France-Japan Stat-Phys

More than 40 years of interactions

Leticia F. Cugliandolo

Sorbonne Université

leticia@lpthe.jussieu.fr

www.lpthe.jussieu.fr/~leticia

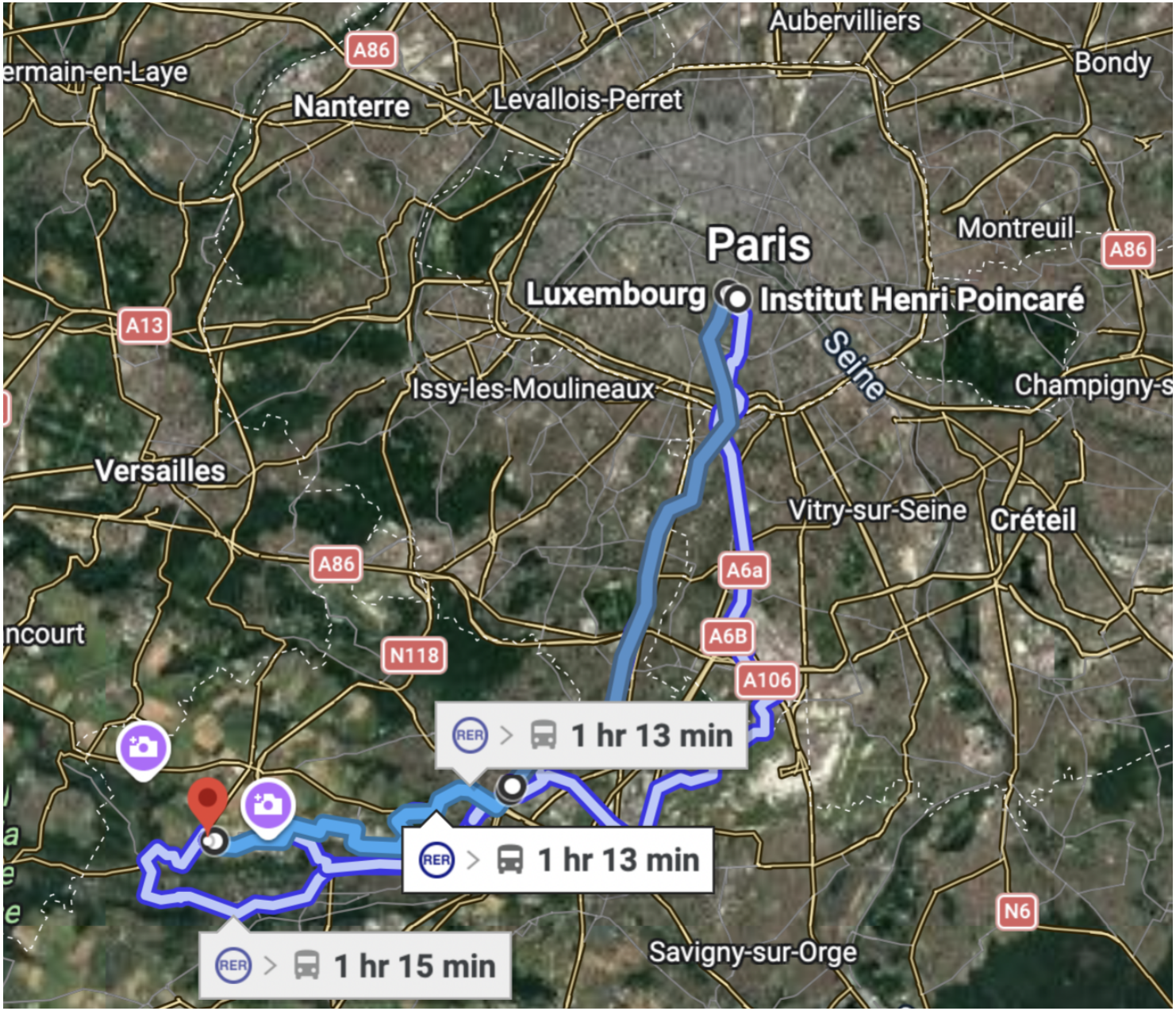
Thanks to Eric Vincent

IHP, Paris, Sept. 2026

A bit of History of

the French-Japanese meetings

The French-Japanese regular meetings on statistical physics started in the 80s, when the **SPEC CEA Saclay** experimental group studying spin-glasses initiated a collaboration with **Prof. Miyako's** experimental group in Japan.


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Savigny-sur-Orge

A bit of History of

the French-Japanese meetings

The French-Japanese regular meetings on statistical physics started in the 80s, when the Saclay experimental group studying spin-glasses initiated a collaboration with Prof. Miyako's experimental group in Japan.

It has since evolved from being focused on **spin-glass physics** to treating **out of equilibrium statistical physics** in general.

Some of the main Japanese organizers/leaders/participants* have been
Prof. Miyako (Hokkaido/Osaka), Prof. Mme. Ito (Ochanomizu/RIKEN ?), Prof. Takayama (Tokyo), Prof. Miyashita**(Kyoto/Tokyo), Prof. Kawamura (Nagoya/Osaka), Prof. Nishimori (Tokyo Tech), Prof. Sasa (Tokyo/Kyoto).
Currently Prof. Hayakawa (Yukawa Inst)
Next **Hajime Yoshino** (Osaka)

*sorry if I miss some names

**Currently, president of the Physical Society of Japan

The 80s–90s

Experimental Spin-Glases

80s Jaques Hammann, **Miguel Ocio**, Eric Vincent SPEC CEA Saclay

'80s Contacts with Prof. **Yoshihito Miyako** Hokkaido Univ., next Osaka Univ.

'88 2nd Yukawa International Symposium on *Cooperative Dynamics in Complex Physical Systems*

'94 – '96 – '98 – '00 Monbu-sho International Cooperative Research Projects (France, US, Japan) *Transition and Dynamics in Complex Systems*. Workshops

1994 Tsukuba

1995 Saclay, Dec. *Physics of frustrated systems* 1 month long total strike + cold

The ’95 Saclay meeting

The 1995 strikes in France paralyzed Paris and the entire country from **November 24 to December 15, 1995**, in massive protest against government austerity measures. Wikipedia +2



Causes

- **The Juppé Plan:** Prime Minister Alain Juppé introduced heavy welfare cuts, health-care overhauls, and tax increases. Wikipedia +2
- **Pension Changes:** Plans aimed to raise public sector retirement requirements and scrap special early retirement options for transport workers. Wikipedia +2
- **Budget Goals:** The austerity package aimed to shrink France's budget deficit to meet European Maastricht Treaty criteria. Wikipedia +1

Time-translation invariance in France

The 80s–90s

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80s Jaques Hammann, **Miguel Ocio**, Eric Vincent SPEC CEA Saclay

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1994 Tsukuba

1995 Saclay, Dec. *Physics of frustrated systems* 1 month long total strike + cold

1996 Lake Arrowhead Ray Orbach* & Peter Young

1997 Kyoto, Holiday Inn, Oct. *Aging phenomena in Complex systems II* bicycle

1998 Abbey Royaumont

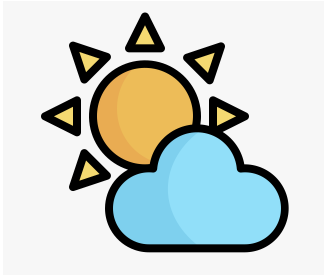
1999 Kyoto

* Director of the Office of Science, Department of Energy, 2002-2006, during G. W. Bush's administration, former Chancellor of UC Riverside, back to spin-glass experiments

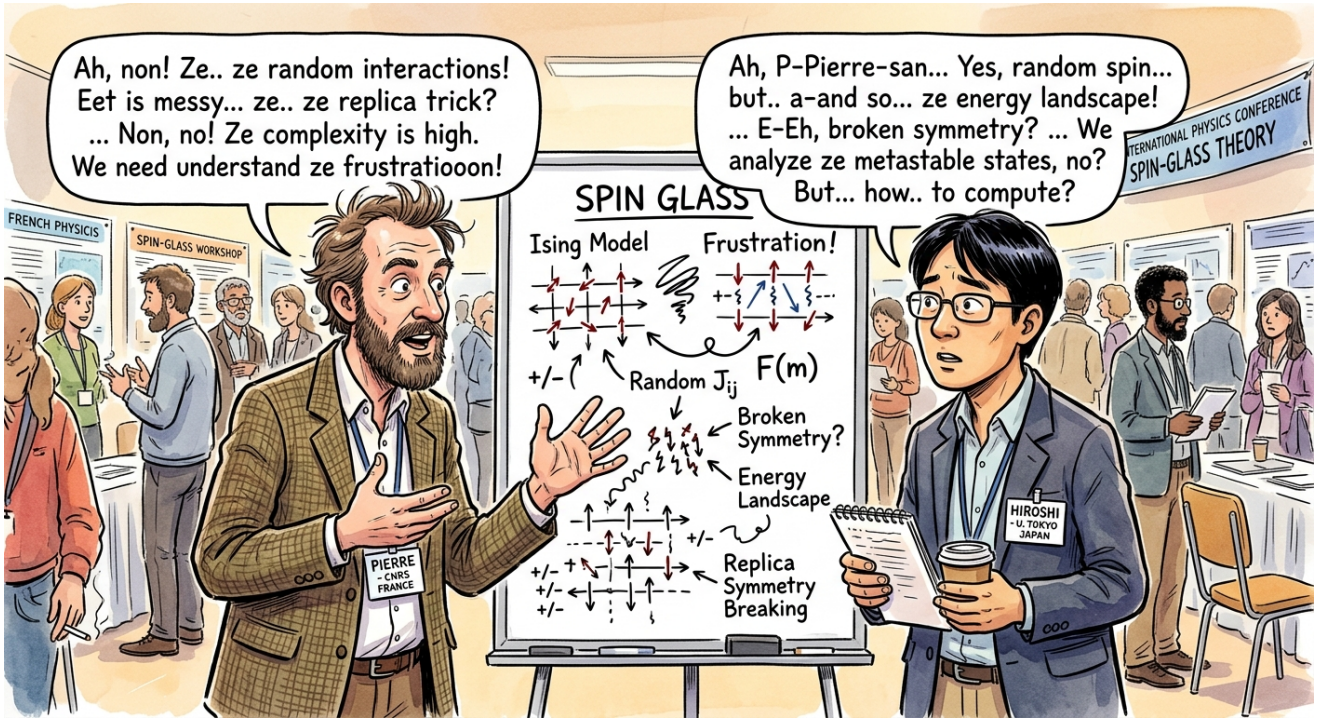
The ’95 Saclay meeting



⇒



1995 ⇒ 2026



The 2000s

French-Japanese Bilateral Seminars

2004 Jan. Yukawa Inst. Kyoto, *Frustrated Magnetism and Slow Dynamics*, organized by H. Kawamura & I. Campbell

2005 Sept. IHP Paris, *Recent Progress in Glassy Physics*, co-organized by H. Takayama, LFC & M. Picco

2008 Yukawa Inst. Kyoto, *Frontiers of Glassy Physics*, Nov. 19-22, LFC, H. Hayakawa & M. Picco

2011, Dec. 7-10, IHP Paris, *Jamming, Glasses and Phase transitions*, co-organized by H. Hayakawa, LFC, F. Krzakala, M. Picco & F. Zamponi

2015, Aug. 11-14, Yukawa Inst. Kyoto, *Physics of Glassy Materials*, co-organized by LFC, H. Hayakawa & M. Picco **COVID**

2023, March 13-16, Yukawa Inst. Kyoto, *Physics of dense and active disordered materials*, co-organized by LFC, H. Hayakawa & M. Picco

2026, Sept. 28-Oct.1, IHP Paris, *Statistical Physics Out of Equilibrium* co-organized by LFC, D. Dean, H. Hayakawa, M. Picco, C. Scalliet & F. van Wijland

Speedy deletion nomination of Draft:French-Japanese Meetings on Statistical Physics [edit]

Latest comment: 4 days ago | 2 comments | 2 people in discussion

A tag has been placed on Draft:French-Japanese Meetings on Statistical Physics requesting that it be speedily deleted from Wikipedia, as it exhibits signs of having been generated by an AI model. Text produced by these applications is prohibited on Wikipedia. For further information, see section G15 of the criteria for speedy deletion.

If this is the first article that you have created, you may want to read the guide to writing your first article.

On random matrices

Leticia F. Cugliandolo

Sorbonne Université

leticia@lpthe.jussieu.fr
www.lpthe.jussieu.fr/~leticia

Work in collaboration with

G. Schehr (LPTHE, Sorbonne Université)

M. Tarzia (LPTMC, Sorbonne Université)

D. Venturelli (SISSA, Trieste, Italia - Erasmus at Sorbonne $\Rightarrow$ post-doc/MdC at LPTMC)

V. Delapalme (Master student at Sorbonne Université $\Rightarrow$ PhD at King's College)

A. Hartmann (University of Oldenbourg)

French Japanese Meeting 2026

Plan

— (Very short) Physical motivation

1. Anderson & Many-Body Localization
2. Neural Networks
3. Pure random matrix theory

— Two kinds of random matrix models

1. Rosenzweig-Porter GOE & Wishart models
2. Weighted Erdos-Renyi graph

— Conclusions

Anderson Localization

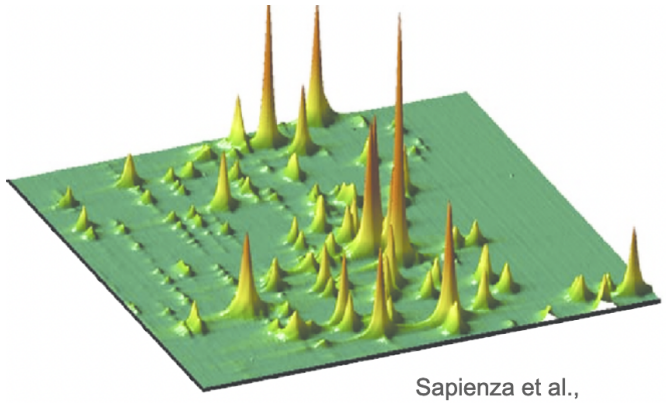
e.g., one in a random potential

$$H = \sum_{ij} \underbrace{t_{ij}}_{j \to i \text{ hopping}} c_i^\dagger c_j + \sum_i \underbrace{\epsilon_i}_{\text{random energies}} \underbrace{c_i^\dagger c_i}_{\text{fermion number}}$$

i.i.d. random local energies from $\epsilon_i \in [-W/2, W/2]$

Competition between kinetic and random disordered terms

no interactions



Sapienza et al.,
Science 327, 1352 (2010)

Real space localization

$\psi(\vec{r}) \sim e^{-|\vec{r}-\vec{r}_o|/\xi}$

$d \leq 2$ for infinitesimal disorder

$d > 2$ metal/insulator transition at W_c

Anderson, *Absence of Diffusion in Certain Random Lattices*, Phys. Rev. 109, 1492 (1958)

Many-body Localization

e.g., many electrons in interaction under a random potential

From the real space Hamiltonian with interactions

$$H = \sum_{ij} t_{ij}\, c_i^\dagger c_j \;+\; \sum_i \epsilon_i\, c_i^\dagger c_i + \sum_{ij} \underbrace{V_{ij}\;\; c_i^\dagger c_i c_j^\dagger c_j}_{\text{fermion}-\text{fermion inter}}$$

to the Fock-space Hamiltonian

$$H' = \sum_{\alpha\neq\beta} \underbrace{T_{\alpha\beta}}_{\text{state change}} \;\; |\alpha\rangle\langle\beta| \;+\; \sum_\alpha \underbrace{E_\alpha}_{\text{energies}} \;\; |\alpha\rangle\langle\alpha|$$

$|\alpha\rangle = |n_1, n_2, \ldots, n_N\rangle$

n_i occupation numbers

$T_{\alpha\beta}$ hierarchical

Fock space is a **graph**

Altshuler, Gefen, Kamenev & Levitov, *Quasiparticle Lifetime in a Finite System : A Nonperturbative Approach*, Phys. Rev. Lett. 78, 2803 (1997)

Many-body Localization

e.g., many electrons in interaction under a random potential

Fock-space Hamiltonian

$$H' = \sum_{\alpha \neq \beta} \underbrace{T_{\alpha\beta}}_{\text{state change}} |\alpha\rangle\langle\beta| + \sum_{\alpha} \underbrace{E_{\alpha}}_{\text{energies}} |\alpha\rangle\langle\alpha|$$

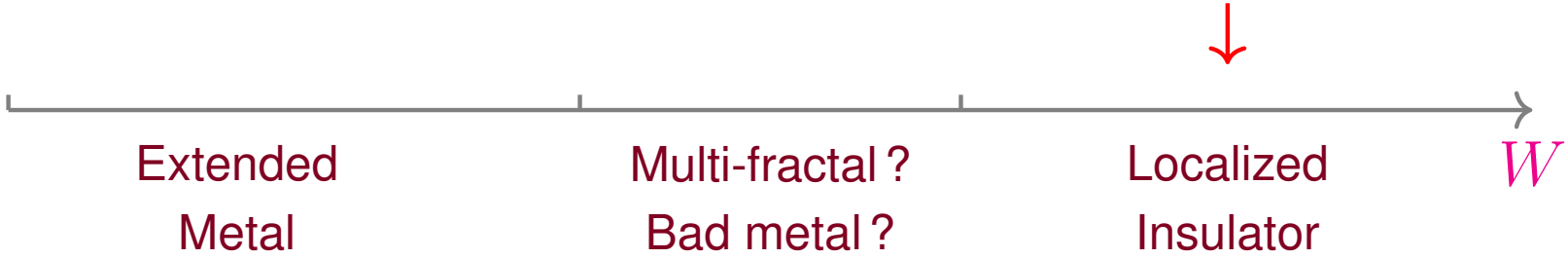
$|\alpha\rangle = |n_1, n_2, \ldots, n_N\rangle$

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Fock space is a graph

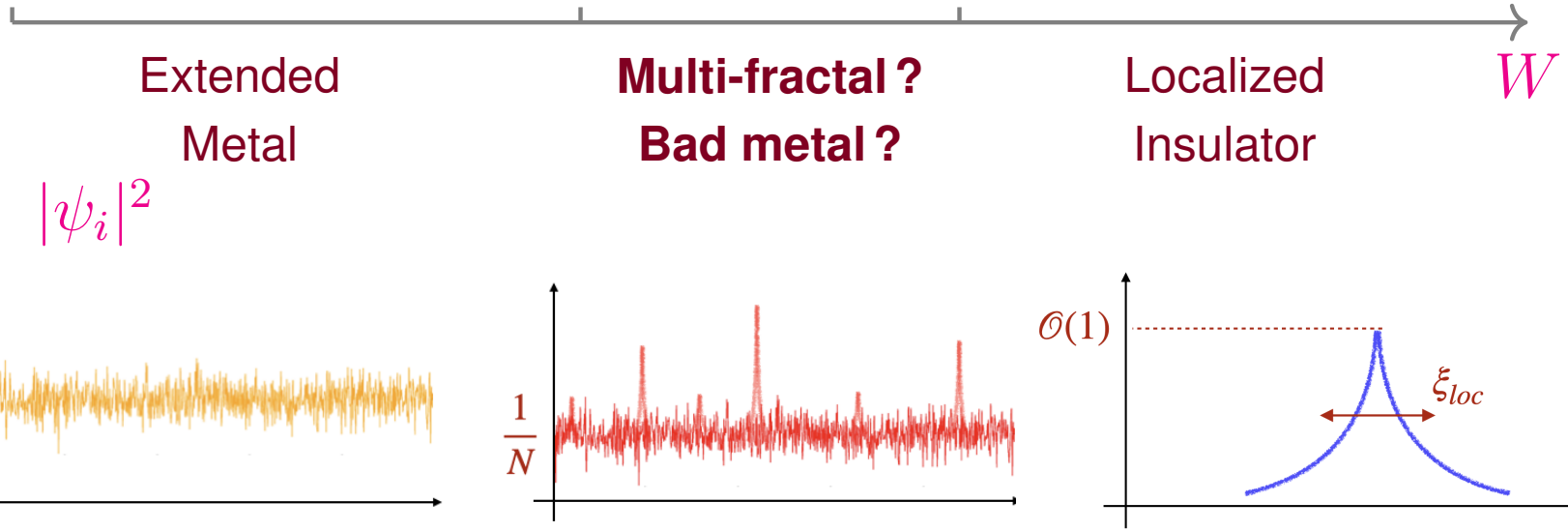
Localization in Fock space



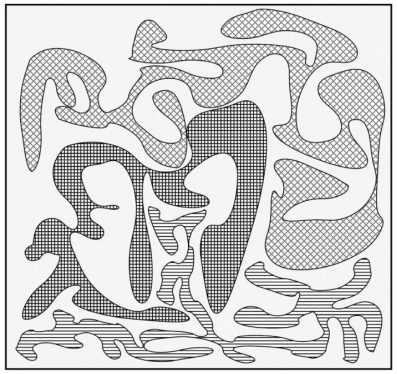
Basko, Aleiner & Altshuler, *Metal-insulator transition in a weakly interacting many-electron system with localized single-particle states*, Ann. Phys. 321, 1126 (2006)

Many-body Localization

Wave functions : Extended, Multifractal, Localized



$|\psi_i|^2$



Random Matrix Model ?

Inverse participation ratio

$$I_q = \sum_i |\psi_i|^{2q} \propto N^{-(q-1)D_q}$$

$D_q = 0$ Localized

$D_q = 1$ Extended

$0 < D_q < 1$ Multifractal

Plan

— (Very short) Physical motivation

1. Anderson & Many-Body Localization
2. Neural Networks[1]
3. Pure random matrix theory[2]

— Two kinds of random matrix models

1. Generalized Rosenzweig-Porter models
2. Weighted Erdos-Renyi graph

— Conclusions

Two kinds of random matrices

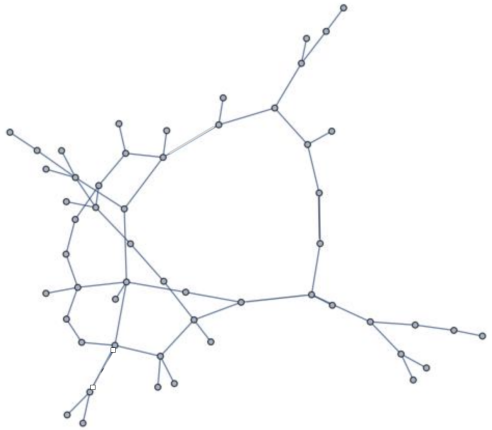
- **Rosenzweig-Porter (RP)**

  Sum of random matrices

$$\mathbb{H} = \mathbb{A} + \frac{\nu}{N^{\gamma/2}}\,\mathbb{B}$$

- **Weighted Erdös-Rényi graph**

  Fluctuating connectivity & random hops



Initial Motivation

Search of random matrices with (multi)fractal eigenvectors

Venturelli, LFC, Schehr, Tarzia, *Replica approach to the generalized Rosenzweig-Porter model*, SciPost Phys. 14, 110 (2023) **Delapalme, LFC, Schehr, Tarzia, Venturelli**, *The Wishart–Rosenzweig–Porter random matrix ensemble*, J. Stat. Mech. 2026, 013301 (2026), **Delapalme, LFC, Hartmann, Tarzia, Venturelli**, *Level statistics in the fractal phase of generalized Rosenzweig–Porter models*, arXiv:2607.09444

LFC, Schehr, Tarzia, Venturelli, *Multifractal phase in the adjacency matrices of random Erdös-Rényi graphs*, Phys. Rev. B 110, 174202 (2024)

Two kinds of random matrices

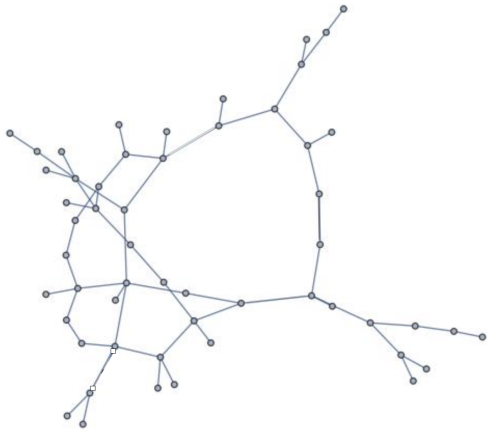
- **Rosenzweig-Porter (RP)**

  Sum of random matrices

$$\mathbb{H} = \mathbb{A} + \frac{\nu}{N^{\gamma/2}}\,\mathbb{B}$$

- **Weighted Erdös-Rényi graph**

  Fluctuating connectivity & random hops



Initial Motivation & Surprises

Search of random matrices with **(multi)fractal eigenvectors**

Venturelli, LFC, Schehr, Tarzia, *Replica approach to the generalized Rosenzweig-Porter model*, SciPost Phys. 14, 110 (2023) **Delapalme, LFC, Schehr, Tarzia, Venturelli**, *The Wishart–Rosenzweig–Porter random matrix ensemble*, J. Stat. Mech. 2026, 013301 (2026), **Delapalme, LFC, Hartmann, Tarzia, Venturelli**, *Level statistics in the fractal phase of generalized Rosenzweig–Porter models*, arXiv:2607.09444

The GOE RP Model

Sum of random matrices

Take a diagonal $N \times N$ matrix $\mathbb{A}$ with *i.i.d.* real elements $a_i = \mathcal{O}(1)$ taken from a $p_a(a_i)$

$$\mathbb{A} = \begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ & \dots & \dots & \dots & \\ 0 & \dots & 0 & a_{N-1\,N-1} & 0 \\ 0 & \dots & 0 & 0 & a_{NN} \end{pmatrix}$$

and a real symmetric $N \times N$ matrix $\mathbb{B}$ from the Gaussian Orthogonal Ensemble $2\mathbb{B} = \mathbb{C} + \mathbb{C}^T$

$$\mathbb{B} = \begin{pmatrix} b_{11} & b_{12} & & \dots & b_{1N} \\ b_{12} & b_{22} & b_{23} & \dots & b_{2N} \\ & \dots & \dots & \dots & \\ b_{1\,N-1} & \dots & \dots & b_{N-1\,N-1} & b_{N-1\,N} \\ b_{1\,N} & & \dots & b_{N-1\,N} & b_{NN} \end{pmatrix}$$

with $b_{ij} = \mathcal{O}(1) \in \mathbb{R}$ taken from $p_b(b_{i \neq j}) \propto e^{-b_{ij}^2/2}$ and $p_b(b_{ii}) \propto e^{-b_{ii}^2/4}$

The GOE RP Model

Sum of random matrices

Add them in the form

$$\mathbb{H} = \mathbb{A} + \underbrace{\frac{\nu}{N^{\gamma/2}}\,\mathbb{B}}_{\gamma>1 \text{ ``perturbation''}}$$

with ν and γ two $\mathbb{R}$ parameters $\mathcal{O}(1)$

& N the size of the square matrices

Historical motivation: adapt random matrix theory to atomic physics studies

Rosenzweig & Porter, *Repulsion of Energy Levels in Complex Atomic Spectra*, Phys. Rev. 120, 1698 (1960)

More recently: many-body localization and the hypothetical bad metal phase

Kravtsov, Khaymovich, Cuevas & Amini, *A random matrix model with localization and ergodic transitions*, New. J. Phys. 17, 122002 (2015)

Sum of random matrices & applications - free probability

A. Zee, *Law of addition in random matrix theory*, Nucl. Phys. B 474, 726 (1996)

Density of eigenvalues

Properties of $\mathbb{H} = \mathbb{A} + \nu N^{-\gamma/2}\,\mathbb{B}$

Averaged spectral density $\rho_N(\lambda) \equiv \left[\frac{1}{N}\sum_{i=1}^{N}\delta(\lambda-\lambda_i)\right]_{\mathbb{H}}$

Limits

λ_i the eigenvalues

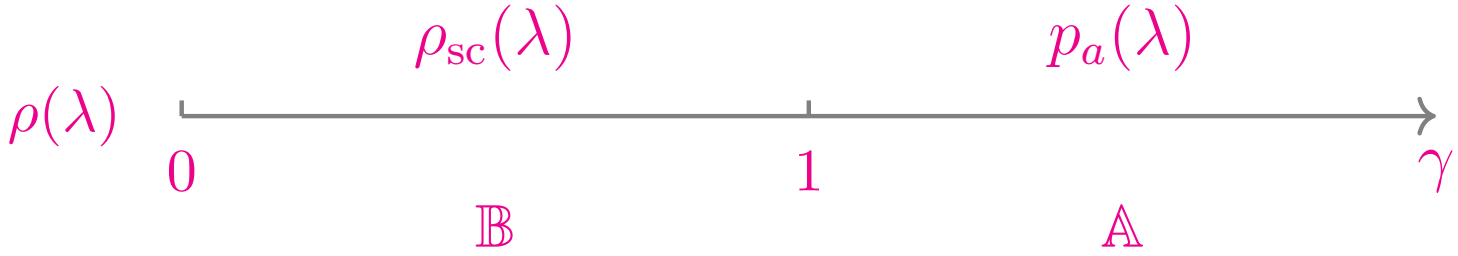
– For $\gamma \gg 1$ the effect of $\mathbb{B}$ is negligible and

$\rho_N(\lambda)$ is just $p_a(\lambda)$

– In the Gaussian Orthogonal Ensemble (GOE), $\gamma = 1$ and $p_a(a_{ii}) = \delta(a_{ii})$, only $\mathbb{B}$ counts, and

$\rho(\lambda) \equiv \lim_{N\to\infty} \rho_N(\lambda)$ is the semi-circle law $\rho_{\rm sc}(\lambda) = \frac{1}{2\pi}\sqrt{4-\lambda^2}$

In general



Level spacings

Properties of $\mathbb{H} = \mathbb{A} + \nu N^{-\gamma/2}\, \mathbb{B}$

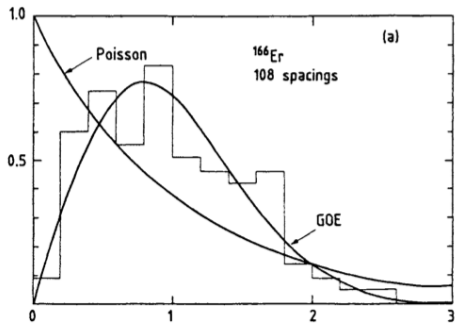
Level statistics The level spacings $s_i = \lambda_{i+1} - \lambda_i$, normalized by their mean $\langle s_i \rangle$, are distributed according to

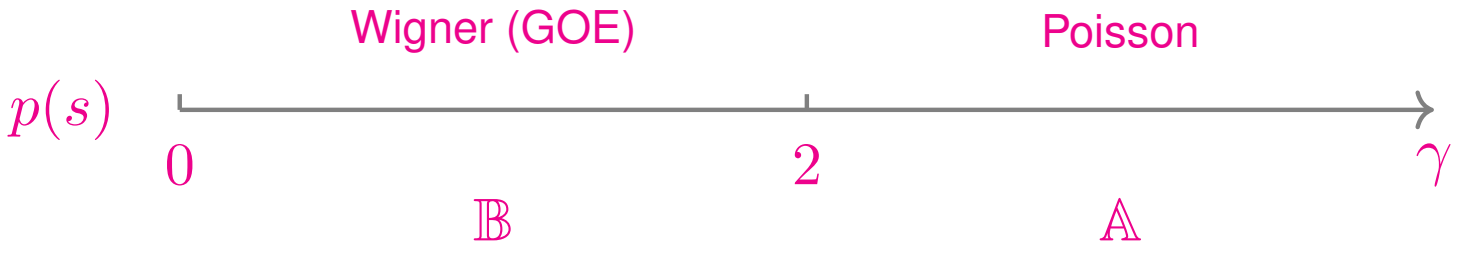
Limits

– for independent levels, Poisson's p.d.f. $p(s) = e^{-s}$

– in the GOE case, the Wigner surmise

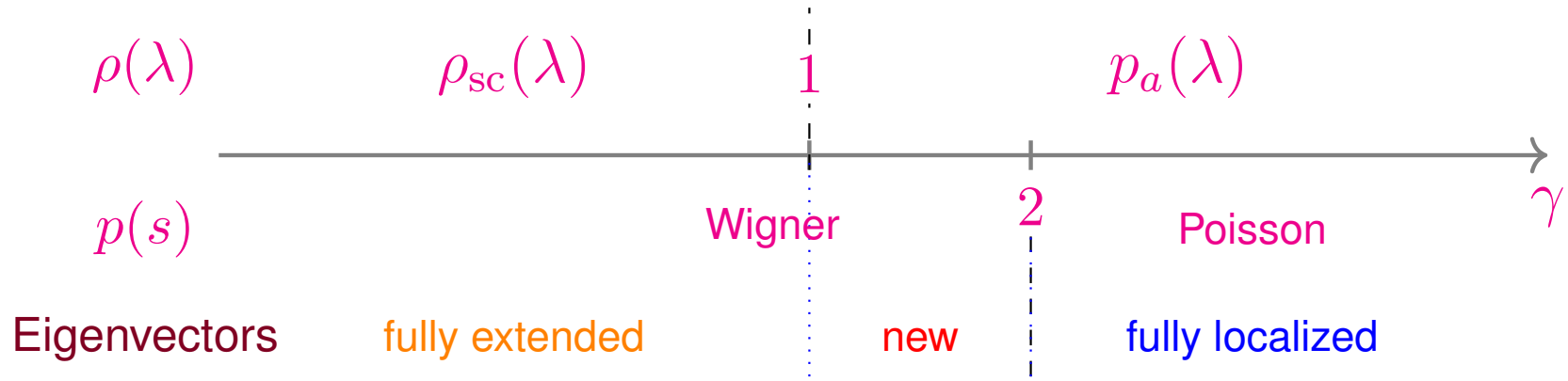
$$p(s) = \frac{\pi}{2}\, s\, e^{-\pi s^2/4}$$





Eigenvectors

Properties of $\mathbb{H} = \mathbb{A} + \nu N^{-\gamma/2}\, \mathbb{B}$



– For $\gamma > 2$, $\mathbb{H} \sim \mathbb{A}$ and the eigenvectors (wave functions) ψ are fully localized

$n = \mathcal{O}(1)$ non-zero component

– For $\gamma < 1$, $\mathbb{H} \sim \nu N^{-\gamma/2}\, \mathbb{B}$ and the eigenvectors ψ are fully extended

$n = \mathcal{O}(N)$ non-zero components

– For $1 < \gamma < 2$, the eigenvectors ψ are localized over a fractal number of sites

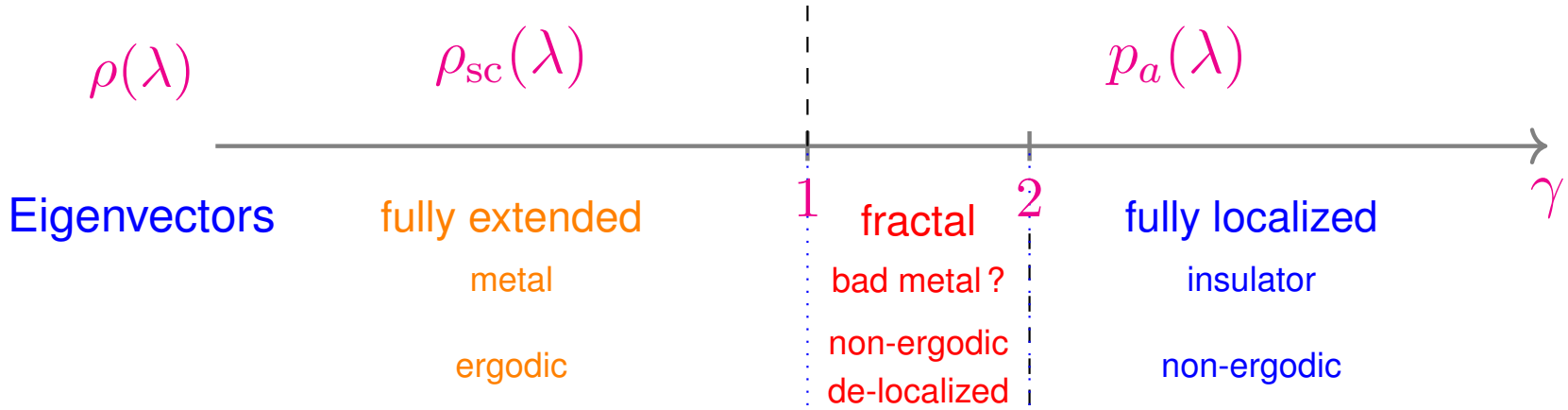
$1 \ll n = \mathcal{O}(N^{2-\gamma}) \ll N$ non-zero components

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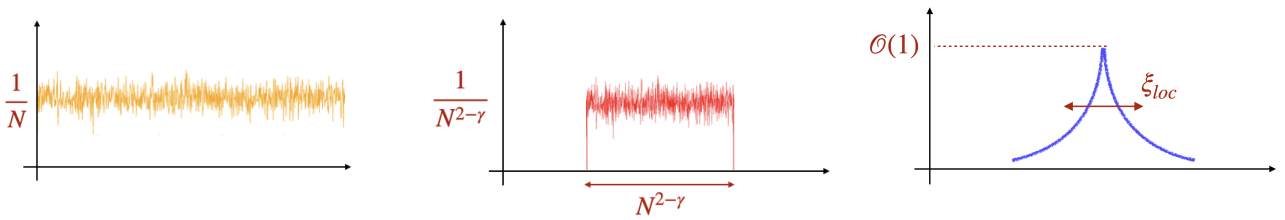
Kravtsov, Khaymovich, Cuevas & Amini, *A random matrix model with localization and ergodic transitions*, New. J. Phys. 17, 122002 (2015)

Eigenvectors

Properties of $\mathbb{H} = \mathbb{A} + \nu N^{-\gamma/2}\,\mathbb{B}$



$|\psi_i^{(\alpha)}|^2$



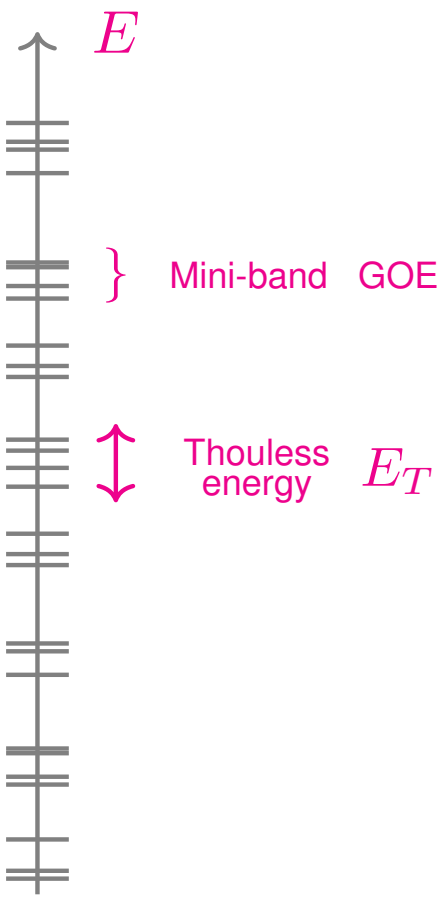
Horizontal axis: "site" i index ordered according to $a_1 \leq \cdots \leq a_N$

Kutlin & Khaymovich, *Anatomy of the eigenstates distribution...* SciPost Phys. 16, 008 (2024)

de Tomasi, Amini, Bera, Khaymovich & Kravtsov, *Survival probability in Generalized Rosenzweig-Porter random matrix ensemble*, SciPost Phys. 6, 014 (2019) $R(t) \equiv [|\langle\psi(t)|\psi(0)\rangle|^2]_{\mathbb{H}} \xrightarrow[t\to\infty]{} N^{-D_f}$

Picture

Mini-bands and the fractal dimension for $1 < \gamma < 2$



Perturbation theory $\Rightarrow \lambda_i \sim a_i + 4\zeta \, \frac{1}{N} \sum_{j(\neq i)} \frac{b_{ij}^2}{a_i - a_j}$

Average spreading of the eigenvalues $\Rightarrow$ **Thouless energy**

$$E_T \equiv \Big[|\lambda_i - a_i| \Big]_{\mathbb{H}} \sim \zeta \equiv (\nu^2/4)\, N^{1-\gamma} \gg 1/N$$

width of the **mini-bands** with GOE statistics

The number of eigenvectors “hybridized" by the perturbation

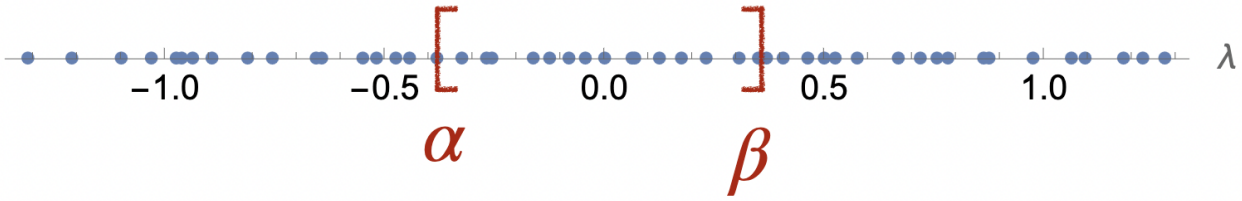
$$\# \sim \frac{E_T}{(N\rho(\lambda))^{-1}} \sim N^{D_f} \quad \text{with} \quad \boxed{D_f = 2 - \gamma < 1}$$

support of the eigenvectors of the perturbed matrix $\mathbb{H}$

But no evidence for **multifractality** in this model

Full counting statistics

The level compressibility



$I_N(\alpha,\beta) = N\int_\alpha^\beta d\lambda\,\rho_N(\lambda)$ counts how many eigenvalues fall in the interval

Large deviation function $\mathcal{F}_{[-E,E]}(s) = \lim\limits_{N\to\infty} \frac{1}{N} \ln\left[e^{-sI_N(-E,E)}\right]_{\mathbb{H}}$ can be calculated with the **replica method** and then get the moments $\left[I^k(-E,E)\right]_{\mathbb{H}}^c$

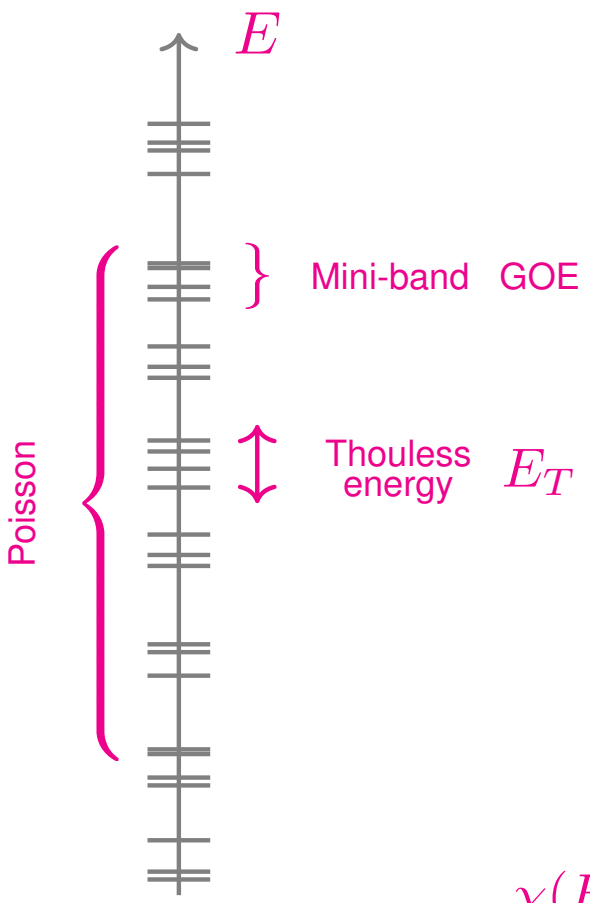
and the **level compressibility** $\boxed{\chi(E) = \dfrac{\left[I^2(-E,E)\right]_{\mathbb{H}}^c}{\left[I(-E,E)\right]_{\mathbb{H}}}}$

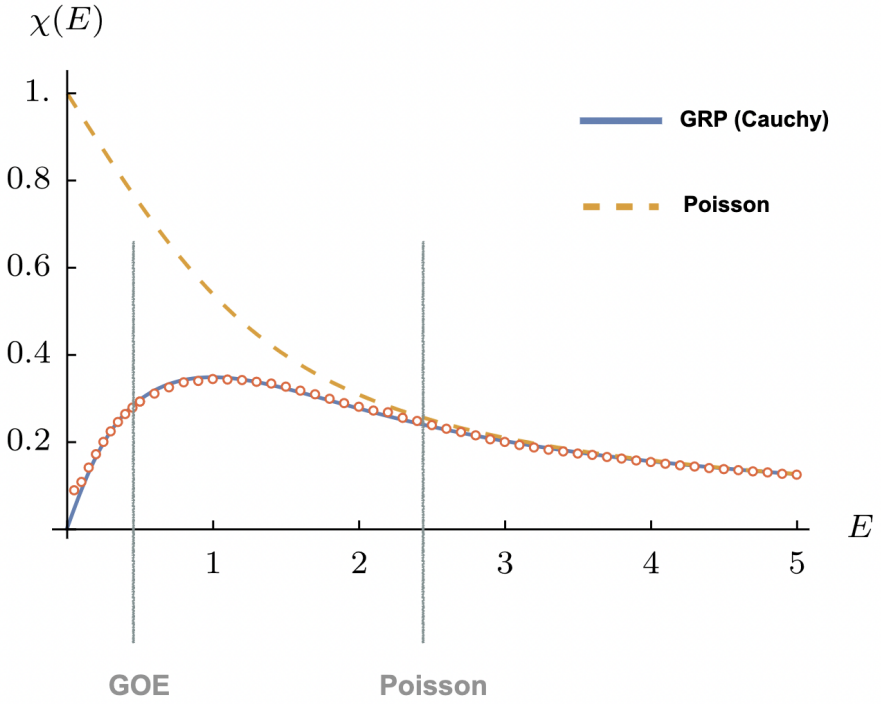
Metz & Pérez Castillo, *Large Deviation Function for the Number of Eigenvalues of Sparse Random Graphs Inside an Interval*, Phys. Rev. Lett. 117, 104101 (2016)

Metz, *Replica-symmetric approach to the typical eigenvalue fluctuations of Gaussian random matrices*, J. Phys. A 50, 495002 (2017)

Picture recovered

Mini-bands in the intermediate $1 < \gamma < 2$ regime





Within the bands $\sim E_T$ Well beyond the bands

$$\chi(E) = \frac{\left[I^2(-E,E)\right]_{\mathbb{H}}^{c}}{\left[I(-E,E)\right]_{\mathbb{H}}} \mapsto \frac{2}{\pi}\arctan y - \frac{1}{\pi y}\ln(1+y^2)$$

$$y = (2\pi p_a(0))\, E/E_T$$

Results so far & beyond

Rosenzweig Porter models

— We rederived $\rho_N(\lambda)$ using replicas and a rotational symm. *Ansatz* and derived the Zee formula in a simple way (with finite N dependencies) for the RP-GOE

— We derived the level compressibility $\chi(E)$, with spectral correlations at all scales

— In the RP-GOE it satisfies a **scaling form** which is **universal** with respect to p_a

— **Is it also universal with respect to p_b ?**

first, verified in the **RP-Wishart** model (also GUE[1])

later, proven perturbatively (large N) around $E_T \sim N^{1-\gamma}$ for $\mathbb{B}$ such that $P[\mathbb{B}] = P[\mathbb{O}\mathbb{B}\mathbb{O}^T]$ ($\mathbb{B} = \mathbb{O}\mathbb{D}\mathbb{O}^T$) and Wigner matrices (finite moments*)

— Full counting statistics - cumulant generating functional of I_N

— Is it always valid ? Not in the **QREM** (or quantum p-spin with $p \to \infty$) model[2] which shows collapse to different scaling functions.

Is it because QREM has trully multifractal states ?

---

[1] **Kravtsov, Khaymovich, Cuevas & Amini**, New J. Phys. 17 122002 (2015)

[2] **Biroli, Facoetti, Schirò, Tarzia & Vivo**, Phys. Rev. B 103, 01420 (2021)

* **Safonova, Lunkin & Feigel'man**, SciPost Phys. 20 003 (2026) got the same function in the RP-Lévy model

Two kinds of random matrices

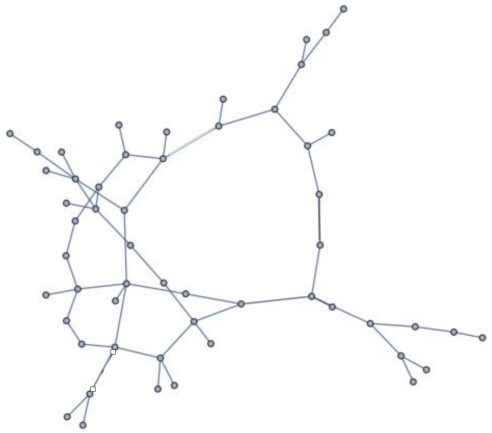
- Rosenzweig-Porter (RP)

  Sum of random matrices

$$\mathbb{H} = \mathbb{A} + \frac{\nu}{N^{\gamma/2}}\,\mathbb{B}$$

- **Weighted Erdös-Rényi graph**

  Fluctuating connectivity & random hops



Motivation

Search of random matrices with **(multi)fractal eigenvectors**

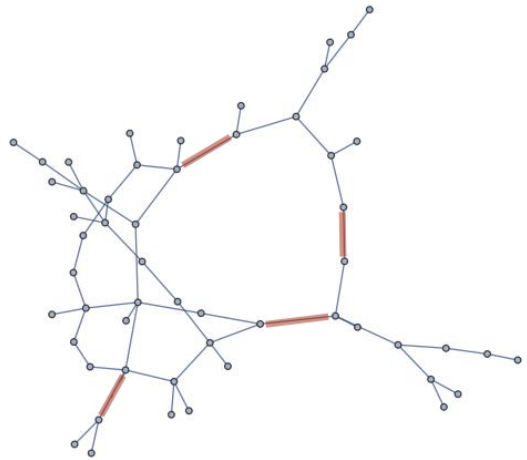
LFC, Schehr, Tarzia, Venturelli, *Multifractal phase in the adjacency matrices of random Erdös-Rényi graphs*, Phys. Rev. B 110, 174202 (2024)

Weighted random graph

Erdös-Rényi

Random graph Hopping

$$H_{ij} = \frac{1}{\sqrt{p}}\,\sigma_{ij} t_{ij} \qquad \sigma_{ij} = \begin{cases} 1 & \text{prob} = p/N \\ 0 & \text{prob} = 1 - p/N \end{cases} \qquad t_{ij} \;\; \text{GOE}$$



A sketch with $p = 3$

$p \geq 1$: a giant component with $\tilde{N} \propto N$ sites and $\mathcal{O}(N)$ finite size clusters.

Focus on the giant component only $\tilde{p} \propto p$

Note that the red links can have $t_{ij} \sim 0$

For the random graph: **Rodgers & Bray**, *Density of states of a sparse random matrix*, Phys. Rev. B 37, 3557 (1988). **Semerjian & LFC**, *Sparse random matrices : the eigenvalue spectrum revisited*, J. Phys. A 35, 4837 (2002). **Kuhn**, *Spectra of sparse random matrices*, J. Phys. A 41, 295002 (2008).

Eigenvectors

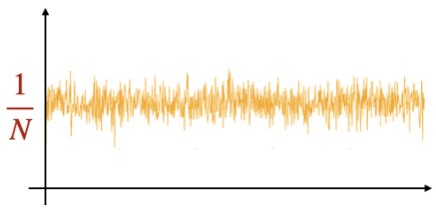
Inverse participation ratios & fractal dimensions

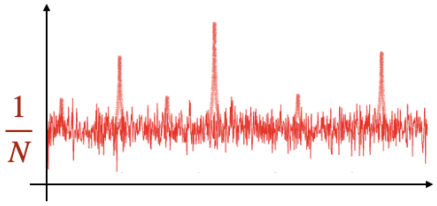
Take the αth eigenvector $\psi_i^{(\alpha)}$ with $i = 1, \ldots, N$ at a given energy E and calculate the disorder average IPR $\Rightarrow$ Fractal dimensions D_q

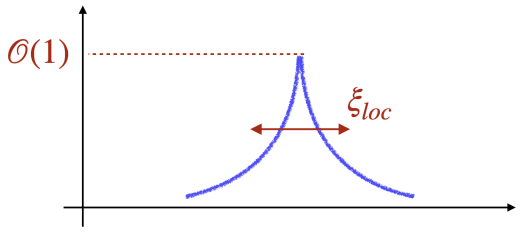
$$I_q = \left[\sum_{i=1}^{N} |\psi_i^{(\alpha)}|^{2q} \right]_{\mathbb{H}} \propto N^{(1-q)D_q} \qquad \text{(IPR)}$$

$D_q = 1 \ \forall\, q$	$D_q \neq 1, 0$	$D_q = 0 \ \forall\, q$
Extended	Special	Localized

$|\psi_i^{(\alpha)}|^2$

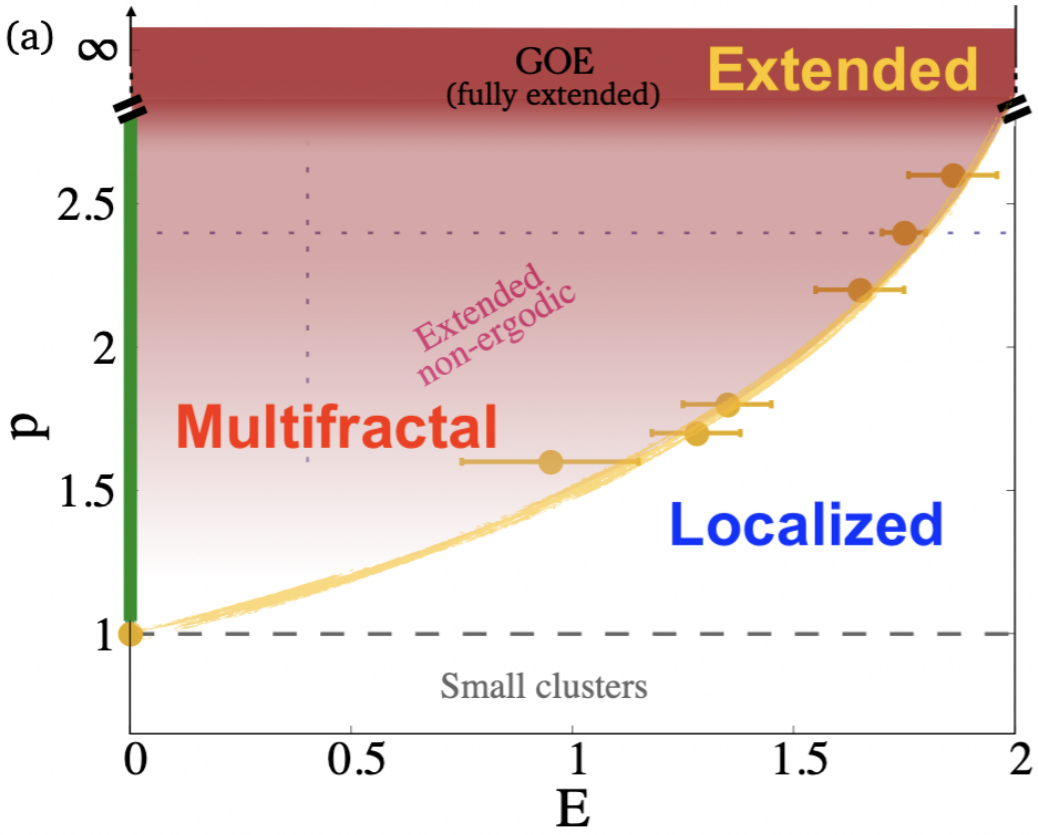






Weighted random graph

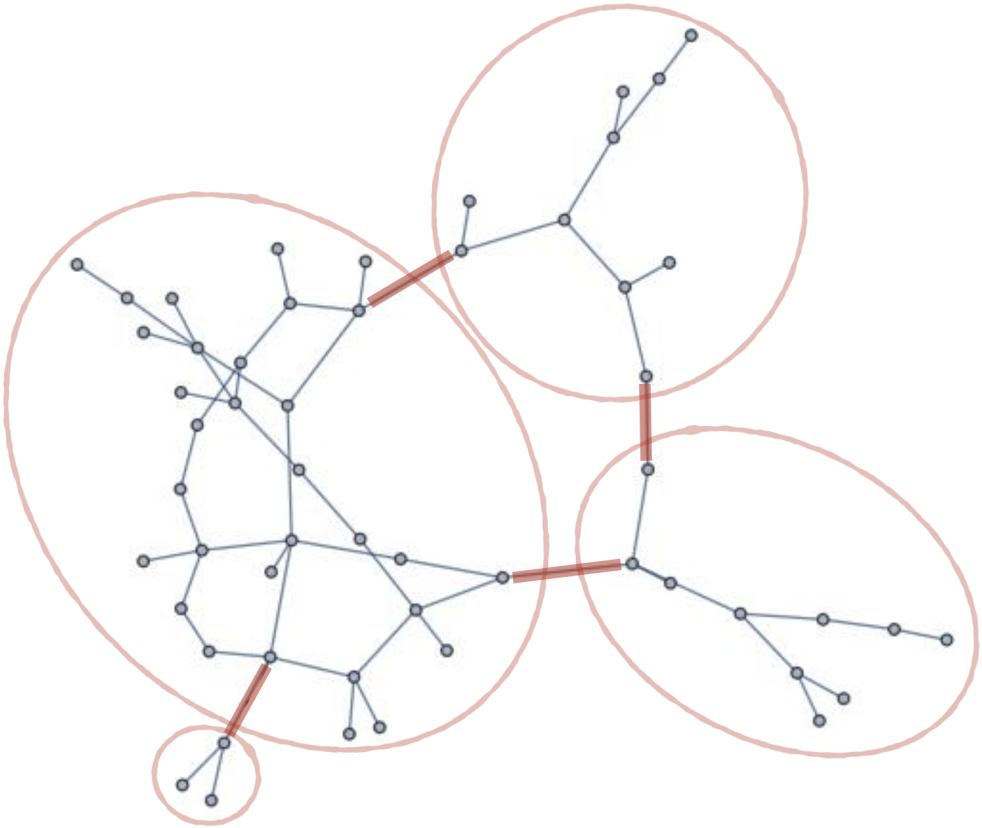
The phase diagram - cavity method



LFC, Schehr, Tarzia & Venturelli, *Multifractal phase in the weighted adjacency matrices of random Erdös-Rényi graphs*, Phys. Rev. B 110, 174202 (2024)

Multifractal phase

Mechanism

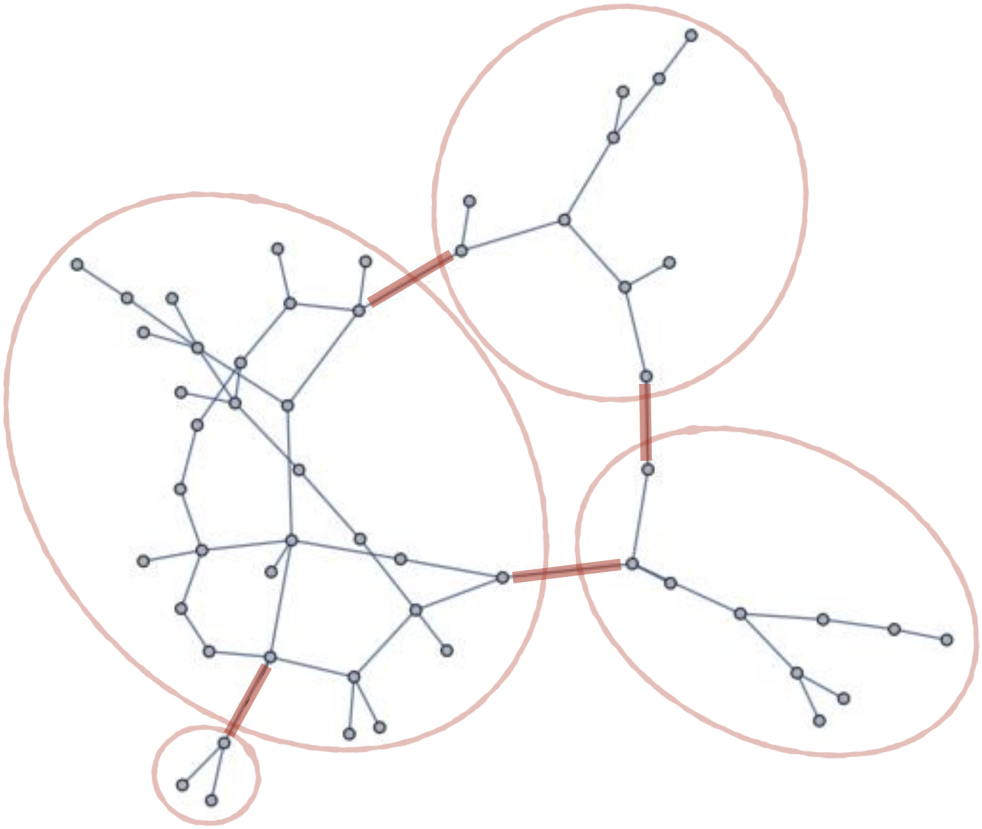


Graph heterogeneity $\Rightarrow$

effective fragmentation due to very weak links

Multifractal phase

Robustness & checks



Re-wiring $\Rightarrow$ eliminate rooted trees

Re-weighting $\Rightarrow$ re-draw $h_{ij} < \nu$ into $h_{ij} > \nu$

Summary

- **General Rosenzweig-Porter**

Sum of random matrices

$$\mathbb{H} = \mathbb{A} + \frac{\nu}{N^{\gamma/2}}\,\mathbb{B}$$

$$1 < \gamma < 2$$

Properties

Level compressibility

$$\tilde{\chi}(E/E_T)$$

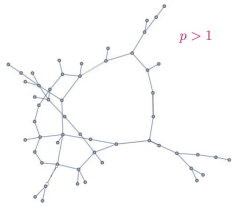
Full counting statistics

$$I_N[-E, E]$$

Universal

- **Weighted Erdös-Rényi graph**

Fluctuating connectivity & random hops



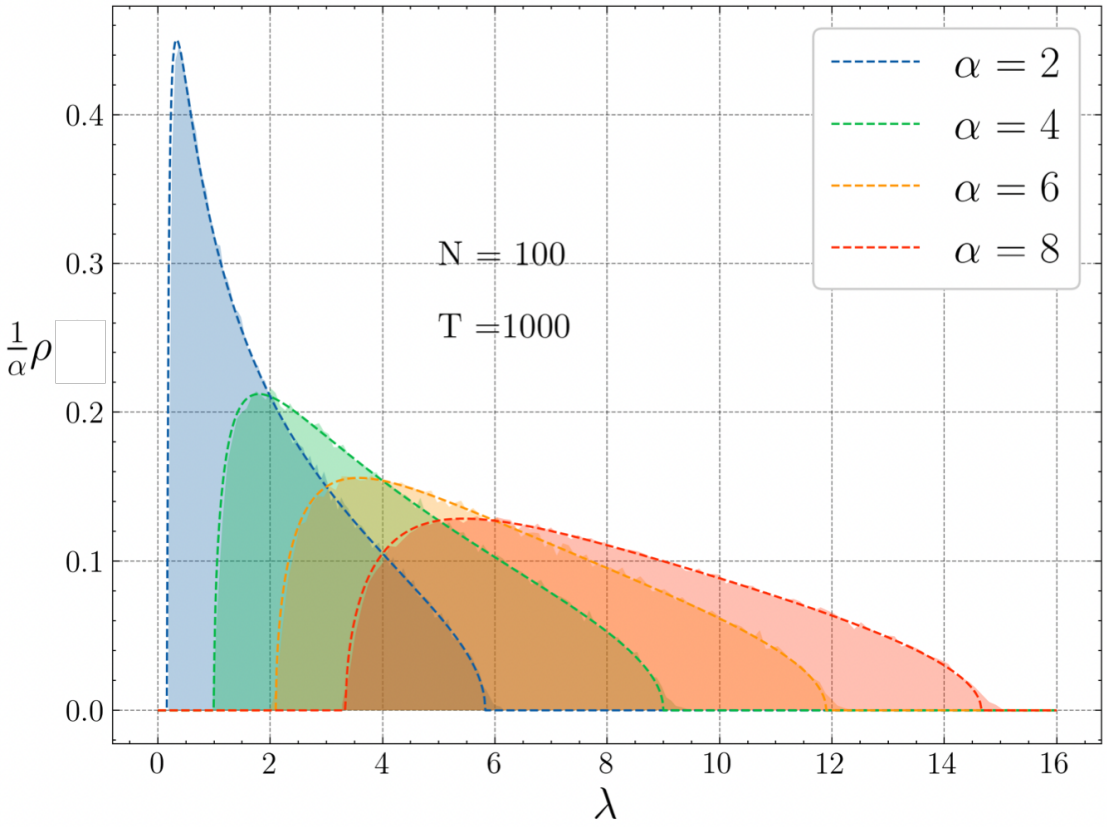
$$\left[|\psi_i^{(\alpha)}|^{2q}\right]_{\mathbb{H}} \propto N^{(1-q)D_q}$$

$$D_q = \begin{cases} \frac{\beta-1}{q-1} & q \geq \beta(p,E) \\ 1 & q < \beta(p,E) \end{cases}$$

Multifractal phase

Wishart matrices

Density of eigenvalues



Numerics vs. analytic expression

Multifractal phase

Return probability & wave function overlap

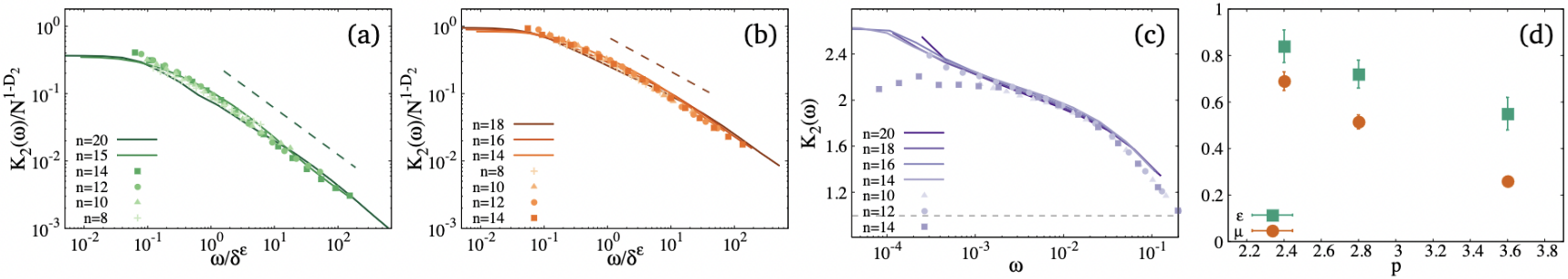


FIG. 10. Overlap correlation function $K_2(\omega; E)$, see Eq. (36), for $E = 0.4$ and $p = 2.4$ (a), $p = 2.8$ (b), and $p = 5$ (c). In panels (a) and (b) the energy separation ω is rescaled by the Thouless energy δ^ϵ — with a p-dependent exponent $\epsilon \in [0, 1]$ — and the y-axis is rescaled by N^{1-D_2} — where $D_2 = \beta - 1$, with β given in Fig. 6(a). Continuous lines correspond to the results obtained by solving the self-consistent cavity equations, while symbols show the exact diagonalization results. The dashed lines in panels (a) and (b) represent the power-law decay of correlations for $\omega \gg \delta^\epsilon$ as $K_2(\omega) \propto (\omega/\delta^\epsilon)^{-\mu}$, with a p-dependent exponent μ. The horizontal gray line in panel (c) corresponds to the Wigner-Dyson behavior for GOE matrices, $K_2(\omega) = 1$. In panel (d) we plot our numerical estimates for the exponents ϵ (squares) and μ (circles) varying the average degree p, showing that a standard fully-delocalized behavior (with $\epsilon, \mu \to 0$) is progressively reached upon increasing p.

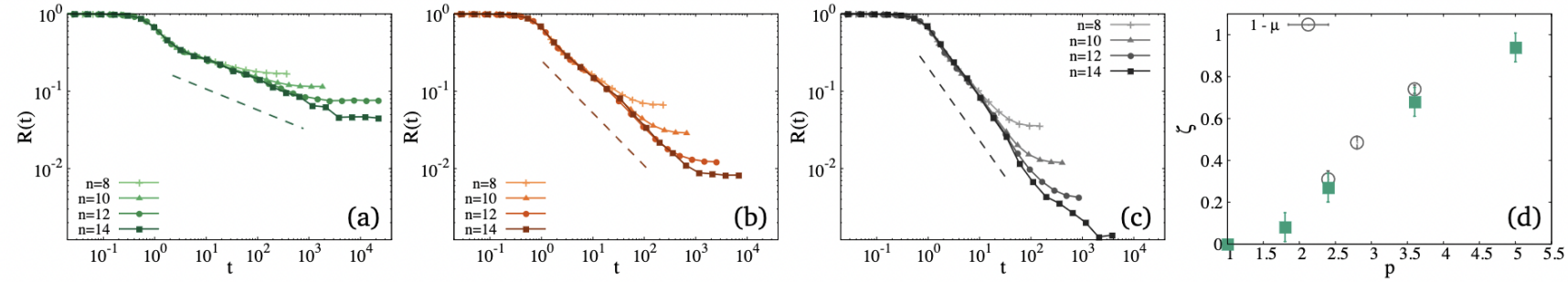
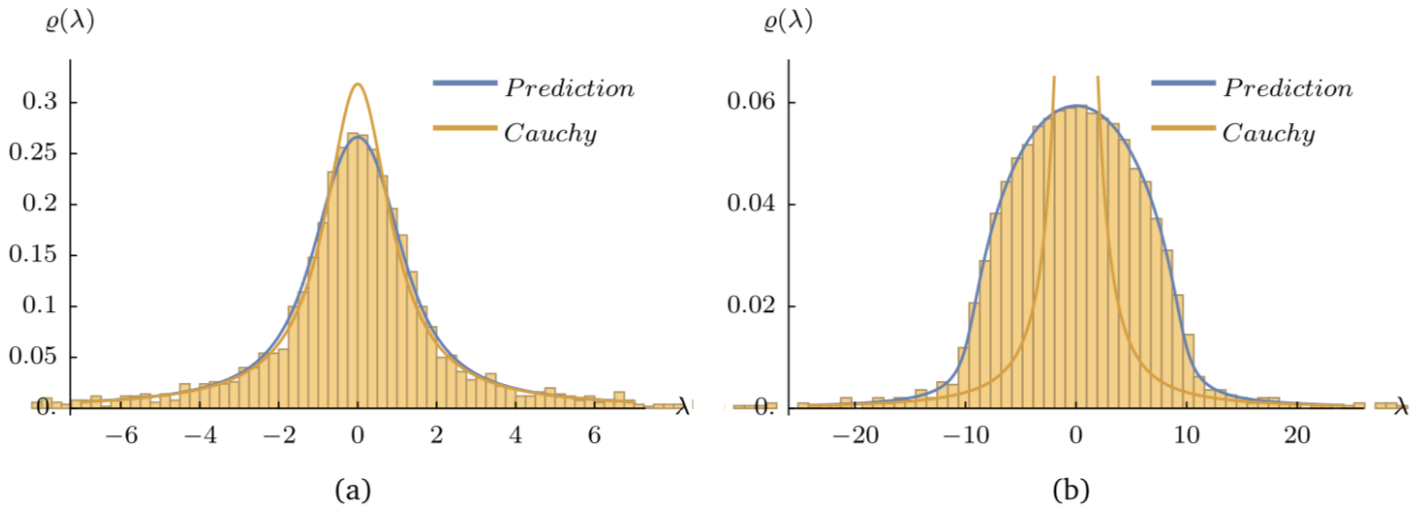


FIG. 12. Return probability, see Eq. (41), as a function of time, for $p = 2.4$ (a), $p = 3.6$ (b), and $p = 5$ (c). The dashed lines show the power-law decay of the return probability as $R(t) \propto t^{-\zeta}$. In panel (d) we plot the exponent ζ upon varying the average degree p, showing that its value is very close to $\zeta \simeq 1 - \mu$ within our numerical accuracy, μ being the exponent that describes the power-law decay of the overlap correlation function $K_2(\omega)$ — see Figs. 10(a–b).

Results

The averaged density of eigenvalues



$\zeta = 0.12$
$\nu = 1$

$\zeta = 11.7$
$\nu = 10$

$$\zeta = \frac{\nu^2}{4N^{\gamma-1}}$$

$N = 2000,\ \gamma = 1.1$

Results

The level compressibility

The # of eigenvalues in the interval $[\alpha, \beta]$ is the random variable

$$I(\alpha, \beta) = \int_{\alpha}^{\beta} d\lambda' \, \rho(\lambda')$$

The level compressibility is defined as

$$\chi(\lambda) = \frac{\kappa_2(\lambda)}{\kappa_1(\lambda)} = \frac{[I^2(-\lambda, \lambda)]_{\mathbb{H}} - [I(-\lambda, \lambda)]^2_{\mathbb{H}}}{[I(-\lambda, \lambda)]_{\mathbb{H}}}$$

Limits

– Pure Poisson, typical level spacing $\mathcal{O}(1)$; $\chi(\lambda) \sim 1$ for small λ and $\chi(\lambda) \sim 0$ for large λ

– Pure GOE ($\mathbb{A} = 0$), $\chi(\lambda \gg N^{-1/2-\gamma/2}) \to 0$ and $\chi(\lambda \ll N^{-1/2-\gamma/2}) \to 1$

Metz & Pérez Castillo, *Large Deviation Function for the Number of Eigenvalues of Sparse Random Graphs Inside an Interval*, Phys. Rev. Lett. 117, 104101 (2016)

Metz, *Replica-symmetric approach to the typical eigenvalue fluctuations of Gaussian random matrices*, J. Phys. A 50, 495002 (2017)

Results

The level compressibility

We calculated the cumulant generating function $\boxed{\mathcal{F}_\lambda(s) = \frac{1}{N}\ln[e^{-sI(-\lambda,\lambda)}]_{\mathbb{H}}}$

In the interesting regime $1 < \gamma < 2$ we found

– $\chi(\lambda) \sim 0$ for $\lambda \ll E_T$ within mini-bands (like not-too-small λ GOE)

– $\chi(\lambda) \sim 1$ for $\lambda \gg E_T$ across mini-bands (like small λ Poisson)

– In the scaling limit $y = \dfrac{\lambda}{2\pi p_a(0)\zeta}$ with $\zeta = \dfrac{\nu^2}{4N^{\gamma-1}}$, a universal form

$$\boxed{\overline{\chi}(y) = \frac{1}{\pi y}[2y\,\mathrm{arctan}(y) - \ln(1+y^2)]}$$

with $\overline{\chi}(y \to 0) = 0$ and $\overline{\chi}(y \to \infty) = 1$

Numerical tests of universality (independence of p_a) are under way

Details

On level compressibility

The # of eigenvalues in the interval $[\alpha, \beta]$ is

$$I(\alpha, \beta) = \sum_{i=1}^{N} [\theta(\beta - \lambda_i) - \theta(\alpha - \lambda_i)]$$

The Heaviside function can be represented as

$$\theta(-x) = \frac{1}{2\pi \mathrm{i}} \lim_{\eta \to 0^+} [\ln(x + \mathrm{i}\eta) - \ln(x - \mathrm{i}\eta)]$$

Then

$$\sum_{i=1}^{N} \theta(\alpha - \lambda_i) = \frac{1}{2\pi \mathrm{i}} \lim_{\eta \to 0^+} \{\ln \det[\mathbb{H} - (\alpha - \mathrm{i}\eta)\mathbb{I}] - \ln \det[\mathbb{H} - (\alpha + \mathrm{i}\eta)\mathbb{I}]\}$$

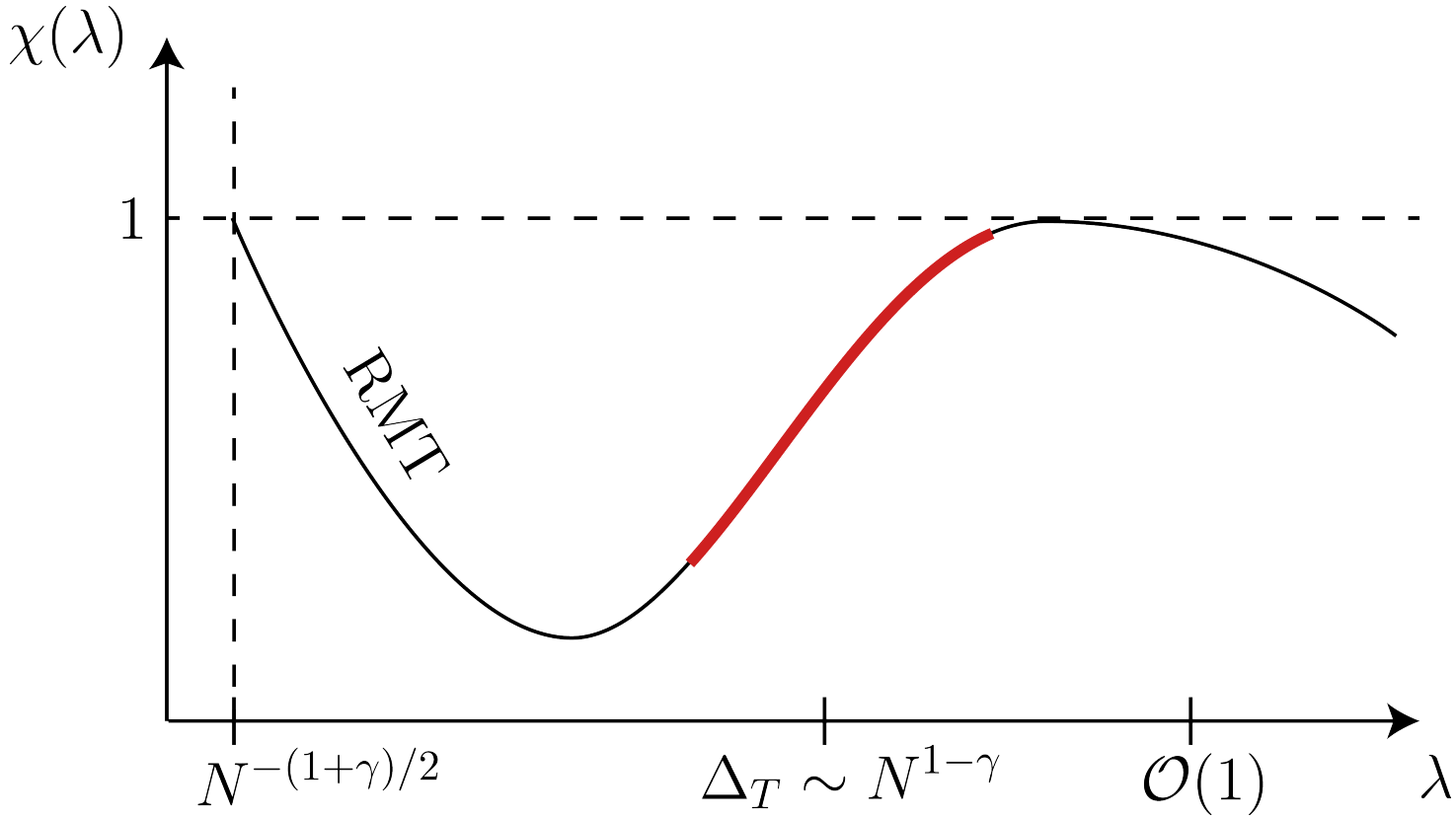
and

$$I(-\alpha, \beta) = -\frac{1}{\pi \mathrm{i}} \lim_{\eta \to 0^+} \ln \frac{\mathcal{Z}(\beta - \mathrm{i}\eta)\, \mathcal{Z}(\alpha + \mathrm{i}\eta)}{\mathcal{Z}(\beta + \mathrm{i}\eta) \mathcal{Z}(\alpha - \mathrm{i}\eta)}$$

+ replica trick

Level compressibility

Sketch of the various scales



$\tilde{\chi}(y)$

Methods

The trace of the **resolvent matrix** $\mathbb{G}(z) = (z\mathbb{I} - \mathbb{H})^{-1}$ is the **global resolvent**

$$G(z) \equiv \tfrac{1}{N} \operatorname{Tr} \mathbb{G}(z) = \tfrac{1}{N} \sum_{i=1}^{N} (z - \lambda_i)^{-1} \xrightarrow[N\to\infty]{} \int d\lambda' \, \frac{\rho(\lambda')}{z - \lambda'}$$

Inverting

$$\rho(\lambda) = \frac{1}{\pi} \lim_{\eta \to 0^+} \operatorname{Im} \lim_{N\to\infty} G(\lambda - \mathrm{i}\eta)$$

$$= \frac{1}{\pi} \lim_{\eta \to 0^+} \operatorname{Im} \lim_{N\to\infty} \tfrac{1}{N} \frac{\partial}{\partial \lambda} \sum_{i=1}^{N} \ln(\lambda - \mathrm{i}\eta - \lambda_i)$$

With the **Edwards-Jones** Gaussian representation

$$[\rho(\lambda)]_{\mathbb{H}} = -\frac{1}{\pi} \lim_{\eta \to 0^+} \operatorname{Im} \frac{\partial}{\partial \lambda} [\ln \mathcal{Z}(\lambda - \mathrm{i}\eta)]_{\mathbb{H}}$$

$$\mathcal{Z}(z) = \frac{1}{(2\pi \mathrm{i})^{N/2}} \int_{\mathbb{R}^N} \mathrm{d}^N r \; e^{-\frac{1}{2} \mathbf{r}^{\mathrm{T}} (z\mathbb{I} - \mathbb{H}) \, \mathbf{r}}$$

Edwards & Jones, *The eigenvalue spectrum of a large symmetric random matrix*, J. Phys. A 9, 1595 (1976)

Methods

The replica trick

$$[\ln \mathcal{Z}]_{\mathbb{H}} = \lim_{n\to 0} \frac{[\mathcal{Z}^n]_{\mathbb{H}} - 1}{n} = \lim_{n\to 0} \frac{1}{n} \ln [\mathcal{Z}^n]_{\mathbb{H}}$$

For $N \to \infty$ the calculation reduces to the saddle-point evaluation of $[\mathcal{Z}^n]_{\mathbb{H}}$

– it can be done with a replica symmetric Ansatz on $NQ_{ab} = \langle \mathbf{r}^a \cdot \mathbf{r}^b \rangle$ as usual

– with a **rotationally invariant Ansatz** in replica space for the density

$$\mu(\vec{r}) = \mu(r^1, \ldots, r^a) = \frac{1}{N} \sum_{i=1}^{N} \prod_{a=1}^{n} \delta(r^a - r_i^a)$$

such that at the saddle point level it only depends on the modulus $\mu(\vec{r}) = \overline{\mu}(r)$

The second path turns out to be more convenient

Livan, Novaes & Vivo, *Introduction to Random Matrices – Theory and Practice*, arXiv : 1712.07903, SpringerBriefs in Mathematical Physics 26 (2018)

Methods & our results

More details on the eigenvalues

The **resolvent**

$$G_N(z) \equiv \frac{1}{N} \operatorname{Tr} (z\mathbb{I} - \mathbb{H})^{-1} = \frac{1}{N} \sum_i \frac{1}{z - \lambda_i} \qquad \text{with } z \in \mathbb{C}$$

N poles at $z = \lambda_i$ the real eigenvalues of $\mathbb{H}$

Average over $\mathbb{H}$ and take the large N limit

$$\lim_{N\to\infty} [G_N(z)]_{\mathbb{H}} = \int d\lambda' \, \frac{\rho(\lambda')}{z - \lambda'}$$

which is the Stieltjes transform of the av. density of eigenvalues ρ

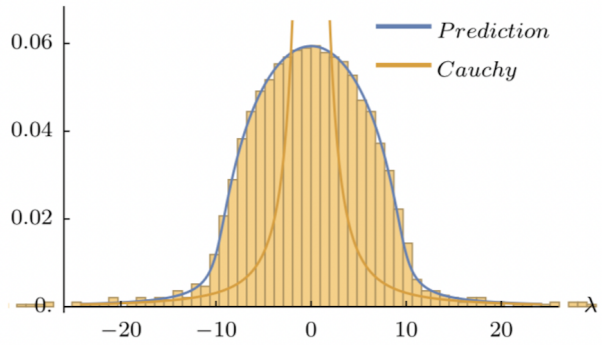
$$\boxed{\rho(\lambda) = \frac{1}{\pi} \lim_{\eta\to 0^+} \operatorname{Im} \lim_{N\to\infty} [G_N(\lambda - \mathrm{i}\eta)]_{\mathbb{H}}}$$

The Zee formula

The averaged density of eigenvalues - derived with replicas

After some lengthy but simple steps

$$\rho_N(\lambda) = -\frac{1}{\pi\zeta} \lim_{\eta\to 0^+} \mathrm{Re}\, C(\lambda - \mathrm{i}\eta)$$
$$C(\lambda) = \mathrm{i}\zeta\, G_a(\lambda + 2\mathrm{i}\, C(\lambda))$$



Solution for Cauchy p_a

$N = 2000, \gamma = 1.1$

$\nu = 10, \zeta = 11.7$

$NG_a(z) = \mathrm{Tr}(\mathbb{A} - z\mathbb{I})^{-1}$ the global resolvent of $\mathbb{A}$

and $\zeta = \frac{\nu^2}{4} N^{1-\gamma}$

Evaluate numerically, leading finite size corrections N captured &

approximate analytic expression for $\rho_N(\lambda)$ in the limit $\zeta \ll 1$ and any p_a

---

Generalization of the **Zee formula** in *Law of addition in random matrix theory*, Nucl. Phys. B 474, 726 (1996)

Krajenbrink, Le Doussal & O'Connell, *Tilted elastic lines with columnar and point disorder, non-Hermitian quantum mechanics, and spiked random matrices: pinning and localization*, Phys. Rev. E 103, 042120 (2021)

Methods & our results

Replica trick

$$\rho_{_N}(\lambda) = \frac{1}{\pi} \lim_{\eta\to 0^+} \mathrm{Im}\left[\frac{1}{N}\sum_{i=1}^{N}\frac{1}{\lambda - \mathrm{i}\eta - \lambda_i}\right]_{\mathbb{H}}$$

$$= -\frac{1}{\pi} \lim_{\eta\to 0^+} \mathrm{Im}\,\frac{\partial}{\partial\lambda}\Big[\ln \mathcal{Z}_N(\lambda - \mathrm{i}\eta)\Big]_{\mathbb{H}}$$

with the partition function $\mathcal{Z}_N(z) = \frac{1}{(2\pi\mathrm{i})^{N/2}} \int_{\mathbb{R}^N} \mathrm{d}^N r\; e^{-\frac{1}{2}\mathbf{r}^{\mathrm{T}}(z\mathbb{I}-\mathbb{H})\,\mathbf{r}}$

Using replicas $[\ln \mathcal{Z}_N]_{\mathbb{H}} = \lim_{n\to 0} \frac{1}{n} \ln [\mathcal{Z}_N^n]_{\mathbb{H}}$ & saddle-point for $N \to \infty$

– usual replica symmetric Ansatz on $NQ_{ab} = \langle \mathbf{r}^a \cdot \mathbf{r}^b \rangle$ (difficult) or

– **rotationally invariant Ansatz** in replica space for the density (simpler !)

$$\mu(\vec{r}) = \mu(r^1, \ldots, r^a) = \frac{1}{N}\sum_{i=1}^{N}\prod_{a=1}^{n}\delta(r^a - r_i^a) = \overline{\mu}(r)$$

Edwards & Jones, *The eigenvalue spectrum of a large symmetric random matrix*, J. Phys. A 9, 1595 (1976)
Livan, Novaes & Vivo, *Introduction to Random Matrices – Theory and Practice*, arXiv:1712.07903, SpringerBriefs in Mathematical Physics 26 (2018)

Method & our results

Cavity Method

The trace of the **resolvent matrix** $\mathbb{G}_N(z) = (\mathbb{H} - z\mathbb{I}_N)^{-1}$

$$G_N(z) \equiv \mathrm{Tr}\, \mathbb{G}_N(z) = \sum_{i=1}^{N} (\lambda_i - z)^{-1} \qquad \Rightarrow \qquad \rho_N(\lambda)$$

The **cavity Green's function** is the diagonal element on node i of the resolvent of the Hamiltonian $\mathbb{H}^{(j)}$ with its neighbor j removed

$$G_{i\to j}(z) = (\mathbb{H}^{(j)} - z\mathbb{I}_{N-1})^{-1}_{ii}$$

It satisfies the recursion relation

$$G_{i\to j}(z) = \left(H_{ii} - z - \sum_{m\in \partial i\setminus j} H_{mi}^2\, G_{m\to i}(z) \right)^{-1}$$

and the solution is used as an estimate of the diagonal elements $G_{ii}(z)$

which yield the IPR

$$I_q(E) \propto \lim_{\eta\to 0^+} \eta^{q-1} \frac{1}{N} \sum_i |G_{ii}(E - \mathrm{i}\eta)|^q$$

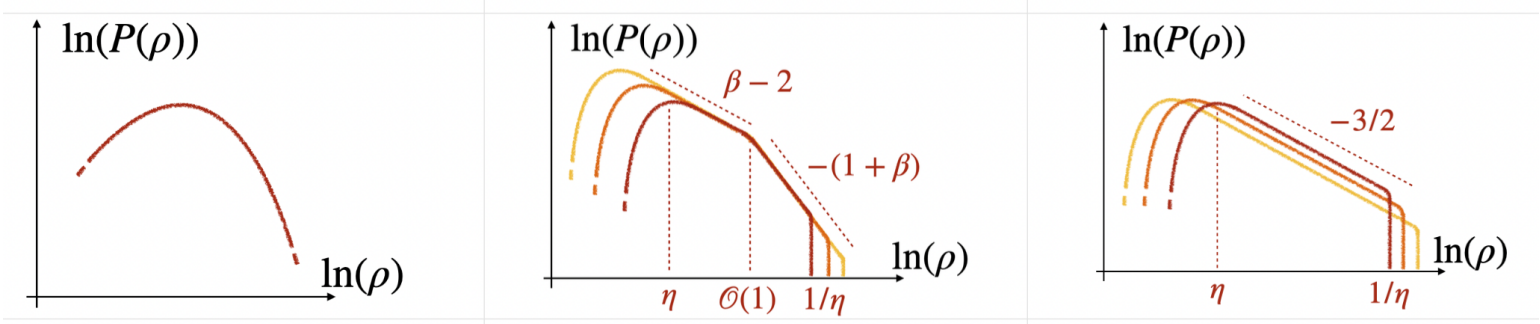
Local density of states

Definition & properties

$$\rho_i(E) \equiv \sum_\alpha |\psi_i^{(\alpha)}|^2\, \delta(E-\lambda_\alpha) = \frac{1}{\pi} \lim_{\eta\to 0^+} \mathrm{Im} G_{ii}(E-\mathrm{i}\eta)$$

$$P(\rho, E) = \left[\frac{1}{N} \sum_{i=1}^{N} \delta\Big(\rho - \pi^{-1} \lim_{\eta\to 0^+} \mathrm{Im} G_{ii}(E-\mathrm{i}\eta)\Big) \right]_{\mathbb{H}}$$

Extended Special Localized



Symmetry $P(\rho) = \rho^{-3} P(1/\rho)$ respected

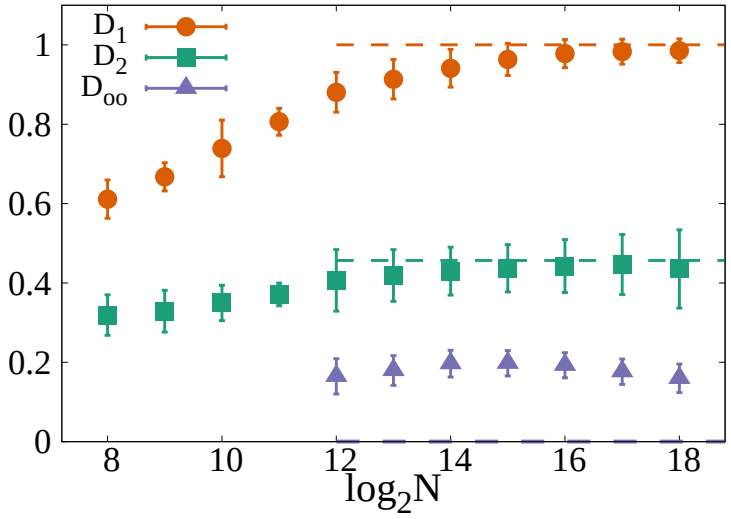
Mirlin, Fyodorov, Mildenberger & Evers, *Exact Relations between Multifractal Exponents at the Anderson Transition*, Phys. Rev. Lett. 97, 046803 (2006)

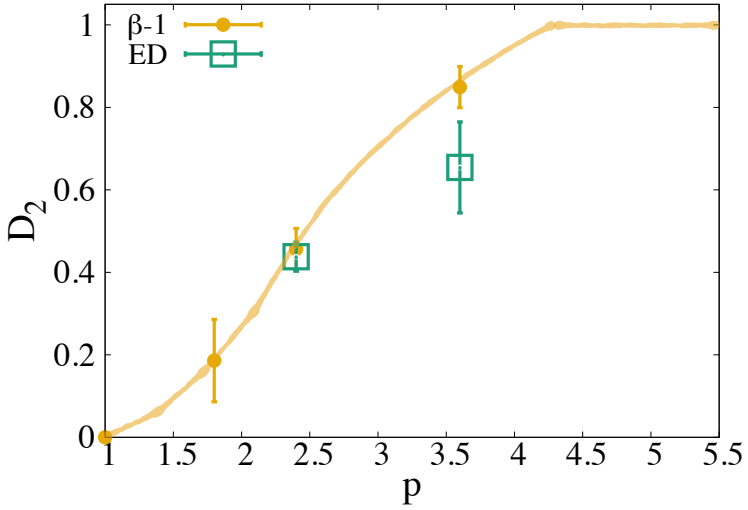
Fractal dimension

Cavity method vs. exact diagonalization

Cavity method vs. exact diagonalization

Horizontal dashed line vs. data points





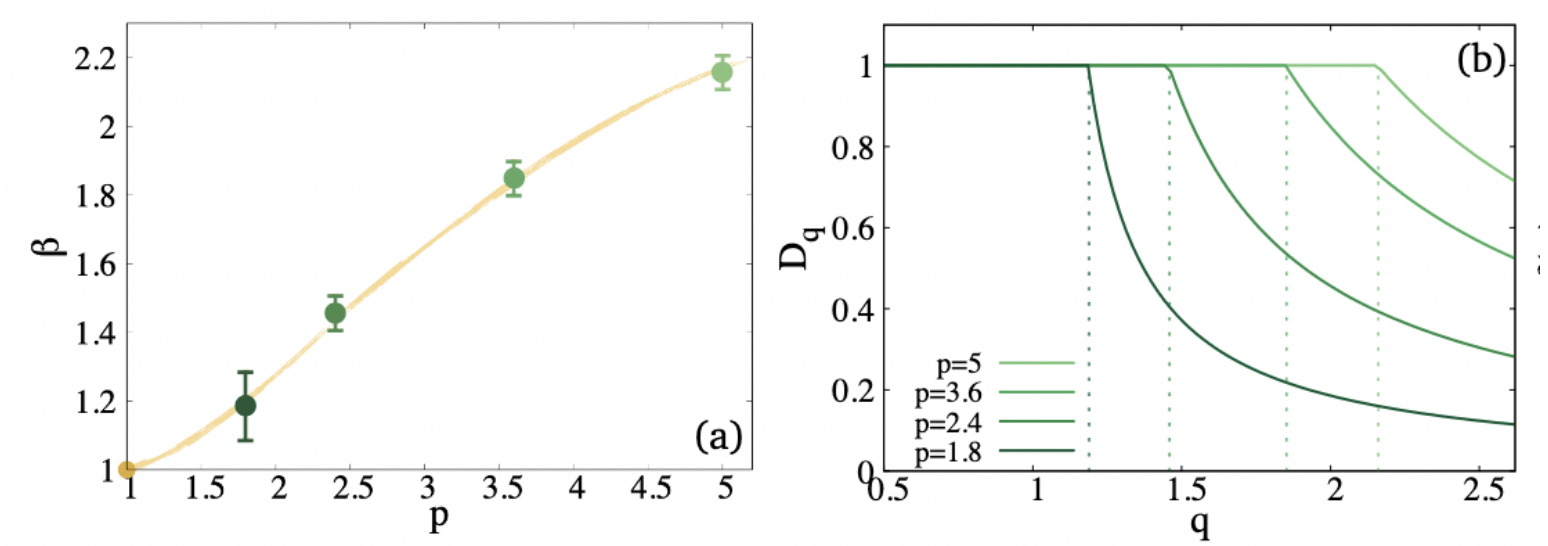
$p = 2.4, \quad E = 0.4, \quad \beta \sim 1.5$

Fractal dimensions

Special regime (p, E) = occupation, energy level in the spectrum

From the analysis of the tail obtained with the cavity method

$$P(\rho) \sim \rho^{-(1+\beta)} \quad \Rightarrow \quad D_q = \begin{cases} \dfrac{\beta - 1}{q - 1} & q \geq \beta \\ 1 & q < \beta \end{cases}$$



As p increases the graph becomes more and more connected, β increases, and the GOE fully extended behavior is approached with $D_q \to 1$ until larger and larger q

Wishart matrices

Universality of χ with respect to $\mathbb{B}$?

Take a rectangular $N \times M$ random matrix $\mathbb{D}$ with $p_d(d_{ij}) \propto e^{-d_{ij}^2/2}$

(Ginibre ensemble)

Build the square $N \times N$ symmetric random matrix $\boxed{\mathbb{W} = M^{-1}\, \mathbb{D}\mathbb{D}^T}$

Could be an estimate of a correlation matrix

The averaged density of eigenvalues of $\mathbb{W}$ is the **Marcenko-Pastur** law

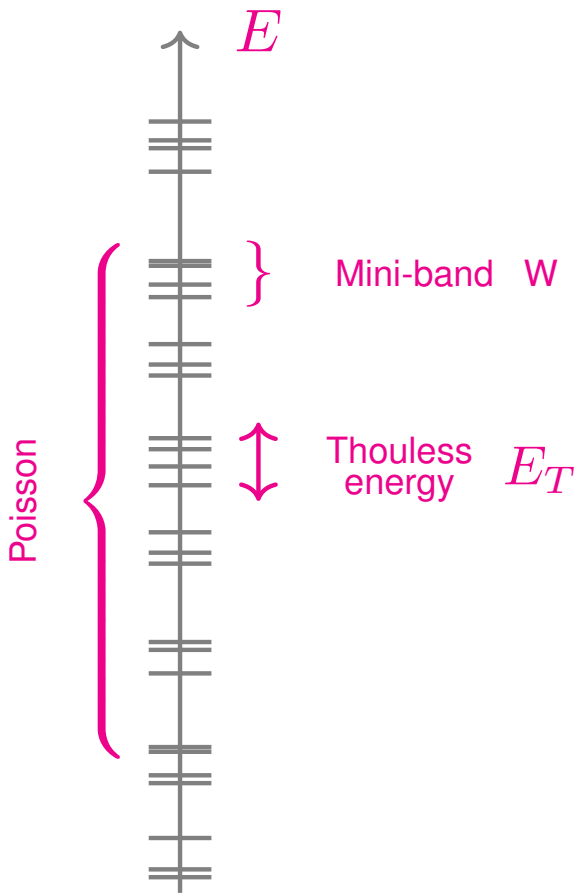
$$\boxed{\rho_{\rm MP}(\lambda) = \alpha \frac{\sqrt{(\lambda - \xi_+)(\xi_- - \lambda)}}{2\pi\lambda} + (1-\alpha)\delta(\lambda)\theta(1-\alpha)}$$

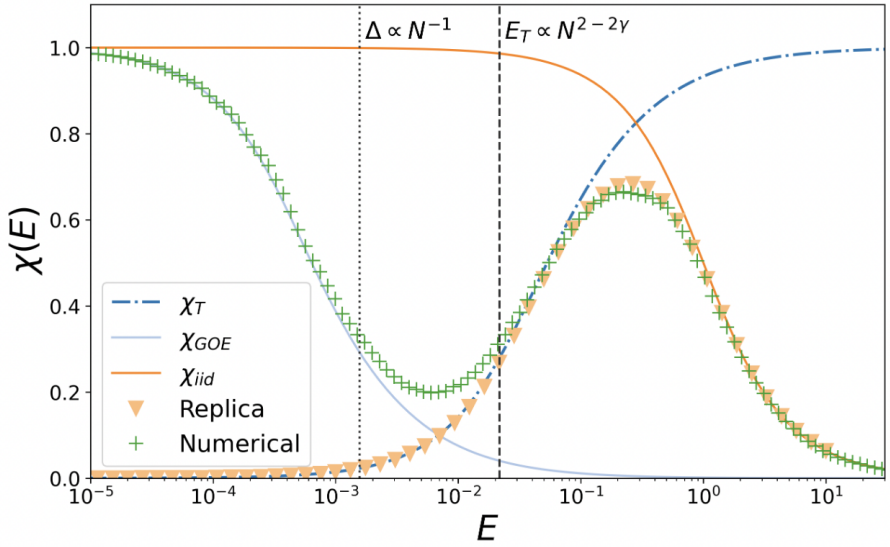
with $\xi_\pm \equiv \left(1 \pm \alpha^{-1/2}\right)^2$ and $\alpha \equiv M/N$

Wishart *The generalised product moment distribution in samples from a normal multivariate population*, Biometrika, 20A, 32 (1928). **Marchenko & Pastur**, *Distribution of eigenvalues for some sets of random matrices*, Matematicheskii Sbornik 114, 507 (1967). **Zavatone-Veth & Cengiz Pehlevan**, *Replica method for eigenvalues of real Wishart product matrices*, arXIv:2209.10499

Picture recovered

Mini-bands in the intermediate regime





$E \sim E_T$ Scaling limit $y = \frac{1}{2\pi p_a(0)} \frac{E}{E_T}$

$$\tilde{\chi}(y) = \frac{1}{\pi y}\left[2y \arctan y - \ln(1+y^2)\right]$$

RP Wishart: same scaling function as for RP-GOE