A classical integrable interacting model with a GGE description

Leticia F. Cugliandolo

Sorbonne Université

Laboratoire de Physique Théorique et Hautes Energies

leticia@lpthe.jussieu.fr

www.lpthe.jussieu.fr/˜leticia/seminars

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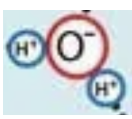
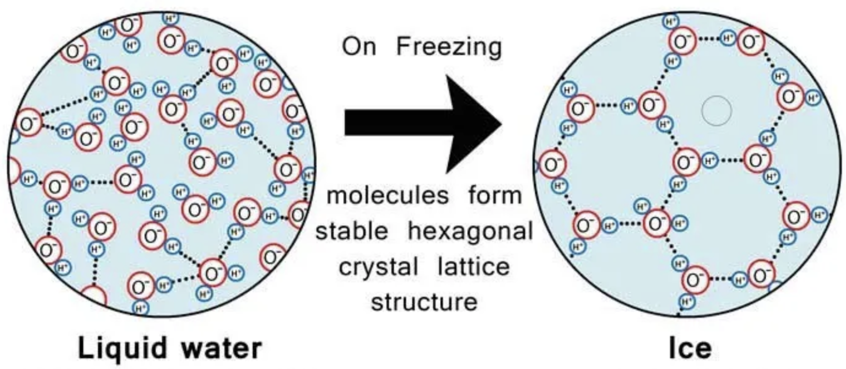
Statistical Physics

From microscopic to macroscopic

Proposes simple models and mathematical methods to go from

microscopic $\mapsto$ macroscopic





Probability theory and Statistics are central $1 \mapsto N \gg 1$

Statistical physics

Advantage

No need to solve the dynamic equations!

Under the *ergodic hypothesis*, after some *equilibration time* $t_{\rm eq}$, *macroscopic observables* can be, on average, obtained with a *static* calculation, as an average over all configurations in phase space weighted with a probability distribution function $P(\{\vec{p}_i, \vec{s}_i\})$

$$\langle A \rangle = \int \prod_i d\vec{p}_i d\vec{s}_i \; P(\{\vec{p}_i, \vec{s}_i\}) \; A(\{\vec{p}_i, \vec{s}_i\})$$

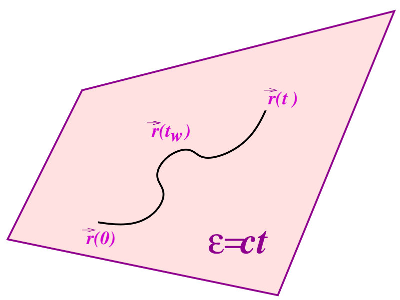
$\langle A \rangle$ should coincide with $\overline{A} \equiv \lim_{\tau\to\infty} \frac{1}{\tau} \int_{t_{\rm eq}}^{t_{\rm eq}+\tau} dt' \, A(\{\vec{p}_i(t'), \vec{s}_i(t')\})$

the *time average* typically measured experimentally

Boltzmann, late XIX

Statistical physics

Ensembles: recipes for $P(\{\vec{p}_i, \vec{s}_i\})$ according to circumstances



Isolated system w/energy conserved

$$\mathcal{E} = H(\{\vec{p}_i, \vec{s}_i\}) = ct$$

Microcanonical distribution

$$P(\{\vec{p}_i, \vec{s}_i\}) \propto \delta(H(\{\vec{p}_i, \vec{s}_i\}) - \mathcal{E})$$

Flat probability density

$$S_{\mathcal{E}} = k_B \ln g(\mathcal{E}) \qquad \beta \equiv \frac{1}{k_B T} = \left.\frac{\partial S_{\mathcal{E}}}{\partial \mathcal{E}}\right|_{\mathcal{E}}$$

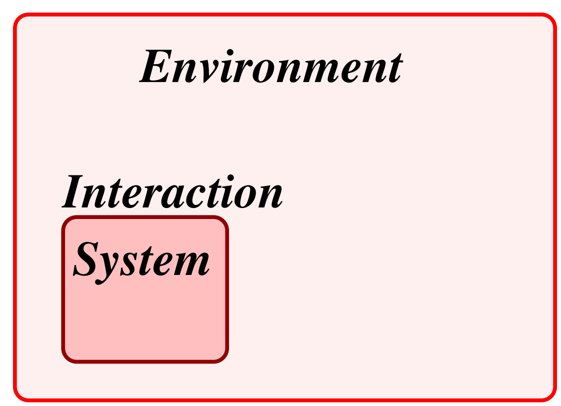
Entropy — Temperature

$$\mathcal{E} = \mathcal{E}_{syst} + \mathcal{E}_{env} + \mathcal{E}_{int}$$

Neglect $\mathcal{E}_{int}$ (short-range interact.)

$$\mathcal{E}_{syst} \ll \mathcal{E}_{env} \qquad \beta = \frac{\partial S_{\mathcal{E}_{env}}}{\partial \mathcal{E}_{env}}$$

$$P(\{\vec{p}_i, \vec{s}_i\}) \propto e^{-\beta H(\{\vec{p}_i, \vec{s}_i\})}$$



Canonical ensemble

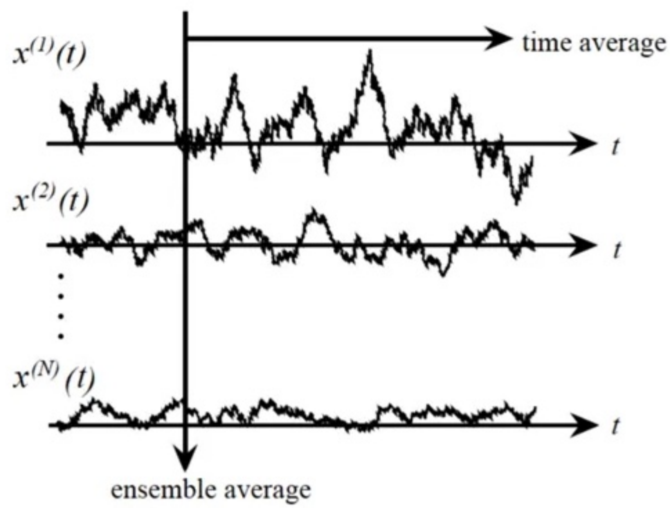
Ergodicity

Dynamical = Statistical averages in the $N \to \infty$ limit

Time averages $\overline{A} \equiv \lim_{\tau \to \infty} \frac{1}{\tau} \int_{t_{\rm st}}^{t_{\rm st}+\tau} dt' \; A(\vec{p}_i(t'), \vec{s}_i(t'))$

& statistical averages $\langle A \rangle \equiv \int \prod_i d\vec{s}_i \prod_i d\vec{s}_i \; A(\vec{p}_i, \vec{s}_i) \; P(\vec{p}_i, \vec{s}_i)$

should be equal $\boxed{\overline{A} = \langle A \rangle}$ for an adequate P



Statistical physics

Classical Closed non-Integrable vs. Integrable Systems

In the usual **closed non-integrable** systems, there are a few constants of motion, e.g. energy H, total linear momentum $\vec{P}$, total angular momentum $\vec{L}$.

In an **closed integrable system**, there are as many constants of motion as degrees of freedom

$$I_i(\{\vec{p}_j, \vec{s}_j\}) \qquad i = 1, \ldots, N$$

fixed by the initial conditions

$$\boxed{I_i(\{\vec{p}_j(0), \vec{s}_j(0)\}) = \mathcal{I}_i}$$

In the **Generalized Microcanonical Ensemble** the measure is

$$\boxed{P(\vec{p}_i, \vec{s}_i) \propto \prod_{i=1}^{N} \delta(I_i(\{\vec{p}_j, \vec{s}_j\}) - \mathcal{I}_i)}$$

Statistical physics

Classical Open non-Integrable vs. Integrable Systems

Easier to work in canonical - open - conditions

Focus on a **non-integrable** system with just energy H conserved

Equivalence of ensembles implies that (non-conserved) observables can also be calculated with Boltzmann's canonical measure

$$P_{\rm GB}(\{\vec{p}_i, \vec{s}_i\}) = \mathcal{Z}_{\rm GB}^{-1} \, e^{-\beta H}$$

and β be such that $\mathcal{E} = \langle H \rangle_{\rm GB}$

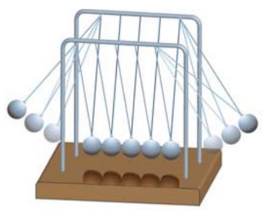
In **open integrable systems**, a Generalized Canonical Distribution is proposed

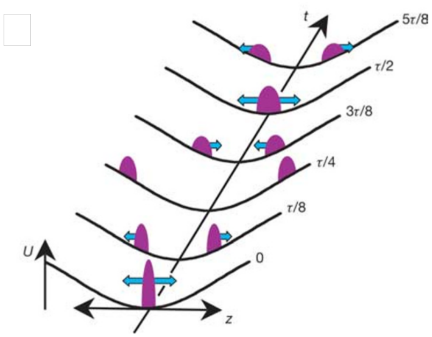
$$P(\{\vec{p}_i, \vec{s}_i\}) = \mathcal{Z}_{\rm GGE} \, e^{-\sum_i \gamma_i I_i(\{\vec{p}_j, \vec{s}_j\})}$$

and $\{\gamma_i\}$ be such that $\mathcal{I}_i = \langle I_i \rangle_{\rm GGE}$

Origin

Isolated quantum systems: experiments and theory $\sim$ 15y ago





A quantum Newton's cradle

experiment

cold atoms in isolation

Kinoshita, Wenger & Weiss 06

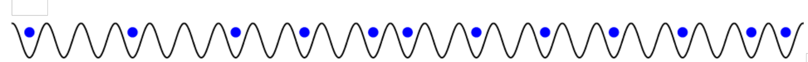
(Conformal) field **theory** methods for

quantum quenches

Calabrese & Cardy 06

Numerical study of

lattice hard core bosons



Rigol, Dunjko, Yurovsky & Olshanii 07

Mostly 1d systems

And many others

Questions

Does an isolated quantum system reach some kind of equilibrium ?

Boosted by recent interest in

— the dynamics after quantum quenches of cold atomic systems

rôle of interactions (integrable *vs.* non-integrable)

— many-body localisation

novel effects of quenched disorder

And, an isolated classical system ?

The (old) ergodicity question revisited

Our contribution **Barbier, LFC, Lozano, Nessi, Picco & Tartaglia 17-22**

More recent **+ Stariolo**

Classical quenches

Definition & questions

- Take an **isolated** classical system with Hamiltonian H_0, evolve with H
- Initialize it in, say, ψ_0 a configuration, *e.g.* $\{\vec{q}_i, \vec{p}_i\}_0$ for a particle system

  ψ_0 could be drawn from a probability distribution, *e.g.* $\mathcal{Z}^{-1}\, e^{-\beta_0 H_0(\psi_0)}$

Does an $N \to \infty$ system reach a steady state ?

Is it described by a thermal equilibrium probability $e^{-\beta H}$?

Do at least some local observables behave as thermal ones ?

Does the evolution occur as in Boltzmann equilibrium ?

If not, GGE $e^{-\sum_i \gamma_i I_i}$ for integrable cases ?

Classical quenches

Interest in integrable models: strategy & goals

– Choose a sufficiently simple classical *integrable interacting* model with (not just harmonic oscillators) an interesting *phase diagram* to investigate different *initial conditions* and *quenches* across the *phase transition(s)*

– Solve the *dynamics* after the quenches

– Build a *Generalised Gibbs Ensemble* (GGE)

– Prove that the asymptotic limit of *local observables* is given by the GGE

Classical quenches

Strategy

Choose a sufficiently simple classical *integrable interacting* model

(not just harmonic oscillators)

with an interesting *phase diagram* to investigate different *initial conditions* and *quenches* across the *phase transition(s)*

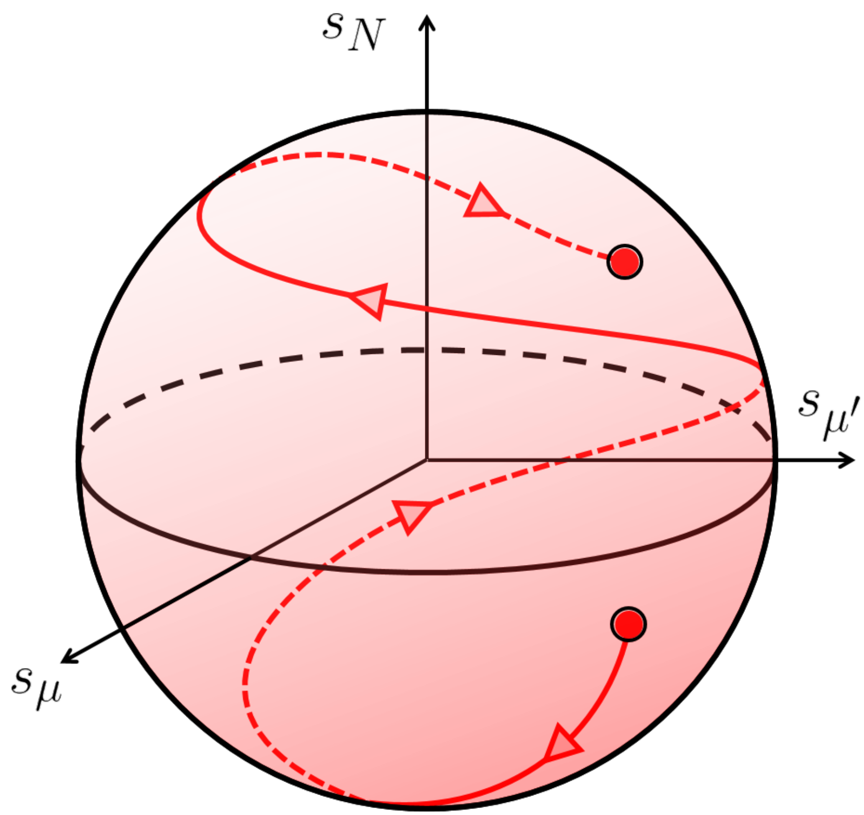
Solve the dynamics after a quench

Build a Generalised Gibbs Ensemble

Prove that the asymptotic limit of local observables is given by the GGE

Classical dynamics

A particle moving on the sphere



Strict constraints

$$\phi : \sum_\mu s_\mu^2 - N = 0$$

$$\phi' : \sum_\mu s_\mu p_\mu = 0$$

$\mu = 1, \ldots, N$ label the coordinates

Neumann's model

1859

Journal

für die

reine und angewandte Mathematik.

In zwanglosen Heften.

Als Fortsetzung des von

A. L. Crelle

gegründeten Journals

herausgegeben

unter Mitwirkung der Herren

Steiner, Schellbach, Kummer, Kronecker, Weierstrass

von

C. W. Borchardt.

Mit thätiger Beförderung hoher Königlich-Preufsischer Behörden

Sechs und funfzigster Band.

In vier Heften.

Berlin, 1859.

Druck und Verlag von Georg Reimer.

A particle on a sphere under an anisotropic harmonic potential

$$-\frac{1}{2}\sum_\mu \lambda_\mu s_\mu^2$$

with spring constants

$$\lambda_\mu \neq \lambda_\nu$$

De problemate quodam mechanico, quod ad primam integralium ultraellipticorum classem revocatur.

(Auctore *C. Neumann*, Hallae.)

§. 1.

Problema proponitur.

Sint puncti mobilis Coordinatae orthogonales x, y, z; sit

$$x^2+y^2+z^2 = 1$$

Journal of Pure & Applied Math.

Crelle Journal

Neumann's model

Integrability

N constants of motion in involution $\{I_\mu, I_\nu\} = 0$ fixed by the initial conditions

$$I_\mu = s_\mu^2 + \frac{1}{mN} \sum_{\nu(\neq\mu)} \frac{(s_\mu p_\nu - s_\nu p_\mu)^2}{\lambda_\nu - \lambda_\mu}$$

K. Uhlenbeck 80s

Modified angular momentum.

Constraints

(using $\sum_\mu s_\mu^2 = N$ & $\sum_\mu s_\mu p_\mu = 0$)

$$H = E_{\text{kin}} + E_{\text{pot}} = -\frac{1}{2} \sum_\mu \lambda_\mu I_\mu$$

$$N = \sum_\mu I_\mu$$

Studies by **Avan, Babelon and Talon 90s** and many others for finite N

Thermodynamic $N \to \infty$ limit ?

Model choice

The simplest non-trivial one

Our inspiration from

– *disordered systems*

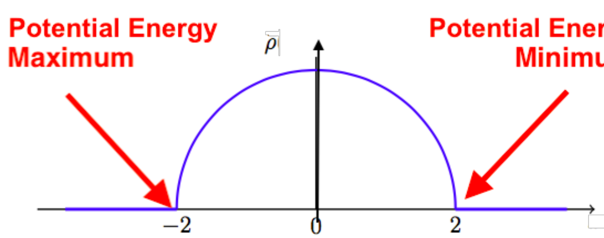
The spherical SK ($p = 2$) model **Kosterlitz, Thouless & Jones 76**

reason for calling s_μ the coordinates

– relaxation dynamics & phase ordering kinetics

Bray 80s-90s, LFC & Dean 95

knowledge Wigner density of spring constants $\rho(\lambda/J)$



The process

Initial conditions and dynamics

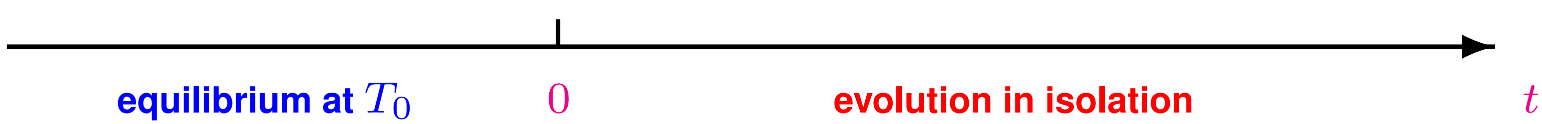
We plan to choose *initial conditions* drawn from the *canonical Gibbs-Boltzmann* equilibrium measure

Physically:

– the spin system or Neumann particle is in thermal equilibrium with a bath at temperature T_0 until $t = 0^-$ (*initial conditions*)

– the coupling to the bath is switched off at this instant $t = 0$

– it further evolves in *isolation* after a *quench* at $t > 0$



The process

Initial conditions and dynamics

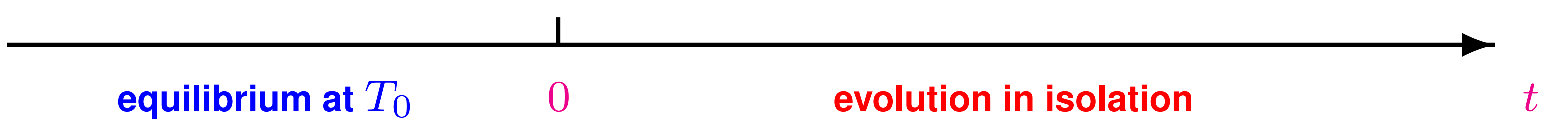
We plan to choose *initial conditions* drawn from the *canonical Gibbs-Boltzmann* equilibrium measure

Why?

– There are two phases for the spin system or Neumann particle in thermal equilibrium, with a continuous phase transition (interesting *initial conditions*)

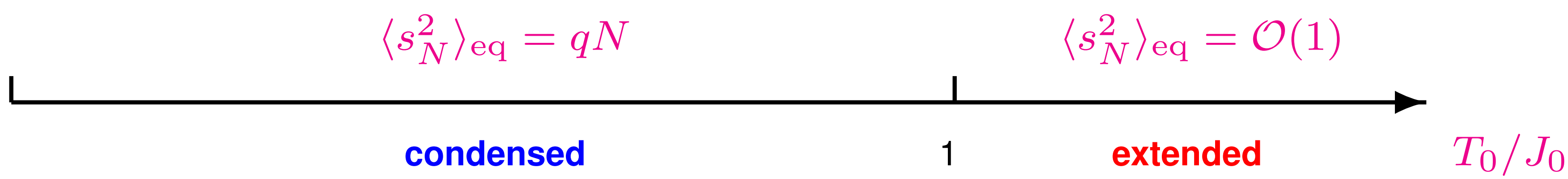
– the coupling to the bath is switched off at this instant $t = 0$

– a *quench* is necessary since a Boltzmann distribution is ‘conserved’ by Newton dynamics

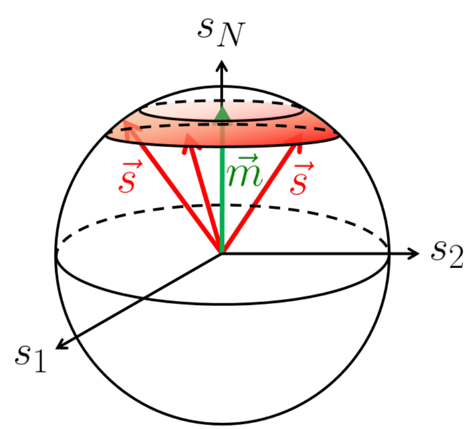


Initial conditions

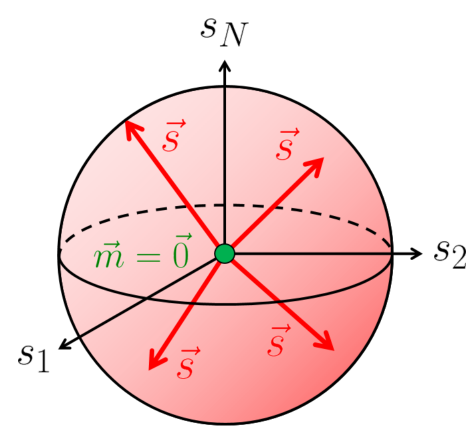
Drawn from canonical equilibrium



$T_0/J_0 < 1$



$T_0/J_0 > 1$



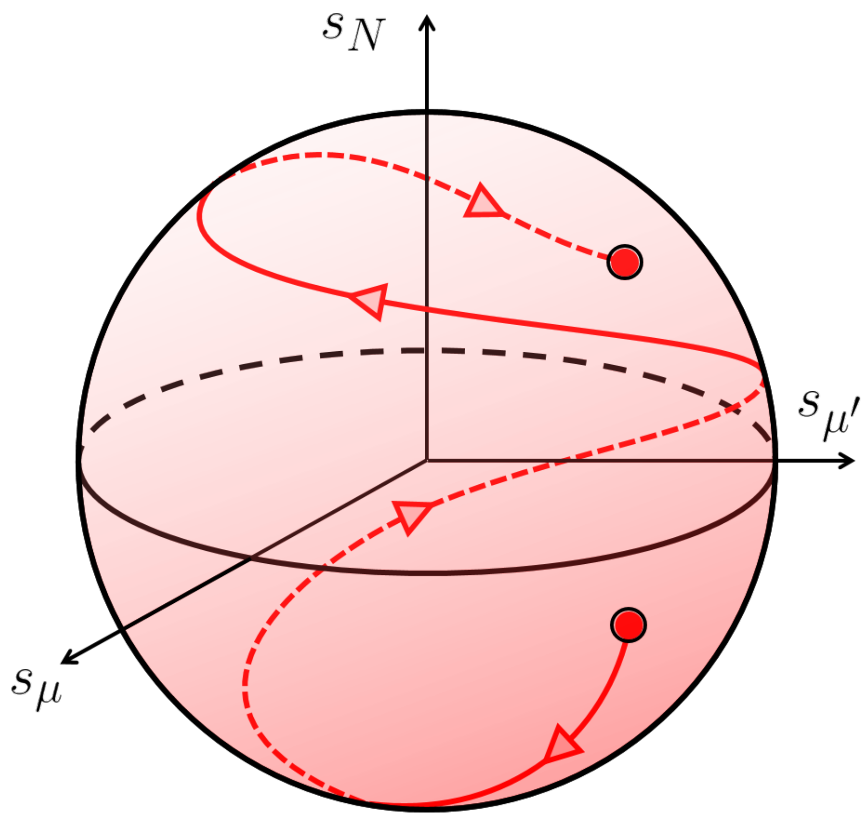
Condensed symmetry broken

Extended

$$\vec{m} \equiv \langle \vec{s} \rangle_{\rm eq} \neq 0$$

Classical dynamics

Interacting spins or a particle moving on the sphere



Initial conditions averaged constraints

$$\langle \phi \rangle_{i.c.} : \sum_\mu \langle s_\mu^2 \rangle_{i.c.} - N = 0$$

Average over initial conditions

$$\langle \phi' \rangle_{i.c.} : \sum_\mu \langle s_\mu p_\mu \rangle_{i.c.} = 0$$

Classical dynamics

Interacting spins or a particle moving on the sphere

Coordinate-momenta pairs $\{\vec{s}, \vec{p}\}$ and Hamiltonian (const w/Lagrange mult.)

$$H_J^{(z)} = E_{\rm kin}(\vec{p}) + V_J^{(z)}(\vec{s})$$

with kinetic energy $E_{\rm kin}(\vec{p}) = \frac{1}{2m} \sum_{\mu=1}^{N} p_\mu^2$ and

$$V_J^{(z)}(\vec{s}) = V_J(\vec{s}) + \frac{z(\vec{s}, \vec{p})}{2} \sum_{\mu=1}^{N} (s_\mu^2 - N)$$

but $V_J^{(z)}(\vec{s})$ is **quartic** due to $z(\vec{s}, \vec{p})$

Newton-Hamilton equations

$$\dot{s}_\mu = p_\mu / m \qquad \dot{p}_\mu = -\frac{\delta V_J(\vec{s})}{\delta s_\mu} - z(\vec{s}, \vec{p}) s_\mu$$

Instantaneous quench

Global rescaling of all coupling constants

At time $t = 0$ to keep some memory of the initial conditions

$$\boxed{\lambda_\mu^{(0)} \mapsto \lambda_\mu = \frac{J}{J_0}\lambda_\mu^{(0)}}$$

in $V_J(\vec{s}\,)$

No change in configuration $\{s_\mu(0^-) = s_\mu(0^+),\ p_\mu(0^-) = p_\mu(0^+)\}$

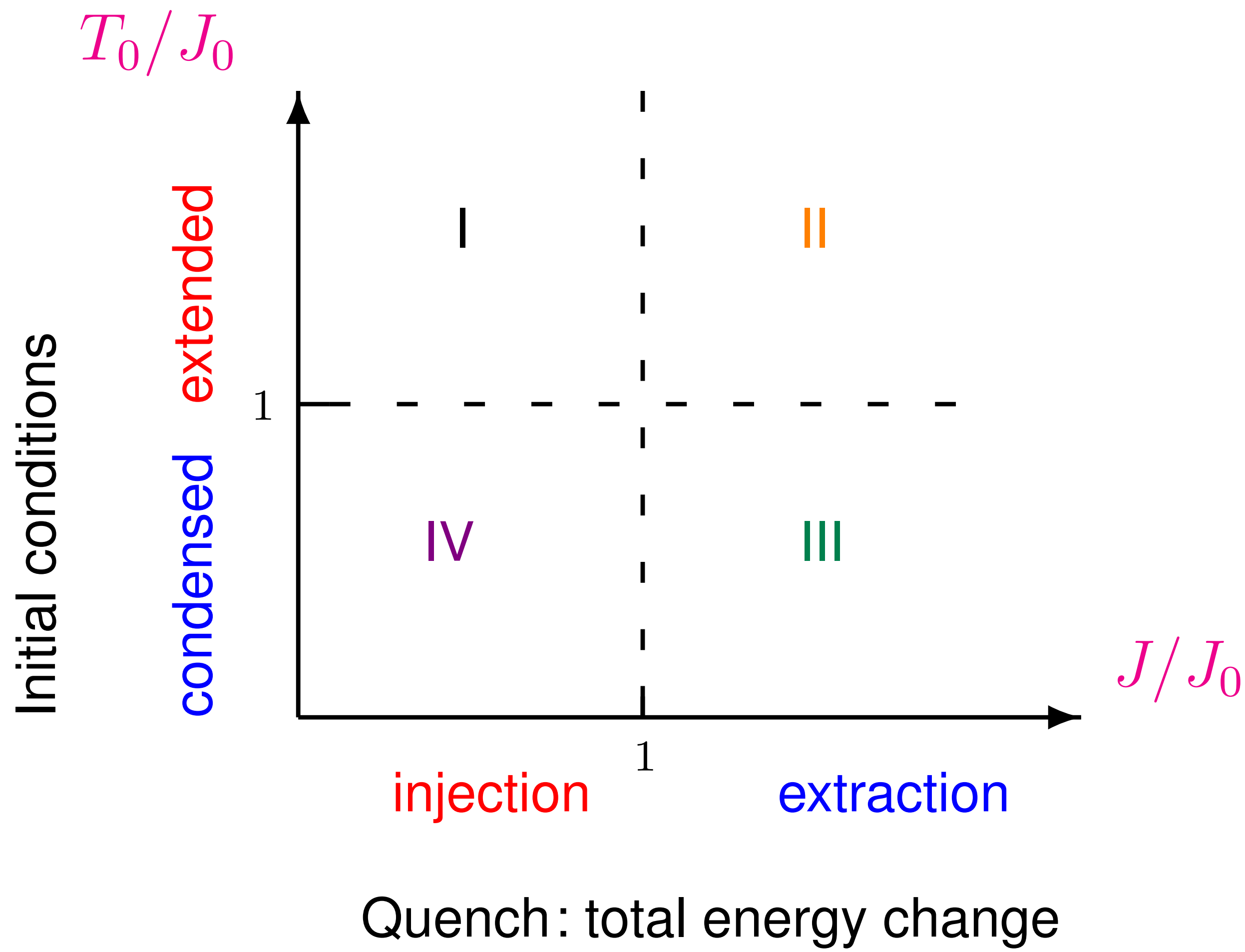
but macroscopic energy change

$$\Delta E = \begin{cases} > 0 \\ < 0 \end{cases} \quad \text{for} \quad \frac{J}{J_0} \begin{cases} < 1 & \textbf{Injection} \\ > 1 & \textbf{Extraction} \end{cases}$$

The edge mode $\mu = N$ is the softer one, with $\delta e_N \propto N$

Control parameters

Total energy change & initial conditions



Classical quenches

Strategy

We have chosen a simple enough classical *integrable interacting* model

(not just harmonic oscillators)

with an interesting *phase diagram* to investigate different *initial conditions* and *quenches* across the *phase transition(s)* Next :

Solve the dynamics after a quench

Build a Generalised Gibbs Ensemble

Prove that the asymptotic limit of local observables is captured by the GGE

Barbier, LFC, Lozano, Nessi, Picco & Tartaglia 18-22

How to study the large N dynamics ?

Firstly, analysis of **global – macroscopic – observables**

Conservative dynamics

on average over randomness & the initial measure

In the $N \to \infty$ limit exact Schwinger-Dyson (DMFT) equations for the global self-correlation and linear response averaged over the $\{\lambda_\mu\}$, denoted $[\dots]_J$, and the initial conditions, noted $\langle \dots \rangle_{i.c.}$,

$$NC(t,t') = \sum_\mu [\langle s_\mu(t) s_\mu(t') \rangle_{i.c.}]_J \qquad \text{Self-correlation}$$

$$NC(t,0) = \sum_\mu [\langle s_\mu(t) s_\mu(0) \rangle_{i.c.}]_J \qquad \text{"Fidelity"}$$

$$NR(t,t') = \sum_\mu [\langle \frac{\delta s_\mu(t)}{\delta h_\mu(t')}\Big|_{\vec{h}=0} \rangle_{i.c.}]_J \qquad \text{Linear response}$$

Coupled causal integro-differential equations

$$(m\partial_t^2 - z_t) R(t,t') = \int dt'' \, \Sigma(t,t'') R(t'',t') + \delta(t-t')$$

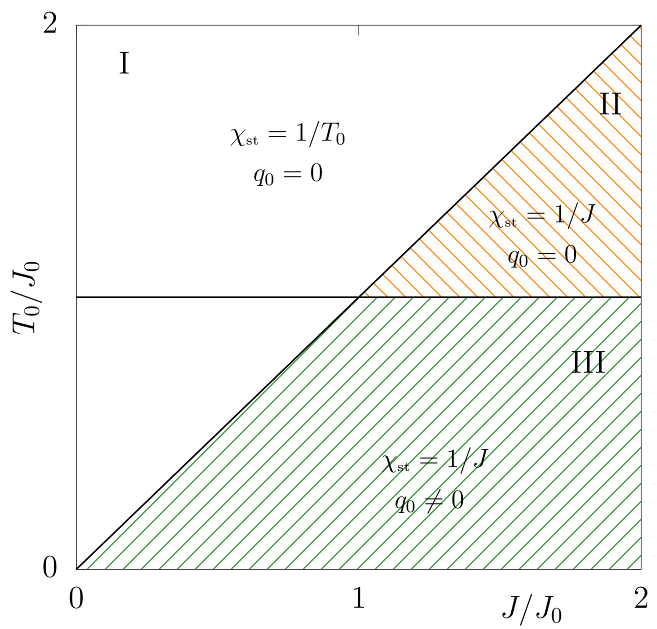
+ three other ones, with terms fixing the initial conditions

Solvable numerically & analytically at long times

Dynamic phase diagram

from Schwinger-Dyson equations

$$\chi_{\rm st} = \lim_{t \gg t_0} \int_0^t dt' \, R(t,t')$$
$$z_f = \lim_{t \gg t_0} z(t)$$



Initial conditions

Injection Extraction

I	$\chi_{\rm st} = 1/T_0$	$z_f > \lambda_N = 2J$	$\lim_{t \gg t_0} C(t,0) = q_0 = 0$
II	$\chi_{\rm st} = 1/J$	$z_f = \lambda_N = 2J$	$\lim_{t \gg t_0} C(t,0) = q_0 = 0$
III	$\chi_{\rm st} = 1/J$	$z_f = \lambda_N = 2J$	$\lim_{t \gg t_0} C(t,0) = q_0 > 0$

Stationary limit

of macroscopic – global – one-time quantities

The Lagrange multiplier approaches (algebraically) a constant,

$$z(t) = 2[e_{\rm kin}(t) - e_{\rm pot}(t)] \to z_f$$

so do the kinetic & potential energies,

$$e_{\rm kin}(t) \to e^f_{\rm kin} \quad \text{and} \quad e_{\rm pot}(t) \to e^f_{\rm pot}$$

The correlation with the initial condition as well

$$C(t,0) \to q_0$$

in all phases (q_0 vanishes in some)

Non-conserved global one-time observables reach constants

Stationary dynamics ? Is this GB equilibrium ? No, FDT not respected

Not surprising since the model is integrable.

Secondly, **dynamic single mode analysis**

to better understand the steady state

Mode dynamics

Non-linear coupling, no average over disorder, any N

The s_μ with $\mu = 1, \dots, N$ obey parametric oscillator equations

$$m\ddot{s}_\mu(t) = -[z(t) - \lambda_\mu] s_\mu(t)$$

with $z(t) = 2[e_{\rm kin}(t) - e_{\rm pot}(t)]$

The solution is

$$s_\mu(t) = s_\mu(0) \sqrt{\frac{\Omega_\mu(0)}{\Omega_\mu(t)}} \; \cos \int_0^t dt' \; \Omega_\mu(t') + \frac{\dot{s}_\mu(0)}{\Omega_\mu(0)\Omega_\mu(t)} \; \sin \int_0^t dt' \; \Omega_\mu(t')$$

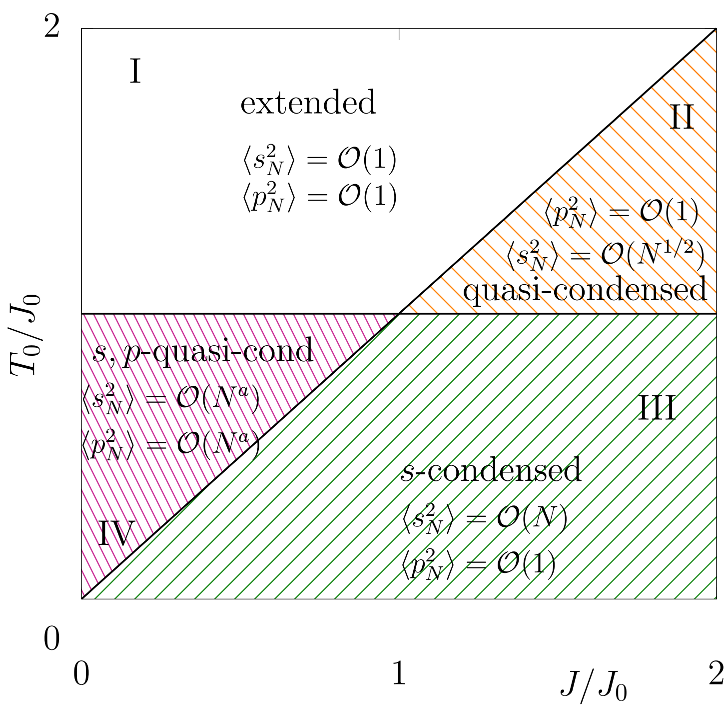
+ equations for the time-dependent frequencies $\Omega_\mu(t)$ and $z(t)$.

Similar to **Sotiriadis & Cardy 10** for the quantum O(N) model

Solvable numerically for finite N

Dynamic phase diagram

Looking more carefully at the condensation phenomena



$\overline{\langle s_N^2(t) \rangle_{i.c.}}$ & $\overline{\langle p_N^2(t) \rangle_{i.c.}}$

Initial conditions

Injection Extraction

For all parameters $\lim_{t \gg t_{\rm st}} \lim_{N \gg 1} \overline{\langle s_\mu^2(t) \rangle_{i.c.}}, \overline{\langle p_\mu^2(t) \rangle_{i.c.}}$ reach constants

The average over *i.c.* and λ_μ are collectively noted $\langle \ldots \rangle$ in the plot

Motion on the sphere

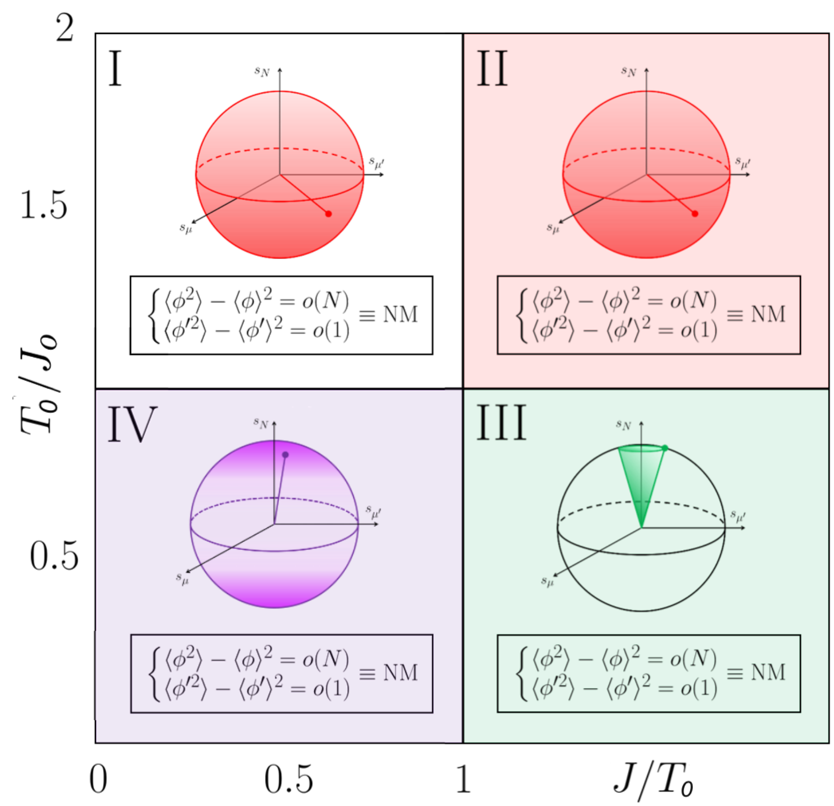
in the four phases of the dynamic phase diagram

Phase I & **Phase II** — extended initial conditions

Phase III **Phase IV** — condensed initial conditions

Extended

De-condensing



quasi-condensed

Condensed

Is there a stationary asymptotic measure ?

Thirdly, establish **the GGE ensemble** and compute averages

Asymptotic measure

Is the Generalized Gibbs Ensemble the good one ?

The GGE "canonical" measure is

$$\rho_{\rm GGE}(\vec{s},\vec{p}) = \mathcal{Z}^{-1}(\{\gamma_\mu\})\; e^{-\sum_{\mu=1}^{N} \gamma_\mu I_\mu(\vec{s},\vec{p})}$$

with **K. Uhlenbeck 80s**

$$I_\mu = s_\mu^2 + \frac{1}{mN} \sum_{\nu(\neq\mu)} \frac{(s_\mu p_\nu - s_\nu p_\mu)^2}{\lambda_\nu - \lambda_\mu} \qquad \mu = 1, \dots, N$$

(quartic & non-local) and we fix the γ_μ **on average** by imposing

$$\langle I_\mu \rangle_{\rm GGE} = \langle I_\mu \rangle_{i.c.} \qquad \forall\, \mu$$

NB in interacting quantum integrable models the charges are usually not known. But we do know them all for this model !

The GGE

Harmonic *Ansatz*

$$\rho_{\rm GGE}(\{\vec{s},\vec{p}\}) = \mathcal{Z}^{-1}(\{\gamma_\mu\})\; e^{-\sum_{\mu=1}^N \gamma_\mu I_\mu(\{\vec{s},\vec{p}\})}$$

Extensive expression in the exponential $\sum_{\mu=1}^N \gamma_\mu I_\mu = \mathcal{O}(N)$ if $\gamma_\mu = \mathcal{O}(1)$

GB measure recovered for $J = J_0$ with $2\gamma_\mu = -\beta_0 \lambda_\mu$ since $\sum_{\mu=1}^N \lambda_\mu I_\mu = -2H_J$

How to calculate $\langle s_\mu^2\rangle_{\rm GGE}$ and $\langle p_\mu^2\rangle_{\rm GGE}$? A plausible **Ansatz**

$$\langle s_\mu^2\rangle_{\rm GGE} = \frac{T_\mu}{z_{\rm GGE}-\lambda_\mu} \qquad\qquad \langle p_\mu^2\rangle_{\rm GGE} = m\,T_\mu$$

with spherical constraint for $z_{\rm GGE}$ & the mode-temperature spectrum fixed by

$$\langle I(\lambda)\rangle_{i.c.} = \langle I(\lambda)\rangle_{\rm GGE} = \frac{2T(\lambda)}{z_{\rm GGE}-\lambda}\left[1 - \fint d\lambda' \,\frac{\rho(\lambda')T(\lambda')}{\lambda-\lambda'}\right]$$

another eq. for the N-th mode for condensed $i.c.$ &

eqs. for $\{\gamma(\lambda)\}$ in the $N\to\infty$ limit

Exact for $N\to\infty$

The GGE

Harmonic *Ansatz* : exact in the $N \to \infty$ limit

$$\rho_{\rm GGE}(\{\vec{s},\vec{p}\}) = \mathcal{Z}^{-1}(\{\gamma_\mu\})\; e^{-\sum_{\mu=1}^N \gamma_\mu I_\mu(\{\vec{s},\vec{p}\})}$$

Extensive expression in the exponential $\sum_{\mu=1}^N \gamma_\mu I_\mu = \mathcal{O}(N)$ if $\gamma_\mu = \mathcal{O}(1)$

GB measure recovered for no quench $J = J_0$ with $2\gamma_\mu = -\beta_0 \lambda_\mu$ since $\sum_{\mu=1}^N \lambda_\mu I_\mu = -2H_J$

Analytically solvable ! (methods typical of random matrix theory)

The spectrum of mode temperatures in Phases I & II

$$2\pi\,(\rho(\lambda))^2\, T^2(\lambda) = -(1+G(\lambda)) + [(1+G(\lambda))^2 + 4(g(\lambda))^2]^{1/2}$$

with $2g(\lambda) = \pi\rho(\lambda)\;(z_{\rm GGE}-\lambda)\langle I(\lambda)\rangle_{i.c.}$ and $G(\lambda) = \frac{2}{\pi} \fint d\lambda' \, \frac{g(\lambda')}{\lambda-\lambda'}$

Another explicit expression for $\gamma(\lambda)$ and $z_{\rm GGE}$

In condensed phases Nth mode treated separarely

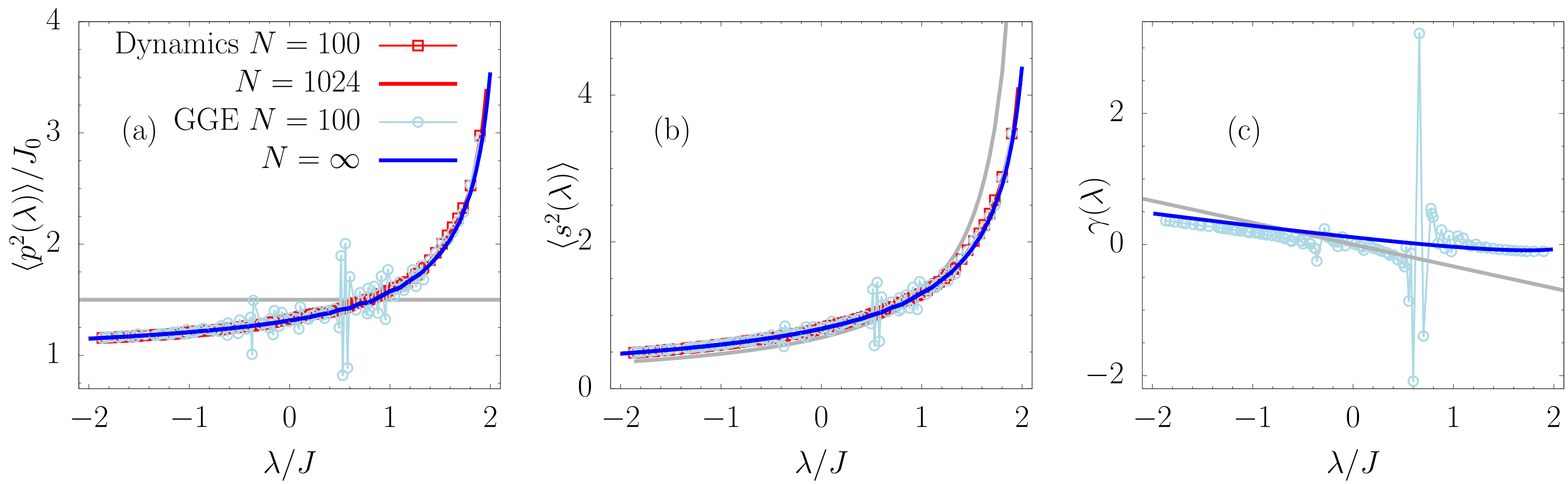
Dynamics vs GGE

$$\langle s_\mu^2 \rangle_{\rm GGE} = \overline{\langle s_\mu^2(t) \rangle_{i.c.}} \quad \text{and} \quad \langle p_\mu^2 \rangle_{\rm GGE} = \overline{\langle p_\mu^2(t) \rangle_{i.c.}} \quad ?$$

Dynamics vs GGE

e.g., comparison for quenches in Phase I

$T(\lambda)$



In gray, the initial functions

Similar coincidence in Phases II, III & IV

Interesting features linked to "fluctuations catastrophe" in Phase IV

Harmonic Ansatz $\equiv$ saddle-point evaluation of the GGE

Fourthly, can one obtain the **mode temperatures** T_μ with a **global dynamic measurement**?

Correlation and linear response

Fluctuation-dissipation theorem in Boltzmann equilibrium

$$C(t,t') = \frac{1}{N}\sum_{\mu=1}^{N} \langle s_\mu(t) s_\mu(t')\rangle_{i.c.} \quad \text{self correlation}$$

$$R(t,t') = \frac{1}{N}\sum_{\mu=1}^{N} \left.\frac{\delta\langle s_\mu(t)\rangle_{i.c.}}{\delta h_\mu(t')}\right|_{h=0} \quad \text{linear response}$$

Stationary limit $C(t,t') \mapsto C_{\rm st}(t-t')$ and $R(t,t') \mapsto R_{\rm st}(t-t')$

Fourier transforms

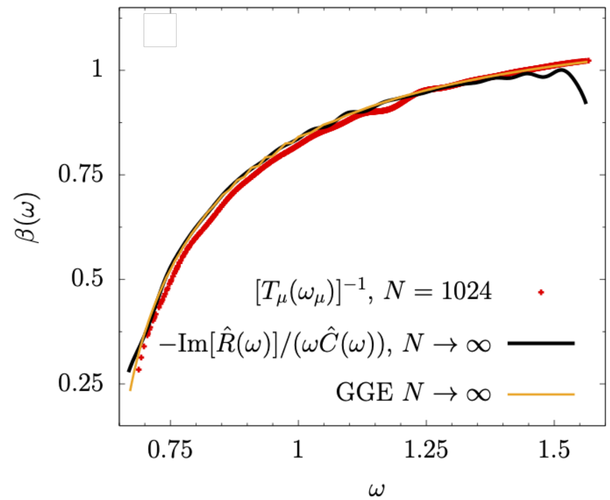
$$\hat{C}(\omega) = \text{F.T. } C_{\rm st}(t-t')$$
$$\hat{R}(\omega) = \text{F.T. } R_{\rm st}(t-t')$$

Fluctuation-dissipation thm

$$-\frac{\text{Im}\hat{R}(\omega)}{\omega\hat{C}(\omega)} = \beta$$

Frequency domain FDR

The T_μs from the FDR at $\omega_\mu = [(z_f - \lambda_\mu)/m]^{1/2}$ **Phase I**



A way to measure the mode temperatures with a single measurement

$$\beta_{\text{eff}}(\omega_\mu) = -\text{Im}\hat{R}(\omega_\mu)/(\omega_\mu \hat{C}(\omega_\mu)) = \beta_\mu$$

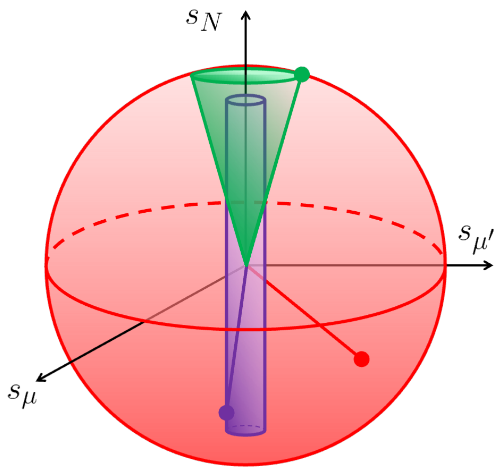
No "partial equilibration" contradiction from the effective temperature perspective. The modes are uncoupled, they do not exchange energy, and can then have different T_μs

Idea in **LFC, de Nardis, Foini, Gambassi, Konik & Panfil 17** for **quantum**

Summary

Goals achieved

In the late times limit taken after the large N limit



We solved

– the *global dynamics* with Schwinger-Dyson/ DMFT eqs.

– the *mode dynamics* with parametric oscillator techniques

of the *(soft) Neumann model*

With the *GGE measure*

$$\rho_{\rm GGE}(\vec{s},\vec{p}) = \mathcal{Z}^{-1}(\{\gamma_\mu\})\, e^{-\sum_\mu \gamma_\mu I_\mu(\vec{s},\vec{p})}$$

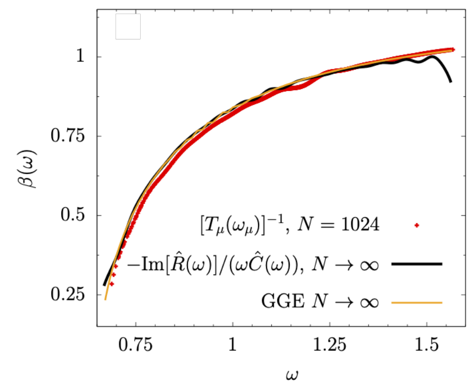
– we calculated & proved

$$\langle s_\mu^2\rangle_{\rm GGE} = \frac{T_\mu}{z_{\rm GGE}-\lambda_\mu} = \overline{\langle s_\mu^2(t)\rangle_{i.c.}}$$

$$\langle p_\mu^2\rangle_{\rm GGE} = T_\mu = \overline{\langle p_\mu^2(t)\rangle_{i.c.}}$$

obtaining also $\{T_\mu, \gamma_\mu\}$

– The $\{T_\mu\}$ are accessed by the *FDR*



$$\beta_{\rm eff}(\omega_\mu) = -\mathrm{Im}\frac{\hat{R}(\omega_\mu)}{(\omega_\mu \hat{C}(\omega_\mu))} = \beta_\mu$$

Goals achieved

– We exhibited a *classical interacting integrable model*, the *Soft Neumann model* or *Hamiltonian spherical SK model*, the *quench dynamics* of which can be elucidated with different means.

– Rich *dynamic phase diagram*.

– We managed to calculate analytically the GGE measure or, better said, all *GGE averaged local observables*

– We showed that *asymptotic dynamic* & *GGE averages* coincide for $N \to \infty$

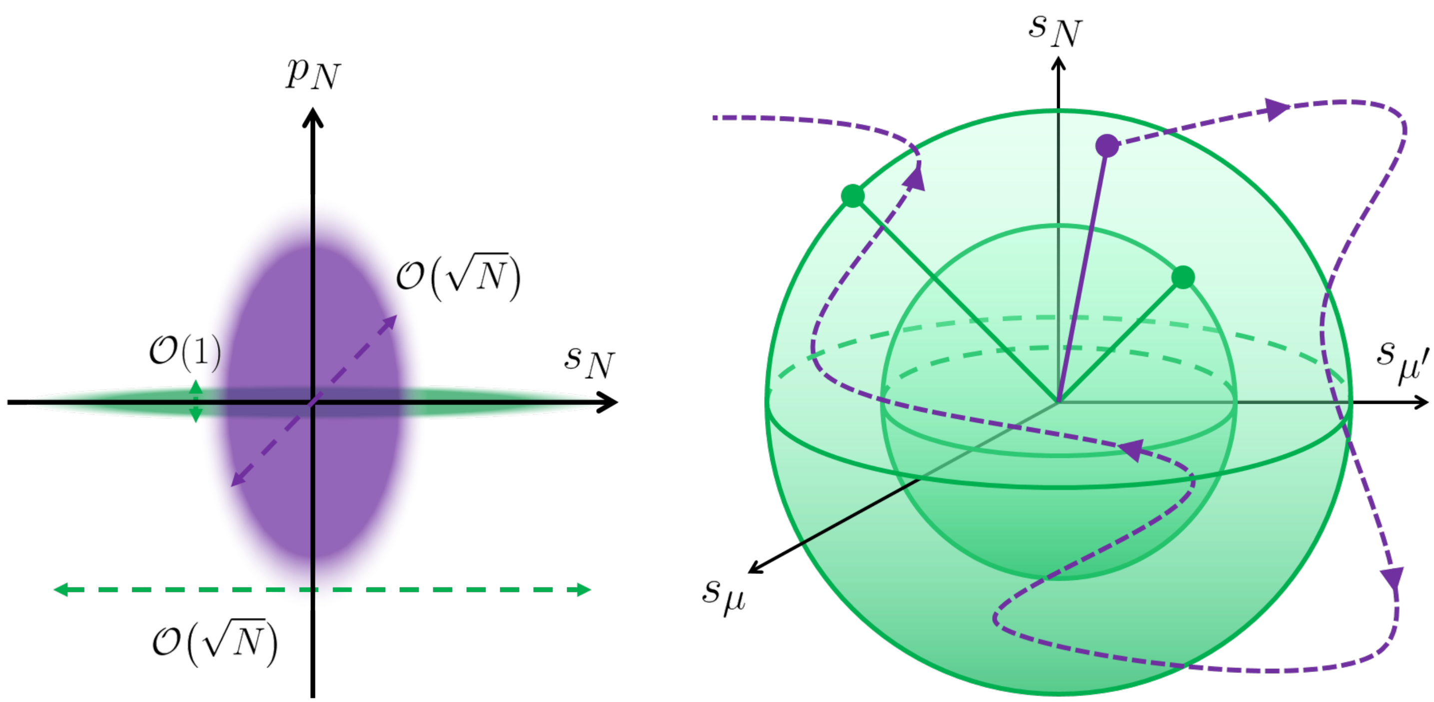
– We can also study the *fluctuations of the constraints* to prove that with symmetry broken initial conditions the **Soft Neumann Model $\equiv$ the Neumann model** with the strict spherical constraint.

Extra

Motion on the sphere

and the GGE averages on the Nth mode phase space

Phase III Phase IV



$z > \lambda_N$ $z = \lambda_N$

Integrals of motion

From microcanonical to canonical ?

The microcanonical GGE measure is ensured **Yuzbashyan 16**

$$\rho^{\rm micro}_{\rm GGE}(\{\mathcal{I}_\nu\}) = c \prod_{\mu=1}^N \delta(I_\mu(\{s_\nu, p_\nu\}) - \mathcal{I}_\mu)$$

Two conditions to prove canonical from microcanonical :

(i) additivity of the energy $\to I_\mu = I_\mu^{(1)} + I_\mu^{(2)}$

(ii) extensivity of the energy $\to I_\mu = \mathcal{O}(N)$

e.g., **Campa, Dauxois, Ruffo 09** on in/equivalence of ensembles

not satisfied in our model by the I_μ's, but maybe combinations ?

Still, let's try $\rho_{\rm GGE}(\{s_\nu, p_\nu\}) = \mathcal{Z}^{-1}(\{\gamma_\mu\})\, e^{-\sum_{\mu=1}^N \gamma_\mu I_\mu(\{s_\nu, p_\nu\})}$

scaling with N $\boxed{\sum_{\mu=1}^N \gamma_\mu I_\mu = \mathcal{O}(N)}$ if $\gamma_\mu = \mathcal{O}(1)$

Integrals of motion

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GB equilibrium for no quench $\boxed{\gamma_\mu = -\frac{\beta_0\lambda_\mu}{2}}$ since $\sum_{\mu=1}^N \frac{\lambda_\mu}{2} I_\mu = -H$

Classical dynamics

From spins to a particle moving on an N-dimensional sphere

Coordinate-momenta pairs $\{\vec{s},\vec{p}\}$ and Hamiltonian (const w/Lagrange mult.)

$$H_J^{(z)} = E_{\rm kin}(\vec{p}) + V_J(\vec{s}) + \frac{z(\vec{s},\vec{p})}{2}\sum_{i=1}^{N}(s_i^2 - N)$$

with the kinetic energy $E_{\rm kin}(\vec{p}) = \frac{1}{2m}\sum_{i=1}^{N} p_i^2$

Newton-Hamilton equations

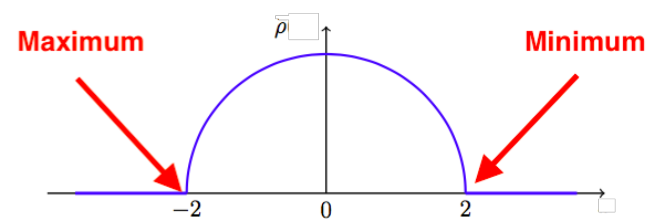
$$\dot{s}_i = p_i/m \qquad\qquad \dot{p}_i = -\delta V_J(\vec{s})/\delta s_i - z(\vec{s},\vec{p})s_i$$

The effective potential energy landscape $2V_J^{(z)}(\vec{s}) = -\sum_{i\neq i} J_{ij}s_is_j + z(s^2-N)$ has

$\mu = 1,\ldots,N$ saddles (including min/max)

the N eigenvectors $\vec{v}_\mu$ of the J_{ij} matrix with

$z = \lambda_\mu$ & pot. energy density $e_{\rm pot}^{(\mu)} = -\lambda_\mu/2$



$$z(\vec{s},\vec{p}) = 2\left[e_{\rm kin}(\vec{p}) - v_J(\vec{s})\right]$$

$-2J \qquad 2J = \lambda_N$

Conservative dynamics

on average over randomness & the initial measure

In the $N \to \infty$ limit exact causal Schwinger-Dyson (DMFT) equations

$$(m\partial_t^2 - z_t)R(t,t_w) = \int dt' \; \Sigma(t,t')R(t',t_w) + \delta(t-t_w)$$

$$(m\partial_t^2 - z_t)C(t,t_w) = \int dt' \left[\Sigma(t,t')C(t',t_w) + D(t,t')R(t_w,t')\right]$$

$$\boxed{+\frac{\beta_0 J_0}{J}\sum_{a=1}^{n} D_a(t,0)C_a(t_w,0)}$$

$$\boxed{(m\partial_t^2 - z_t)C_a(t,0) = \int dt' \, \Sigma(t,t')C_a(t',0) + \frac{\beta_0 J_0}{J}\sum_{a=1}^{n} D_b(t,0)Q_{ab}}$$

$a = 1, \ldots, n \to 0$, replica method to deal with $e^{-\beta_0 H^{(z)}_{J_0}}$ and fix Q_{ab}

Initial cond **Houghton, Jain, Young 86, Franz, Parisi 95, Barrat, Burioni, Mézard 96**

Conservative dynamics

on average over randomness and the initial measure

In the $N \to \infty$ limit exact causal Schwinger-Dyson (DMFT) equations

with the post-quench self-energy and vertex

$$D(t,t_w) = J^2\, C(t,t_w) \qquad NC(t,t') = \sum_i [\langle s_i(t) s_i(t')\rangle_{i.c}]_J$$

$$D_a(t,0) = J^2\, C_a(t,0) \qquad \text{with} \quad NC_a(t,0) = \sum_i [\langle s_i(t) s_i(0)\rangle_{i.c}]_J$$

$$\Sigma(t,t_w) = J^2\, R(t,t_w) \qquad NR(t,t') = \sum_i [\langle \frac{\delta s_i(t)}{h_i(t')}|_{\vec h=0}\rangle_{i.c}]_J$$

The Lagrange multiplier is fixed by $C(t,t) = 1 \Rightarrow z(t) = 2[e_{\rm kin}(t) - e_{\rm pot}(t)]$

Initial conditions:

- Disordered $\quad Q_{ab} = \delta_{ab}$
- Condensed $\quad Q_{ab} = (1-q)\delta_{ab} + q$

Solvable numerically & analytically at long times

Averaged integrals of motion

Properties, scaling and parameter dependence

'Sum rules' $\sum_\mu I_\mu = N$ and $\sum_\mu \lambda_\mu I_\mu = -2H_J$

In the $N \to \infty$ limit

$$\lim_{N\to\infty} I_\mu = I(\lambda) = s^2(\lambda) + \frac{1}{m}\int d\lambda' \, \rho(\lambda') \, \frac{[s(\lambda)p(\lambda') - s(\lambda')p(\lambda)]^2}{\lambda - \lambda'}$$

For GB equilibrium initial conditions

$$\langle I(\lambda)\rangle_{i.c.} = \langle s^2(\lambda^{(0)})\rangle_{i.c.} + \frac{1}{m}\int d\lambda' \rho(\lambda') \, \frac{\langle s^2(\lambda^{(0)})\rangle_{i.c.}\langle p^2(\lambda^{(0)\prime})\rangle_{i.c.} + \lambda^{(0)} \leftrightarrow \lambda^{(0)\prime}}{\lambda - \lambda'}$$

with $\langle s^2(\lambda^{(0)})\rangle_{i.c.} = k_B T_0/(z_0 - \lambda^{(0)})$ and $\langle p^2(\lambda^{(0)})\rangle_{i.c.} = m\, k_B T_0$

and N-th mode :

Extended	$\langle I_N\rangle_{i.c.} = \mathcal{O}(1)$
Condensed	$\langle I_N\rangle_{i.c.} = \mathcal{O}(N)$

The constants of motion

$\langle I_\mu(0^+)\rangle_{i.c.}$ **averaged over the initial measure**

Extended

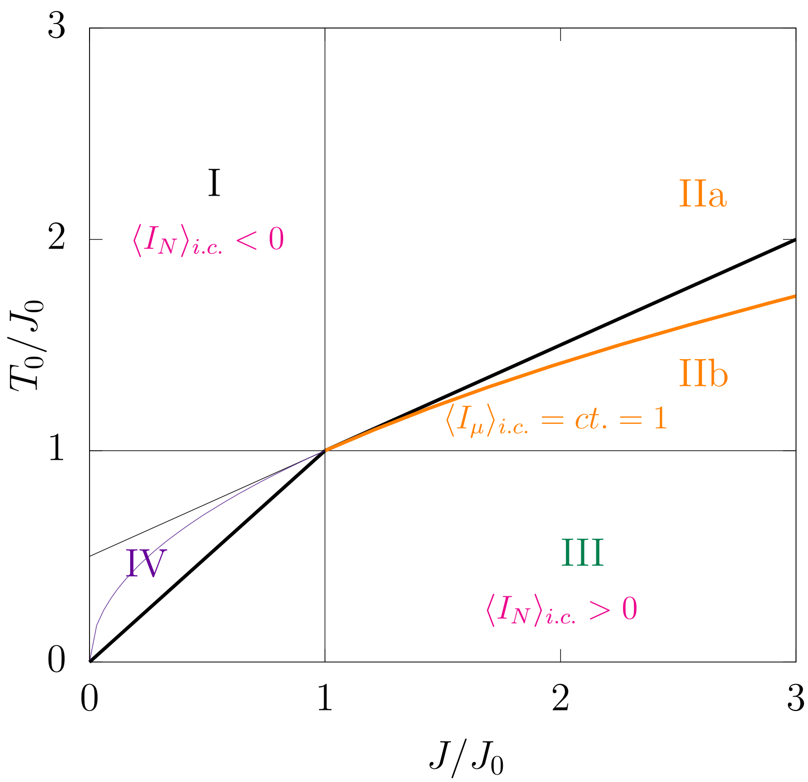
$$\frac{{T_0}^2}{J_0 J}\;\frac{J(J_0+J)/T_0-\lambda_\mu}{J(J_0+{T_0}^2/J_0)/T_0-\lambda_\mu}$$

Condensed & $\mu \neq N$

$$\frac{T_0^2}{J_0 J}\;\frac{J(J_0+J)/T_0-\lambda_\mu}{2J-\lambda_\mu}$$

Condensed & $\mu = N$

$$\left(1-\frac{T_0}{J_0}\right)\left(1-\frac{T_0}{J}\right)N+\mathcal{O}(1)$$

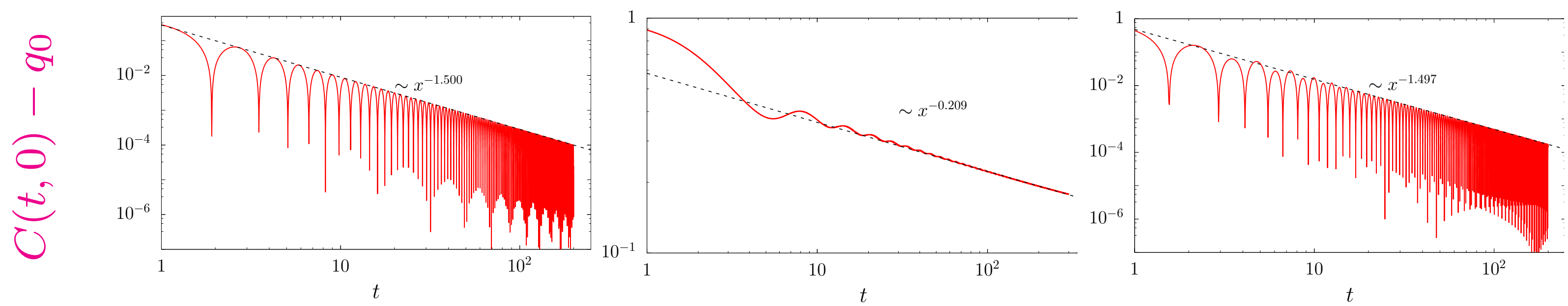


NB On the thick orange line the constants are all equal !

Asymptotic analysis

Algebraic approach to $q_0 = \lim_{t \gg t_0} C(t,0)$ **- fidelity**

condensed critical extended



$T_0/J_0 < 1 \quad J > J_0$

$C(t,0) = q_0 + c\,t^{-1.5} g(t)$

the exponent is independent

$T_0/J_0 > 1$

$C(t,0) = c\,t^{-0.2} g(t)$

dependent of parameters

Similar time-dependencies & asymptotics for $z(t)$

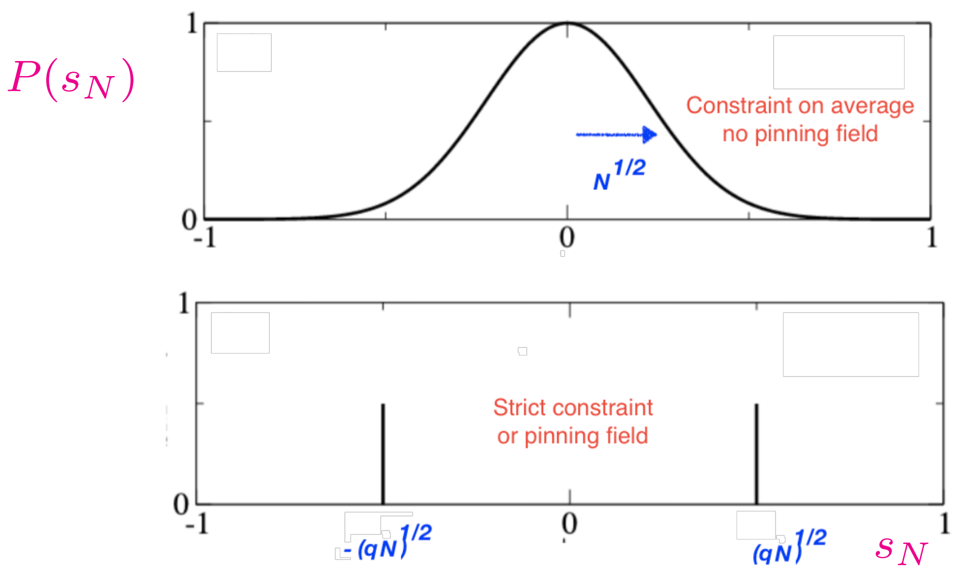
Initial conditions

Drawn from canonical equilibrium

$$\mathcal{Z}(\beta_0 J_0) \propto \int dz \int \prod_\mu ds_\mu \, e^{-\beta V^{(z)}_{J_0}(\{s_\mu\})}$$

The spherical constraint fixes $\langle s_N^2 \rangle_{\rm eq} = qN$ *via* $\int d\lambda^{(0)} \, \rho(\lambda^{(0)}) \, \frac{T_0}{2J_0 - \lambda^{(0)}_\mu} + \frac{\langle s_N^2 \rangle_{\rm eq}}{N} = 1$

Two possibilities



At $T_0 \leq T_c = J_0$ the mode

$s_N = \vec{s} \cdot \vec{v}_N$ is **massless**

$$\lim_{N \to \infty} (\lambda^{(0)}_N - z) = 0$$

and **condenses**

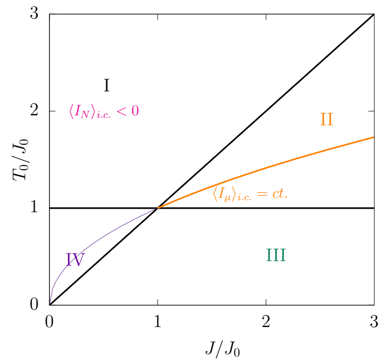
$$\langle s_N^2 \rangle_{\rm eq} = q(T_0/J_0) N$$

Kac & Thompson 71, Zannetti 15, Crisanti, Sarracino & Zannetti 19

Dynamics vs GGE

A special case : GGE $\mapsto$ GB

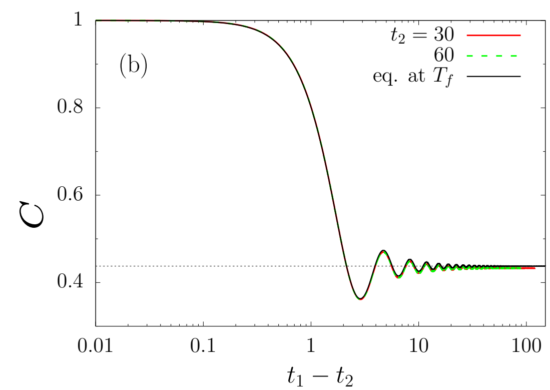
For $T_0/J_0 > 1$ and $(T_0/J_0)^2 = J/J_0$

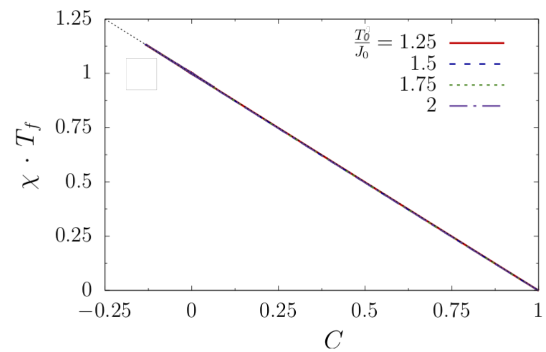


the constants of motion

are all equal

$\langle I_\mu \rangle_{i.c.} = 1$





The GGE construction yields

$$T_\mu = J \quad \text{and} \quad \gamma_\mu = -\lambda_\mu/(2J)$$

Therefore

$$-\sum_\mu \gamma_\mu I_\mu = \frac{1}{2J} \sum_\mu \lambda_\mu I_\mu = -\frac{1}{J} H$$

and the GGE reduces to the GB measure

at $T_f = J$

Stationarity & FDT OK

A spin model with randomness

The spherical SK ($p=2$) model — **Kosterlitz, Thouless & Jones 76**

$$V^{(z)}_{J_0}(\{s_i\}) = -\frac{1}{2}\sum_{i\neq j} J^{(0)}_{ij} s_i s_j + \frac{z(\vec{s})}{2}\left(\sum_i s_i^2 - N\right)$$

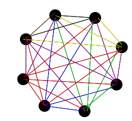
Fully connected interactions & $s_i \in \mathbb{R}$

Global spherical constraint $|\vec{s}|^2 = \sum_i s_i^2 = N$

imposed on average by a Lagrange multiplier $z(\vec{s})$

Gaussian distributed interaction strengths

$J^{(0)}_{ij} = J^{(0)}_{ji}, \;\; [J^{(0)}_{ij}] = 0$ & $[(J^{(0)}_{ij})^2] = \frac{J_0^2}{2N}$



J_{ij} has N eigenvalues $\lambda^{(0)}_\mu$ and eigenvectors $\vec{v}_\mu$

$\rho(\lambda^{(0)}_\mu) \propto \sqrt{(2J_0)^2 - (\lambda^{(0)}_\mu)^2}$

Diagonalised effective potential, basis of eigenvectors $s_\mu = \vec{v}_\mu \cdot \vec{s}$

$$V^{(z)}_{J_0}(\{s_\mu\}) = -\frac{1}{2}\sum_{\mu} \lambda^{(0)}_\mu s_\mu^2 + \frac{z(\vec{s})}{2}\left(\sum_\mu s_\mu^2 - N\right)$$

The model

The spherical SK ($p = 2$) model **Kosterlitz, Thouless & Jones 76**

If we add Newton classical dynamics on the s_μ

$$\dot{s}_\mu = p_\mu / m \qquad \dot{p}_\mu = -\delta V_J(\vec{s})/\delta s_\mu - z(\vec{s}, \vec{p}) s_\mu$$

it is **Neumann's model** $\uparrow$

with the *spherical constraint imposed on average*

About strict vs. averaged constraints in the stat-phys context

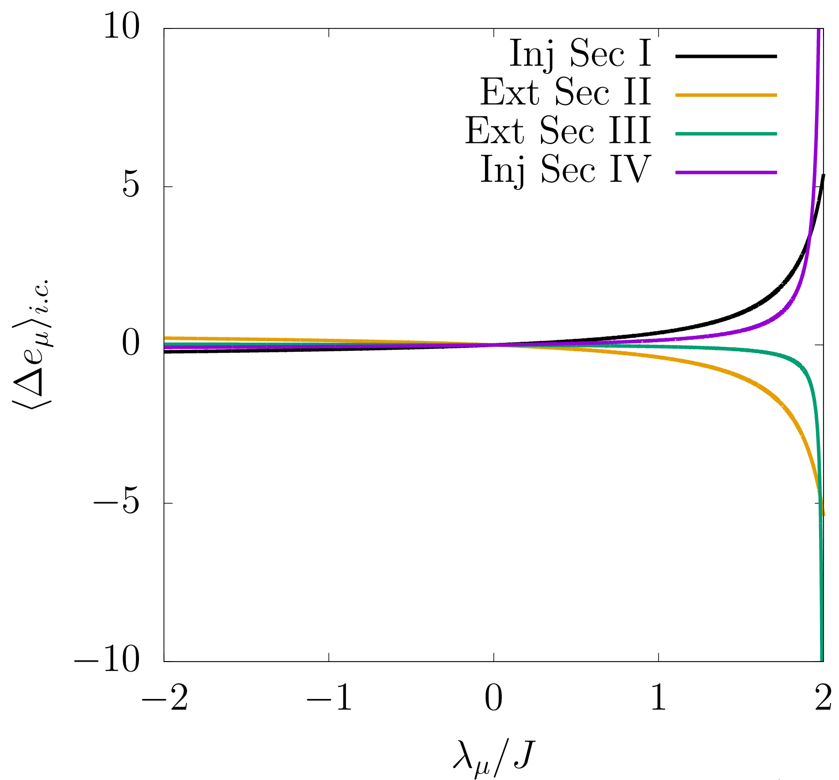
Kac & Thompson 71, ... , Zannetti et al 2000s

and the λ_μ drawn from **Wigner's semi-circle** law ($\lambda_\mu \neq \lambda_\nu$)

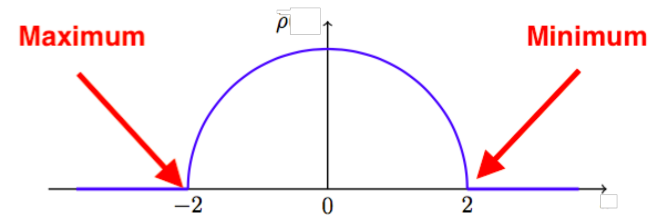
We just need to choose the *initial conditions*

Instantaneous quench

Mode energy change under $J_0 \mapsto J$: hard & soft



Wigner density of spring constants $\rho(\lambda/J)$



Mode energy spectrum

$$e_\mu = \frac{1}{2m} p_\mu^2 - \frac{1}{2} \lambda_\mu s_\mu^2$$

Mode energy change

$$\langle \Delta e_\mu \rangle_{i.c.} = \langle e_\mu(0^+) - e_\mu(0^-) \rangle_{i.c.}$$

↑

The energies of the modes at the **right edge** of the λ_μ spectrum are the more affected ones

These are the **softer modes**

The constants of motion

$\langle I_\mu(0^+)\rangle_{i.c.}$ **averaged over the initial measure**

Extended

$$\frac{T_0{}^2}{J_0 J}\;\frac{J(J_0+J)/T_0-\lambda_\mu}{J(J_0+T_0{}^2/J_0)/T_0-\lambda_\mu}$$

Condensed & $\mu \neq N$

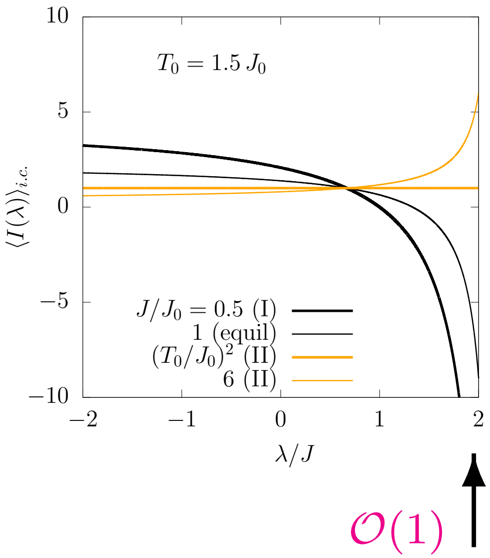
$$\frac{T_0^2}{J_0 J}\;\frac{J(J_0+J)/T_0-\lambda_\mu}{2J-\lambda_\mu}$$

Condensed & $\mu = N$

$$\left(1-\frac{T_0}{J_0}\right)\left(1-\frac{T_0}{J}\right)N+\mathcal{O}(1)$$

$T_0/J_0 > 1$

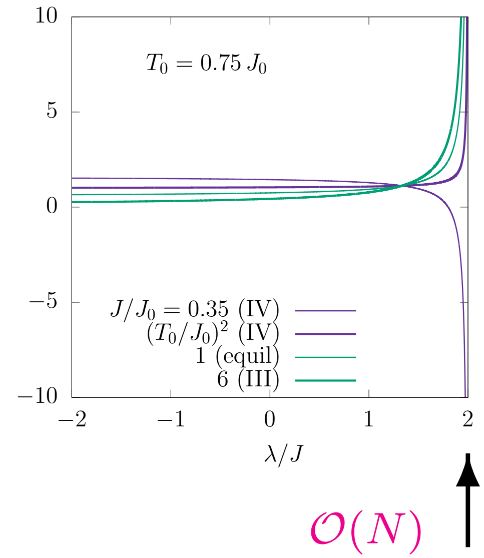
Extended



$\mathcal{O}(1)$

$T_0/J_0 < 1$

Condensed



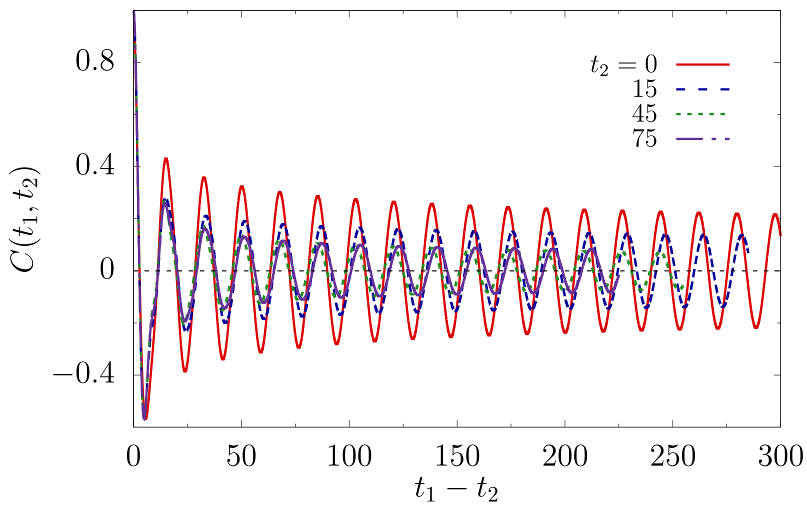
$\mathcal{O}(N)$

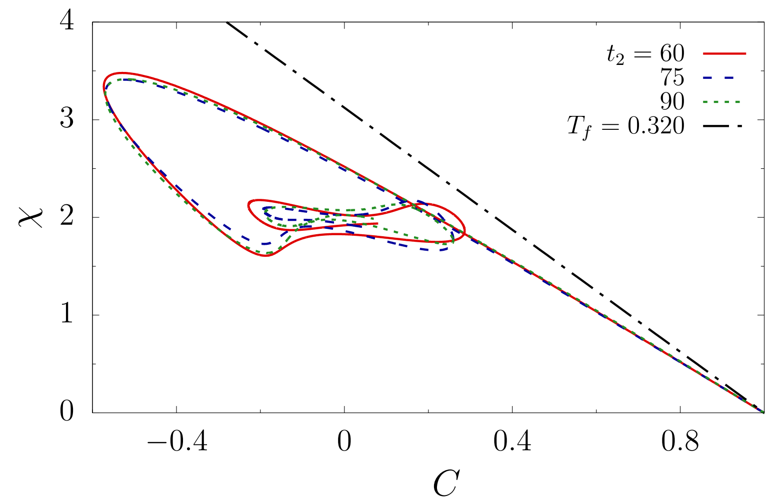
NB for $T_0/J_0 > 1$ and $(T_0/J_0)^2 = J/J_0$ the constants are all equal

No Gibbs-Boltzmann equilibrium

e.g. large energy injection on a condensed state (IV)

$C(t,t') \to C_{\rm st}(t-t')$ but





$$\chi_{\rm st}(t-t') \equiv \int_{t'}^{t} dt''\, R_{\rm st}(t,t'') \neq -\beta_f \left[C_{\rm st}(t-t') - C_{\rm st}(0)\right]$$

Stationary dynamics but no FDT at a single temperature **no GB equilibrium**