Theory of Disordered Systems

Statics and Dynamics

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Plan

A. Equilibrium

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2. Spin-Glass Experiments
3. The finite d EA model
4. Properties
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   (b) Self-averageness
   (c) Frustration
5. Order parameters
6. The mean-field SK model

Plan

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   (b) Free-energy landscape
   (c) Statistical averages
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8. The replica method
9. Mean-field classes
10. Quantum spin-glasses
11. Composite systems
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A. Equilibrium

1. Equilibrium Framework

Equilibrium Statistical Physics

Classical

No need to solve Newton dynamics!

Under the *ergodic hypothesis*, after some *equilibration time* $t_{\rm eq}$, a *macroscopic observable* A of a *macroscopic system* can, on average, be obtained with a *static calculation*, as an average over all configurations in phase space weighted with a probability distribution function $\rho(\{p_i, x_i\})$

$$\langle A \rangle = \int \prod_i dp_i dx_i \, \rho(\{p_i, x_i\}) \, A(\{p_i, x_i\})$$

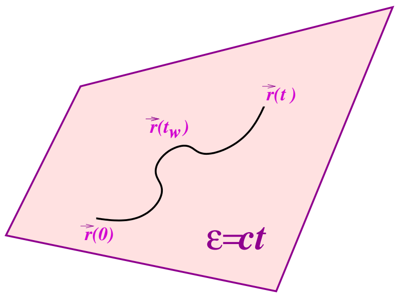
$\langle A \rangle$ should coincide with $\overline{A} \equiv \lim_{\tau \to \infty} \frac{1}{\tau} \int_{t_{\rm eq}}^{t_{\rm eq}+\tau} dt' \, A(\{p_i(t'), x_i(t')\})$ the *time average* typically measured experimentally

Ergodicity

Boltzmann, late XIX

Equilibrium Statistical Physics

Recipes for $\rho(\{p_i, x_i\})$ according to circumstances



Microcanonical ensemble

$$\boxed{\rho(\{p_i, x_i\}) \propto \delta(\mathcal{H}(\{p_i, x_i\}) - \mathcal{E})}$$

Flat probability density

Isolated system

$$\mathcal{E} = \mathcal{H}(\{p_i, x_i\}) = ct$$

$$S_{\mathcal{E}} = k_B \ln g(\mathcal{E}) \qquad \beta \equiv \frac{1}{k_B T} = \left. \frac{\partial S_{\mathcal{E}}}{\partial \mathcal{E}} \right|_{\mathcal{E}}$$

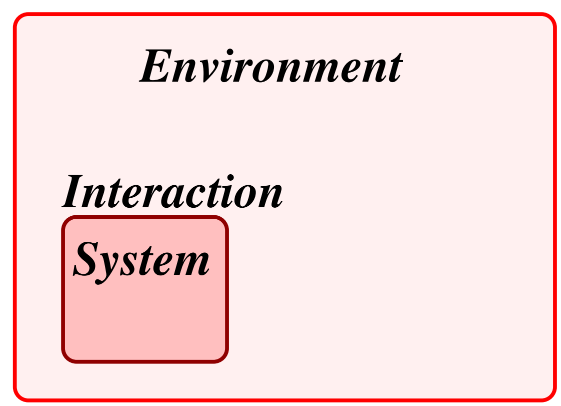
Entropy | Temperature

$$\mathcal{E} = \mathcal{E}_{syst} + \mathcal{E}_{env} + \mathcal{E}_{int}$$

Neglect $\mathcal{E}_{int}$ (short-range interact.)

$$\mathcal{E}_{syst} \ll \mathcal{E}_{env} \qquad \beta = \left. \frac{\partial S_{\mathcal{E}_{env}}}{\partial \mathcal{E}_{env}} \right|_{\mathcal{E} \approx \mathcal{E}_{env}}$$

$$\boxed{\rho(\{p_i, x_i\}) \propto e^{-\beta \mathcal{H}(\{p_i, x_i\})}}$$



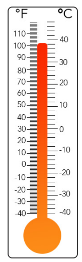
Canonical ensemble

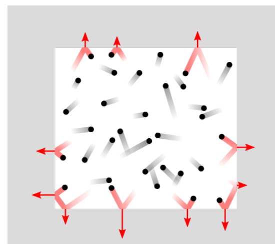
Equilibrium Statistical Physics

Accomplishments

- Microscopic definition & derivation of thermodynamic concepts

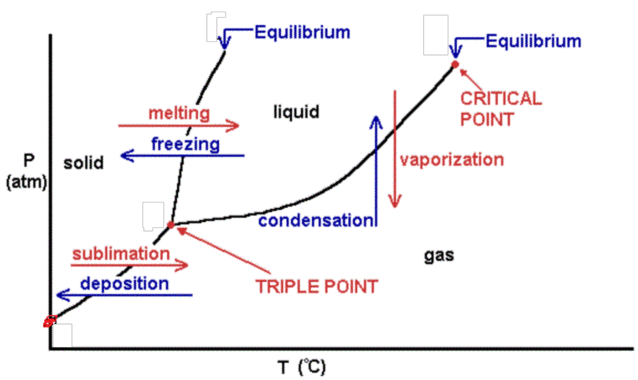
  (temperature, pressure, *etc.*) and laws (equations of state, *etc.*)





$$PV = nRT$$

- Theoretical understanding of collective effects $\Rightarrow$ phase diagrams



Phase transitions : sharp changes in the macroscopic behavior when an external (*e.g.* the temperature of the environment) or an internal (*e.g.* the interaction potential) parameter is changed

- Calculations can be difficult but the theoretical frame is set beyond doubt

Equilibrium Statistical Physics

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Ergodicity

Equilibrium Statistical Physics

Quantum

For an isolated quantum system with Hamiltonian $\hat{\mathcal{H}}$ in an initial state $|\psi(0)\rangle$

The constant energy is $\mathcal{E} = \langle\psi(0)|\hat{\mathcal{H}}(\{\hat{p}_i, \hat{x}_i\})|\psi(0)\rangle$

The long-time average of an observable $\hat{A}$ is

$$\overline{\langle\hat{A}\rangle} \equiv \lim_{\tau\to\infty} \frac{1}{\tau} \int_{t_{\rm eq}}^{t_{\rm eq}+\tau} dt' \, \langle\psi(t')|\hat{A}(\{\hat{p}_i, \hat{x}_i\})|\psi(t')\rangle = \sum_n |c_n|^2 A_{nn}$$

with $|\psi(0)\rangle = \sum_n c_n |E_n\rangle$ in the energy eigenbasis and $A_{nn} = \langle E_n|\hat{A}|E_n\rangle$ (assuming a non-degenerate spectrum, so cross terms dephase away).

This diagonal ensemble depends explicitly on $|\psi(0)\rangle$ through $|c_n|^2$ unlike a genuine thermal (microcanonical) average, which only depends on energy $\mathcal{E}$.

Equilibrium Statistical Physics

Eigenstate Thermalization Hypothesis (ETH)

The long-time average of an observable $\hat{A}$ is

$$\overline{\langle\hat{A}\rangle} \equiv \lim_{\tau\to\infty}\frac{1}{\tau}\int_{t_{\text{eq}}}^{t_{\text{eq}}+\tau} dt'\,\langle\psi(t')|\hat{A}(\{\hat{p}_i,\hat{x}_i\})|\psi(t')\rangle = \sum_n |c_n|^2 A_{nn}$$

For a chaotic (non-integrable) system, the individual matrix elements themselves are <u>assumed</u> to behave thermally :

$$A_{nm} = A(\overline{\mathcal{E}})\,\delta_{mn} + e^{-S(\overline{\mathcal{E}})/2}\,f_O(\overline{\mathcal{E}},\omega)\,R_{nm}$$

$\overline{\mathcal{E}} = (E_n + E_m)/2$ and $\omega = E_n - E_m$, $A(\overline{\mathcal{E}})$ a smooth function
$S(\mathcal{E})$ the entropy, $f_O(\overline{\mathcal{E}},\omega)$ a smooth envelope, R_{nm} random $O(1)$ numbers.

$$\overline{\hat{A}} = \sum_n |c_n|^2 A_{nn} \approx A(\overline{\mathcal{E}})\sum_n |c_n|^2 = A(\overline{\mathcal{E}})$$

Equilibrium Statistical Physics

Eigenstate Thermalization Hypothesis (ETH)

If initially mixed state, density operator $\hat{\rho}(\{\hat{p}_i, \hat{x}_i\}, 0)$

$$\overline{\hat{A}} \equiv \lim_{\tau\to\infty} \frac{1}{\tau} \int_{t_{\rm eq}}^{t_{\rm eq}+\tau} dt' \,\mathrm{Tr}\Big(\hat{A}(\{\hat{p}_i, \hat{x}_i\})\hat{\rho}(\{\hat{p}_i, \hat{x}_i\}(t'))\Big) = \sum_n \rho_{nn}(0) A_{nn}$$

For a chaotic (non-integrable) system, the individual matrix elements themselves are <u>assumed</u> to behave thermally :

$$A_{nm} = A(\overline{\mathcal{E}})\,\delta_{mn} + e^{-S(\overline{\mathcal{E}})/2}\, f_O(\overline{\mathcal{E}}, \omega)\, R_{nm}$$

$\overline{\mathcal{E}} = (E_n + E_m)/2$ and $\omega = E_n - E_m$, $A(\overline{\mathcal{E}})$ a smooth function
$S(\mathcal{E})$ the diagonal entropy $-\sum_n \rho_{nn} \ln \rho_{nn}$,
$f_O(\overline{\mathcal{E}}, \omega)$ a smooth envelope, R_{nm} random $O(1)$ numbers.

$$\overline{\hat{A}} = \sum_n |\rho_{nn}|^2 A_{nn} \approx A(\overline{\mathcal{E}}) \sum_n |\rho_{nn}|^2 = A(\overline{\mathcal{E}})$$

Equilibrium Statistical Physics

Eigenstate Thermalization Hypothesis (ETH)

For a chaotic (non-integrable) system, the individual matrix elements themselves are <u>assumed</u> to behave thermally :

$$A_{nm} = \underbrace{A(\overline{\mathcal{E}})\delta_{mn}}_{\text{asymptotic limit}} + \underbrace{e^{-S(\overline{\mathcal{E}})/2} f_O(\overline{\mathcal{E}},\omega)\, R_{nm}}_{\text{approach to asymptotic limit}}$$

The random-looking off-diagonal elements R_{nm} weighted by $e^{-S(\mathcal{E})/2}$, control the fluctuations and relaxation toward the diagonal-ensemble value, playing the role classical chaos plays in generating effective randomness

R_{nm} **Random Matrix Theory**

Deutsch 91, Srednicki 94 & 99, Rigol, Dunjko & Olshanii 08

Review **D'Alessio, Kafri, Polkovnikov & Rigol 16**

2. Spin Glass Experiments

in Equilibrium

The compounds

Dilute magnetic impurities in some non-magnetic host

Metallic spin glasses

— $Cu_{1-x}Mn_x$ the archetypal one

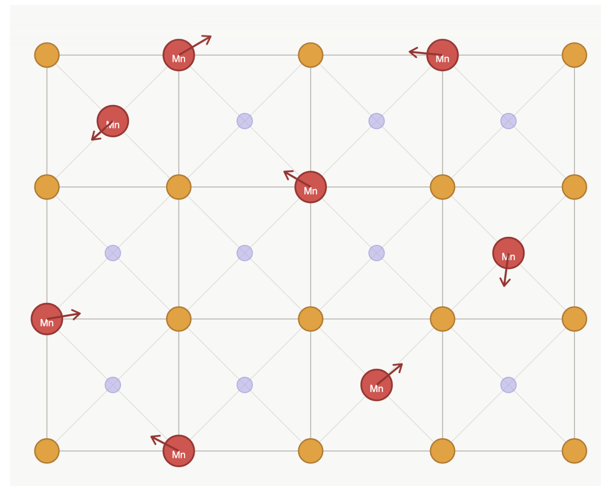
— $Au_{1-x}Fe_x$ extensively studied

— $Ag_{1-x}Mn_x$ similar to CuMn

Insulating spin glasses

— Eu_xSr_{1-x} a semiconductor

— $Fe_{1-x}Mn_xTiO_3$ a frustrated insulator



Low but not too low temperature, $\sim 10\mathrm{K}$, **classical physics**

How much spin-glass material is there in physics labs ? **Grams**

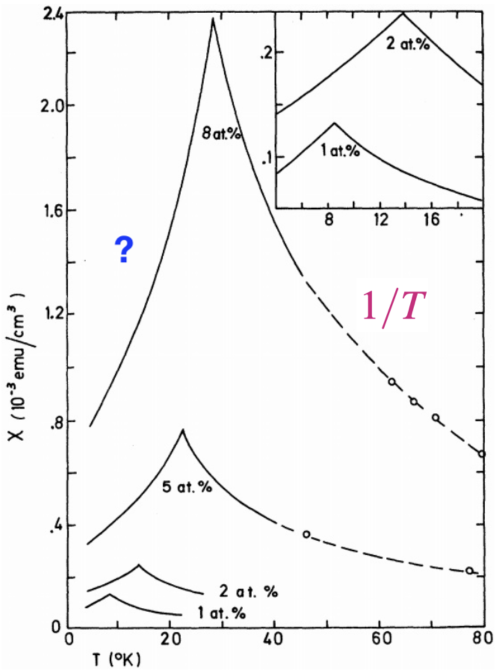
Are they useful materials ? **No**

Have they been inspirational ? **So much so - Fundamental research**

The anomaly

The experiment & the phase transition – 70s

ac Magnetic Susceptibility



Temperature

Canella & Mydosh 1972

A cusp indicates a phase transition

towards **which phase ?**

The anomaly

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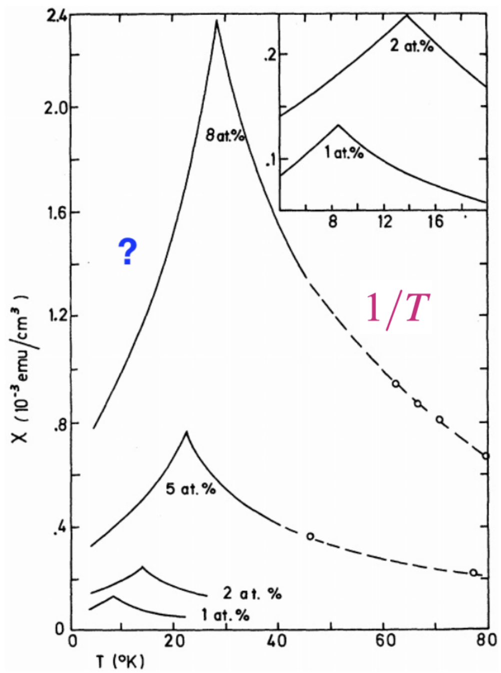

Temperature

Canella & Mydosh 1972

A cusp indicates a phase transition towards **which phase ?**

Hypothesis : **canonical equilibrium**

for simplicity, Ising spins $s_i = \pm 1$ or $\uparrow, \downarrow$

partition function $Z = \sum_{\rm conf} e^{-\beta \mathcal{H}}$

linear susceptibility $\chi = \frac{\partial m_h}{\partial h}|_{h=0}$

magnetization density $m_h = N^{-1} \sum_i \langle s_i \rangle_h$

average with Z_h and $\mathcal{H}_h = \mathcal{H} - h \sum_i s_i$

$\langle s_i \rangle_h = \sum_{\rm conf} s_i \, e^{-\beta \mathcal{H}_h} / Z_h$

Taking the derivative explicitly & $h \to 0$

Fluctuation-dissipation theorem **FDT** **Ex. 1**

$$\boxed{\chi = N\beta \, \langle (\frac{1}{N} \sum_i s_i - \frac{1}{N} \sum_i \langle s_i \rangle)^2 \rangle}$$

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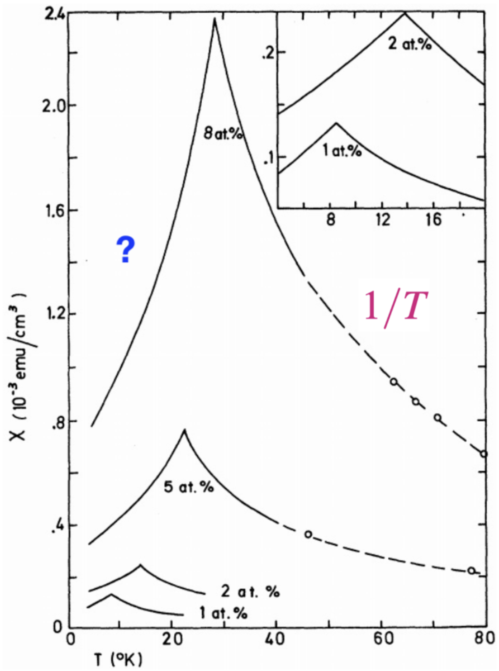

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Fluctuation-dissipation theorem **FDT**

$$\chi = N\beta \left\langle \left(\frac{1}{N}\sum_i s_i - \frac{1}{N}\sum_i \langle s_i\rangle\right)^2 \right\rangle$$
$$= \frac{\beta}{N}\sum_{ij}\langle (s_i s_j - \langle s_i\rangle\langle s_j\rangle)\rangle$$
$$= \frac{\beta}{N}\sum_{ij} C_{ij}^{\text{conn}}$$

At high T :

$\langle s_i\rangle = 0$ no magnetic order **paramagnet**

$$\sum_{ij}\langle s_i s_j\rangle = \sum_i \langle s_i^2\rangle + \sum_{i\neq j}\langle s_i s_j\rangle$$

$s_i^2 = 1 \Rightarrow$ 1st term $= N$

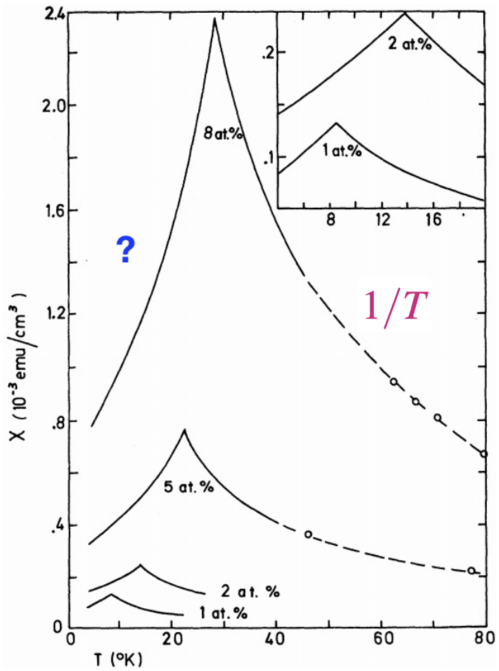
No spatial correlations : 2nd term negligible

$$\boxed{\chi = \beta = 1/T}$$ **Curie law**

The anomaly

The experiment & the phase transition – 70s

ac Magnetic Susceptibility



Temperature

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$$= \frac{\beta}{N} \sum_{ij} \langle (s_i s_j - \langle s_i \rangle \langle s_j \rangle) \rangle$$

$$= \frac{\beta}{N} \sum_{ij} C_{ij}^{\mathrm{conn}}$$

At low T :

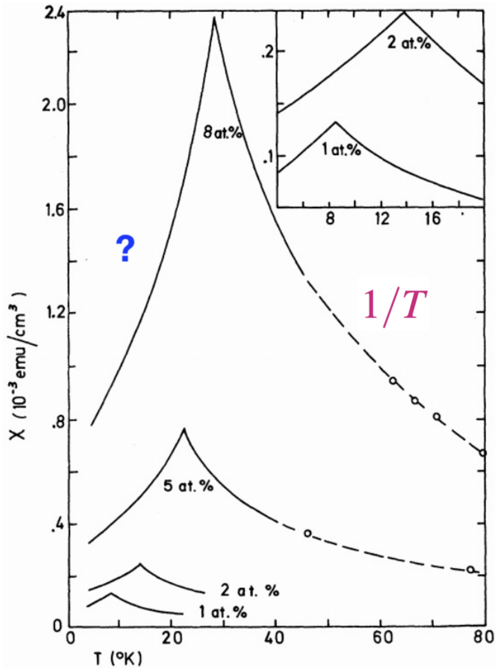
?

3. The finite d EA model

Edwards-Anderson Model

The experiment, the paper & the authors – 70s

Magnetic susceptibility



Temperature

Canella & Mydosh 1972

A cusp indicates a phase transition towards **which phase ?**

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Theory of spin glasses

S F Edwards† and P W Anderson‡
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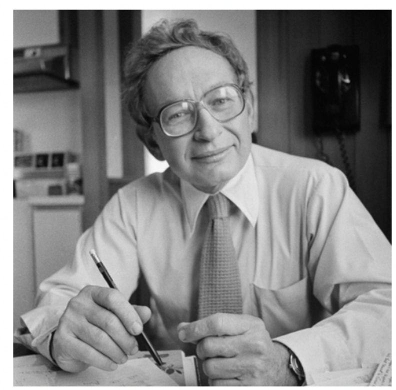
Received 14 October 1974, in final form 13 February 1975



Edwards-Anderson Model

Disordered low temperature equilibrium

Theory of spin glasses

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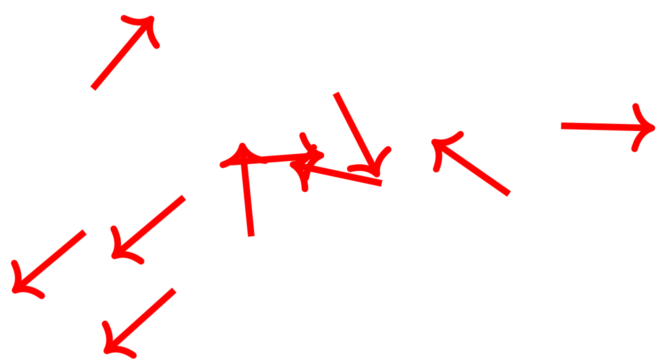
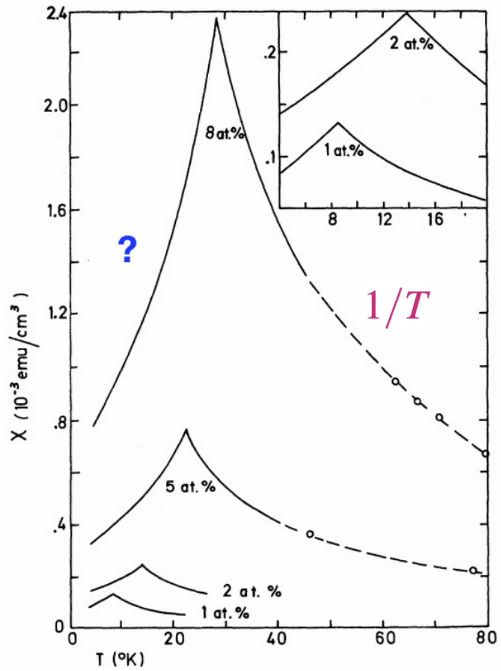
Abstract. A new theory of the class of dilute magnetic alloys, called the spin glasses, is proposed which offers a simple explanation of the cusp found experimentally in the susceptibility. The argument is that because the interaction between the spins dissolved in the matrix oscillates in sign according to distance, there will be no mean ferro- or antiferromagnetism, but there will be a ground state with the spins aligned in definite directions, even if these directions appear to be at random. At the critical temperature, the existence of these preferred directions affects the orientation of the spins, leading to a cusp in the susceptibility. This cusp is smoothed by an external field. If the potential



The anomaly

The experiment & the phase transition – 70s

ac Magnetic Susceptibility



Temperature

Canella & Mydosh 1972

A cusp indicates a phase transition towards **which phase ?**

Hypothesis : **canonical equilibrium**

FDT $\chi = \frac{\beta}{N} \sum_{ij} \langle (s_i s_j - \langle s_i \rangle \langle s_j \rangle) \rangle$

$$\approx \beta \left(1 - \frac{1}{N} \sum_i \langle s_i \rangle^2\right) \qquad i = j$$

The low T phase cannot be ferro since

$m = N^{-1} \sum_{i=1}^{N} \langle s_i \rangle = 0$ measured

but $m_i = \langle s_i \rangle$ could be $\neq 0$ for each i

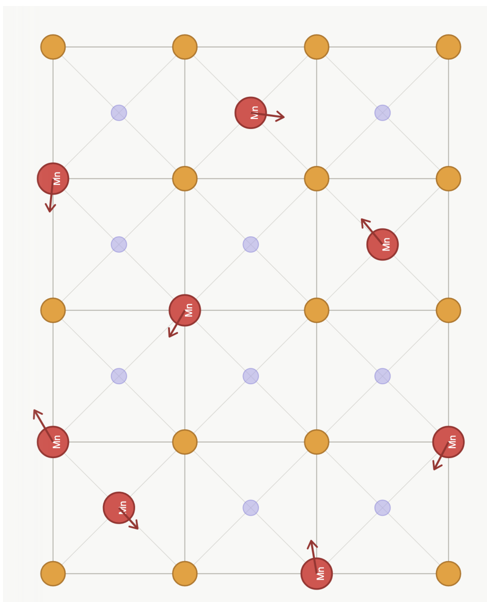
decrease in χ implies

$$q_{\mathrm{EA}} \equiv \frac{1}{N} \sum_i \langle s_i \rangle^2 \begin{cases} = 0 & T \geq T_c \\ \neq 0 & T < T_c \end{cases}$$

EA order parameter

Interpretation

Dilute magnetic impurities in some non-magnetic host



In experimental times,

each magnetic moment

does not move in space but

fluctuates – rotates, say **flips**

$$t_{\text{flip}} \ll t_{\text{exp}} \ll t_{\text{diffusion}}$$

It has tendency to point in some local direction

$$m_i \neq 0$$

decided by the material's structure

$\{m_i\}$ different real values

such that

$$m = \frac{1}{N}\sum_i m_i \to 0 \quad \text{but} \quad q_{\text{EA}} = \frac{1}{N}\sum_i m_i^2 \neq 0 \quad \text{for} \quad N \to \infty$$

Interpretation

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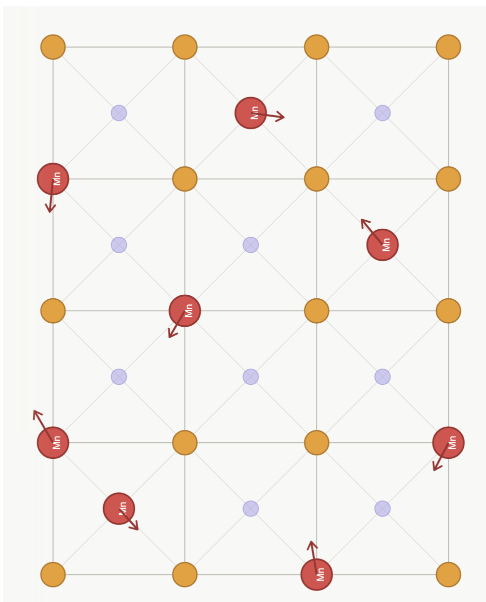


In experimental times,

each magnetic moment

does not move in space but flips

$$t_{\text{flip}} \ll t_{\text{exp}} \ll t_{\text{diffusion}}$$

For **EA** there is only one (& its reversed - $\mathbb{Z}_2$)

possibility for each of the

$$m_i \neq 0$$

decided by the material's structure

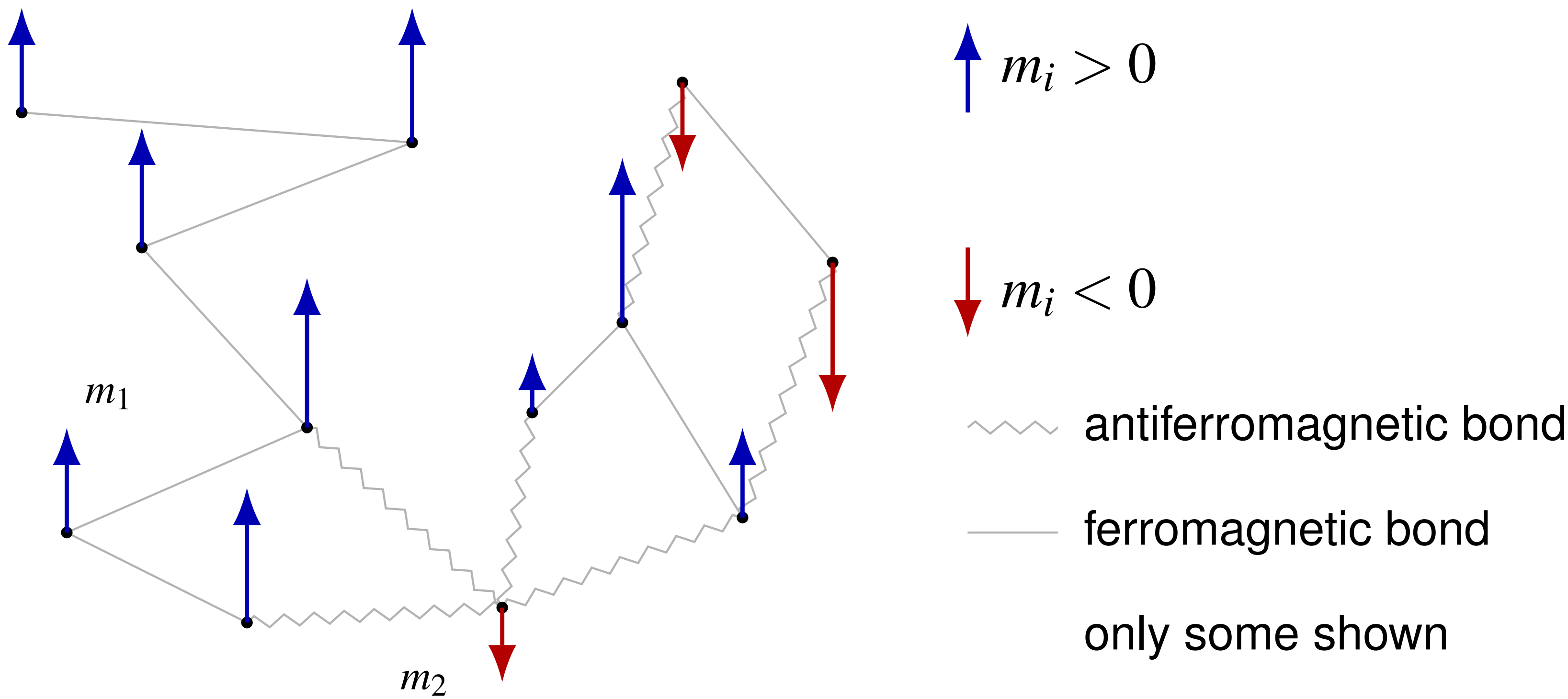
$\{m_i\}$ different real values

such that

$$m = \frac{1}{N}\sum_i m_i \to 0 \quad \text{but} \quad q_{\text{EA}} = \frac{1}{N}\sum_i m_i^2 \neq 0 \quad \text{for} \quad N \to \infty$$

Example

Focus on Ising magnetic impurities



to be explained

Pair interactions

RKKY between spins

Indirect exchange coupling between **localized classical magnetic moments** (nuclear spins or d-f-shell electronic spins) mediated by the surrounding sea of conduction electrons in a metal.

$$\mathcal{H}_J = -J_{\mathrm{RKKY}}(r_{ij})\,\vec{s}_i \cdot \vec{s}_j$$

$$J_{\mathrm{RKKY}}(r_{ij}) = J_0 \frac{\sin(2k_F r_{ij}) - 2k_F r_{ij} \cos(2k_F r_{ij})}{(2k_F r_{ij})^4} \qquad d = 3$$

For $2k_F r_{ij} \gg 1$, simplifies to $J_{\mathrm{RKKY}}(r_{ij}) \sim J_0 \frac{\cos(2k_F r_{ij})}{(2k_F r_{ij})^3}$

Sign depends on r_{ij} compared to Fermi wavelength $\lambda_F = 2\pi/k_F$.

Threshold between short and long range in $d = 3$ **Ex. 2**

Ruderman & Kittel 54 Kasuya & Yosida 56-57

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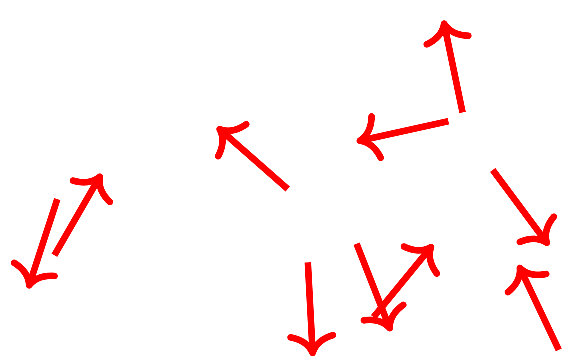


$$\mathcal{H} = -\frac{1}{2}\sum_{i\neq j} J_{ij}\,\vec{s}_i\cdot\vec{s}_j$$

with

$$J_{ij} = J_0\frac{\cos(2k_F r_{ij})}{r_{ij}^3}$$ RKKY

Edwards-Anderson Model

A minimal model

$$\boxed{\mathcal{H}_J = -\frac{1}{2}\sum_{\langle ij\rangle} J_{ij} s_i s_j}$$

$s_i = \pm 1$ **Ising spins** - the variables

from magnetic moments to bimodal variables

$\langle ij \rangle$ nearest-neighbours on a **cubic lattice**

from random positions to the vertices of a regular lattice

J_{ij} **quenched random**, taken from a probability distribution - the couplings

$P[\{J_{ij}\}]$ bimodal, Gaussian, or other with no fat tails, with

$[J_{ij}] = 0$ and $[J_{ij}^2] = J^2$

from RKKY to first neighbour random interactions

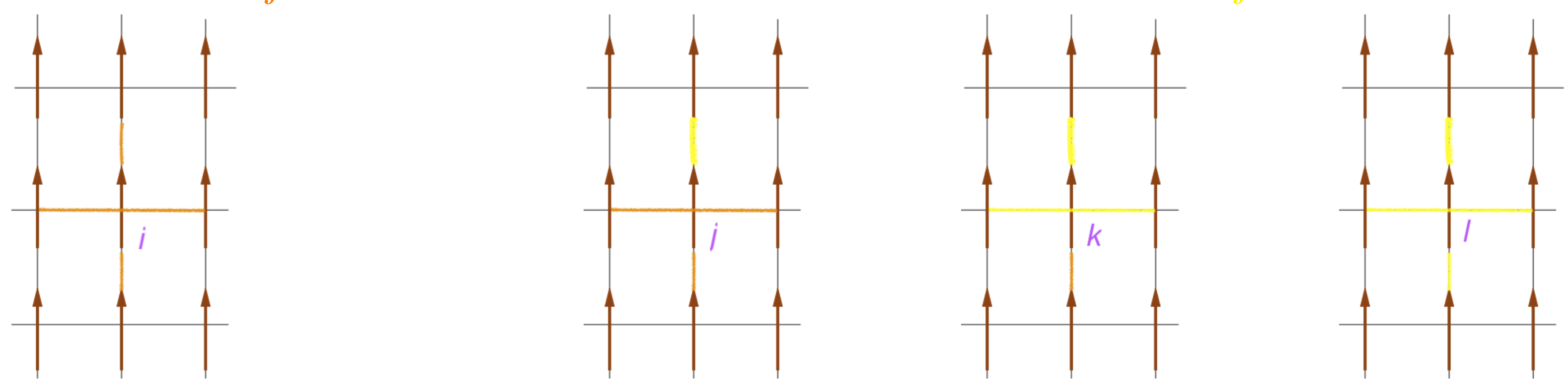
4. Properties

Heterogeneity

Induced by disorder

Each variable, spin or other, feels a different local field, $h_i = \sum_{j=1}^{z} J_{ij} s_j$, contrary to what happens in a ferromagnetic sample, for instance.

$J_{ij} > 0$ $\qquad$ $J_{ij} < 0$



Homogeneous $\qquad$ **Heterogeneous**

$h_i = 4J \ \forall\, i$ $\qquad$ $h_j = 2J$ $\qquad$ $h_k = -2J$ $\qquad$ $h_l = -4J$.

Each **sample** is *a priori* different but,

do they all have a different thermodynamic and dynamic behavior ?

Self-averageness

The question

Given two samples with different quenched randomness

(e.g. different interaction strengths J_{ij}s or random fields h_i)

but drawn from the same (kind of) distribution

is their behaviour going to be totally different ?

Which quantities are expected to be the same and which not ?

Self-averageness

The statement

Given a quantity A_J, which depends on the quenched randomness J_{ij}, it is distributed according to

$$P(A) = \int \prod_{ij} dJ_{ij}\, P(\{J_{ij}\})\, \delta(A - A_J)$$

For "additive" quantities, $P(A)$ expected to be more peaked as $N \to \infty$

Therefore, one will observe $A = A_{\rm typ}$ such that $P(A_{\rm typ}) = \max_A P(A)$

However, it is difficult to calculate $A_{\rm typ}$

What about calculating $[A] = \int \prod_{ij} dJ_{ij}\, P(\{J_{ij}\})\, A$?

Self-averageness for $N \gg 1$ $\boxed{[A] = A_{\rm typ}}$ **Ex. 3**

Self-averageness

Example : the disordered Ising chain

$$\mathcal{H}_J = -\sum_i J_i s_i s_{i+1} \qquad J_i \text{ } \textit{i.i.d.} \text{ with any pdf } P(J_i)$$

Compute the partition function Z by introducing $\sigma_i = s_i s_{i+1}$

$$\mathcal{Z}_J = \sum_{s_i=\pm 1} e^{\beta \sum_i J_i s_i s_{i+1}} = \sum_{\sigma_i=\pm 1} e^{\beta \sum_i J_i \sigma_i} = \prod_{i=1}^{N} 2\coth(\beta J_i)$$

(boundary condition effects are negligible for $N \to \infty$)

$\mathcal{Z}_J$ is a **product** of N *i.i.d.* random numbers $2\coth(\beta J_i)$

The free-energy is $-\beta F_J = \sum_{i=1}^{N} \ln\coth(\beta J_i) + N \ln 2$

First term in rhs is a **sum** of N *i.i.d.* random numbers

Self-averageness

Example : the disordered Ising chain

$$\mathcal{H}_J = -\sum_i J_i s_i s_{i+1} \qquad J_i \text{ i.i.d. with any pdf } P(J_i)$$

The partition function & the free energy density are different objects

$$\mathcal{Z}_J = \prod_{i=1}^{N} 2\coth(\beta J_i) \qquad -\beta f_J = \frac{1}{N}\sum_{i=1}^{N} \ln\coth(\beta J_i) + \ln 2$$

Take J_i to be *i.i.d.* with zero mean $[J_i] = 0$ & finite variance $[J_i^2] = \sigma_J^2$ and use

Central Limit Theorem (CLT) :

$\Longrightarrow$ $X = \frac{1}{N}\sum_i x_i$ is Gaussian distributed with average $\mu_X = [X] = [x_i]$, variance $\sigma_X^2 = [(X-[X])^2] \propto N$ and relative dispersion $\sigma_X/\mu_X \propto 1/\sqrt{N}$

f_J is Gaussian distributed and

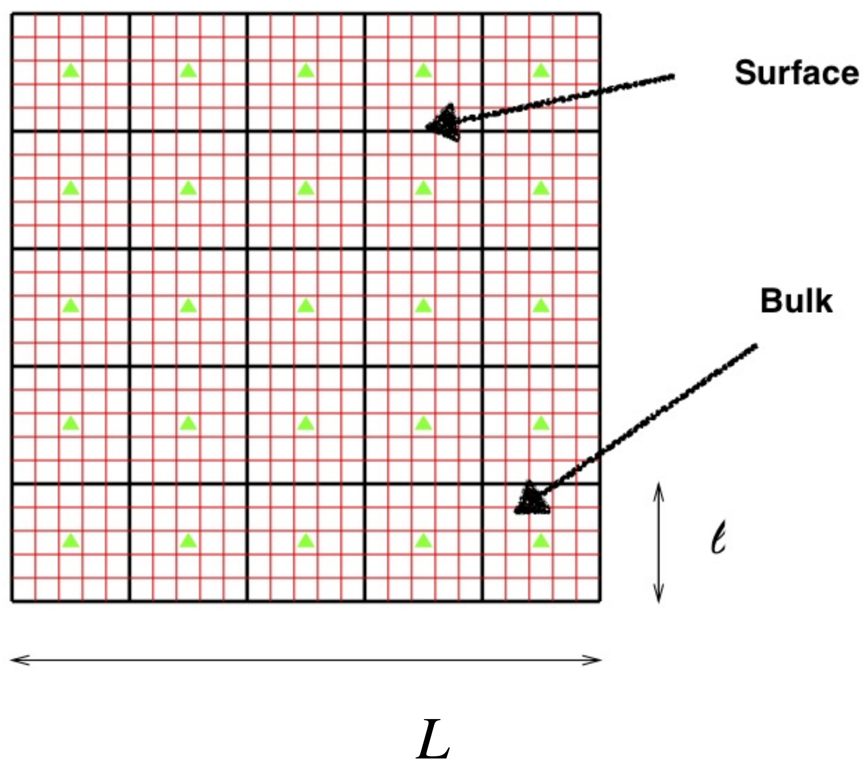
$$\boxed{f_J^{\rm typ} \underset{N\to\infty}{=} [f_J]}$$

Self-averageness

Systems with short-range interactions

Divide a, say, cubic system of volume $V = L^d$ in n_B boxes with volume $v = \ell^d$

$V = n_B v$ and the number of spins is $N = V/a^d$ with a the lattice spacing



$$a \ll \ell \ll L$$

$$N = (L/a)^d = n_B(\ell/a)^d \qquad n = (\ell/a)^d$$

Self-averageness

Proof for short-range interacting systems

$$\mathcal{H}_J(\text{conf}) = \mathcal{H}_J(\text{bulk}) + \mathcal{H}_J(\text{int})$$

The total free-energy is the sum of two terms, a contribution from the bulk of the subsystems and a contribution from the interactions between the subsystems :

$$\begin{aligned} -\beta F_J &= \ln Z_J = \ln \sum_{\text{conf}} e^{-\beta \mathcal{H}_J(\text{conf})} = \ln \sum_{\text{conf}} e^{-\beta \mathcal{H}_J(\text{bulk}) - \beta \mathcal{H}_J(\text{int})} \\ &\approx \ln \sum_{\text{bulk}} e^{-\beta \mathcal{H}_J(\text{bulk})} + \ln \sum_{\text{surf}} e^{-\beta \mathcal{H}_J(\text{surf})} = -\beta F_J^{\text{bulk}} - \beta F_J^{\text{surf}} \end{aligned}$$

where the $\approx$ indicates that we dropped the contributions of far away interactions between different bulks/boxes (short-range assumption) and we kept only surface interactions between neighbouring bulks

Self-averageness

Proof for short-range interacting systems

If the interaction extends over a short distance l and the linear size of the boxes is $\ell \gg l$, we also assume that the surface energy is negligible with respect to the bulk one

$$-\beta F_J \approx \ln \sum_{\text{bulk}} e^{-\beta \mathcal{H}_J(\text{bulk})}$$

The disorder-dependent free-energy is a sum of $n_B = (L/\ell)^d$ independent random numbers, each one being the disorder dependent free-energy of the bulk of each subsystem :

$$-\beta F_J \approx \sum_{k=1}^{n_B} \ln \sum_{\text{bulk}_k} e^{-\beta \mathcal{H}_J(\text{bulk}_k)}$$

In the limit of a very large number of subsystems ($L \gg \ell$ or $n_B \gg 1$) the **CLT** $\Rightarrow$ the free-energy is Gaussian distributed with

$$F_J^{\text{typ}} = [F_J]$$

Self-averageness

Scalings

The free-energy $-\beta F_J \approx \sum_{k=1}^{n_B} \ln \sum_{\text{bulk}_k} e^{-\beta \mathcal{H}_J(\text{bulk}_k)} = \sum_{k=1}^{n_B} (-\beta F_{Jk})$ is extensive : $F_J = O(n_B n) = O(N)$ (number of boxes times number of spins in each box = total number of spins)

The variance $\sigma^2_{F_J} = [F_J^2] - [F_J]^2$ and the mean $\mu_{F_J} = [F_J]$ grow as N

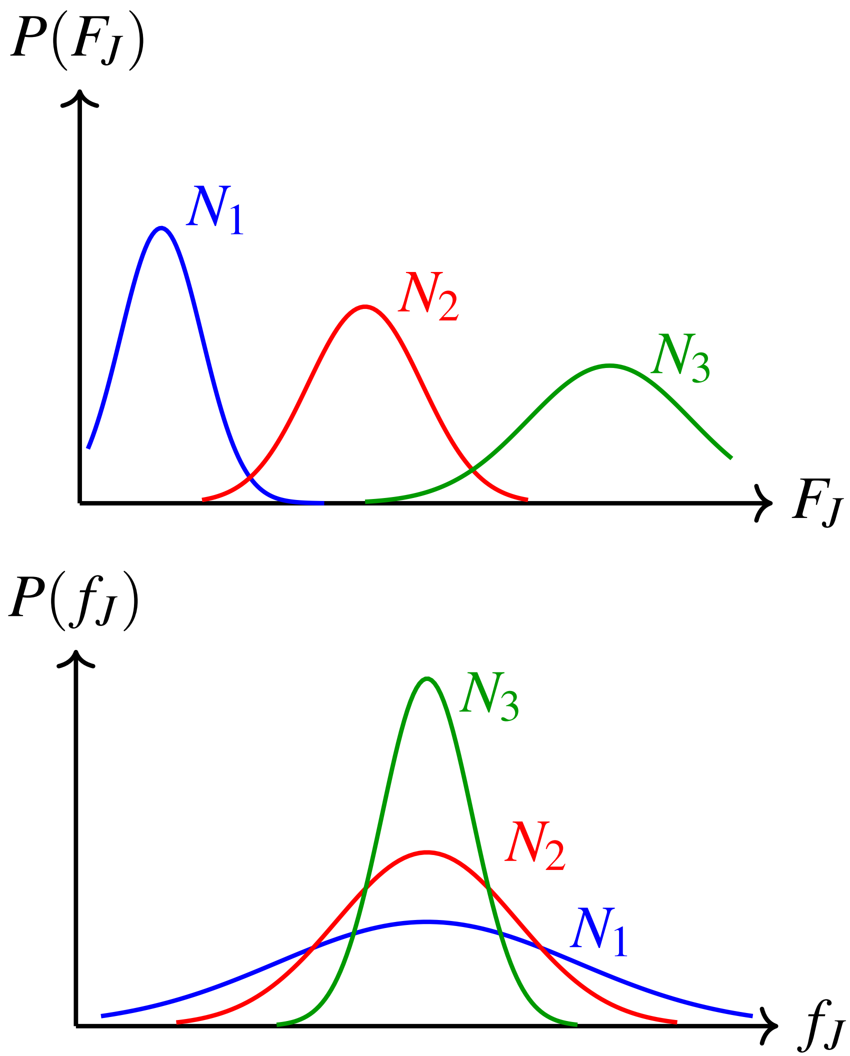
The relative fluctuations of the total free-energy vanishes in the large N limit, $\sigma_{F_J}/[F_J] \propto \sqrt{N}/N = N^{-1/2} \to 0$

The mean of the intensive $f_J = F_J/N = O(1)$ is $O(1)$, its standard deviation is $\sigma_{f_J} = \sigma_{F_J}/N = N^{-1/2}$ and the relative fluctuations also vanish as $N^{-1/2}$

For large N the typical free-energy density f_J^{typ} is very close to the averaged $[\, f_J \,]$. The latter is easier to compute.

Self-averageness

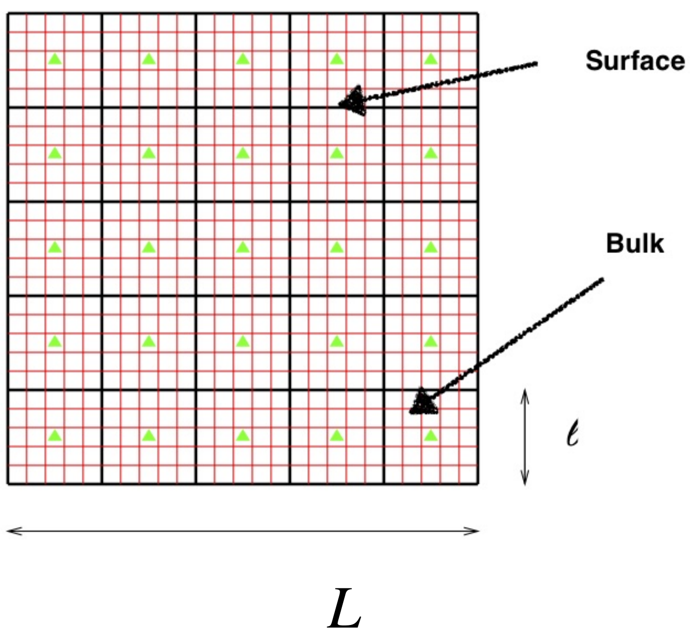
Scaling of the free-energy $N_1 < N_2 < N_3$



Self-averageness

Systems with short-range interactions - summary

Divide a, say, cubic system of volume $V = L^d$ in n_B boxes of volume $v = \ell^d$ with $V = n_B v$



$$-\beta F_J \approx \sum_{k=1}^{n_B} \ln \sum_{\text{bulk}_k} e^{-\beta \mathcal{H}_J(\text{bulk}_k)}$$

For $L \gg \ell$ the **CLT**

$\Rightarrow f_J$ is Gaussian distributed with

$\sigma_{f_J} = N^{-1/2}$ and

$$f_J^{\text{typ}} = [\, f_J\,]$$

Quenched vs annealed averages

Definitions

Go back to the one dimensional disordered Ising chain and show that

the partition function and the spatial correlations

are not self-averaging **Ex. 3**

The annealed free-energy is defined as $-\beta F^{\text{annealed}} \equiv \ln[Z_J]$

The quenched free-energy is defined as $-\beta F^{\text{quenched}} \equiv [\ln Z_J]$ **Ex. 4**

Jenssen's inequality applied to the convex function $-\ln x$ implies

$$-\ln[Z_J] \leq -[\ln Z_J]$$

and for the free-energies one deduces

$$F^{\text{annealed}} = -\beta^{-1} \ln[Z_J] \leq -\beta^{-1}[\ln Z_J] = F^{\text{quenched}}$$

Jenssen's inequality

Deterministic Form

Convex function f and $\lambda \in [0,1]$

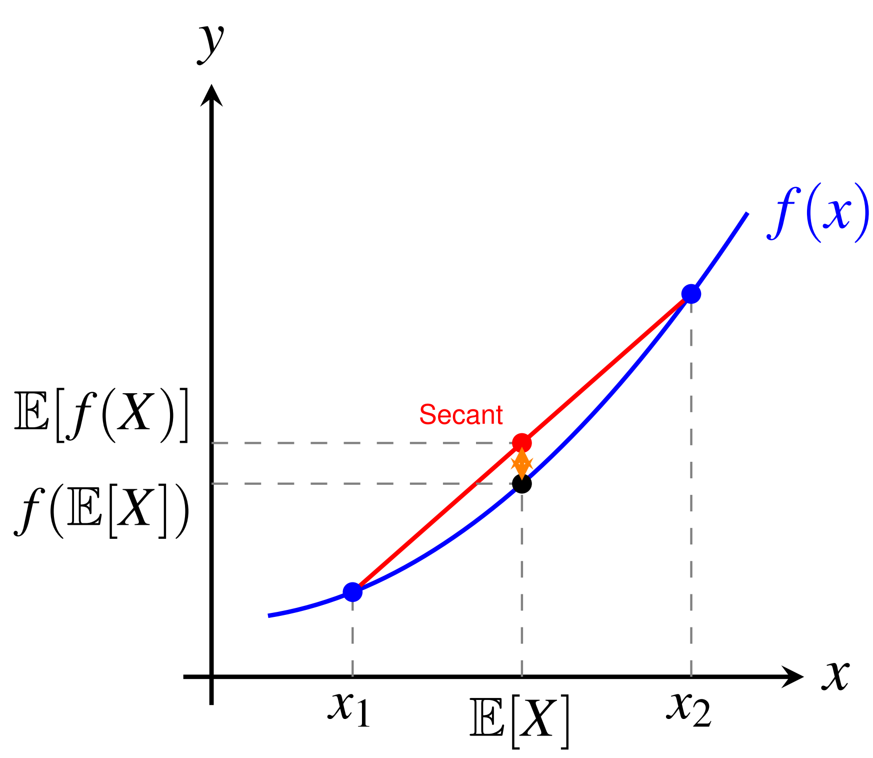
$$f\left(\lambda x_1 + (1-\lambda)x_2\right) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

For any convex curve, the secant line connecting $(x_1, f(x_1))$ and $(x_2, f(x_2))$ lies entirely **above or on** the function graph.

Probabilistic Form

Random variable X

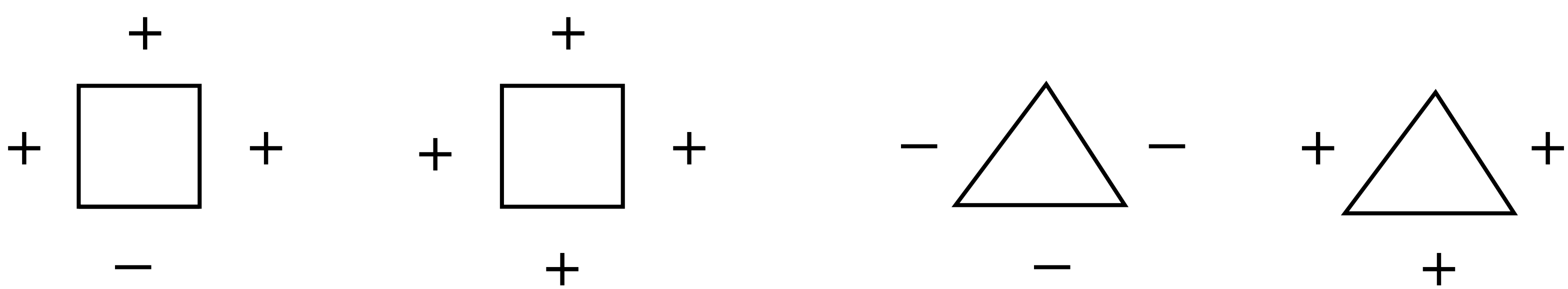
$$f\left(\mathbb{E}[X]\right) \leq \mathbb{E}\left[f(X)\right]$$



Frustration

Examples & definition

$$\boxed{\mathcal{H}_J = -\sum_{\langle ij \rangle} J_{ij} s_i s_j \qquad \textbf{Ising model}}$$



Disordered **Geometric** (spin ices)

$$\boxed{E_{\rm GS}^{\rm frust} > E_{\rm GS}^{\rm FM} \quad \text{and} \quad S_{\rm GS}^{\rm frust} > S_{\rm GS}^{\rm FM}}$$

Frustration enhances the **ground-state** energy and **degeneracy**

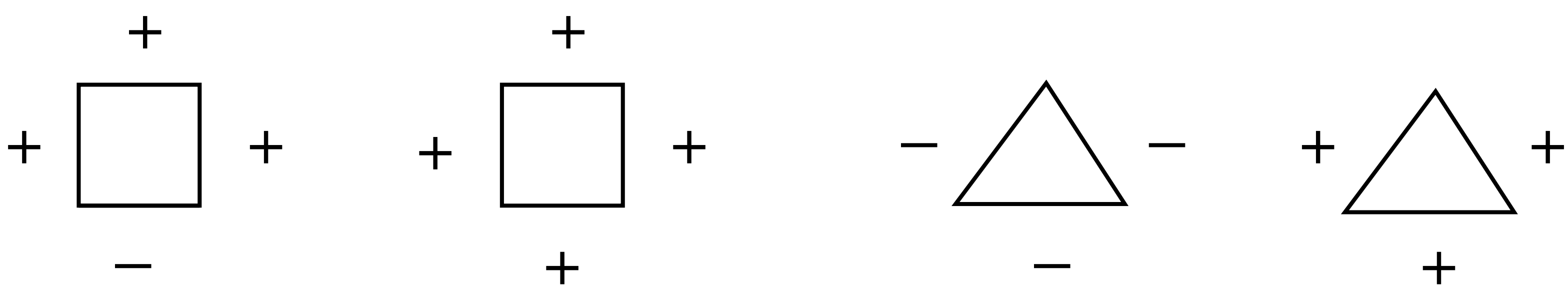
One can expect to have many **metastable states** too

One cannot satisfy all couplings simultaneously if $\boxed{\prod_{\rm loop} J_{ij} < 0}$

Frustration

Examples

$$\boxed{\mathcal{H}_J = -\sum_{\langle ij\rangle} J_{ij}\, s_i s_j \qquad \textbf{Ising model}}$$



Disordered **Geometric** (spin ices)

$$\boxed{E_{\mathrm{GS}}^{\mathrm{frust}} > E_{\mathrm{GS}}^{\mathrm{FM}} \qquad \text{and} \qquad S_{\mathrm{GS}}^{\mathrm{frust}} > S_{\mathrm{GS}}^{\mathrm{FM}}}$$

Disordered example

$$E_{\mathrm{GS}}^{\mathrm{frust}} = -2J \quad > \quad E_{\mathrm{GS}}^{\mathrm{FM}} = -4J$$

$$S_{\mathrm{GS}}^{\mathrm{frust}} = \ln 4 \quad > \quad S_{\mathrm{GS}}^{\mathrm{FM}} = \ln 1$$

Weak disorder

Disorder with no frustration

Does not modify the nature of the phases but can affect the one of the transition

- Random field disorder destroys a phase transition in $D \leq 2$ **Ex. 6**

  Imry & Ma 75

- Random bond disorder modifies the critical exponents of 2nd order ones

  random-mass-like perturbation to the pure critical point

  Harris' criterium $D\nu < 2$ **Harris 74 Chayes, Chayes, Fisher & Spencer 86**

  Review **Vojta 06**

- Quenched disorder coupling to the energy density rounds 1st order transitions, they become 2nd order ones in $D \leq 2$

  Math-Phys proof **Azeimann & Wehr 89** Field Theory/CFT **Cardy 96**

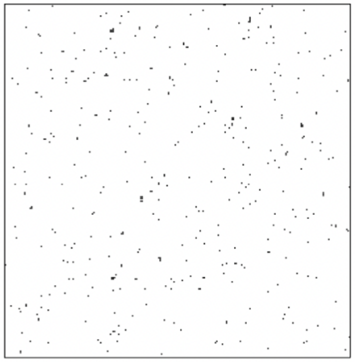
  CFT & numerics **Dotsenko, Picco & Pujol 90s**

5. Order parameters

Low temperature phase

Ferromagnetic Ising model

$$\mathcal{H}_J = -\frac{J}{2}\sum_{\langle ij \rangle} s_i s_j \qquad\qquad J > 0$$



Instantaneous snapshot

of an equilibrium configuration

at $0 < T < T_c$

Spontaneous symmetry breaking at $0 \leq T < T_c$ towards a

negatively magnetized $-1 < m(T) < 0$ state

The reverse state with $0 < m(T) < 1$ is equally probable

Low temperature phase

Phenomenology : homogeneity vs inhomogeneity

In a **ferromagnet in equilibrium** at temperature $T < T_c$, $\langle s_i \rangle = m(T)\ \forall i$ or $\langle s_i \rangle = -m(T)\ \forall i$ in the two homogeneous, symmetric and degenerate equilibrium states

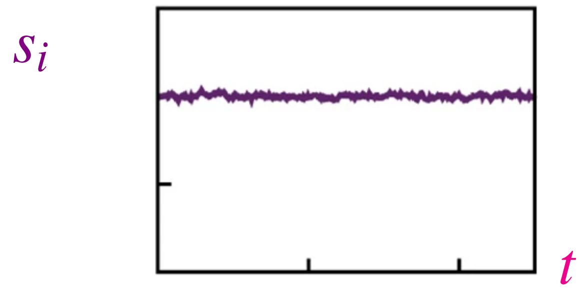
$m(T)$ is the global magnetization density

Low temperature phase

Phenomenology : homogeneity vs inhomogeneity

In a **ferromagnet in equilibrium** at temperature $T < T_c$, $\langle s_i \rangle = m(T)\ \forall i$ or $\langle s_i \rangle = -m(T)\ \forall i$ in the two homogeneous, symmetric and degenerate equilibrium states

If one were to follow the time evolution of each spin in one of the two equilibrium states at $T < T_c$ and measure a running time average $\overline{s_i}(t) = \tau^{-1} \int_t^{t+\tau} dt'\, s_i(t')$ it would be $\sim m(T)$ with small fluctuations due to finite τ

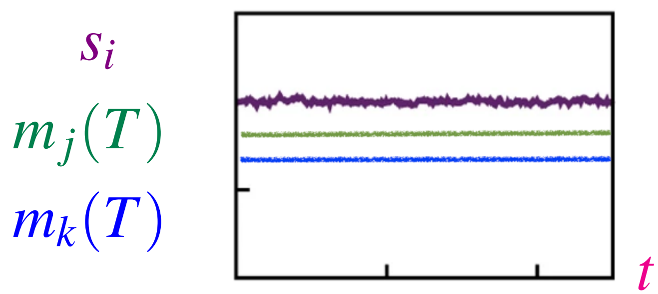


Low temperature phase

Phenomenology : homogeneity vs inhomogeneity

In a **spin-glass in equilibrium** at temperature $T < T_c$, one expects $\langle s_i \rangle = m_i(T)$, with a different value for each i, in the inhomogeneous equilibrium state.

If one were to follow the time evolution of each spin in an equilibrium state at $T < T_c$, one would measure the running time average
$\overline{s_i}(t) = \tau^{-1} \int_t^{t+\tau} dt' \, s_i(t') \sim m_i$



Low temperature phase

Phenomenology : homogeneity vs inhomogeneity

In a **spin-glass in equilibrium** at temperature $T < T_c$, one expects $\langle s_i \rangle = m_i(T)$, with a different value for each i, in each inhomogeneous and degenerate equilibrium state.

Local order parameters

There may be many different ensembles $\{m_i(T)\}$ that are equilibrium states (degeneracy, similar to what happens in frustrated magnets for the ground states but here in the full low T phase) **More later**

There is also the up-down symmetry $\{m_i(T)\} \mapsto \{-m_i(T)\}$

Edwards-Anderson Model

Dynamic order parameter

Theory of spin glasses

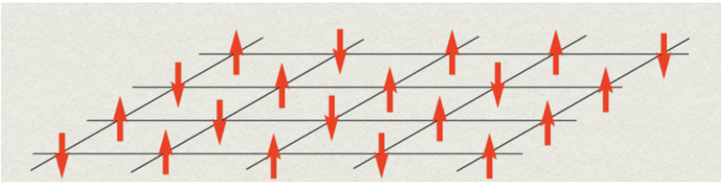
S F Edwards† and P W Anderson‡
Cavendish Laboratory, Cambridge, UK

Received 14 October 1974, in final form 13 February 1975

viewed later will be the *same* random coil. Thus what we must argue is that if on one observation a particular spin is $\mathbf{s}_i^{(1)}$ then if it is studied again a long time later, there is a nonvanishing probability that $\mathbf{s}_i^{(2)}$ will point in the same direction, ie

$$q = \langle \mathbf{s}_i^{(1)} . \mathbf{s}_i^{(2)} \rangle \neq 0.$$

$$\boxed{q_{\mathrm{EA}} = \lim_{t \gg t'} \lim_{N \to \infty} \langle s_i(t) s_i(t') \rangle \neq 0}$$



Here the angular brackets represent average over different runs of the evolution

Spatially disordered but dynamically frozen, on average

$$\chi = \beta(1 - q_{\mathrm{EA}})$$

More later on dynamics

6. The mean-field SK model

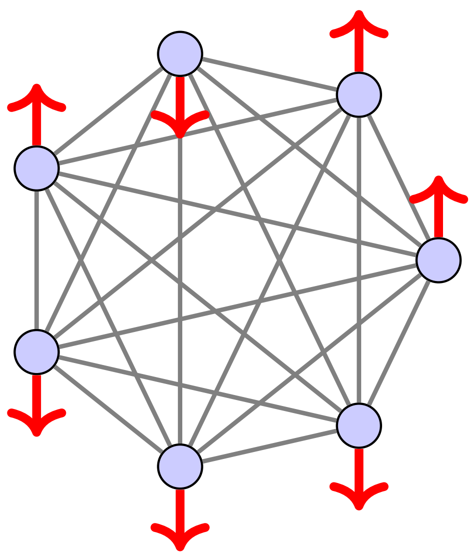
Sherrington-Kirkpatrick Model

The mean-field extension – all to all couplings – 70s

Solvable Model of a Spin-Glass

David Sherrington* and Scott Kirkpatrick
IBM Thomas J. Watson Research Center, Yorktown Heights, New York 10598
(Received 16 October 1975)

We consider an Ising model in which the spins are coupled by infinite-ranged random interactions independently distributed with a Gaussian probability density. Both "spin-glass" and ferromagnetic phases occur. The competition between the phases and the type of order present in each are studied.



$$\mathcal{H}_J = -\frac{1}{2}\sum_{i\neq j} J_{ij}\, s_i s_j$$

$$[J_{ij}] = 0 \qquad [J_{ij}^2] = \frac{J^2}{N}$$

Justification

Mean-field theory

Fully connected Ising models

General model

$$\mathcal{H}_J = -\frac{1}{2} \sum_{i \neq j} J_{ij} s_i s_j \quad \text{with Ising variables} \quad s_i = \pm 1$$

$O(1)$ scaling of the local fields $\Rightarrow$ scaling of J_{ij}

What is a local field ?

It is the field felt by a selected site

$$h_i = \frac{1}{2} \sum_{j(\neq i)} J_{ij} s_j$$

and we require it to be $O(1)$

Mean-field theory

Fully connected Ising models

General model

$$\mathcal{H}_J = -\frac{1}{2}\sum_{i\neq j} J_{ij} s_i s_j \qquad \text{with Ising variables} \qquad s_i = \pm 1$$

$O(1)$ **scaling of the local fields** $\Rightarrow$ **scaling of** J_{ij}

In the **Curie-Weiss ferromagnetic case**

$$J_{ij} = \frac{J}{N} \ \text{such that}\ h_i = \frac{J}{2N} \sum_{j(\neq i)} s_j = O(1)$$

in the two ferromagnetic $s_i = 1\ \forall i$ or $s_i = -1\ \forall i$ phases

Mean-field theory

Fully connected Ising models

General model

$$\mathcal{H}_J = -\frac{1}{2}\sum_{i\neq j} J_{ij} s_i s_j \qquad \text{with Ising variables} \qquad s_i = \pm 1$$

$O(1)$ **scaling of the local fields** $\Rightarrow$ **scaling of** J_{ij}

In the Curie-Weiss ferromagnetic case

$$J_{ij} = \frac{J}{N} \;\; \text{such that} \;\; h_i = \frac{J}{2N}\sum_{j(\neq i)} s_j = O(1)$$

in the ferromagnetic $s_i = 1 \;\forall i$ or $s_i = -1 \;\forall i$ phases

In the **Sherrington-Kirkpatrick disordered case**

$$J_{ij} = O(\frac{J}{\sqrt{N}}) \;\; \text{such that} \;\; h_i \sim \frac{J}{2\sqrt{N}}\sum_{j(\neq i)} s_j = O(1)$$

in the PM or spin-glass phases $s_i = \pm 1 \;\forall i$

Mean-field theory

Fully connected Ising models

General model

$$\mathcal{H}_J = -\frac{1}{2}\sum_{i\neq j} J_{ij}s_is_j \quad \text{with Ising variables} \quad s_i = \pm 1$$

$O(1)$ **scaling of the local fields $\Rightarrow$ scaling of J_{ij}**

In the **Sherrington-Kirkpatrick disordered case**

$$J_{ij} = O(\frac{J}{\sqrt{N}}) \text{ such that } h_i \sim \frac{J}{2\sqrt{N}} \sum_{j(\neq i)} s_j = O(1)$$

in the PM or spin-glass phases, say, $s_i = \pm 1$ with equal probability

One can use a Gaussian pdf

$$P(J_{ij}) = (2\pi\sigma^2)^{-1/2} \exp[-J_{ij}^2/(2\sigma^2)] \text{ with } \sigma^2 = J^2/N$$

7. TAP & landscapes

Thouless Anderson Palmer (TAP)

Working at fixed disorder

PHILOSOPHICAL MAGAZINE, 1977, VOL. 35, NO. 3, 593–601

Solution of 'Solvable model of a spin glass'

By D. J. THOULESS
Department of Mathematical Physics, University of Birmingham, Birmingham, England

and P. W. ANDERSON†‡ and R. G. PALMER
Department of Physics, Princeton University, Princeton, New Jersey 08540, U.S.A.‡

[Received 12 October 1976]

ABSTRACT

The Sherrington–Kirkpatrick model of a spin glass is solved by a mean field technique which is probably exact in the limit of infinite range interactions. At and above T_c the solution is identical to that obtained by Sherrington and Kirkpatrick (1975) using the $n \to 0$ replica method, but below T_c the new result exhibits several differences and remains physical down to $T = 0$.

$$\boxed{\langle s_i \rangle = m_i = \tanh\{\beta \textstyle\sum_{j(\neq i)} [J_{ij} m_j - \beta J_{ij}^2 (1 - m_j^2) m_i]\}}$$

Curie-Weiss

Fully connected ferromagnetic Ising model

Normalize J by the size of the system N to have $O(1)$ local fields

$$\boxed{\mathcal{H}_J = -\frac{J}{2N}\sum_{i\neq j} s_i s_j - h\sum_i s_i}$$

The partition function reads $\mathcal{Z}_J = \int_{-1}^{1} du\; e^{-\beta N \mathrm{f}(u)}$ with $Nu = \sum_i s_i$

$$\boxed{\mathrm{f}(u) = -\frac{J}{2}u^2 - hu + T\left[\frac{1+u}{2}\ln\frac{1+u}{2} + \frac{1-u}{2}\ln\frac{1-u}{2}\right]}$$

Energy terms + entropic contribution stemming from $\mathcal{N}(\{s_i\})$ yielding the same u value.

Use the **saddle-point** $u^*(\beta J)$ from

$$\boxed{u^* = \tanh\left(\beta J u^* + \beta h\right) = \langle u \rangle = m}$$

to obtain $\lim_{N\to\infty} f_N(\beta J, \beta h) = \mathrm{f}(u^*)$

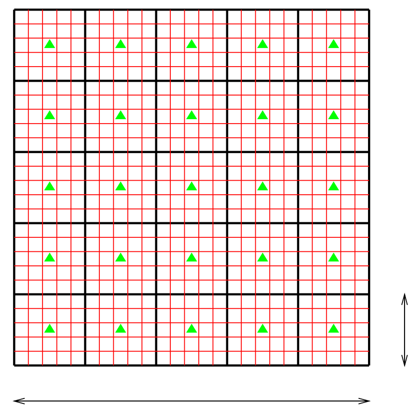
Ginzburg-Landau

Continuous scalar statistical field theory with local aspects

Coarse-grain the spin

$\phi(\vec{r}) = V_{\vec{r}}^{-1} \sum_{i \in V_{\vec{r}}} s_i$

Set $h = 0$



The partition function is $\mathcal{Z}_J = \int \mathcal{D}\phi \; e^{-\beta V \mathrm{f}(\phi)}$ with V the volume and

$$\boxed{\mathrm{f}(\phi) = \int d^d r \; \left\{ \frac{1}{2}[\nabla\phi(\vec{r})]^2 + \frac{T-J}{2}\phi^2(\vec{r}) + \frac{\lambda}{4}\phi^4(\vec{r}) \right\}}$$

Elastic + potential energy with the latter inspired by the results for the fully-connected model (entropy around $\phi \sim 0$ and symmetry arguments.

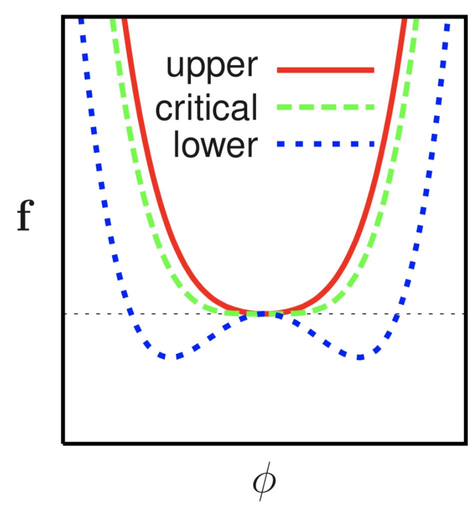
Uniform saddle point in the $V \to \infty$ limit : $\phi^*(\vec{r}) = \langle \phi(\vec{r}) \rangle = m$

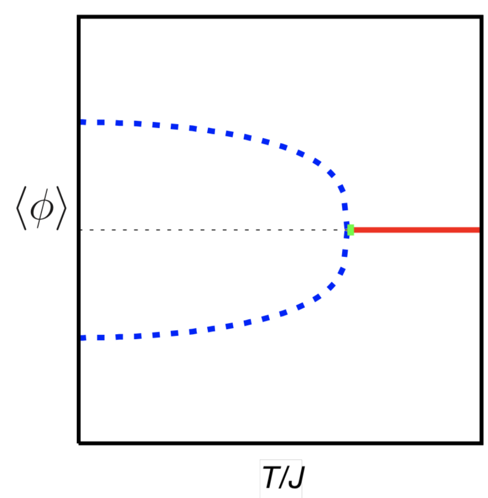
The free-energy density is $\lim_{V\to\infty} f_V(\beta J) = \mathrm{f}(\phi^*)$

2nd order phase-transition

Continuous scalar statistical field theory

bi-valued equilibrium states related by symmetry





Ginzburg-Landau free-energy

Scalar order parameter

Features

Phases and phase transition

- **Spontaneous symmetry breaking** below T_c
- Two equilibrium states related by **symmetry** $\phi \to -\phi$
- The state is chosen by a **pinning field**
- If the partition sum is performed over the whole phase space $\langle \phi \rangle = 0$
  (a consequence of the symmetry of the action)
- **Restricted statistical averages** over *half* phase space $\langle \phi \rangle \neq 0$
- Under a magnetic field the free-energy landscape is tilted and one of
  the minima becomes a **metastable state**
- The barrier in the **free-energy landscape** between the two states
  diverges with the size of the system implying **ergodicity breaking**

Features

Phases and phase transition

- $\mathrm{f}(u)$ and $\mathrm{f}(\phi(\vec{r}))$ are **large deviation function(al)s** determining the probability of finding an equilibrium system with u or $\phi(\vec{r})$
- The system spends $t_{\pm} \simeq e^{N\tau}$ close to each minima and it makes rapid transitions between the two

These results were not fully accepted as realistic at the time

Recall. the discussion on phase transitions & ergodicity breaking

- With $p > 2$-uplet ferromagnetic interactions one finds **first order phase transitions** (relevant for glasses & K-sat like problems)

More later on dynamics and p-spin interactions

Thouless-Anderson-Palmer

Fully connected Sherrington-Kirkpatrick model

$$\boxed{\mathcal{H}_J = -\frac{1}{2}\sum_{i\neq j} J_{ij} s_i s_j}$$

with J_{ij} i.i.d. Gaussian variables, $[J_{ij}] = 0$ and $[J_{ij}^2] = J^2/N = O(1/N)$.

One finds the **naive free-energy landscape**

$$\boxed{N\mathrm{f}(\{m_i\}) = -\frac{1}{2}\sum_{i\neq j} J_{ij} m_i m_j + k_B T \sum_{i=1}^{N} \frac{1+m_i}{2}\ln\frac{1+m_i}{2} + \frac{1-m_i}{2}\ln\frac{1-m_i}{2}}$$

and the naive TAP equations (including random fields for completeness)

$$\boxed{{m_i}^* = \tanh(\beta \textstyle\sum_{j(\neq i)} J_{ij} {m_j}^* + \beta h_i) = \langle m_i \rangle}$$

that determine the averages $m_i = \langle s_i \rangle = {m_i}^*$

Thouless-Anderson-Palmer

Fully connected SK model : a simple proof

The more traditional one assumes independence of the spins,

$$P(\{s_i\}) = \prod_i p_i(s_i)$$

with $p_i(s_i) = \frac{1+m_i}{2}\delta_{s_i,1} + \frac{1-m_i}{2}\delta_{s_i,-1}$

and uses this form to express $\langle H \rangle - T\langle S \rangle$ with $S = k_B \ln \mathcal{N}(\{s_i\})$

The energetic contribution is straightforward to evaluate

The entropic contribution is the one we already computed for the Curie-Weiss model, taking care of keeping the indices i **Ex. 7**

A more powerful proof expresses f as the **Legendre transform** of $-\beta F(h_i)$ with $m_i = N^{-1}\partial[-\beta F(h_i)]/\partial h_i$ and takes care of a “problem” to be solved in the next slides

Georges & Yedidia 91

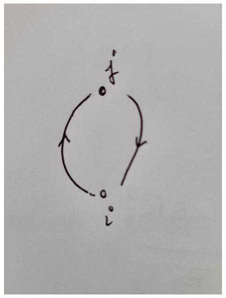
Thouless-Anderson-Palmer

The Onsager reaction term

These equations are not completely correct.

The **Onsager reaction term** is missing.

This term represents the reaction of the spin i to itself



The magnetisation in i produces a field $h'_{j(i)} = J_{ji} m_i = J_{ij} m_i$ on spin j

This field induces a magnetisation $m'_{j(i)} = \chi_{jj} h'_{j(i)} = \chi_{jj} J_{ij} m_i$ on the spin j.

This magnetisation produces a field $h'_{i(j)} = J_{ij} m'_{j(i)} = J_{ij} \chi_{jj} J_{ij} m_i$ on site i.

The equilibrium fluctuation-dissipation relation between susceptibilities and connected correlations implies $\chi_{jj} = \beta \langle (s_j - \langle s_j \rangle)^2 \rangle = \beta(1 - m_j^2)$ and one then has
$h'_{i(j)} = \beta(1 - m_j^2) J_{ij}^2 m_i$

Thouless-Anderson-Palmer

The Onsager reaction term

The idea of Onsager – or **cavity method** – is that one has to study the ordering of the spin i in the absence of its own effect on the rest of the system.

The total field produced by the sum of $h'_{i(j)} = \beta(1-m_j^2)J_{ij}^2 m_i$ over all the spins j with which it can connect, has to be subtracted from the mean-field created by the other spins in the sample, *i.e.* the total local field should be

$$h_i^{\mathrm{loc}} = \sum_{j(\neq i)} J_{ij} m_j - \beta m_i \sum_{j(\neq i)} J_{ij}^2 (1-m_j^2) + \beta h_i$$

recall that $J_{ij} = O(1/\sqrt{N})$. Finally, the TAP equations read

$$m_i = \tanh\left\{ \sum_{j(\neq i)} \left[\beta J_{ij} m_j - \beta^2 m_i J_{ij}^2 (1-m_j^2) \right] + \beta h_i \right\}$$

TAP equations

Orders of magnitude - SK model

$$m_i = \tanh\left\{ \sum_{j(\neq i)} [\beta J_{ij} m_j - \beta^2 m_i J_{ij}^2 (1 - m_j^2)] + \beta h_i \right\}$$

Recall that $m_i = \langle s_i \rangle$

The first term in the rhs $\sum_{j(\neq i)} J_{ij} m_j \simeq \frac{1}{\sqrt{N}} \sqrt{N} = O(1)$ because of the central limit theorem

The second term $\sum_{j(\neq i)} J_{ij}^2 (1 - m_j^2) \simeq \frac{1}{N} N = O(1)$ because all terms in the sum are semi-positive definite ($0 \leq m_j^2 \leq 1 \; \forall j$)

Onsager reaction term eliminates back reaction

No higher order terms

TAP equations

Orders of magnitude - Curie-Weiss

$$\mathcal{H}_J = -\frac{J}{N}\sum_{i\neq j} s_i s_j \qquad\qquad h = 0$$

The **TAP equations** become the mean-field equation

$$\underbrace{m_i}_{m} = \tanh\left\{ \underbrace{\sum_{j(\neq i)} \beta \underbrace{J_{ij}}_{J/N} \underbrace{m_j}_{m}}_{O(1)} - \beta^2 \underbrace{m_i}_{m} \underbrace{\sum_{j(\neq i)} \underbrace{J_{ij}^2}_{J^2/N^2} \underbrace{(1-m_j^2)}_{(1-m^2)}}_{O(1/N)} + \beta h_i \right\}$$

Recall that $m_i = \langle s_i \rangle = m$ if ferromagnetic interactions

$$m = \tanh(\beta J m + \beta h)$$

The **Onsager reaction term** is negligible in this case

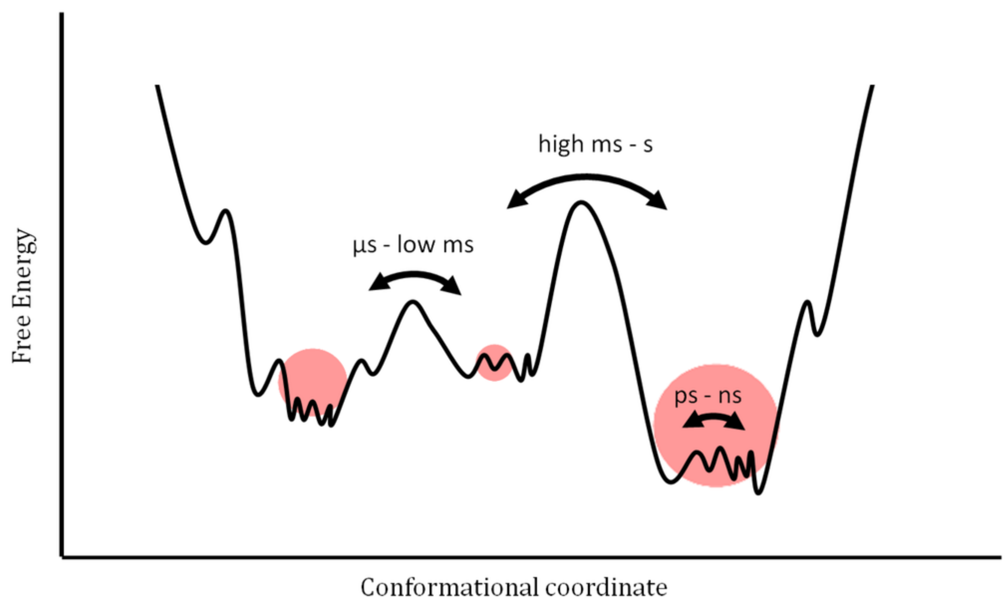
Free-energy Landscapes

Free-energy landscapes

Free-energy density at fixed randomness

The TAP equations are the extremization conditions on the *TAP free-energy*

$$F_J^{\mathrm{tap}}(\{m_i\}) = -\frac{1}{2}\sum_{i\neq j} J_{ij} m_i m_j - \sum_i h_i m_i - \frac{\beta}{4}\sum_{i\neq j} J_{ij}^2 (1-m_i^2)(1-m_j^2)$$
$$+ k_B T \sum_{i=1}^{N}\left[\frac{1+m_i}{2}\ln\frac{1+m_i}{2} + \frac{1-m_i}{2}\ln\frac{1-m_i}{2}\right]$$



At low temperatures $\{m_i\}$

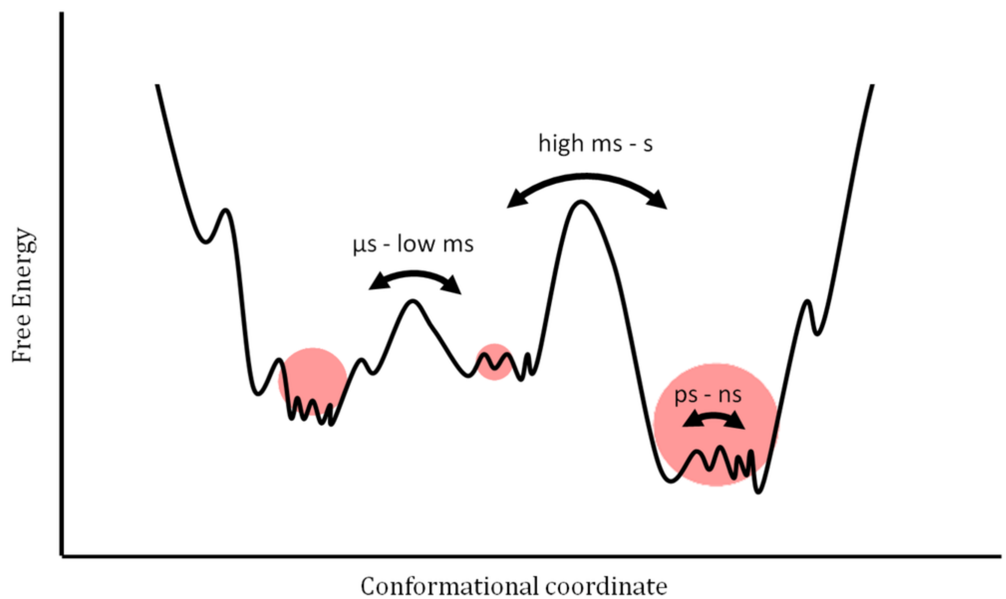
Free-energy landscapes

Free-energy density at fixed randomness

The TAP equations are the extremization conditions on the *TAP free-energy*

$$\frac{\partial F_J^{\text{tap}}(\{m_i\})}{\partial m_j} = 0$$

The stability of the solutions is determined by the Hessian $\frac{\partial^2 F_J^{\text{tap}}(\{m_i\})}{\partial m_j \partial m_k}$



At low temperatures $\{m_i\}$

Free-energy landscapes

Free-energy density at fixed randomness

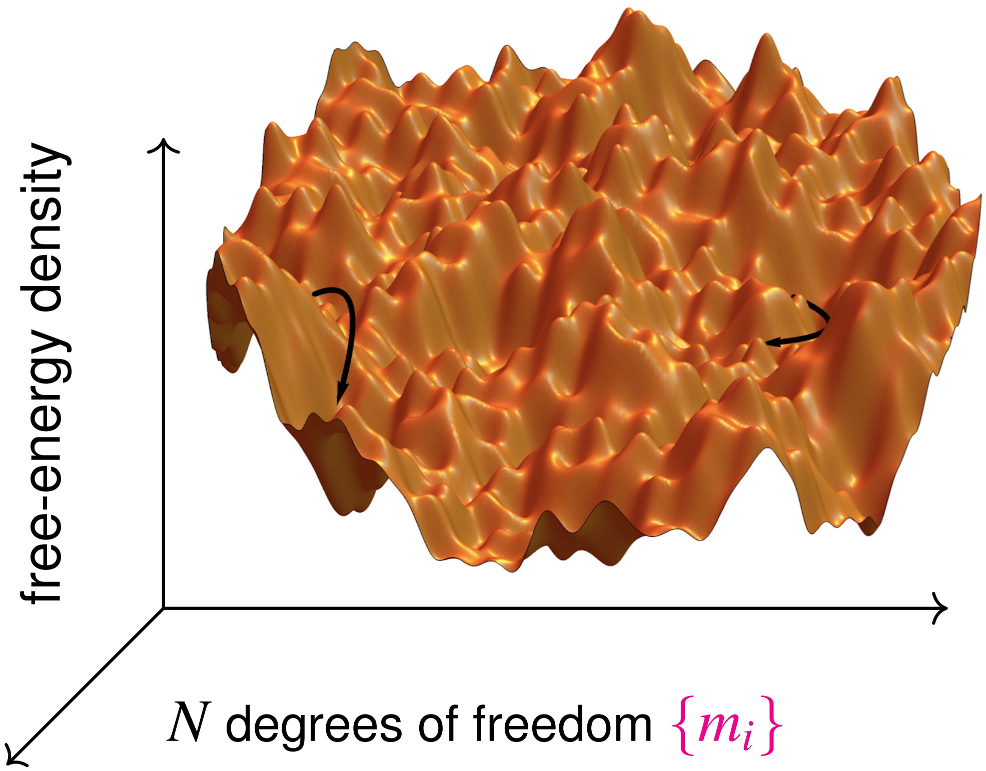


Figure adapted from a picture by **C. Cammarota**

For each $\{J_{ij}\}$ realization the landscape is different
but the **statistical properties** are the same **Bray & Moore**, **Cavagna, Giardina & Parisi**, **Fyodorov & Ros**, **Kurchan & Kent-Dobias**, etc.

Free-energy landscapes

Free-energy density at fixed randomness

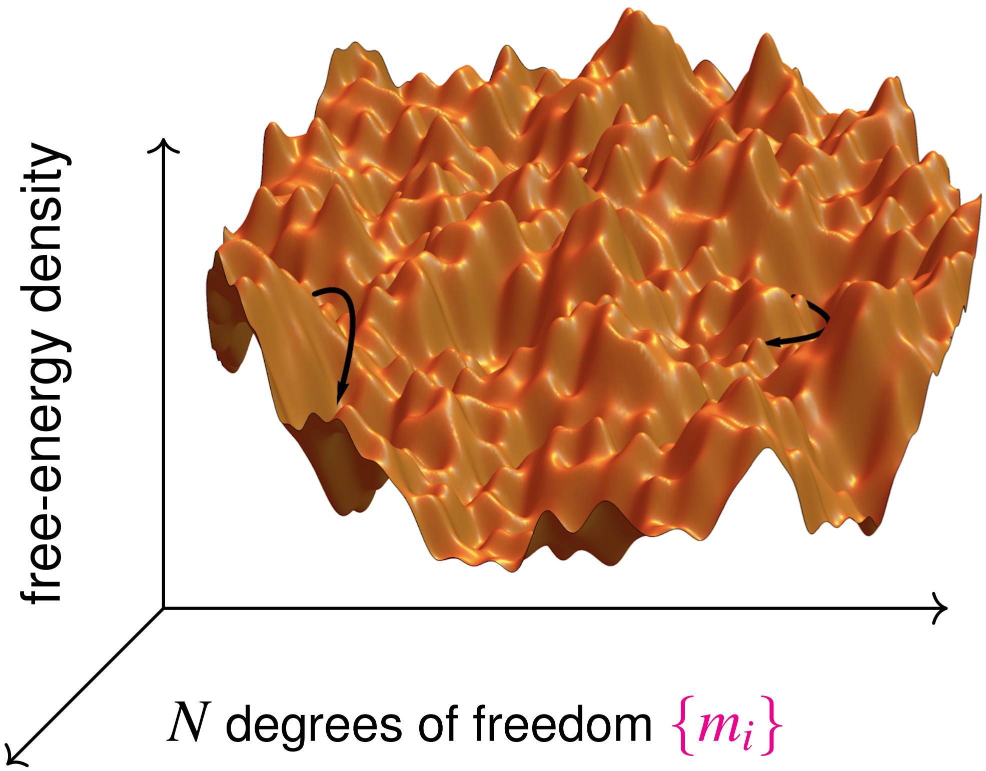


Figure adapted from a picture by **C. Cammarota**

Topography of the landscape on the N-dimensional substrate made by the N order parameters ?

Features

At fixed randomness

- There are N **local order parameters**, $m_i, \; i = 1, \ldots, N$
- The saddle-points are **heterogeneous** : m_i differ from site to site
- At high temperatures only one trivial solution $\{m_i = 0\}$
- At low temperatures the TAP equations have **many solutions** $\{m_i{}^\alpha\}$, which are extrema of the TAP free-energy landscape, *i.e.* saddles of all types, $\alpha = 1, \ldots, \mathcal{N}_J$
- For each solution $\{m_i{}^\alpha\}$, there is also $\{-m_i{}^\alpha\}$ but apart from this trivial doubling, the remaining solutions are not related by symmetry
- The TAP free-energy can take different values at different $\{m_i{}^\alpha\} \Rightarrow f^\alpha_{\mathrm{tap}}$

Features

All this is reshuffled for another realization of disorder

- Still N **local order parameters**, $m_i, \; i = 1, \ldots, N$
- The TAP equations have other solutions $\{m_i^\alpha\}$, extrema of the TAP free-energy landscape, F_J^{tap}, labelled by $\alpha = 1, \ldots, \mathcal{N}_J$
- A **global order parameter** ? The simplest guess $\frac{1}{N} \sum_{i=1}^{N} m_i^\alpha$ cannot be since it is $= 0$ One expects as many positive as negative m_is and similarity in all respects. Another try

$$q_{\text{EA}}^\alpha = \frac{1}{N} \sum_{i=1}^{N} (m_i^\alpha)^2$$

- **Dependence on α** ? **On the disorder configuration ?**

Features

Numbers of metastable states

- N **local order parameters**, $m_i, \; i = 1, \dots, N$

- The TAP equations have many solutions $\{m_i{}^\alpha\}$, extrema of the TAP free-energy landscape, $\alpha = 1, \dots, \mathcal{N}_J$

- One can count how many saddles of each kind exist and their **complexity**

$$\mathcal{N}_J = \prod_{i=1}^{N} \int_{-1}^{1} dm_i \, \delta(m_i - m_i^\alpha) \qquad\qquad \Sigma_J = k_B \ln \mathcal{N}_J$$

- How many of these at each level of free-energy density, by inserting a delta-function $\delta(f_J^{\text{tap}}(\{m_i^\alpha\}) - f) \Rightarrow \mathcal{N}_J(f)$

- How many with a given stability $\mathcal{N}_J(f, K)$ with K the number of positive eigenvalues of the Hessian, with adequate delta-functions

Summary

Local & global order parameters

$$m_i^\alpha \equiv \langle s_i \rangle_\alpha$$

$= 0$ at $T \geq T_c$

$\neq 0$ at $T < T_c$

α labels the TAP state

Magnetization

$m^\alpha = \frac{1}{N} \sum_i m_i^\alpha = 0$ at all temperatures and for all α

Edwards-Anderson order parameter

Single state property

$$q_{\mathrm{EA}}^\alpha \equiv \frac{1}{N} \sum_i (m_i^\alpha)^2 = \frac{1}{N} \sum_i \langle s_i \rangle_\alpha^2$$

$= 0$ at $T \geq T_c$

$\neq 0$ at $T < T_c$

Statistical Averages

role played by

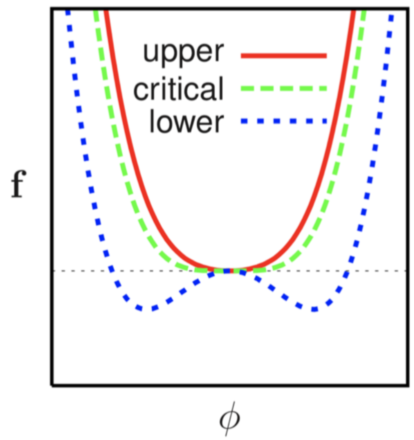
the existence of many states

Statistical averages

At fixed interactions

The average of a generic observable is $\boxed{\langle O \rangle = \sum_\alpha w_\alpha \langle O \rangle_\alpha}$

In the **FM case**, each state ($\langle \phi \rangle = \pm\phi_0$) has weigth $w_\pm = 1/2$ and the sum is $\langle O \rangle = \frac{1}{2}\langle O \rangle_+ + \frac{1}{2}\langle O \rangle_-$ with $\langle O \rangle_\pm$ the average in each of the states. For instance, the averaged magnetization vanishes if one sums over the $\pm$ states or it is different from zero if one restricts the sum to only one of them.



FM case

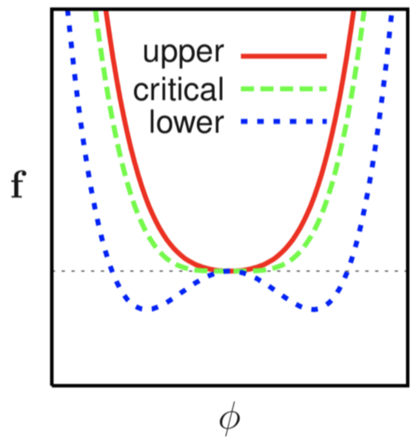
The dashed blue line with two minima $\pm|\phi_0|$

Statistical averages

At fixed interactions

The average of a generic observable is $\boxed{\langle O \rangle = \sum_\alpha w_\alpha \langle O \rangle_\alpha}$

In the **FM case**, each state ($\langle \phi \rangle = \pm\phi_0$) has weigth $w_\pm = 1/2$ and the sum is $\langle O \rangle = \frac{1}{2}\langle O \rangle_+ + \frac{1}{2}\langle O \rangle_-$ with $\langle O \rangle_\pm$ the average in each of the states. For instance, the averaged magnetization vanishes if one sums over the $\pm$ states or it is different from zero if one restricts the sum to only one of them.



FM case $f_+ = f_- < f_0$

$$w_\pm = \frac{e^{-\beta N f_\pm}}{e^{-\beta N f_+} + e^{-\beta N f_-} + e^{-\beta N f_0}} \simeq \frac{1}{2}$$

$$w_0 = \frac{e^{-\beta N f_0}}{e^{-\beta N f_+} + e^{-\beta N f_-} + e^{-\beta N f_0}} \ll w_\pm$$

Statistical averages

At fixed randomness

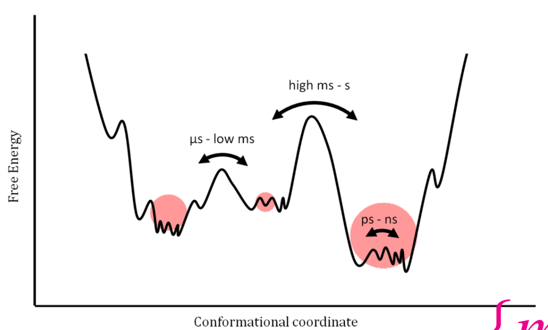
The average of a generic observable is $\boxed{\langle O \rangle = \sum_\alpha w_\alpha \langle O \rangle_\alpha}$

For systems with **quenched randomness** $\boxed{w_\alpha^J = \frac{e^{-\beta N f_\alpha^J}}{\sum_\gamma e^{-\beta N f_\gamma^J}}}$

where we added a super-script to the weight w

^{J} indicates that the weights depend on the disorder realization

and α is a label that identifies the TAP solution



$\{m_i\}$

One can sum over all saddles irrespectively of their stability. Higher lying ones will be exponentially suppressed or will dominate depending on $\Sigma_J(f, K)$

Statistical averages

At fixed randomness

The average of a generic observable is $\boxed{\langle O \rangle = \sum_\alpha w_\alpha \langle O \rangle_\alpha}$

For systems with **quenched randomness** $\boxed{w_\alpha^J = \dfrac{e^{-\beta N \mathrm{f}_\alpha^J}}{\sum_\gamma e^{-\beta N \mathrm{f}_\gamma^J}}}$

The sum over α, in the case in which there are an exponential in N number of TAP solutions, can be replaced by an integral over f

$$\boxed{\langle O \rangle = \mathcal{Z}^{-1}(\beta, J) \int d\mathrm{f}\, e^{-\beta[N\mathrm{f} - k_B T \ln \mathcal{N}_J(\mathrm{f},\beta)]}\, O(\mathrm{f},\beta)}$$

$\mathcal{N}_J$ is the number of solutions to the TAP eqs. with free-energy density f.

For $N \to \infty$ the integral is dominated by the saddle point f^*

$$\frac{1}{k_B T} = \frac{1}{N} \frac{\partial \ln \mathcal{N}_J(\mathrm{f},\beta)}{\partial \mathrm{f}}\bigg|_{\mathrm{f}^*} = \frac{1}{N} \frac{\partial \Sigma_J(\mathrm{f},\beta)}{\partial \mathrm{f}}\bigg|_{\mathrm{f}^*} \qquad \textbf{complexity}$$

Statistical averages

Consequences

The **equilibrium free-energy** f is given by the saddle-point evaluation of the partition sum :

$$\boxed{f = \mathrm{f}^* - \frac{T}{N} \ln \mathcal{N}_J(\mathrm{f}^*, \beta)}$$

The rhs is the **Landau free-energy** of the problem, with f^* playing the role of the "energy" (but it includes de usual intra-state entropy !) and $N^{-1} \ln \mathcal{N}_J(\mathrm{f}^*, \beta)$ of the entropy

The contribution of the **complexity** or **configurational entropy** contribution is negative and in some cases higher lying extrema (metastable states) can dominate the partition sum with respect to lower lying ones if $\ln \mathcal{N}_J(\mathrm{f}^*, \beta) \propto N$

This feature is proposed to describe **super-cooled liquids** & **fragile glasses**.

A global observable

Effect of multi-states

What is the expression of the global order parameter once one takes into account the multi-states ?

$$q \equiv \frac{1}{N}\sum_i \langle s_i\rangle^2 = \frac{1}{N}\sum_i\Big(\sum_\alpha w^J_\alpha m^\alpha_i\Big)^2 = \frac{1}{N}\sum_i\sum_\alpha w^J_\alpha m^\alpha_i \sum_\beta w^J_\beta m^\beta_i$$

note that this is different from $q^\alpha_{\mathrm{EA}} = \frac{1}{N}\sum_i (m^\alpha_i)^2$

Defining $\boxed{q_{\alpha\beta} \equiv \frac{1}{N}\sum_i m^\alpha_i m^\beta_i}$ an overlap between different states and

$$\left.\begin{array}{l}\boxed{P_J(q') \equiv \sum_{\alpha\beta} w^J_\alpha w^J_\beta\, \delta(q' - q_{\alpha\beta})} \\ \text{we obtain} \\ \boxed{q \equiv \frac{1}{N}\sum_i \langle s_i\rangle^2 = \int_0^1 dq'\, P_J(q')\, q'}\end{array}\right\}$$

which can depend on J_{ij}

How to check the existence of many states？

Real replicas

Real replicas

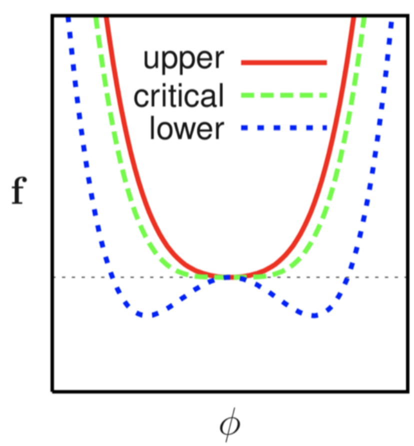
How to see the TAP states ? overlaps between replicas

Take one sample and run it, with e.g. Monte Carlo, until it reaches **equilibrium**, measure the spin configuration $\{s_i\}$.

Re-initialize the same sample (same J_{ij}), run it again until it reaches **equilibrium**, & measure the spin configuration $\{\sigma_i\}$.

Construct the overlap $q_{s\sigma} \equiv N^{-1} \sum_{i=1}^{N} s_i \sigma_i$.

In a **PM system** the overlap will typically vanish as, say, $N^{-1/2}$



Many repetitions

for a system with $N \gg 1$

$$\boxed{P(q_{s\sigma}) = \delta(q_{s\sigma})}$$

Real replicas

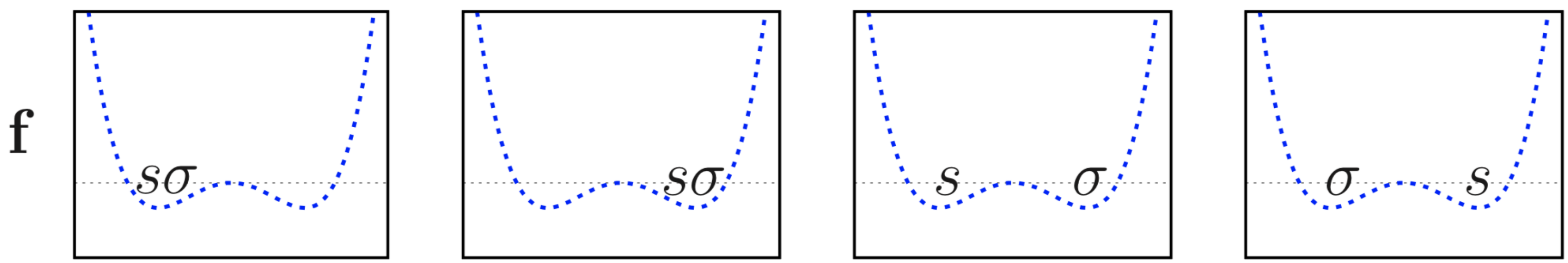
How to see the TAP states ? overlaps between replicas

Take one sample and run it, with e.g. Monte Carlo, until it reaches **equilibrium**, measure the spin configuration $\{s_i\}$.

Re-initialize the same sample (same J_{ij}), run it again until it reaches **equilibrium**, & measure the spin configuration $\{\sigma_i\}$.

Construct the overlap $q_{s\sigma} \equiv N^{-1} \sum_{i=1}^{N} s_i \sigma_i$.

In a **FM system** there are four possibilities



$q_{s\sigma} = m^2$ $\quad m^2 \quad -m^2 \quad -m^2$

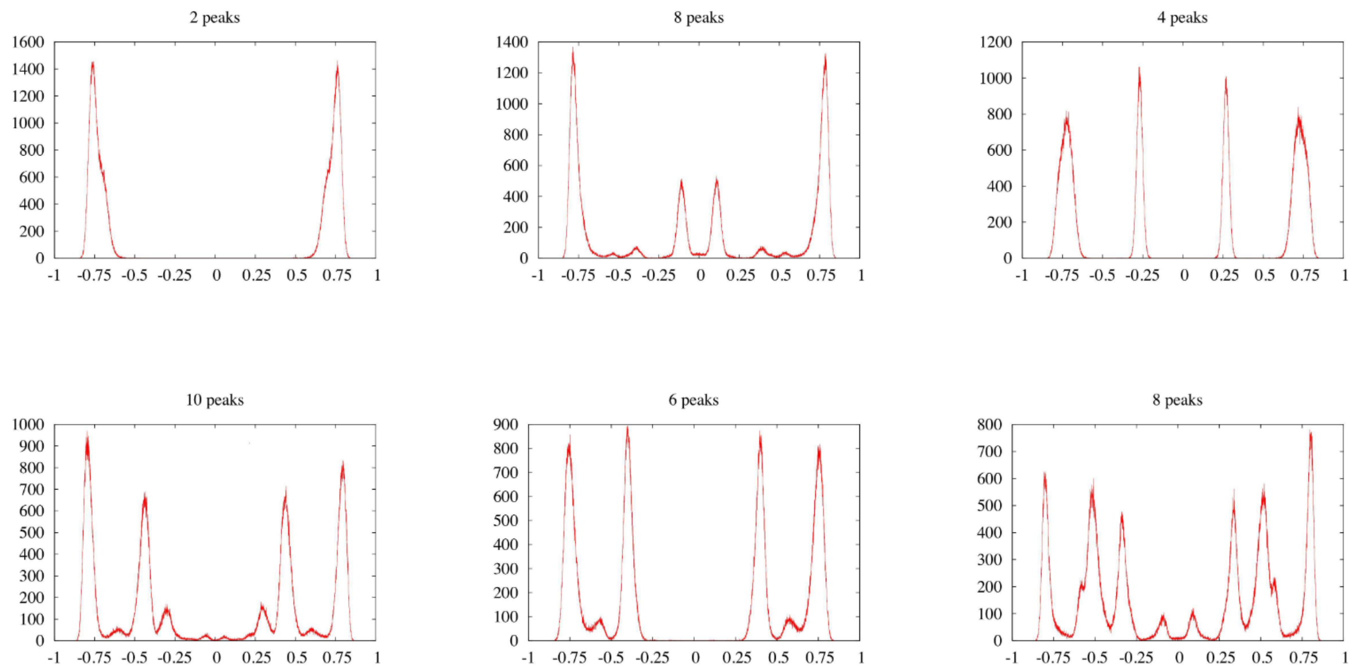
Many repetitions $\boxed{P(q_{s\sigma}) = \frac{1}{2}\delta(q_{s\sigma} - m^2) + \frac{1}{2}\delta(q_{s\sigma} + m^2)}$

Real replicas

Pdf of overlaps between replicas for different randomness

Sherrington-Kirkpatrick model with $N = 4096$ at $T = 0.4\,T_c$

$$\mathcal{H}_J = -\frac{1}{2}\sum_{i\neq j} J_{ij} s_i s_j \qquad q_{s\sigma} = \frac{1}{N}\sum_i s_i \sigma_i \qquad P_J(q_{s\sigma})$$



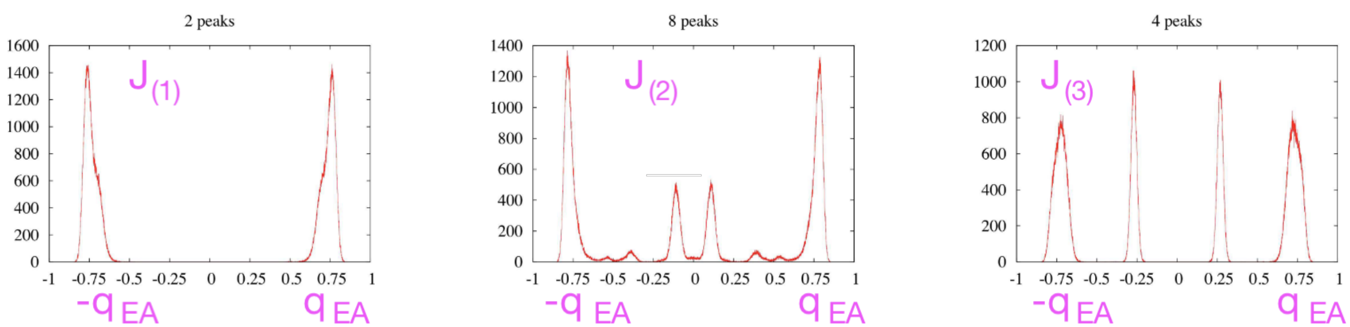
Six samples, i.e. different J_{ij} **Aspelmeier, Billoire, Marinari & Moore 07**

Real replicas

Ooverlaps between replicas at fixed randomness

Sherrington-Kirkpatrick model with $N = 4096$ at $T = 0.4\,T_c$

$$\mathcal{H}_J = -\frac{1}{2}\sum_{i\neq j} J_{ij} s_i s_j \qquad q_{s\sigma} = \frac{1}{N}\sum_i s_i \sigma_i \qquad P_J(q_{s\sigma})$$



$J_{ij}^{(1)}$ $\qquad$ $J_{ij}^{(2)}$ $\qquad$ $J_{ij}^{(3)}$

Data in each panel for a different realization of the random couplings

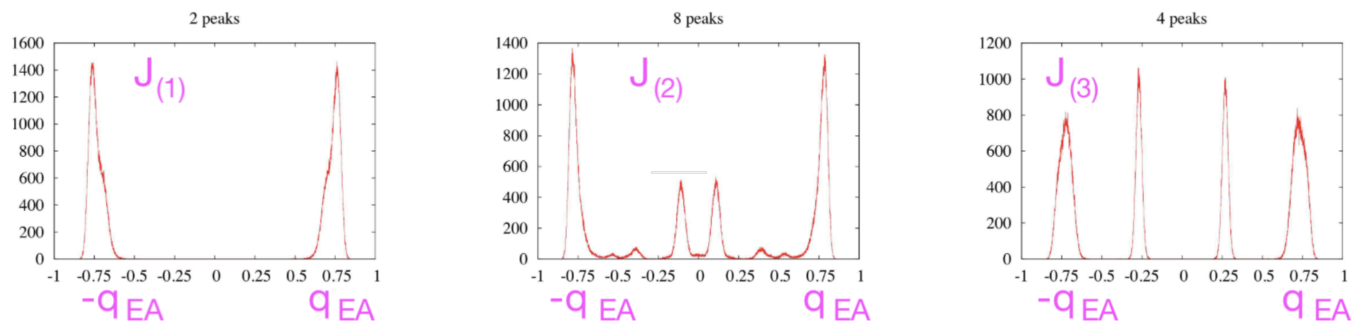
> Each sample has peaks at $q_{s\sigma} = \pm q_{\rm EA} \simeq \pm 0.75$:
>
> two configurations in the same (or the spin-reversed) state

Real replicas

Pdf of overlaps between replicas at fixed randomness

Sherrington-Kirkpatrick model with $N = 4096$ at $T = 0.4\,T_c$

$$\mathcal{H}_J = -\frac{1}{2}\sum_{i\neq j} J_{ij} s_i s_j \qquad q_{s\sigma} = \frac{1}{N}\sum_i s_i \sigma_i \qquad P_J(q_{s\sigma})$$



$$J_{ij}^{(1)} \qquad J_{ij}^{(2)} \qquad J_{ij}^{(3)}$$

Data in each panel for a different realization of the random couplings

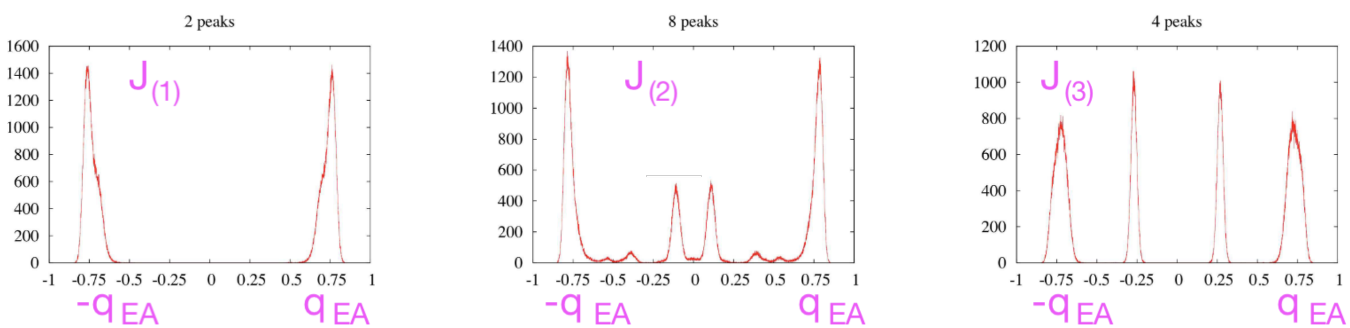
> $0.75 \simeq q_{\rm EA} < 1$ and the width of the peaks at $q_{s\sigma} = \pm q_{\rm EA}$:
>
> due to $0 < T < T_c$ and finite N, respectively

Real replicas

Pdf of overlaps between replicas at fixed randomness

Sherrington-Kirkpatrick model with $N = 4096$ at $T = 0.4\,T_c$

$$H_J[\{s_i\}] = -\frac{1}{2}\sum_{i \neq j} J_{ij} s_i s_j \qquad q_{s\sigma} = \frac{1}{N}\sum_i s_i \sigma_i \qquad P_J(q_{s\sigma})$$



$J_{ij}^{(1)}$ $\qquad$ $J_{ij}^{(2)}$ $\qquad$ $J_{ij}^{(3)}$

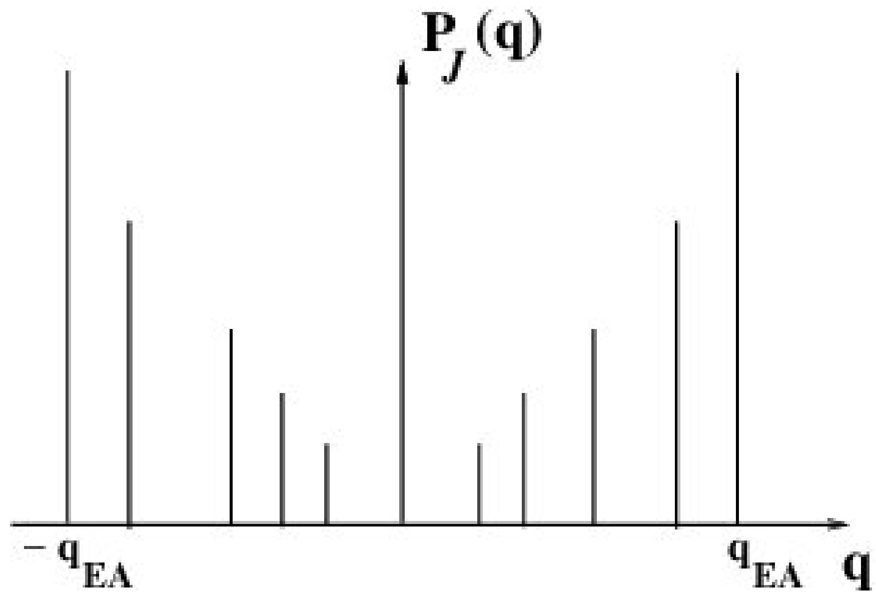
Data in each panel for a different realization of the random couplings

> Most samples also have peaks at $|q_{s\sigma}| < q_{\mathrm{EA}}$:
>
> replicas $\{s_i\}$ and $\{\sigma_i\}$ falling in different states

Real replicas

Pdf of overlaps between replicas at fixed randomness

SK model with $N \to \infty$ at $T < T_c$

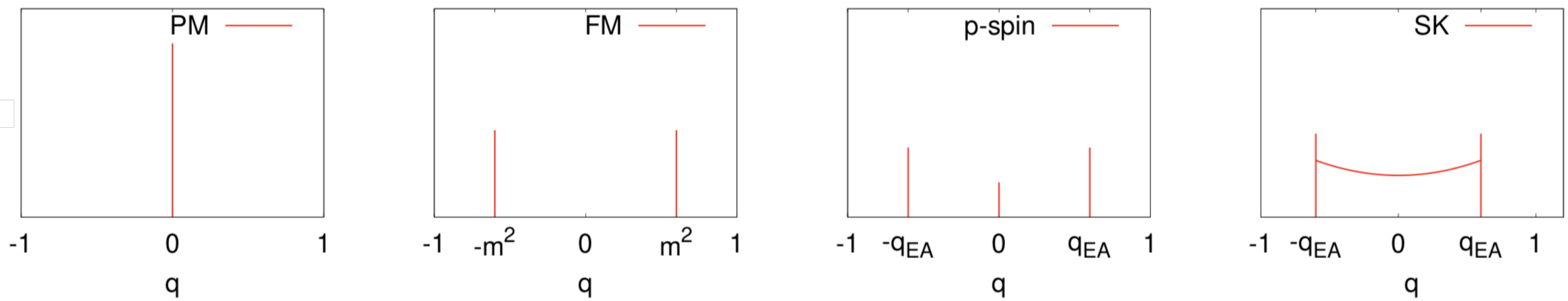


What happens if one averages $P_J(q)$ over disorder

Real replicas

Disordered averaged pdf of overlaps $P(q) = [P_J(q)]$

Parisi 79-82 prescription for the replica symmetry breaking Ansatz yields



High temperature FM Structural glasses Spin-glasses

Thermodynamic quantities, in particular the equilibrium free-energy density are expressed as functions of $P(q)$.

The equilibrium free-energy density predicted by the replica theory was confirmed by **Guerra & Talagrand 00-04** with indepedent mathematical-physics mthods.

8. The replica method

Sherrington-Kirkpatrick Model

Self-averageness & the Replica Method

Take a fully-connected Ising spin $\{s_i = \pm 1\}, i = 1, \ldots, N$, model with quenched random interactions J_{ij} drawn from a probability distribution $P(\{J_{ij}\})$

$$\boxed{\mathcal{H}_J = -\frac{1}{2}\sum_{i \neq j} J_{ij} s_i s_j}$$

In the $N \to \infty$ limit, disorder averaged & typical free-energy densities, coincide

$$\boxed{f_{\text{typ}} \underset{N\to\infty}{=} [f_J] = -\frac{k_B T}{N}[\ln Z_J]}$$

self-averageness

The disorder average can be evaluated with the help of the **replica trick** which uses the identify $x^n = \exp(n \ln x)$ Taylor expanded around $n = 0$

$$x^n \underset{n\to 0}{=} 1 + n \ln x + O(n^2) \quad \Rightarrow \quad \boxed{[\ln Z_J] \underset{n\to 0}{=} \frac{[Z_J^n] - 1}{n}}$$

Sherrington-Kirkpatrick Model

The replica method - a sketch

$$-\beta[f_J] = \frac{[\ln Z_J]}{N} = \lim_{n\to 0} \frac{[Z_J^n]-1}{Nn}$$

Assume that Z_J^n is the partition function of $n \in \mathbb{N}$ independent copies of the system : the **replicas**. Approach $n = 0$ through the natural numbers

$$n = \ldots, 1000, \ldots, 100, \ldots, 10, \ldots, 3, 2, 1, 0$$

Gaussian $P(\{J_{ij}\})$ average over disorder $J_{ij} \Rightarrow a, b$ **replica coupling**

$$\sum_a \sum_{i\neq j} J_{ij} s_i^a s_j^a \;\Rightarrow\; \sum_{i\neq j} \sum_{ab} s_i^a s_j^a s_i^b s_j^b = \sum_{ab} \sum_i s_i^a s_i^b \sum_j s_j^a s_j^b - \underbrace{\sum_{ab} \sum_i \underbrace{(s_i^a)^2}_{=1} \underbrace{(s_i^b)^2}_{=1}}_{\text{constant}=Nn^2}$$

We want to keep only $a \neq b$ cases

$$\sum_{a\neq b} \sum_i s_i^a s_i^b \sum_j s_j^a s_j^b + \sum_a \sum_i \underbrace{(s_i^a)^2}_{=1} \sum_j \underbrace{(s_j^a)^2}_{=1} - Nn^2 = \sum_{a\neq b} \sum_i s_i^a s_i^b \sum_j s_j^a s_j^b + N^2 n \underbrace{-Nn^2}_{\text{Drop}}$$

Replica method

In a nutshell

$\mathcal{Z}_J^n$ partition function of $n \in \mathbb{N}$ independent copies of the system : **replicas**.

$$\mathcal{Z}_J^n = \underbrace{\sum_{\{s_i^{(1)}=\pm 1\}} \cdots \sum_{\{s_i^{(n)}=\pm 1\}}}_{\text{notation } \mathrm{Tr}_{\{s_i^a\}}} e^{\frac{\beta}{2}\sum_{a=1}^{n}\sum_{i\neq j} J_{ij} s_i^a s_j^a}$$

One can exchange the order of the trace and the average over disorder

$$[\mathcal{Z}_J^n] = \mathrm{Tr}_{\{s_i^a\}} \int \prod_{i\neq j} dJ_{ij} P(J_{ij})\, e^{\frac{\beta}{2}\sum_{a=1}^{n}\sum_{i\neq j} J_{ij} s_i^a s_j^a}$$

$$\boxed{[\mathcal{Z}_J^n] = \mathrm{Tr}_{\{s_i^a\}}\, e^{-\beta \mathcal{H}_{\mathrm{eff}}[\{s_i^a\}]}}$$

$\mathcal{H}_{\mathrm{eff}}[\{s_i^a\}]$ does not have any randomness but couples the replicas

Problem, we still have to compute the partition sum over Nn Ising spins

Sherrington-Kirkpatrick Model

Spin decoupling

$[Z_J^n]$ disorder averaged partition function of $n \in \mathbb{N}$ independent copies of the system : the **replicas**

Quadratic decoupling with the Hubbard-Stratonovich (Gaussian) trick, with Q_{ab}

$$Q_{ab} \sum_i s_i^a s_i^b + \frac{1}{2} Q_{ab}^2 \quad \text{for each} \quad a \neq b$$

Q_{ab} is an $n \times n$ matrix with off-diagonal elements only

Sites i are **decoupled !** $\Rightarrow$ factor N in the exponential

$$[Z_J^n] = e^{\frac{(\beta J)^2}{4} N n} \int \prod_{a \neq b} dQ_{ab} \, e^{-N \frac{(\beta J)^2}{4} \sum_{a \neq b} Q_{ab}^2} \, e^{N \ln \zeta(Q_{ab})}$$

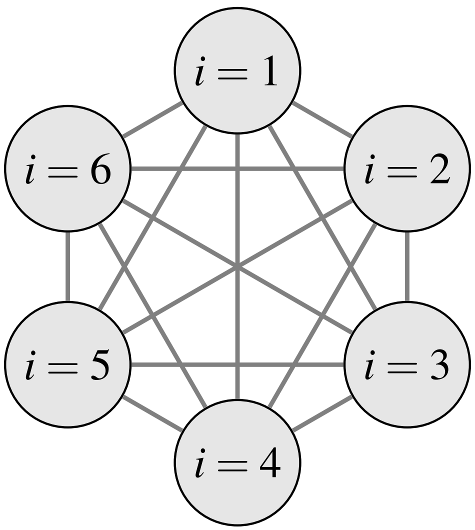
$$\zeta(Q_{ab}) = \sum_{\{s^a\}} e^{\frac{(\beta J)^2}{2} \sum_{a \neq b} Q_{ab} \, s^a s^b}$$

Partition sum of a system of n Ising spins

Replica method

Starting SK model

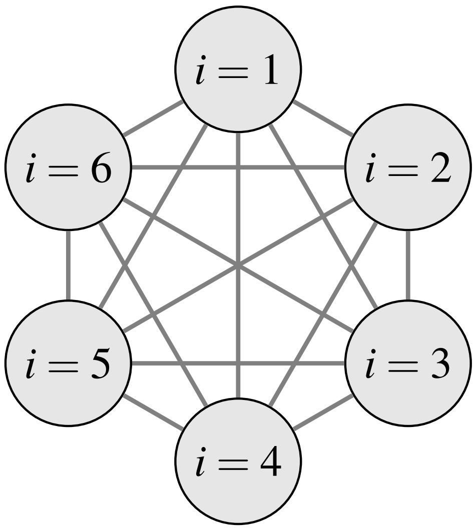
$N = 6$

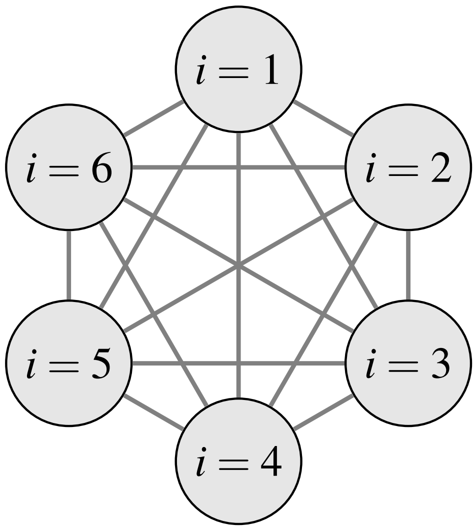


Replica method

Replicating the SK model

$a = 1,\ N = 6$ $\qquad$ $a = 2,\ N = 6$



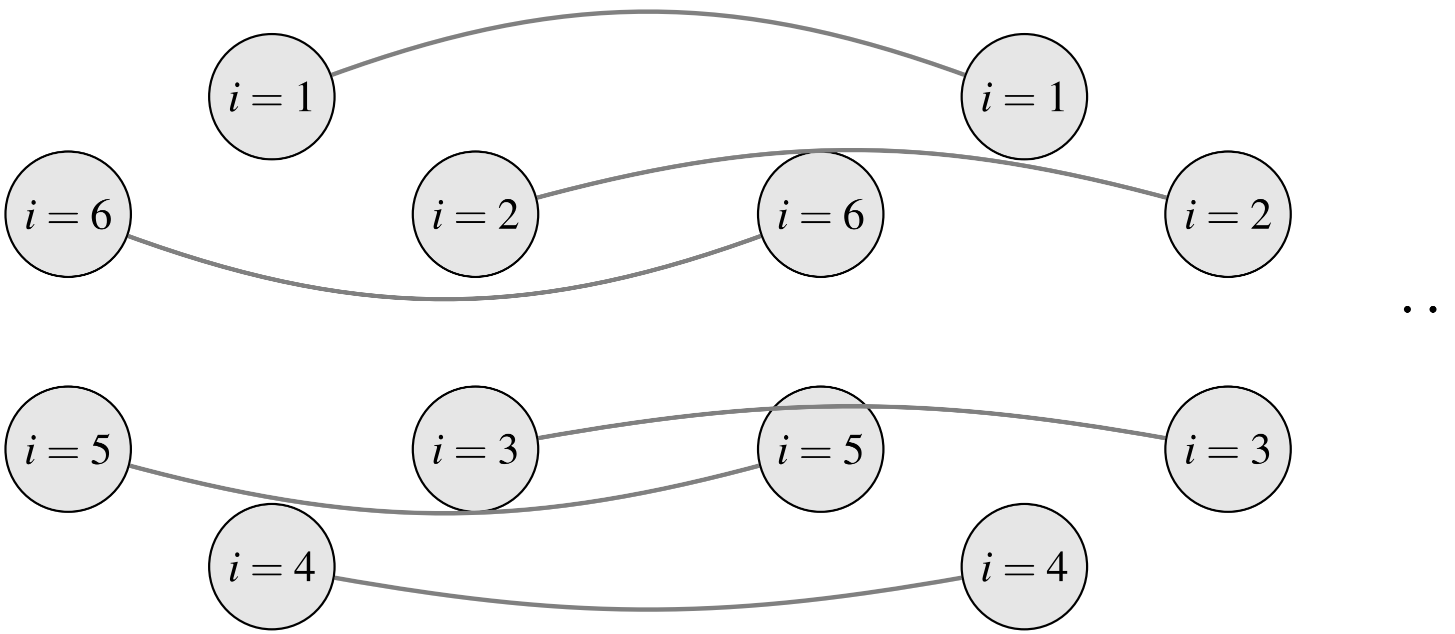


$\ldots$

n independent copies of N systems fully connected via J_{ij}

Replica method

After averaging over disorder : coupling between raplicas



N independent copies of n systems fully connected via Q_{ab}

Replica method

In a nutshell

$$[Z_J^n] = \text{Tr}_{\{s_i^a\}}\, e^{-\beta\mathcal{H}_{\text{eff}}[\{s_i^a\}]}$$

$\mathcal{H}_{\text{eff}}[\{s_i^a\}]$ does not have any randomness but **couples the replicas**

After introducing Q_{ab} $$[Z_J^n] \propto \int \prod_{ab} dQ_{ab}\, e^{-\beta N \mathrm{f}(Q_{ab})}$$

Let us consider the thermodynamic limit and exchange the order of limits

$$\lim_{N\to\infty}\lim_{n\to 0}\ldots \quad\mapsto\quad \lim_{n\to 0}\lim_{N\to\infty}\ldots$$

At the saddle-point $N\to\infty$ the elements of $\mathbb{Q}$ are the local **overlaps** between replicas

$$NQ_{ab}^* = \sum_i [\langle s_i^a s_i^b\rangle]$$ **Ex. 5**

Like a Ginzburg-Landau approach with $\mathrm{f}(Q_{ab})$ the free-energy density

Replica method

In a nutshell

$$[Z_J^n] = \text{Tr}_{\{s_i^a\}} \, e^{-\beta \mathcal{H}_{\text{eff}}[\{s_i^a\}]}$$

$\mathcal{H}_{\text{eff}}[\{s_i^a\}]$ does not have any randomness but **couples the replicas**

After introducing Q_{ab} $$[Z_J^n] \propto \int \prod_{ab} dQ_{ab} \, e^{-\beta N \mathrm{f}(Q_{ab})}$$

Let us consider the thermodynamic limit and exchange the order of limits

$$\lim_{N\to\infty} \lim_{n\to 0} \ldots \;\mapsto\; \lim_{n\to 0} \lim_{N\to\infty} \ldots$$

If one **parametrizes** the matrix $\mathbb{Q}$ in an intelligent way, it can be evaluated at the saddle-point $\mathbb{Q}^*(\beta J)$ and obtain the free-energy density $\mathrm{f}(Q_{ab}^*(\beta J)) = f(\beta J)$

The **spin glass transition** separates a paramagnetic state with $Q_{a\neq b}^* = 0$ at high $T \geq T_c$ and a spin glass state with $Q_{a\neq b}^* \neq 0$ at low $T < T_c$

Replica method

SK model : replica symmetric Ansatz

Permutation symmetry between replicas $\Rightarrow$

Insert $Q_{a \neq b} = q$ in the effective Hamiltonian

Saddle-point with respect to $0 \leq q \leq 1$ and $n \to 0$

$$\boxed{q^* = \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}}\, e^{-z^2/2}\, \tanh(\beta J \sqrt{q^*} z)}$$

Note the similarity with the equation for m in the Curie-Weiss model

$q^* = 0$ for $T \geq T_c = J$

$q^* \neq 0$ for $T < T_c = J$

Problem I Is this solution stable ? No

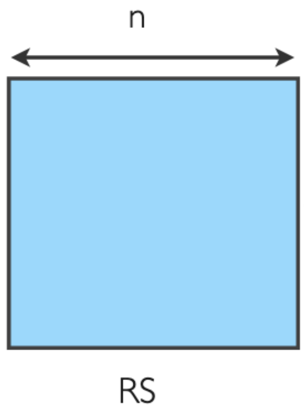
Problem II Does it have a zero-temperature vanishing entropy ? No

Problem III Ground state energy density $e = -0.77 \pm 0.01$ while the replica symmetric value $e = -0.798$, is three standard deviations smaller (in units of J)

Sherrington-Kirkpatrick Model

The structure of the matrix $NQ^*_{ab} = \sum_i [\langle s_i^a s_i^b \rangle]$

Replica Symmetry



Sherrington & Kirkpatrick, *Solvable model of a spin-glass*, PRL 35, 1792 (1975)

but $S(T=0) < 0$ and the saddle-point $Q^*_{ab} = q$ is not stable

de Almeida & Thouless , *Stability of the Sherrington-Kirkpatrick solution of a spin glass model*, J. Phys. A : Math. Gen. **11**, 983 (1978)

Fig. from **Morone, Caltagirone, Harrison & Parisi**, *Replica Theory and Spin Glasses,* Les Houches 2013

Sherrington-Kirkpatrick Model

Parisi's solution - Replica Symmetry Breaking   Nobel 2021

J. Phys. A: Math. Gen. 13 (1980) L115–L121. Printed in Great Britain

LETTER TO THE EDITOR

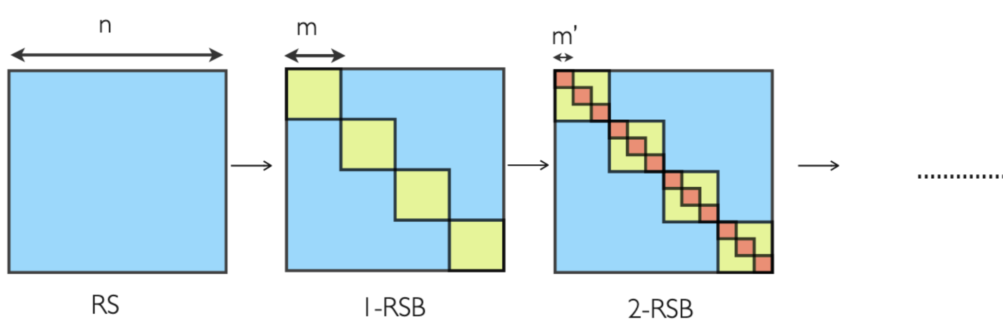
A sequence of approximated solutions to the S–K model for spin glasses

G Parisi

Istituto Nazionale de Fisica Nucleare, Laboratori Nazionali di Frascati, Casella Postale 13, 0004 Frascati, Roma, Italy

Received 4 January 1980

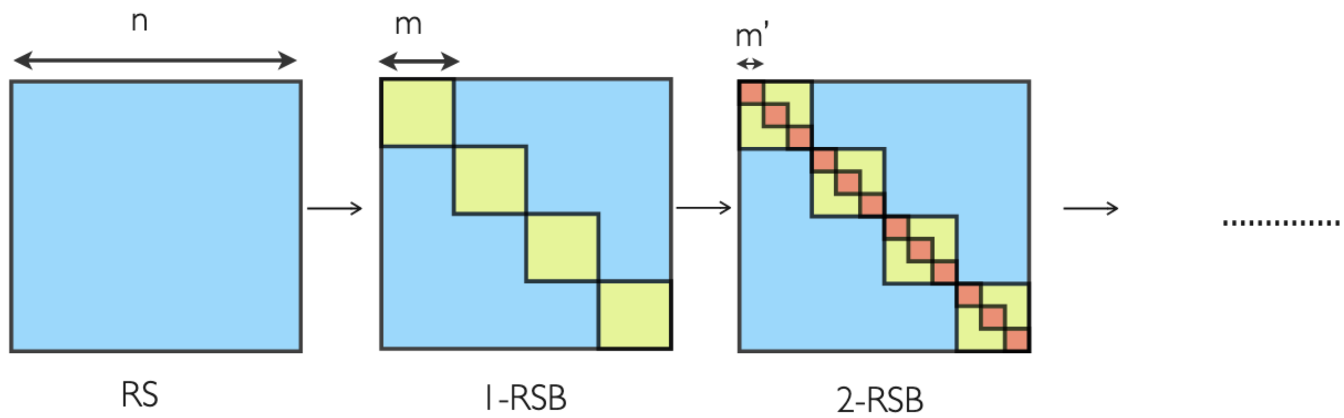
Abstract. In the framework of the new version of the replica theory, we compute a sequence of approximated solutions to the Sherrington–Kirkpatrick model of spin glasses.



Sherrington-Kirkpatrick Model

Parisi's solution - Replica Symmetry Breaking **Nobel 2021**

Replica Symmetry Breaking



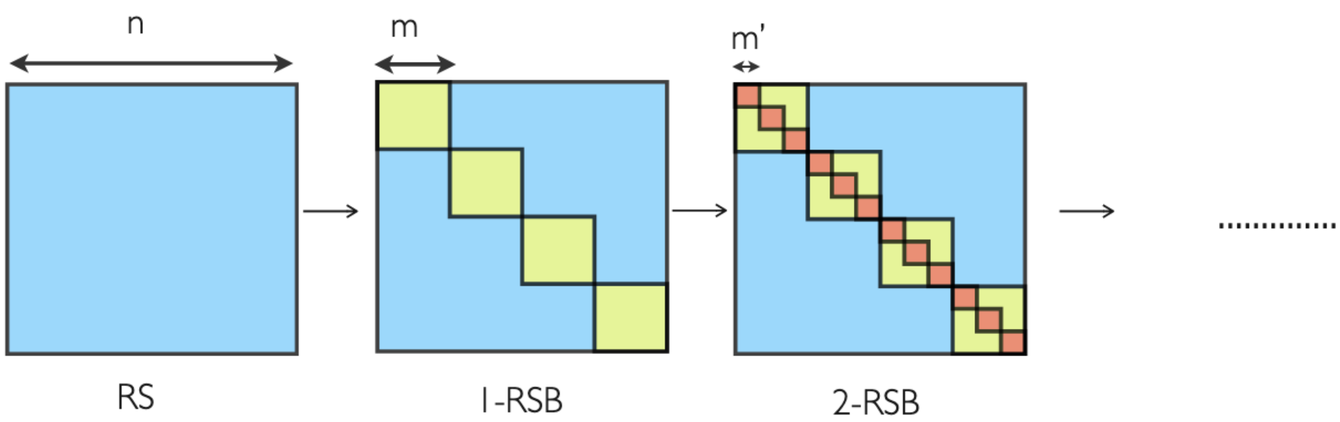
Question : Giorgio, why did you get interested in this problem ?

Answer : It looked like a nice open mathematical problem.

Sherrington-Kirkpatrick Model

The structure of the overlap matrix $NQ^*_{ab} = \sum_i [\langle s_i^a s_i^b \rangle]$

Replica Symmetry Breaking



Now $S(T=0)=0$ and the saddle-point Q^*_{ab} is stable

Fig. from **Morone, Caltagirone, Harrison & Parisi**, *Replica Theory and Spin Glasses,* Les Houches 2013

Replica method

SK model : step by step

1RSB Divide the $n \times n$ matrix in two parts :

n/m_1 diagonal blocks of size $0 < \mathrm{m}_1 < n = \mathrm{m}_0$ with elements q_1

place a parameter $\mathrm{q}_0 < \mathrm{q}_1$ in the rest of the matrix

2RSB Divide the $n \times n$ matrix in three parts :

Break the diagonal $\mathrm{m}_1 \times \mathrm{m}_1$ blocks in

$\mathrm{m}_1/\mathrm{m}_2$ diagonal blocks of size $0 < \mathrm{m}_2 < \mathrm{m}_1 < n$ with elements q_2

place $\mathrm{q}_1 < \mathrm{q}_2$ in the rest of the $\mathrm{m}_1 \times \mathrm{m}_1$ blocks

place $q_0 < \mathrm{q}_1$ in the rest of the matrix

kRSB Continue this way $n \equiv \mathrm{m}_0 > \mathrm{m}_1 > \mathrm{m}_2 > \cdots > \mathrm{m}_\mathrm{k} \to 1$

(Note that m_a is not m_i !!!)

Replica method

SK model : full replica symmetry breaking

Blocks of size $\mathrm{m_a} (< n)$ with parameter $\mathrm{q_a}$ (Note that $\mathrm{m_a}$ is not m_i ! ! !)

e.g. for replica symmetric case one block and a single q.

∞ number of breaking steps, that is, of blocks

$\mathrm{m_a} \mapsto x \in [0,1]$ & the parameter $\mathrm{q_a} \mapsto q(x)$ non-decreasing with x

The **order parameter** is a **function** $q(x)$

Parisi derived a functional equation which fixes $q^*(x)$ for the SK model

A bit complicated, not necessary to write it here

Problem I Stability : solved

Problem II Zero-temperature entropy : solved $S = 0$

Problem III e in agreement with numerical value

within numerical accuracy $e = -0.7633$

Sherrington-Kirkpatrick Model

Multiplicity and similarity of equilibrium states

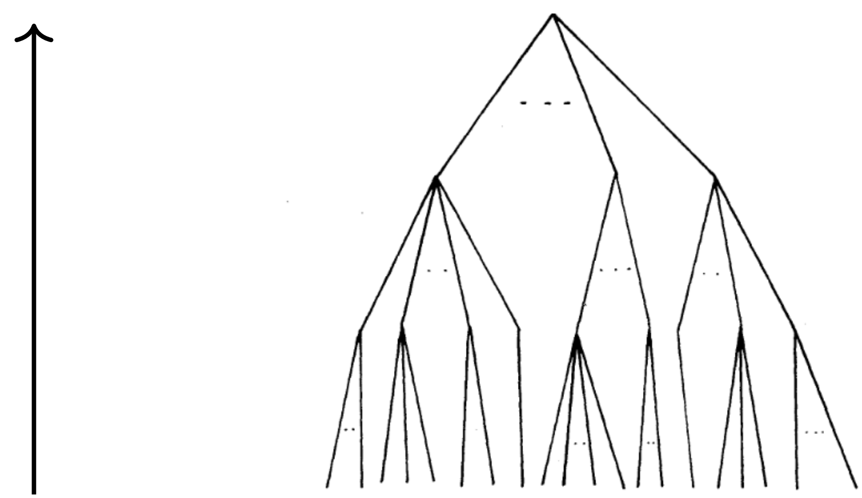
VOLUME 52, NUMBER 13 PHYSICAL REVIEW LETTERS 26 MARCH 1984

Nature of the Spin-Glass Phase

M. Mézard, G. Parisi,[a] N. Sourlas, G. Toulouse,[b] and M. Virasoro[c]
Laboratoire de Physique Théorique de l'Ecole Normale Supérieure, F-75231 Paris-Cedex, France
(Received 12 December 1983)

A probability distribution has been proposed recently by one of us as an order parameter for spin-glasses. We show that this probability depends on the particular realization of the couplings even in the thermodynamic limit, and we study its distribution. We also show that the space of states has an ultrametric topology.

$x = 0 \quad q(0) = \mathrm{q}_{\mathrm{min}}$



$x = 1 \quad q(1) = q_{\mathrm{EA}}$

Sherrington-Kirkpatrick Model

Multiplicity of equilibrium states - order parameter

One has $q(x)$

The value $x \in [0,1]$ counts the fraction of replica pairs that have an overlap less than or equal to q

$x = 0$ associated with $\mathrm{q_{min}}$

$x = 1$ associated with q_{EA}

The inverse $x(q)$ (assuming if exists) is the **cumulative distribution function** of the **probability of having an overlap** q

$$x(q) = \int_0^q dq' \, P(q')$$

Taking the derivative with respect to q $\Rightarrow$ $\boxed{P(q) = \frac{dx(q)}{dq}}$

Sherrington-Kirkpatrick Model

Multiplicity of equilibrium states - order parameter

Take a **1RSB** Q_{ab} matrix.

The matrix is $n \times n$ and the number of off-diagonal elements is $n^2 - n$

There are n/m_1 blocks of size $\mathrm{m}_1 \times \mathrm{m}_1$

The number of off-diagonal elements in each of these blocks is $\mathrm{m}_1^2 - \mathrm{m}_1$

The total off-diagonal elements q_1 in all these blocks is

$$\frac{n}{\mathrm{m}_1}\left(\mathrm{m}_1^2 - \mathrm{m}_1\right) = n\mathrm{m}_1 - n$$

The total number of q_0 entries is $(n^2 - n) - (n\mathrm{m}_1 - n) = n^2 - n\mathrm{m}_1$

The fraction of elements equal to q_0 and q_1 are

$$\mathrm{fr}(q = \mathrm{q}_0) = \frac{n^2 - n\mathrm{m}_1}{n(n-1)} = \frac{n - \mathrm{m}_1}{n-1} \qquad \mathrm{fr}(\mathrm{q} = \mathrm{q}_1) = \frac{n\mathrm{m}_1 - n}{n(n-1)} = \frac{\mathrm{m}_1 - 1}{n-1}$$

Sherrington-Kirkpatrick Model

Multiplicity of equilibrium states - order parameter

Take a **1RSB** Q_{ab} matrix.

The fraction of elements equal to q_0 and q_1 in the $n \to 0$ limit are

$$\mathrm{fr}(q = \mathrm{q}_0) = \frac{n - \mathrm{m}_1}{n-1} \xrightarrow[n \to 0]{} \mathrm{m}_1 \qquad \mathrm{fr}(q = \mathrm{q}_1) = \frac{\mathrm{m}_1 - 1}{n-1} \xrightarrow[n \to 0]{} 1 - \mathrm{m}_1$$

m_1 is the fraction of off-diagonal elements with $q = q_0 < q_1$

The probability distribution function is

$$P(q) = \mathrm{m}_1 \, \delta(q - \mathrm{q}_0) + (1 - \mathrm{m}_1) \, \delta(q - \mathrm{q}_1)$$

The cumulative probability distribution function is a step function

$$x(q) = \int_0^q dq' \, P(q') = \begin{cases} 0 & 0 \le q < \mathrm{q}_0 \\ \mathrm{m}_1 & \mathrm{q}_0 \le q < \mathrm{q}_1 \\ 1 & \mathrm{q}_1 < q < 1 \end{cases}$$

Sherrington-Kirkpatrick Model

Multiplicity of equilibrium states - function order parameter

Sample the equilibrium states

at fixed realization of J_{ij}

Get the configurations $\{s_i^a\}$

calculate the **overlap/correlation**

$q = N^{-1} \sum_i \langle s_i^a s_i^b \rangle$

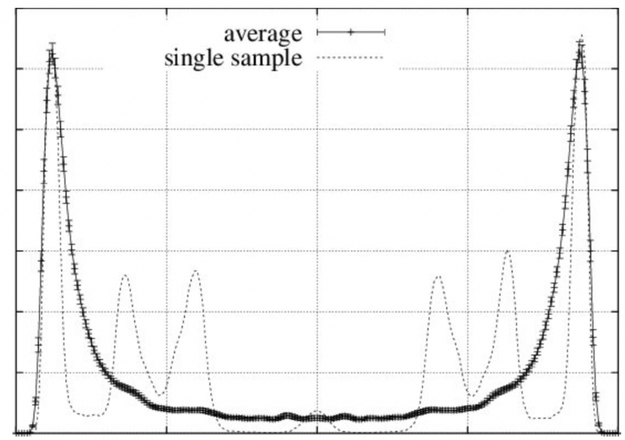
Result : $\pm q_{\text{EA}}$

same state or reversed found

Result : $\neq q_{\text{EA}}$

two different states sampled

P_J



q

Marinari, Martin & Zuliani 01

Real replicas

Bold curve, $[P_J](q)$

Non-zero weight for $q \neq q_{\text{EA}} \implies$

Non-trivial structure of equil states

Replica method

Observables

The **order parameter** is a **function**

Parisi derived a functional equation which fixes $q^*(x)$ for the SK model

A bit complicated, not necessary to write it here

$$[\langle s_i \rangle^2] = \int_0^1 dx\, q(x) = \int dq\, \frac{dx}{dq}\, q(x) = \int dq\, P(q)\, q$$

with $$\boxed{P(q) = \frac{dx}{dq}}$$

Problem I Stability : solved

Problem II Zero-temperature entropy : solved $S = 0$

Problem III e in agreement with numerical value within numerical accuracy $e = -0.7633$

OK with the TAP approach ?

TAP

Phase transition

For large N one expects $J_{ij}^2 \simeq [J_{ij}^2] = J^2/N$ with $J = O(1)$

Simplification $m_i = \tanh\left\{\beta \sum_{j(\neq i)} J_{ij} m_j - \beta^2 m_i \frac{J^2}{N} \sum_{j(\neq i)} (1 - m_j^2)\right\}$

A 2nd order phase transition $\Rightarrow m_i \simeq 0$ at $T \stackrel{<}{\sim} T_c$ then using $\tanh y \sim y$

The TAP equations become $m_i \sim \beta \sum_{j(\neq i)} J_{ij} m_j - \beta^2 J^2 m_i$

Diagonalize this eq. going to the basis of eigenvectors of the J_{ij} matrix

The eqs read $m_\lambda \sim \beta \left(J_\lambda - \beta J^2\right) m_\lambda$

The notation we use is such that

J_λ is an eigenvalue of the J_{ij} matrix associated to the eigenvector $\vec{v}_\lambda$

m_λ represents the projection of $\vec{m}$ on the eigenvector $\vec{v}_\lambda$, $m_\lambda = \vec{v}_\lambda \cdot \vec{m}$

with $\vec{m}$ the N-vector with components m_i, $\vec{m} = (m_1, \ldots, m_N)$

TAP

Phase transition

If we add a weak external field the eqs read $m_\lambda \sim \beta(J_\lambda - \beta J^2) m_\lambda + \beta h_\lambda^{\rm ext}$

The variation with respect to the field at linear order is

$$\left.\frac{\partial m_\lambda}{\partial h_\lambda^{\rm ext}}\right|_{\vec{h}^{\rm ext}=\vec{0}} = \beta(J_\lambda - \beta J^2) \left.\frac{\partial m_\lambda}{\partial h_\lambda^{\rm ext}}\right|_{\vec{h}^{\rm ext}=\vec{0}} + \beta$$

and the *staggered susceptibility* (of the projection on $\vec{v}_\lambda$)

$$\chi_\lambda \equiv \left.\frac{\partial m_\lambda}{\partial h_\lambda^{\rm ext}}\right|_{\vec{h}^{\rm ext}=\vec{0}} = \beta \, \left(1 - \beta J_\lambda + (\beta J)^2\right)^{-1}$$

Random matrix theory tells us that the eigenvalues of the random matrix $J_{ij} = O(1/\sqrt{N})$ are distributed with the Wigner semi-circle law and the largest eigenvalue is $J_\lambda^{\rm max} = 2J$

> The staggered susceptibility of staggered magnetization in the direction of the largest eigenvalue diverges at $\beta_c J = 1$ the value found with replicas

TAP

Phase transition

If we add a weak external field the eqs read $m_\lambda \sim \beta(J_\lambda - \beta J^2) m_\lambda + \beta h_\lambda^{\rm ext}$

The variation with respect to the field at linear order is

$$\left.\frac{\partial m_\lambda}{\partial h_\lambda^{\rm ext}}\right|_{\vec{h}^{\rm ext}=\vec{0}} = \beta(J_\lambda - \beta J^2) \left.\frac{\partial m_\lambda}{\partial h_\lambda^{\rm ext}}\right|_{\vec{h}^{\rm ext}=\vec{0}} + \beta$$

and the *staggered susceptibility* (of the projection on $\vec{v}_\lambda$)

$$\chi_\lambda \equiv \left.\frac{\partial m_\lambda}{\partial h_\lambda^{\rm ext}}\right|_{\vec{h}^{\rm ext}=\vec{0}} = \beta \left(1 - \beta J_\lambda + (\beta J)^2\right)^{-1}$$

Random matrix theory tells us that the eigenvalues of the random matrix J_{ij} are distributed with the Wigner semi-circle law

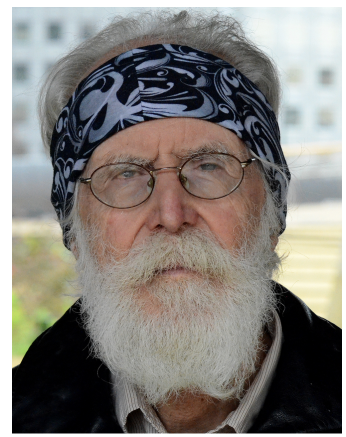
For $J_{ij} = O(1/\sqrt{N})$ the largest eigenvalue is $J_\lambda^{\max} = 2J$

The staggered susceptibility for the largest eigenvalue diverges at $\beta_c J = 1$

Without the reaction term the divergence is at the inexact value $T^* = 2T_c$

Sherrington-Kirkpatrick Model

Mathematics entered the game



t

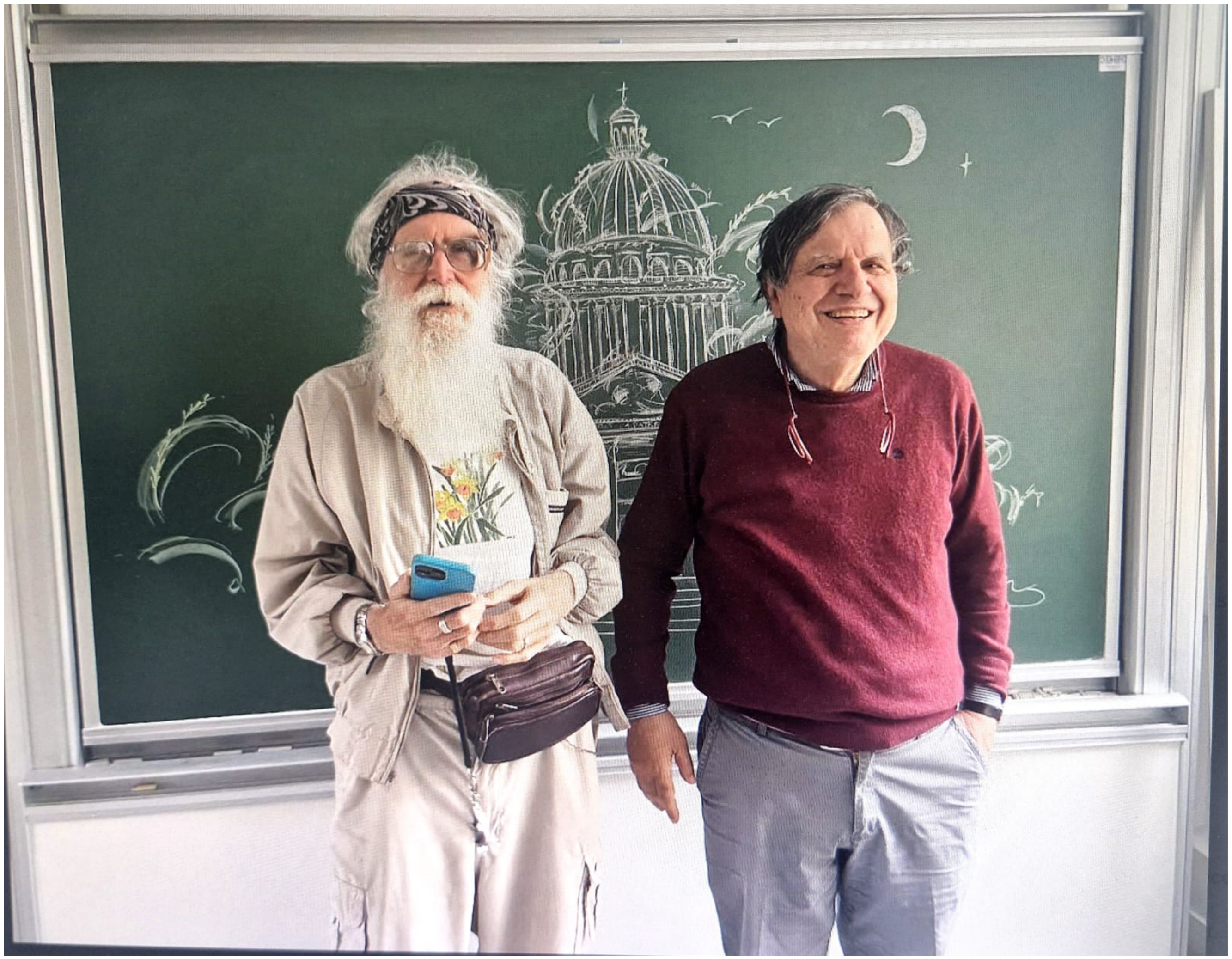
Francesco Guerra 1942 - 2025 **Michel Talagrand - Prix Abel 2024**

$$\boxed{f \;\geq\; f_{\mathrm{FRSB}} \;\geq\; f \qquad \Rightarrow \qquad f = f_{\mathrm{FRSB}}}$$

Guerra 2003 **Talagrand 2006**

A recent picture in Paris

Talagrand & Parisi



9. Mean-field classes

Mean-field classes

According to equilibrium and metastability

$$\boxed{\mathcal{H} = - \sum_{i_1 \neq \ldots \neq i_p} J_{i_1 \ldots i_p} s_{i_1} \ldots s_{i_p}}$$

p integer

quenched random couplings $J_{i_1 \ldots i_p}$ drawn from a Gaussian $P[\{J_{i_1 \ldots i_p}\}]$

with $[J_{i_1 \ldots i_p}] = 0$ and $[J^2_{i_1 \ldots i_p}] = J^2/N^{p-1}$

Choice of spin variables

Ising $s_i = \pm 1$

Globally spherical $\sum_{i=1}^{N} s_i^2 = N$ and $\mathcal{H} \to \mathcal{H} + \frac{z}{2}(\sum_{i=1}^{N} s_i^2 - N)$

$p = 2$ Ising Sherrington-Kirkpatrick **spin-glass** Model **FRSB**

$p = 2$ sherical : ferromagnetic **coarsening** **Berlin-Kac 52** **RS**

$p \geq 3$ Ising or spherical RFOT modelling of **glasses** **1RSB**

Kirkpatrick, Thirumalai & Wolynes 87-89

Mean-field theory

Fully connected Ising models

Even more general models (recall the K-sat problem)

$$\mathcal{H}_J = -\frac{1}{3!}\sum_{i\neq j\neq k} J_{ijk} s_i s_j s_k \quad \text{with Ising variables} \quad s_i = \pm 1$$

$O(1)$ scaling of the local fields $\Rightarrow$ scaling of J_{ijk}

In the $p=3$ **Curie-Weiss ferromagnetic** case

$$J_{ijk} = \frac{J}{N^{p-1}} \quad \text{such that } h_i \sim \frac{J}{2N^{p-1}} \sum_{jk(\neq i)} s_j s_k = O(1)$$

in the ferromagnetic $s_i = 1 \ \forall i$ or $s_i = -1 \ \forall i$ phases

In the $p=3$ **disordered case**

$$J_{ijk} = O\left(\frac{J}{\sqrt{N^{p-1}}}\right) \quad \text{such that } h_i \sim \frac{J}{2\sqrt{N^{p-1}}} \sum_{j\neq k(\neq i)} s_j s_k = O(1)$$

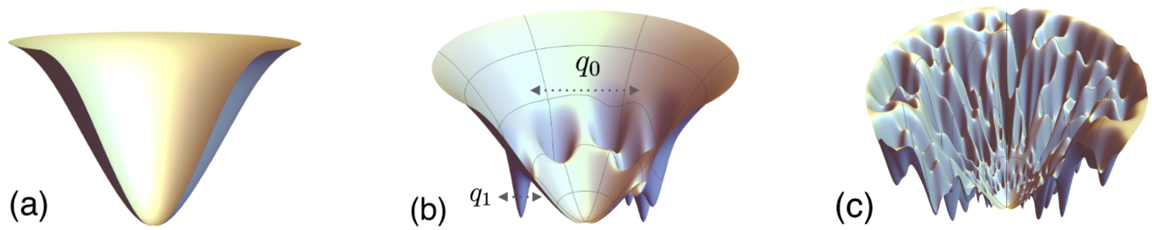
in the PM or spin-glass phases $s_i = \pm 1$ with equal probability

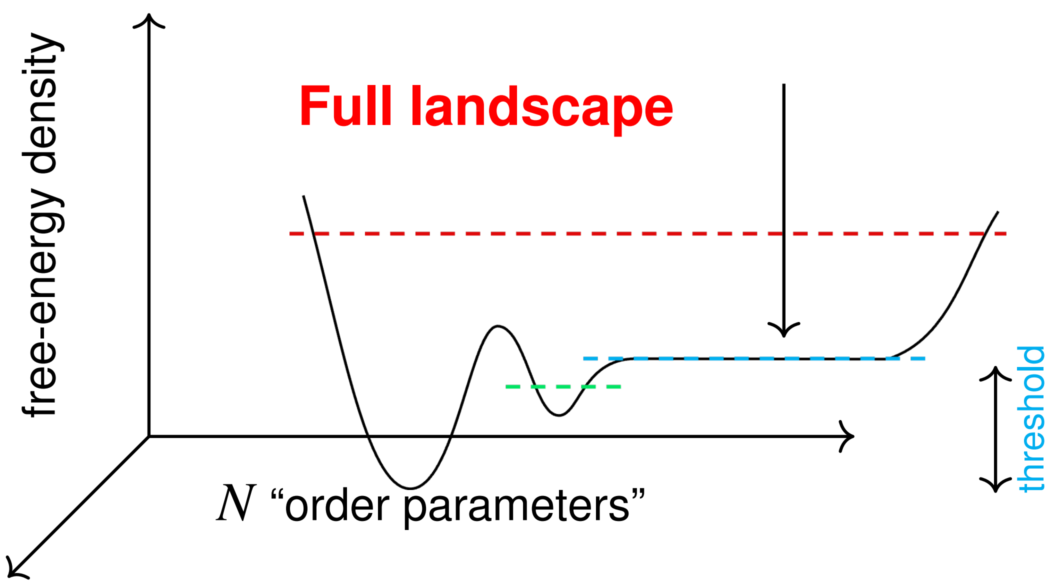
Mean-field classes

According to Replica Theory and TAP analysis

Equilibrium states

RS **1RSB** **FRSB**





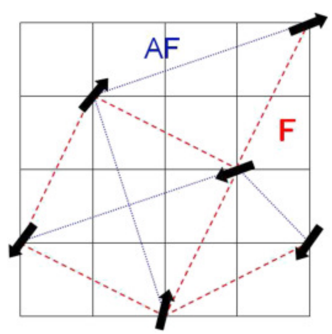
A particle in a random potential

Spin Disordered Potential

$V = -1/2 \sum_{ij} J_{ij} s_i s_j$ or p-spin potential

the exchanges J_{ij}, etc. taken from

a probability distribution



Real variables $s_i \in \mathbb{R}$

Spherical constraint $\boxed{\sum_{i=1}^{N} s_i^2 = N}$

Connection with the following problem

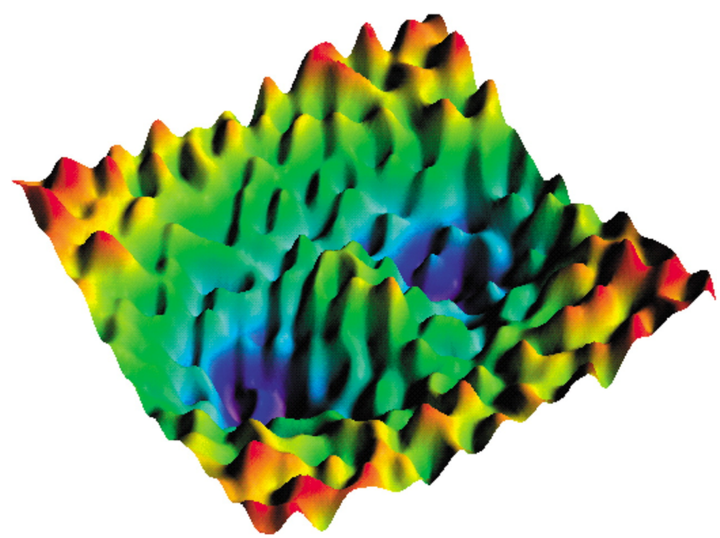
A particle

position $\vec{s} = (s_1, \dots, s_N)$

in an N dimensional space

under a random potential $V(\vec{s})$

Sketch for $N = 2$



but wrapped on the sphere

Finite dimensions

Scaling Theory

Two ground states related by spin reversal $\Longrightarrow$ no replica symmetry breaking.

Compact clusters (droplets) in excitation above the ground state.

The surface of the droplets is fractal, with dimension $d_f > d - 1$.

A droplet of linear size L costs free-energy proportional to L^θ,

where θ is the stiffness exponent.

Unstable in any magnetic field (no AT-line in finite dimensions).

The spin glass is like a magnet with a rough energy landscape but a single ordered phase – excitations cost more and more energy as they grow, unlike the RSB picture where infinitely many competing states exist at no cost.

The alternative view of **Bray & Moore, Fisher & Huse** late 80s

Issue not settled

More history

Texts & Interviews

“For posterity and future study, we are conducting a series of interviews with the original scientific participants and deposit them at the Centre d’Archives de Philosophie, d’Histoire et d’Édition des Sciences (CAPHÉS) of École normale supérieure de Paris (ENS)

`https://caphes.ens.fr/history-of-replica-symmetry-breaking-in-physics/`

The interviews are also assembled on a dedicated site

`https://historyrsb.nakala.fr/en`

Interestingly, two recent Nobel prize winners fall within the scope of this project : Giorgio Parisi (Physics, 2021) and John Hopfield (Physics, 2024).”

10. Quantum spin glasses

Quantum spin glasses

Models

$$\boxed{\hat{\mathcal{H}}_J = -\sum_{ij} J_{i_1 \dots i_p} \hat{\sigma}^z_{i_1} \dots \hat{\sigma}^z_{i_p} + \Gamma \sum_i \hat{\sigma}^x_i - \sum_i h_i \hat{\sigma}^z_i}$$

$\hat{\sigma}^a_i$ with $a = 1, 2, 3$ the Pauli matrices, $[\hat{\sigma}^a_i, \hat{\sigma}^b_j] = 2i\varepsilon_{abc}\delta_{ij}\hat{\sigma}^c_i$.

$J_{i_1 \dots i_p}$ quenched random., e.g. Gaussian pdf conveniently normalized.

Γ transverse field. It measures quantum fluctuations.

In the limit $\Gamma \to 0$ the classical limit should be recovered.

h_i longitudinal local fields

$i_1 \dots i_p$ nearest neighbours on the lattice – $p = 2$ finite d
or fully connected – mean-field

Generalization to other spin algebras, e.g. SU(M)

Quantum spin glasses

Models

Classical spherical models : a particle constrained on a sphere embedded in N-dim space under a random potential

Add kinetic energy and Newtonian dynamics

$p = 2$ integrable Neumann model

$p > 2$ non-integrable model

Hamiltonian

$$\hat{\mathcal{H}} = \frac{1}{2M}\hat{p}^2 + \frac{1}{2}\sum_{i_1 \dots i_p} J_{i_1 \dots i_p} \hat{x}_{i_1} \dots \hat{x}_{i_p} + \frac{z}{2}\left(\sum_i \hat{x}_i^2 - N\right)$$

Quantize

$$[\hat{x}_i, \hat{x}_j] = 0 \qquad [\hat{p}_i, \hat{p}_j] = 0 \qquad [\hat{p}_i, \hat{x}_j] = -\mathrm{i}\hbar$$

The Sachdev-Ye-Kitaev model

Connection with black holes

Based on **holography**, a simple $d = 0$ **quantum model** of a black hole

$$\hat{\mathcal{H}}_J = \underbrace{\sum_{i \neq j \neq k \neq l}}_{\text{sum over all groups of four}} \underbrace{J_{ijkl}}_{\text{interactions}} \underbrace{\hat{\psi}_i \hat{\psi}_j \hat{\psi}_k \hat{\psi}_l}_{\text{variables}}$$

There are $i, j, k, l = 1, \ldots, N$ Majorana fermions, $\hat{\psi}_i^\dagger = \hat{\psi}_i$ and $\{\hat{\psi}_i, \hat{\psi}_j\} = 0$

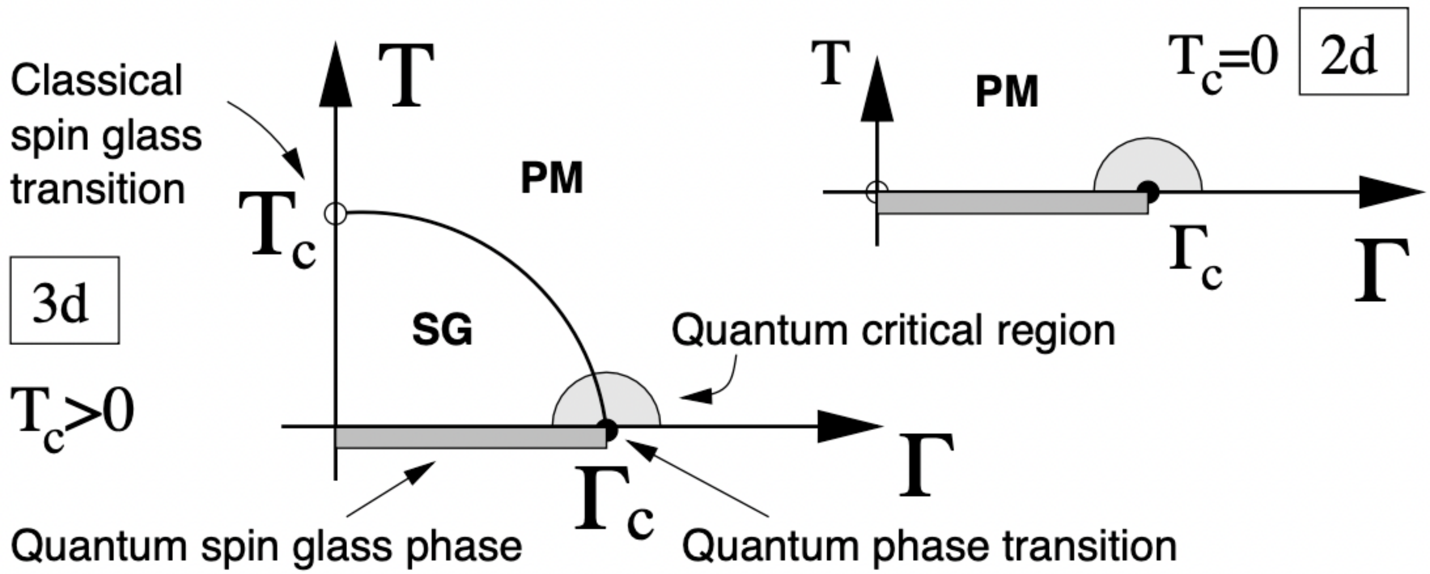
Random interactions J_{ijkl} with $[J_{ijkl}] = 0$ and $[J_{ijkl}^2] = 4! J^2 / N^3$

The **entropy** $S(T) \xrightarrow{T \to 0} a + bT$ & **time evolution** similar to black hole ones

Some other properties to be discussed later

Quantum spin glasses

Phase transitions in $d = 1, 2, 3$ & fully connected



Γ_c Quantum critical point. T_c Classical critical point.

Quantum MC. Overviews in Rieger & Young 95, Kawashima & Rieger 03

Many special features in $d = 1$ obtained with the Dasgupta-Ma RG decimation procedure. **D. S. Fisher 92**

Matsubara replica calculation

A sketch for the $p = 2$ case

$$\boxed{-\beta f = \lim_{N\to\infty} [\ln \mathcal{Z}_J] = \lim_{N\to\infty} \lim_{n\to 0} \frac{[\mathcal{Z}_J^n] - 1}{Nn}}$$

$\mathcal{Z}_J^n$ partition function of n independent copies of the system : **replicas**.

Quantum mech., $\mathcal{Z}_J = \text{Tr}\, e^{-\beta\hat{\mathcal{H}}} = \int_{\{s_i(0)\}}^{\{s_i(\beta\hbar)\}} \mathcal{D}\{s_i(\tau)\}\, e^{-\frac{1}{\hbar}S^e[\{s_i\}]}$

with the Euclidean action

$$\boxed{S^e[\{s_i\}] = \int_0^{\beta\hbar} d\tau \left[\frac{M}{2} \left(\frac{\partial s_i(\tau)}{\partial\tau} \right)^2 + \sum_{ij} J_{ij} s_i(\tau) s_j(\tau) \right]}$$

and the 'mass' given by $M = (\hbar\tau_0)/2\ln[\hbar/(\Gamma\tau_0)]$ as a function of the transverse field Γ.

Feynman-Matsubara construction of functional integral over imaginary time

Matsubara replica calculation

A sketch for the $p = 2$ model

Average over disorder : coupling between replicas

$$\boxed{\sum_a \sum_{i\neq j} J_{ij} \int d\tau \, s_i^a(\tau) s_j^a(\tau) \;\Rightarrow\; N^{-1} \sum_{i\neq j} \left(\sum_a \int d\tau \, s_i^a(\tau) s_j^a(\tau) \right)^2}$$

Decoupling with Hubbard-Stratonovich trick

$$\boxed{Q_{ab}(\tau,\tau') N^{-1} \sum_i s_i^a(\tau) s_i^b(\tau') - \frac{1}{2} Q_{ab}^2(\tau,\tau')}$$

In terms of the replica indices $Q_{ab}(\tau,\tau')$ is still a 0×0 matrix.

$Q_{ab}(\tau,\tau')$ can be evaluated by saddle-point if one exchanges the limits

$N \to \infty$ $n \to 0$ with $n \to 0$ $N \to \infty$

Off-diagonal RSB $Q_{a\neq b}$ & imaginary-time dependent $q_d(\tau)$

Bray & Moore 80, more recent for SK **Andreanov & Müller**

Argument à la Guerra-Talagrand for quantum spin-glasses ?

The spherical p-spin model

Equilibrium in disordered phase $Q_{a\neq b} \neq 0$

$$\boxed{M\hbar \frac{\partial^2 q_d(\tau,\tau')}{\partial\tau^2} + z(\tau) q_d(\tau,\tau') = \delta(\tau-\tau') + \frac{1}{\hbar}\int_0^{\beta\hbar} d\tau'' \, \Sigma(\tau,\tau'') q_d(\tau'',\tau')}$$

with τ the imaginary time, $q_d(\tau,\tau') \equiv \frac{1}{N}\sum_{i=1}^{N} \mathcal{T}[\langle x_i(\tau) x_i(\tau')\rangle]$ the correlation

and $\Sigma(\tau,\tau') \equiv J^2(p/2) q_d^{p-1}(\tau,\tau')$ the self-energy

LFC, Grempel & da Silva Santos 00

Slow dynamics for long $\tau-\tau' \implies z(\tau) = z_\infty$, drop the time-derivative, and

time reparametrization invariance under

$$\tau \mapsto h(\tau) \qquad\qquad q_d(\tau,\tau') \mapsto [h(\tau)h(\tau')]^\Delta \, q_d(h(\tau),h(\tau'))$$

and, by holography, **invariance under diffeomorphisms of general relativity**

Sompolinsky, Feigel'man & Ioffe, Chamon & LFC in classical context

The SYK model

Equilibrium

$$\boxed{\frac{\partial q_d(\tau,\tau')}{\partial \tau} = \delta(\tau-\tau') + \int_0^{\beta\hbar} d\tau'' \, \Sigma(\tau,\tau'') q_d(\tau'',\tau')}$$

with τ the imaginary time, $q_d(\tau,\tau') \equiv \frac{1}{N}\sum_{i=1}^{N} \mathcal{T}[\langle \hat{\psi}_i(\tau)\hat{\psi}_i(\tau')\rangle]$ the correlation

and $\Sigma(\tau,\tau') \equiv J^2 q_d^3(\tau,\tau')$ the self-energy

Slow dynamics for long $\tau-\tau'$ $\Longrightarrow$ drop the time-derivative and then

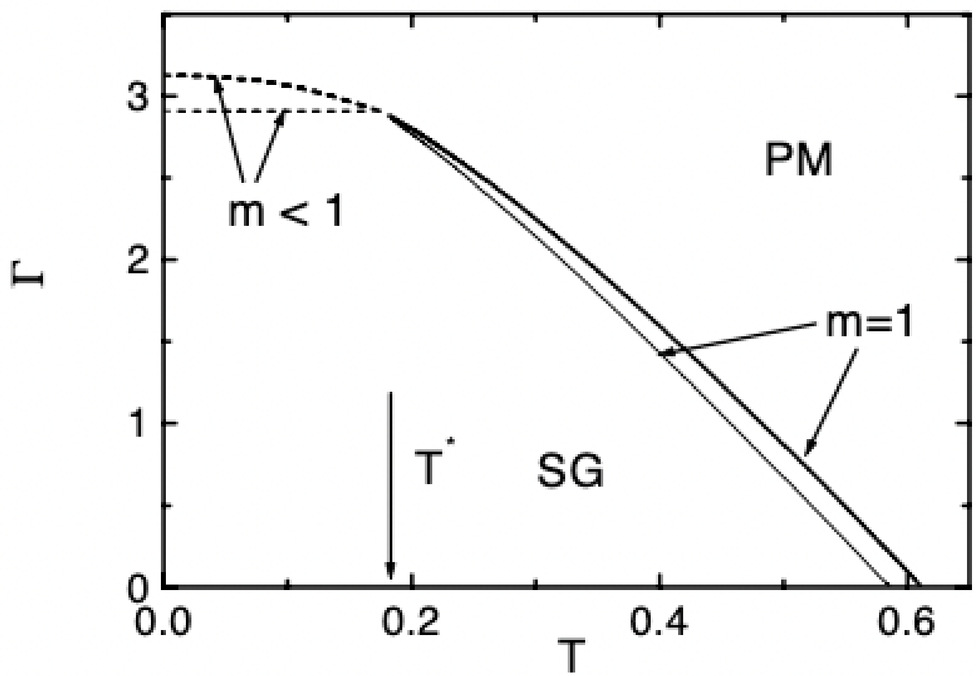
time reparametrization invariance under

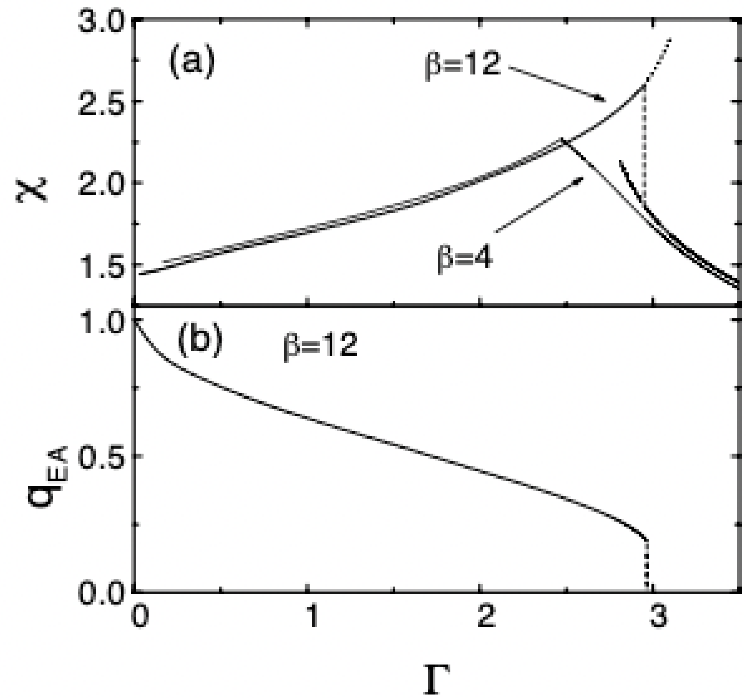
$$\tau \mapsto h(\tau) \qquad\qquad q_d(\tau,\tau') \mapsto [h(\tau)h(\tau')]^{1/4} \, q_d(h(\tau),h(\tau'))$$

and, by holography, **invariance under diffeomorphisms of general relativity**

Matsubara replica calculation

1st order phase transition in the quantum p-spin model





Jump in the susceptibility across the dashed part of the critical line.

LFC, Grempel & da Silva Santos 00

Quantum TAP

A sketch

Legendre transform of $f(\beta, h)$ with respect to $\{m_i(\tau)\}$ and $C(\tau - \tau')$ with $m_i(\tau) = \langle s_i(\tau) \rangle$ and $C(\tau - \tau') = N^{-1} \sum_i \langle s_i(\tau) s_i(\tau') \rangle$.

In fully-connected models one finds an exact free-energy functional.

In **random first order transition** model the complexity vanishes at zero temperature for all Γ; thus, the static transition cannot be due to an entropy crisis as in the classical case.

A more detailed analysis that goes beyond these lectures explains the 1st order transition, see **Biroli & LFC 01**

Quantum cavity methods that allow to deal with **dilute quantum spin models** **Krzakala, Rosso, Semerjian, Zamponi 08, Laumann, Moessner, Scardicchio, Sondhi 09**

11. Composite systems

Composite systems

Aim

Our interest is to describe the **statics** and **dynamics** of a **classical or quantum system** coupled to a **classical or quantum environment**.

The Hamiltonian of the ensemble is

$$\boxed{\mathcal{H} = \mathcal{H}_{syst} + \mathcal{H}_{env} + \mathcal{H}_{int}}$$

Syst.	x, p
Env.	$\{q_a, \pi_a\}$

What is the static and dynamic behaviour of the reduced system ?

Discuss it step by step :

1) equilibrium classical,

2) equilibrium quantum

Reduced system

Model the environment and the interaction

E.g., an ensemble of harmonic oscillators and a bi-linear coupling :

$$\boxed{\mathcal{H}_{env} + \mathcal{H}_{int} = \sum_{\alpha=1}^{\mathcal{N}} \left[\frac{p_\alpha^2}{2m_\alpha} + \frac{m_\alpha \omega_\alpha^2}{2} q_\alpha^2 \right] + \sum_{\alpha=1}^{\mathcal{N}} c_\alpha q_\alpha x}$$

Classically (coupled Newton equations) and **quantum mechanically** (easier in a path-integral formalism) one can integrate out the oscillator variables.

Assuming the environment is coupled to the sample at the initial time, t_0, and that its variables are characterized by a **Gibbs-Boltzmann distribution or density function** at inverse temperature β one finds a **colored Langevin equation** (classical) or a **reduced dynamic generating functional** $\mathcal{Z}_{red}$ (quantum mechanically).

Reduced system

Statics of a classical system

The partition function of the coupled system is

$$\mathcal{Z} = \sum_{osc,\, syst} e^{-\beta \mathcal{H} - \beta \eta x}$$

Integrating out the oscillator variables :

$$\mathcal{Z}_{red}[\eta] = \sum_{syst} e^{-\beta \left(\mathcal{H}_{syst} + \mathcal{H}_{count} + \eta x - \frac{1}{2} \sum_{a=1}^{N_b} \frac{c_a^2 \omega_a^2}{m_a} x^2 \right)}$$

Choosing $\mathcal{H}_{count}$ to cancel the quadratic term in x^2 one recovers

$$\boxed{\mathcal{Z}_{red} = \mathcal{Z}_{syst}}$$

i.e., the partition function of the system of interest.

Reduced system

Statics of a quantum system

The density matrix of the coupled system is

$$\rho(x'', q_a''; x', q'_a) = \langle x'', q_a'' | \hat{\rho} | x', q_a' \rangle$$
$$= \frac{1}{\mathcal{Z}} \int_{x(0)=x'}^{x(\beta\hbar)=x''} \mathcal{D}x(\tau) \int_{q_a(0)=q_a'}^{q_a(\beta\hbar)=q_a''} \mathcal{D}q_a(\tau) e^{-\frac{1}{\hbar} S^e[\eta]}$$

One integrates the oscillator's degrees of freedom to get the reduced density matrix

$$\rho_{red}(x'', x') = \mathcal{Z}_{red}^{-1} \int_{x'}^{x''} \mathcal{D}x(\tau)\; e^{-\frac{1}{\hbar}\left(S^e_{syst} - \int_0^{\beta\hbar} d\tau \int_0^{\tau} d\tau'\; x(\tau) K(\tau-\tau') x(\tau')\right)}$$

The counter-term was chosen to cancel a quadratic term in $x^2(\tau)$, $\mathcal{Z}_{red} = \mathcal{Z}/\mathcal{Z}_{osc}$ and $\mathcal{Z}_{osc}$ the partition function of isolated ensemble of oscillators, but a non-local interaction in the imaginary time with kernel

$K(\tau) = \frac{2}{\pi\hbar\beta} \sum_{n=-\infty}^{\infty} \int_0^{\infty} d\omega\, \frac{I(\omega)}{\omega}\, \frac{\nu_n^2}{\nu_n^2+\omega^2} \exp(i\nu_n\tau)$ remains.

Reduced system

Some consequences

Lamb shift : bath coupling renormalizes system energy levels even at zero temperature (vacuum fluctuations)

Single particle localization - tunneling suppressed for sufficiently strong coupling

Bray & Moore – Caldeira & Leggett 80s

"More coupling to environment = faster approach to equilibrium" is not always true

Markovian approximation breakdown at strong coupling or structured (non-Ohmic) baths, leading to non-Markovian dynamics with memory effects.

Books **Weiss 99 ; Breuer & Petruccione 02**

Change in the location & even environment induced phase transitions

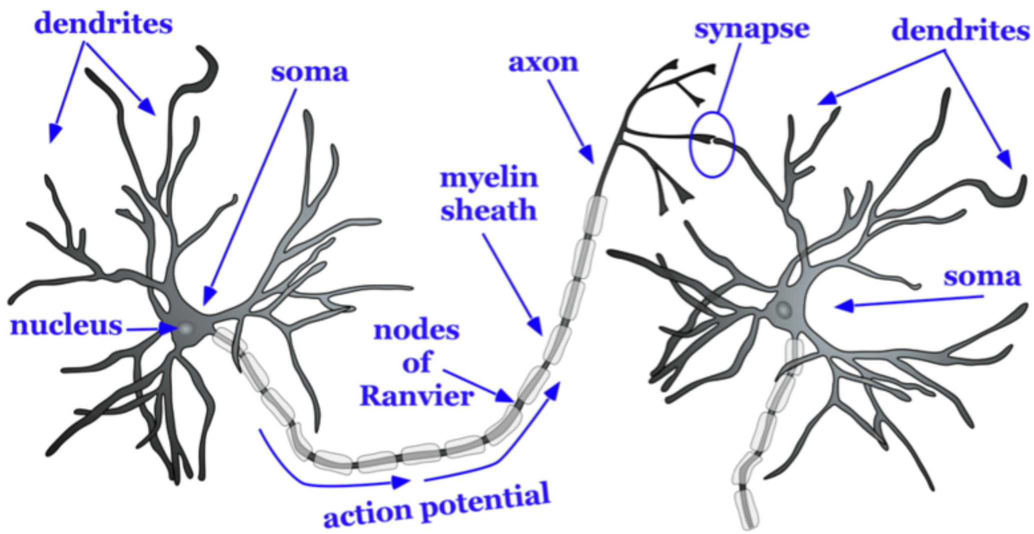
Bath can induce ergodicity/thermalization even in systems that wouldn't thermalize on their own, e.g. Many-Body Localization

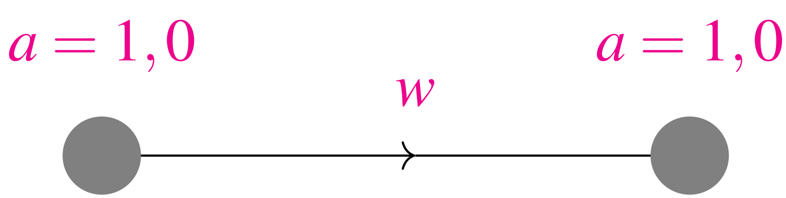
12. Beyond

Neural networks

Neural networks

From biology to physics : simplest possible modelling



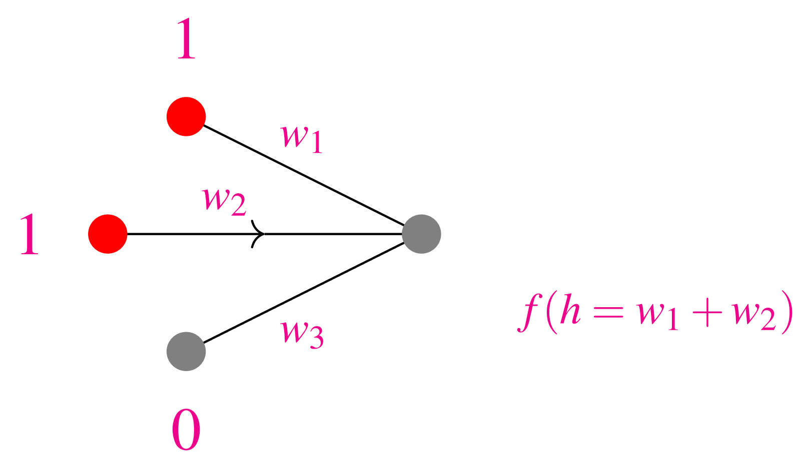


Ramón y Cajal, ca. 1890 Nobel 1906 McCulloch & Pitts 43

(father of neuroscience working in Madrid, neurophysiologist, autodidact math & logic at Chicago)

Neural networks

Random graphs with weighted connections



At each instant, each neuron calculates the sum of the entries sent by its neighbors

weighted by the strength of the synapses, $h_i = \sum_j w_{ij} a_j$, and it applies a function, f.

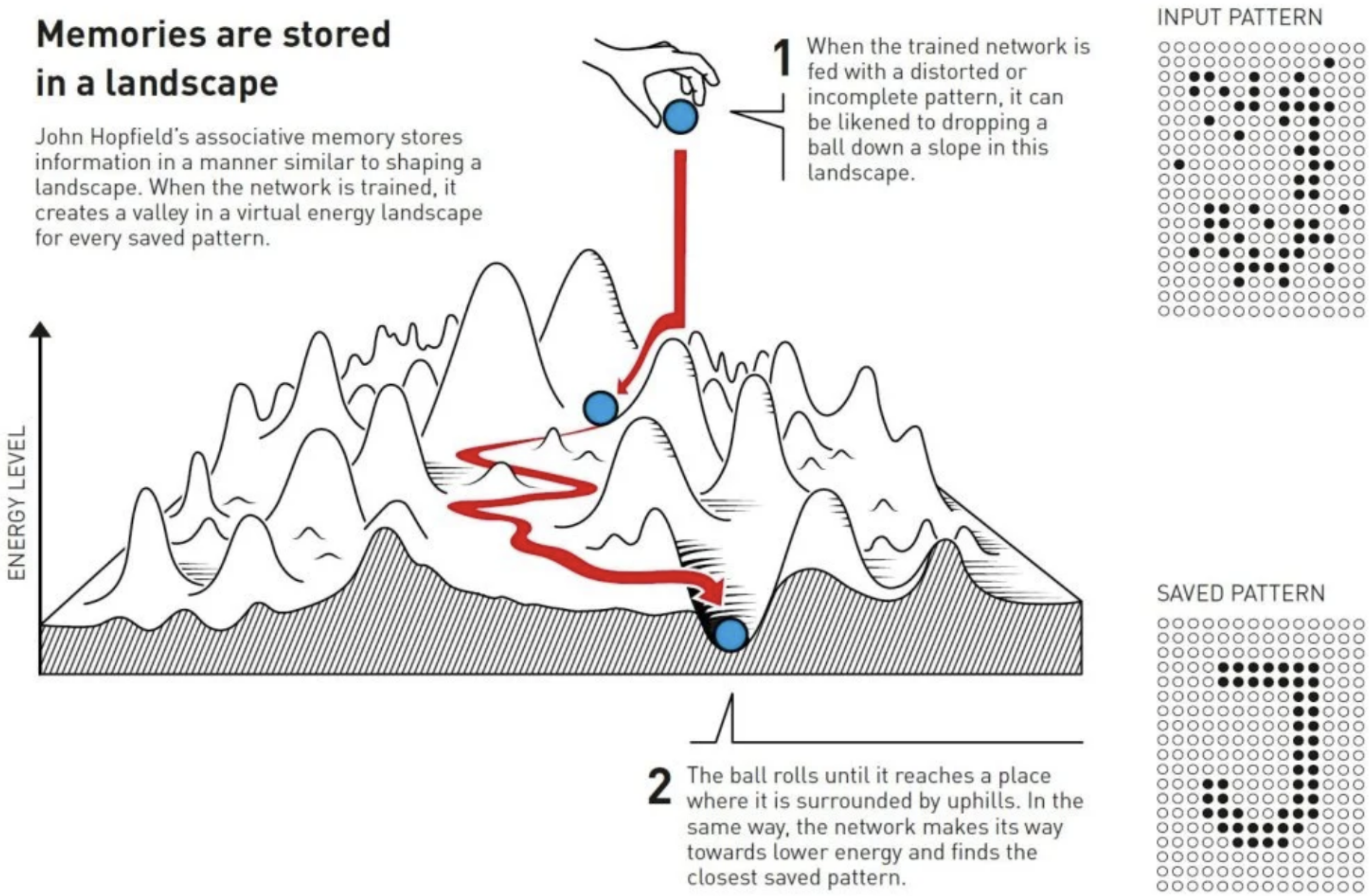
if the result overcomes a threshold, $f(h_i) > \theta$, the receptor neuron spikes, $a_i = 1$.

So on and so forth on the whole network

Details to make precise, e.g. parallel or random sequential dynamics

Hopfield Model

Associative memories : one chooses the synaptic weights w_{ij}



Hopfield 82 Nobel 2024

Image `www.nobelprize.org`

Hopfield Model

Properties

- The **synapses** $\{w_{ij}\}$ shape the landscape & encode the memories **Hebb 49**
- **Basins of attraction** : input objects must sufficiently resemble the network's learned templates for successful recognition.
- The network has a limiting **capacity** : beyond a threshold

$$\left(\frac{\text{number of learnt objets}}{\text{number of neurons}}\right)_c$$

it fails **Phase Transition : order (recall possible) - disorder (failure)**

- The network builds **illegitimate valleys** which do not correspond to learnt objects

Amit, Gutfreund & Sompolinsky 85 maximal capacity of Hopfiled-like networks

Elizabeth Gardner 88 Space of interactions in neural network models : all possible connection weights that could successfully store a given set of memories

with **replica methods** as developed by Parisi **Nobel 2021**

More history

Personal views

A field with early contributions from many Argentine physicists, in part thanks to **Miguel Virasoro's** (Roma Sapienza, ICTP & Buenos Aires) influence.

Néstor Parga (Bariloche $\mapsto$ Madrid)

Roberto Perazzo (Buenos Aires)

Edgardo Ferrán (Buenos Aires $\mapsto$ bioinformatics at Sanofi & others, France)

Walter Theumann (Porto Alegre)

Tato Moscato (La Plata $\mapsto$ Newcastle, Australia)

After the early 90s : Neural networks & AI winter

Some people stayed, e.g. **Geoffrey Hinton** **Nobel 2024**, **Yann LeCun** (Les Houches)

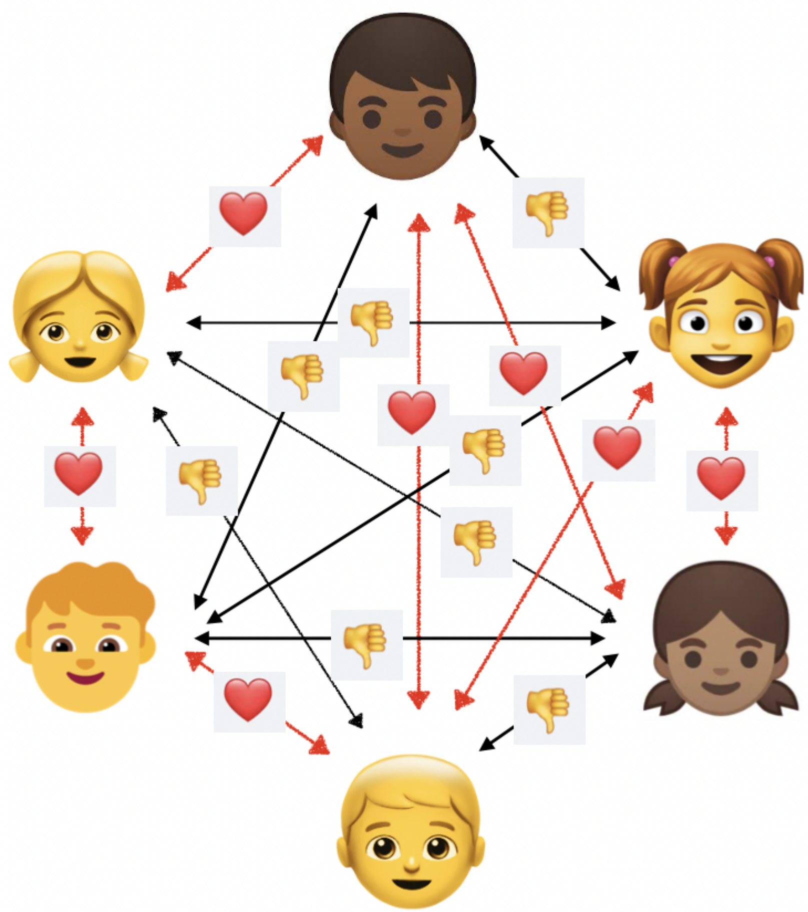
Huge amount of very important work by **Sara Solla** (Chicago, USA)

13. Beyond

Random optimization

Random optimization

How do we split the group in two equal sized parts



and minimize frustration ?

Random optimization

Mapping to a statistical physics model

In the **graph partitioning** - **group splitting** example

- $i, j = 1, \ldots, N$ label the persons.
- Predetermined $J_{ij} = -1$ for ❤️ love or $J_{ij} = 1$ for 👎 hate feelings and $J_{ij} = 0$ for strangers
- $s_i = 1$ if i is in group A or $s_i = -1$ if i is in group B

find the assignment of all the s_i so that they **add up to zero** ($\sum_{i=1}^{N} s_i = 0$) & the **cost function**

$$\mathcal{H} = \text{ sum over all pairs of the love/hate values in the same group}$$

is **minimized**

Random optimization

Mapping to a statistical physics model

In the **graph partitioning** - **group splitting** example

- $i, j = 1, \ldots, N$ label the persons.
- Predetermined $J_{ij} = -1$ for ❤️ love or $J_{ij} = 1$ for 👎 hate feelings
- $s_i = 1$ if i is in group A or $s_i = -1$ if i is in group B

find the assignment of all the s_i so that they **add up to zero** ($\sum_{i=1}^{N} s_i = 0$)

Cost function should be minimised like finding a ground state

$$\mathcal{H} = \underbrace{\sum_{i \neq j}}_{\text{sum over all pairs}} \underbrace{J_{ij}}_{\substack{\text{quenched} \\ \text{love/hate}}} \underbrace{\left(\frac{1 + s_i s_j}{2}\right)}_{\substack{\text{selects pairs in same group} \\ \text{vanishes if } i,j \text{ in different groups}}}$$

Random optimization

Mapping to a statistical physics model

Some successes of the statistical physics approach

- **S. Kirkpatrick, C. D. Gelatt Jr and M. P. Vecchi**, “Optimization by **Simulated Annealing**". Science 220, 671 (1983). **Dynamics**
- **R. Monasson, R. Zecchina, S. Kirkpatrick, B. Selman and L. Troyansky**, “Determining Computational Complexity from Characteristic **Phase Transitions**” Nature 400, 133 (1999)
- **J. S. Yedidia, W. T. Freeman and Y. Weiss**, “Understanding **Belief Propagation** and its Generalizations” Exploring artificial intelligence in the new millennium 8 (236-239), 0018-9448 (2003) **TAP**
- **M. Mézard, G. Parisi and R. Zecchina**, Science 297, 812 (2002) “Analytic and Algorithmic Solution of Random Satisfiability Problems” **TAP++** **Onsager Prize, APS, 2016**

Combinatorial optimization

A sketch

In **Optimization problems** a **cost** must be minimized, e.g. the distance traveled by a salesman to visit each of N cities once and only once.

Mapping on classical spin models on random (hyper-)graphs

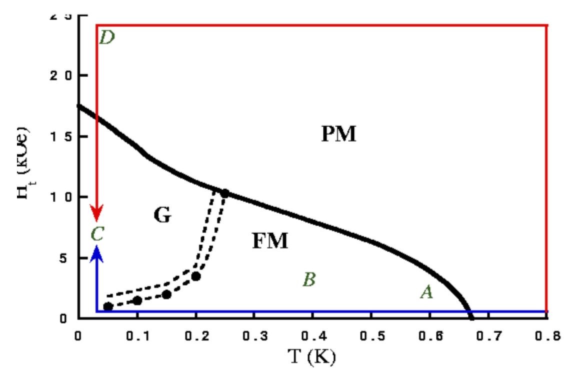
K-satisfiability $\Rightarrow$ $p(\leq K)$-spin models on a random (hyper-)graph.

Quantum annealing

Kadowaki & Nishimori 98

Dipolar spin-glass

Aeppli et al. 90s



1st order transitions : trouble for **quantum annealing** techniques.

Jorg, Krzakala, Kurchan, Maggs & Pujos 09

Short Overview of the
Statics of Disordered Systems

L. F. Cugliandolo,
my webpage `TEACHING/Master2022/disorder-2022.pdf`

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F. Zamponi, Mean field theory of spin glasses, arXiv :1008.4844

Spin Glass Theory and Far Beyond : Replica Symmetry Breaking After 40 Years, eds. P. Charbonneau et al (World Scientific, 2023)