation such that **one-time quantities** [*e.g.* the energy-density $e(t)$] **approach a finite value**, any three correlations

$$\boxed{C(t_1,t_2),\ C(t_2,t_3),\ C(t_1,t_3)}$$

for $t_3 < t_2, < t_1$ behave such that

$$\boxed{C(t_1,t_3) = \min[C(t_1,t_2),C(t_2,t_3)]}$$

Ultrametricity in time.

Also or weakly forced non-equilibrium systems in the limit of small work

LFC & Kurchan 94, Franz & Mézard 94

Statements

SK-like models

Relaxation dynamics after a quench

separation of time-scales, stationary & time-ultrametricity

FDT and its violation $x(C) = \frac{T}{T_{\rm eff}(C)} = \frac{f'''(C)}{f''(C)} \sqrt{\frac{f''(q)}{f''(C)}}$

An increasing $x(C)$ implies $\eta(C) \equiv \frac{f'''(C)}{(f''(C))^{3/2}} < \eta(q)$

Partial thermal equilibration interpretation

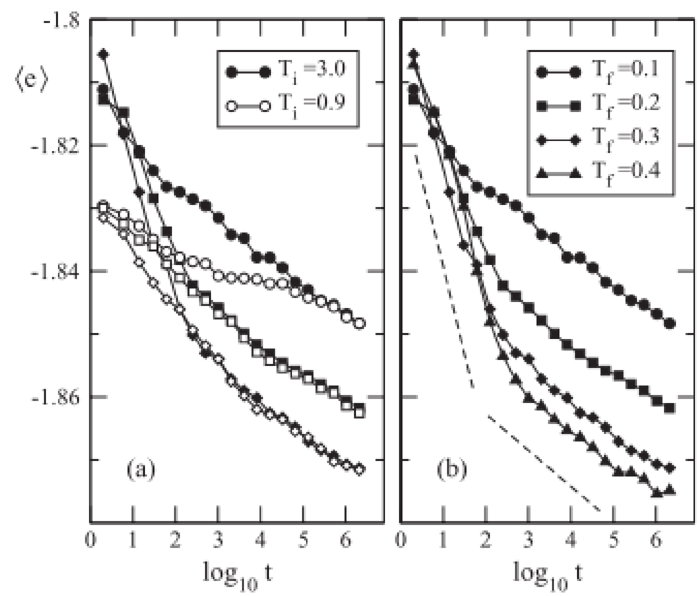
Approach to the equilibrium level (where marginal states are)

$$x(C) = \frac{T}{T_{\rm eff}(C)} = x_{\rm Parisi}(C) \Leftrightarrow P(C) = P_{\rm Parisi}(C)$$

LFC & Kurchan 94, Baldassarri et al. 95, Franz, Mézard, Parisi & Peliti 99

Beyond $N \to \infty$

The random orthogonal model



from algebraic at $t \ll N$ to logarithmic at $t(N)$ energy relaxation

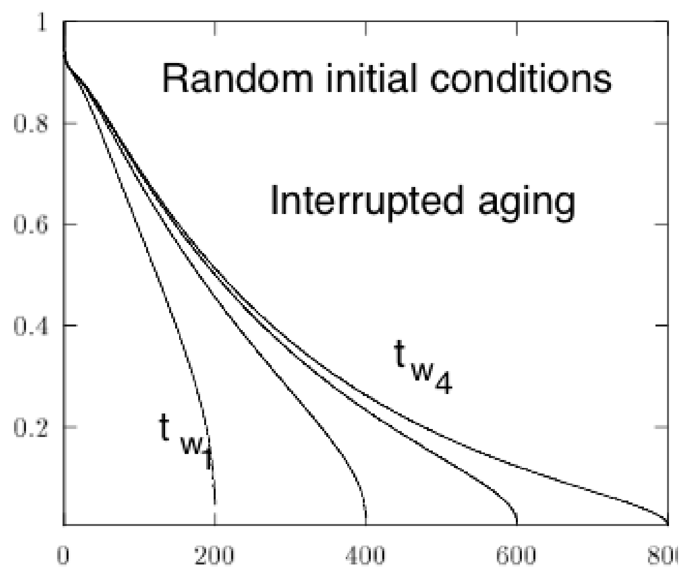
from an approach to the threshold to activation below it

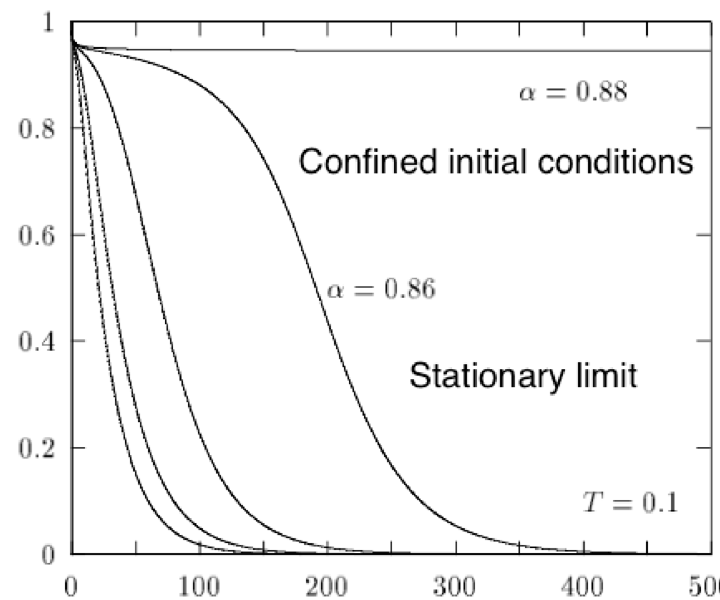
Crisanti & Ritort 00

Can one see the TAP states ?

Non-potential force

Driven $p = 3$ spherical model, $N \to \infty$





Waiting-time dependence (α fixed) and α dependence in steady state

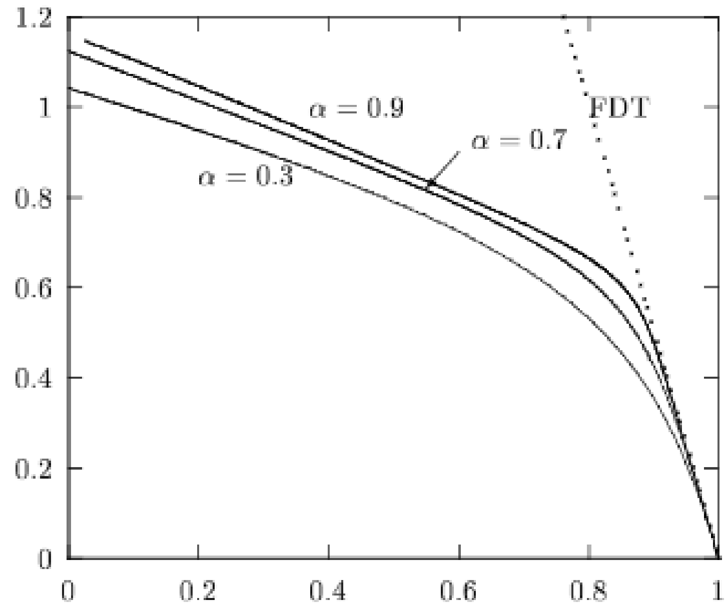
$$\boxed{J_{ii_2i_3\ldots i_p} = J^{\text{symm}}_{ii_2i_3\ldots i_p} + \alpha J^{\text{asymm}}_{ii_2i_3\ldots i_p} \quad \Rightarrow \quad \Sigma \neq D'R}$$

LFC, Kurchan, Le Doussal & Peliti 97

Non-potential force

Driven $p = 3$ Ising spin model with $N \to \infty$

$\chi(t, t_w)$



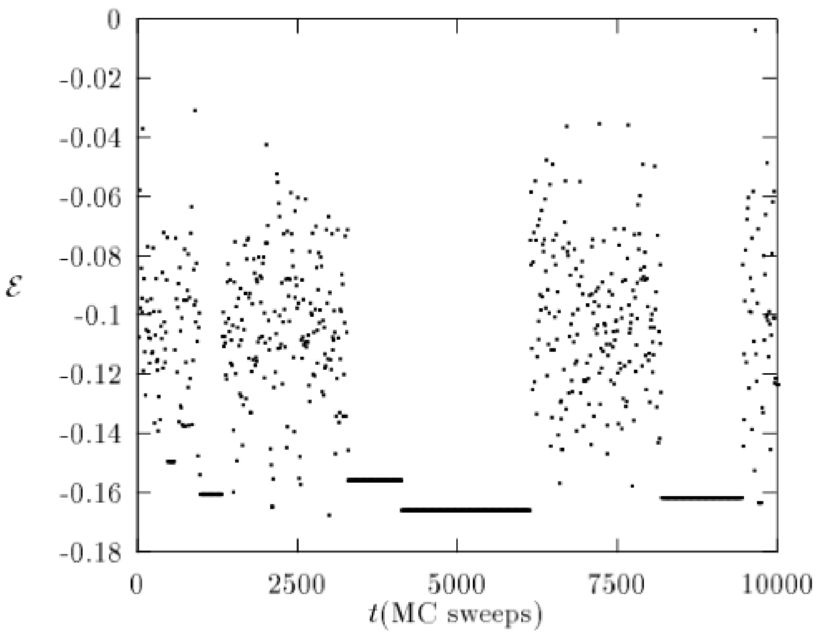
$C(t, t_w)$

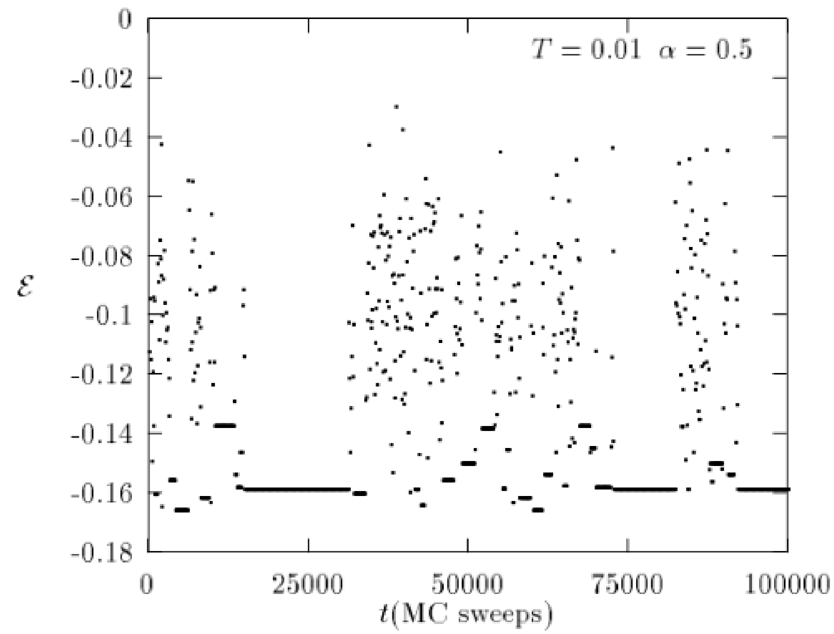
$T_{\rm eff}$ admits the good limit for $\alpha \to 1$

More on this from a rheological viewpoint **Barrat, Berthier, Kurchan 00**

Non-potential force

Trapping in the driven $p = 3$ spherical model, $N = 50$





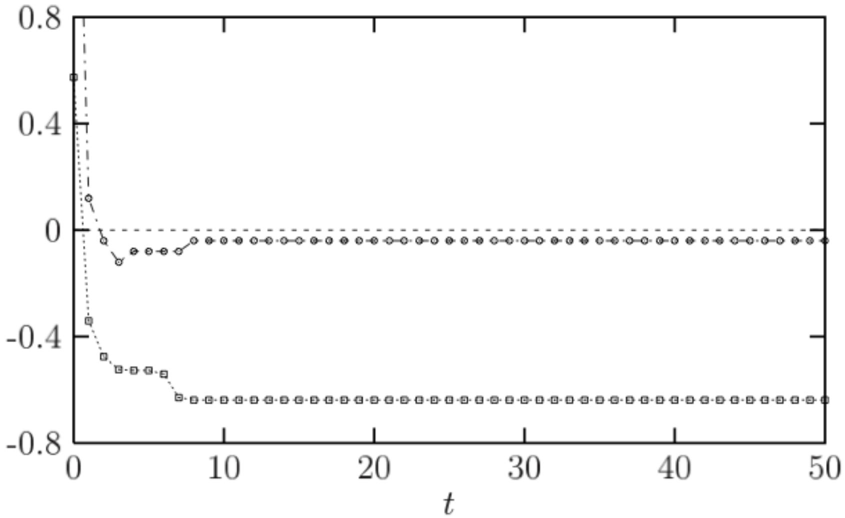
Time dependent energy density

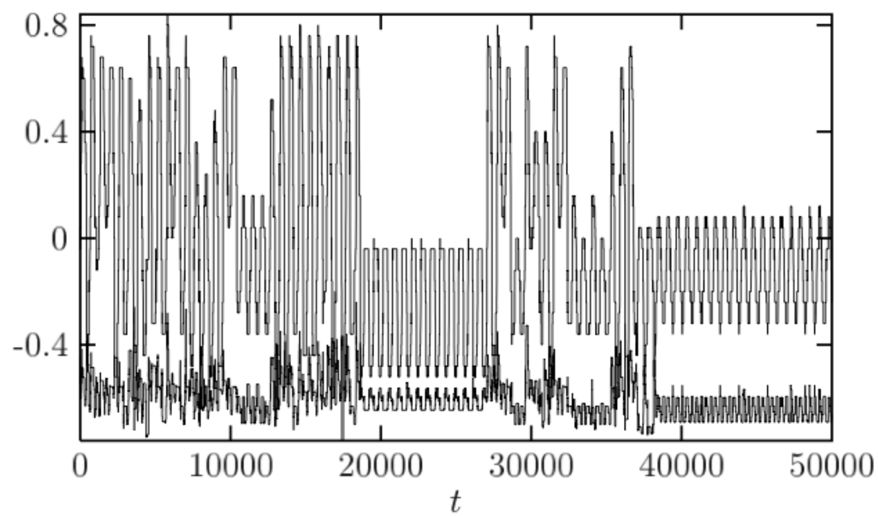
$$\boxed{J_{ii_2i_3\ldots i_p} = J^{\text{symm}}_{ii_2i_3\ldots i_p} + \alpha J^{\text{asymm}}_{ii_2i_3\ldots i_p} \quad \Rightarrow \quad \Sigma \neq D'R}$$

LFC, Kurchan, Le Doussal & Peliti 97

Time-dependent force

Driven $p = 3$ Ising spin model with $N = 50$





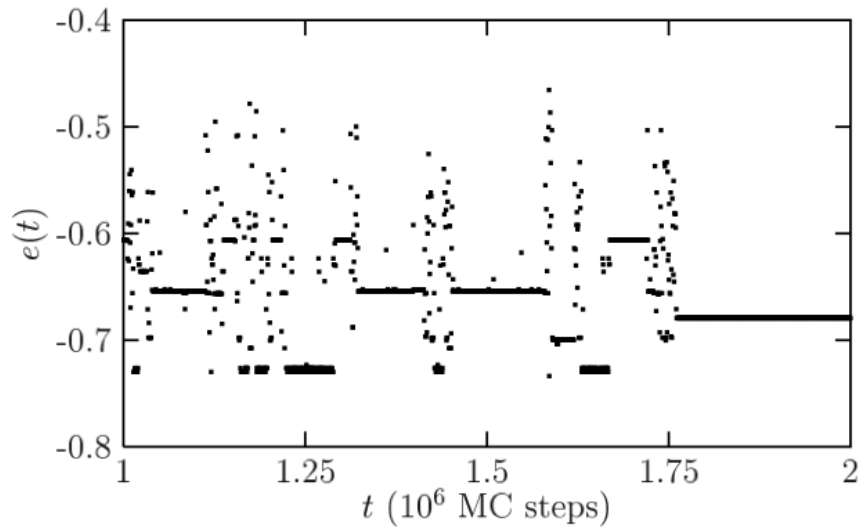
$h = 0$

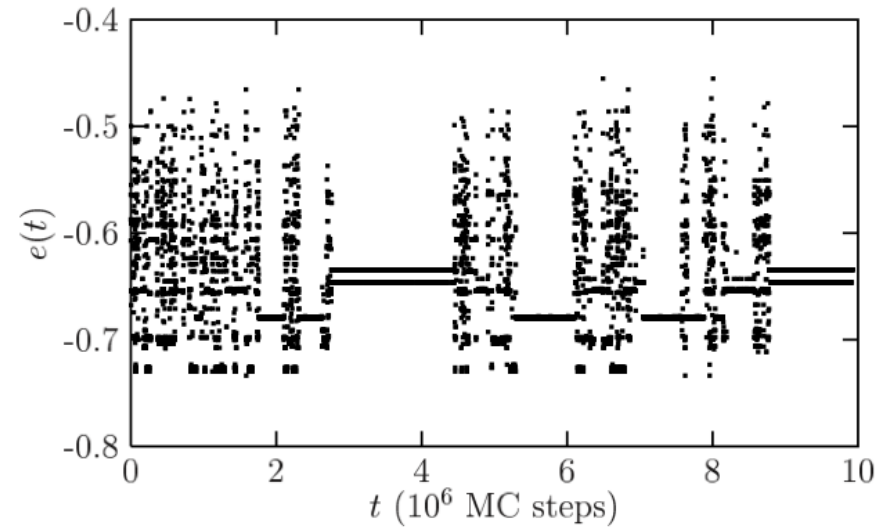
$\boxed{h(\omega, t) = h\cos(\omega t)}$

Time dependent magnetisation and energy density

Time-dependent force

Driven $p = 3$ Ising spin model with $N = 50$





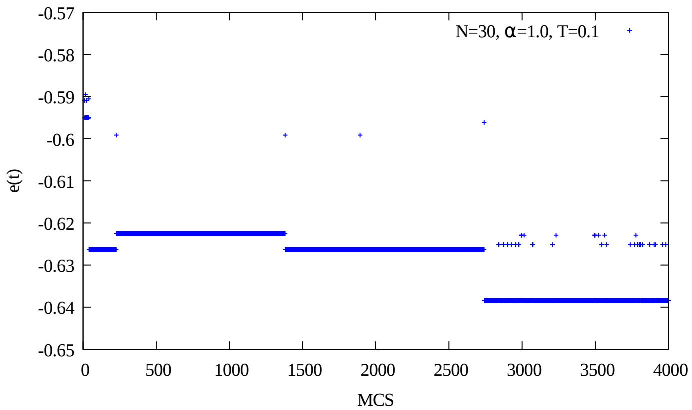
$h(\omega, t) = h\cos(\omega t)$ with $h = 2,\ \omega = 0.01$

Stroboscopic-time dependent magnetisation and energy density

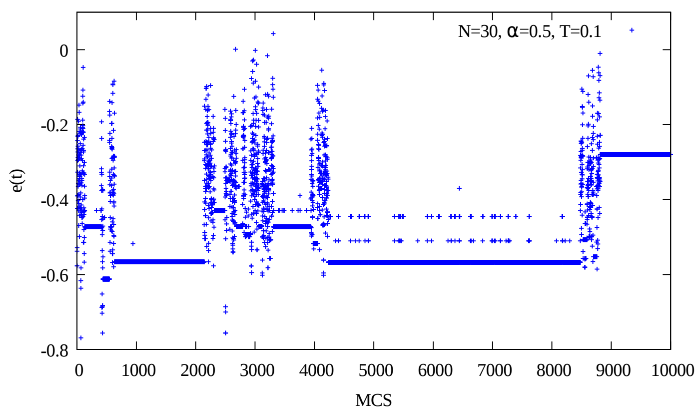
Simulations

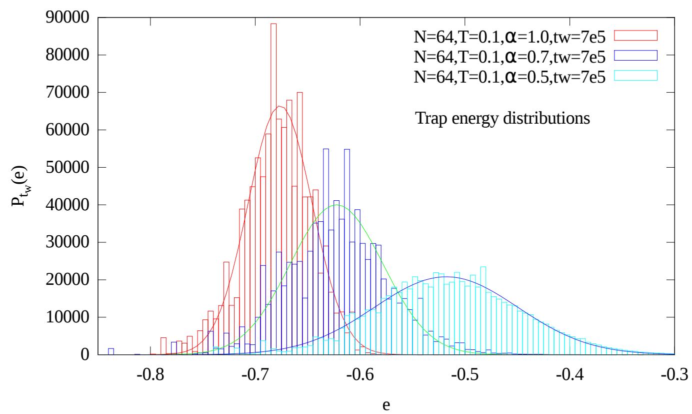
Ising fully-connected p spin models with $N = 30$

Symmetric couplings



Asymmetric couplings





Stariolo & LFC in progress

Statements

Finite N and non-potential forces

Evidence for trapping below threshold

Possibility to test trapping ideas

Better characterisation of trapping times, also barriers ?

Chaotic periods, what is the system actually doing ?

Links to math-phys results on dynamics **Ben Arous et al**

After 2000

Lines of research, out of equil. dynamics of these models

Rheology

Barrat, Berthier & Kurchan 00-04

Fluctuations

Castillo, Chamon, Charbonneau, Corberi, LFC, Kennett, Reichman, Sellitto & Yoshino 02-12

Glassy aspects of active matter

Berthier, Kurchan, Dasgupta, Gov, Szamel, etc. 13-19

Quantum dissipative

Aron, Baldwin, LFC, Biroli, Giamarchi, Grempel, Laumann, Le Doussal, Lozano, Schehr, Schiró, Busiello, Scardicchio, Saburova, Sushkova, etc. 99 -19

Quenches in isolation : integrability vs. thermalisation

LFC, Lozano, Nessi, Picco, Tartaglia 17-18

Ecology, neural nets, etc.

Agoritsas, Biroli, Burin, Fyodorov, Zamponi, Zdebodorová's talk, Fisher's talk

$$x = \frac{T}{T_{\text{eff}}} = \frac{1-q}{q}\left[\frac{q f''(q) - f'(q)}{f'(q)}\right]$$ for 1RSB-like

$$x(C) = \frac{T}{T_{\text{eff}}(C)} = \frac{q}{4}\,\frac{f'''(C)}{f''(C)}\sqrt{\frac{f''(q)}{f''(C)}}$$ for full RSB

$$\eta(C) \equiv \frac{f'''(C)}{[f''(C)]^{3/2}} < \eta(q)$$ to be full RSB