NON EQUILIBRIUM MACROSCOPIC COMPLEX SYST

MANY ADJECTIVES

NON-EQ: WE'LL TALK ABOUT OUT OF EQ DYNAMICS

SOMETIMES QUANTUM: WELL... I'LL GO FROM CLASSICAL TO QUANTUM BACK & FORTH
NOT SO MANY DIFFERENCES, THOUGH
I'LL HIGHLIGHT THEM

MACROSC: VS. SINGLE PARTICLE. MANY-BODY
THERMODYN LIMIT IMPORTANT ⇒
LAW OF LARGE NUMBERS
STAT PHYS.

COMPLEX: HARD, NOT EASY ⇒ INTERESTING

PLAN

1 - INTRODUCTION TO STOCHASTIC PROCESSES

- LANGEVIN EQS.
  - JUSTIFICATION
    - ORIGINAL - GENERAL
  - EXTENSIONS
  - APPLICATIONS

1st LECT.

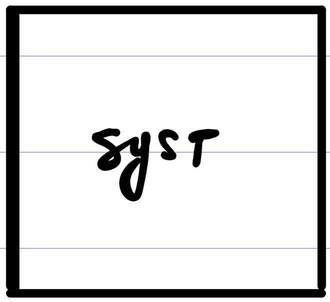
- FOKKER-PLANCK.
- GEN. FUNCT.

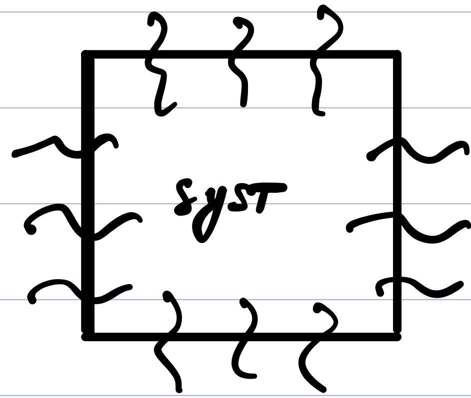
2nd LECT.

2 - SOME FEATURES OF MACROSC. SYST. OUT OF EQ.

3rd LECT.

CLOSED vs. OPEN





CLASS. NOTATION

SURROUNDINGS

ENVIRONMENT - BATH

$E = H(\{\vec{x}_i, \vec{p}_i\}) = \text{cst.}$

E FLUCTUATES

MAY $\exists$ OTHER CONSERVED QUANTITIES, eg MOMENTUM, ANG MOMENTUM, ETC.

EXCHANGES w/ BATH ALSO N OR SOMETHING ELSE

MICROCANONICAL

I WILL NOT TALK ABOUT THESE BUT ALSO VERY INTERESTING

CANONICAL

GRAN. CANONICAL

EQUILIBRIUM MEASURES

MICROCANONICAL – CLOSED

$$\mathcal{P}(\{\vec{x}_i, \vec{p}_i\}) = \prod_{\mu=1}^{N_c} \delta\left(\mathcal{I}_\mu(0) - I_\mu(\{\vec{x}_i, \vec{p}_i\})\right)$$

N_c # CT. MOTION $I_\mu(\{\vec{x}_i, \vec{p}_i\})$ $\mu = 1, \dots, N_c$ $\frac{dI_\mu}{dt} = 0$

$\mathcal{I}_\mu(0)$ THEIR INITIAL VALUES

CANONICAL – OPEN

$$\mathcal{P}(\{\vec{x}_i, \vec{p}_i\}) = \frac{1}{\mathcal{Z}(\{\gamma_\mu\})} e^{-\sum_{\mu=1}^{N_c} \gamma_\mu I_\mu(\{\vec{x}_i; \vec{p}_i\})}$$

e.g. $I_1 = H$, $\gamma_1 = \beta$ INVERSE TEMP

$I_2 = N$, $\gamma_2 = \mu$ CHEMICAL POT.

DYNAMICS

CLOSED

NEWTON - HAMILTON CLASS

HEISENBERG - SCHRÖDINGER QUANTUM

OPEN

LANGEVIN SINGLE NOISY TRAJECTORIES

FOKKER - PLANCK - MASTER EQ $P(\{\vec{x}_i, \vec{p}_i\}, t)$

MSR - JD GENERATING FUNCT TRAJ.

LINDBLAD EQ. MASTER EQ. FOR $\hat{\rho}$

SCHWINGER - KELDYSH CLOSED TIME PATH WITH

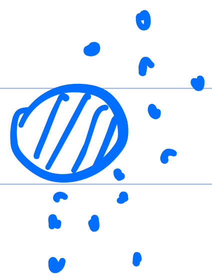
INFLUENCE FUNCTIONALS IN GEN. FUNCT

GEN MSR - JD

15 min

LANGEVIN'S LANGEVIN EQ.

COLLOID in



WATER

BROWN'S EXP

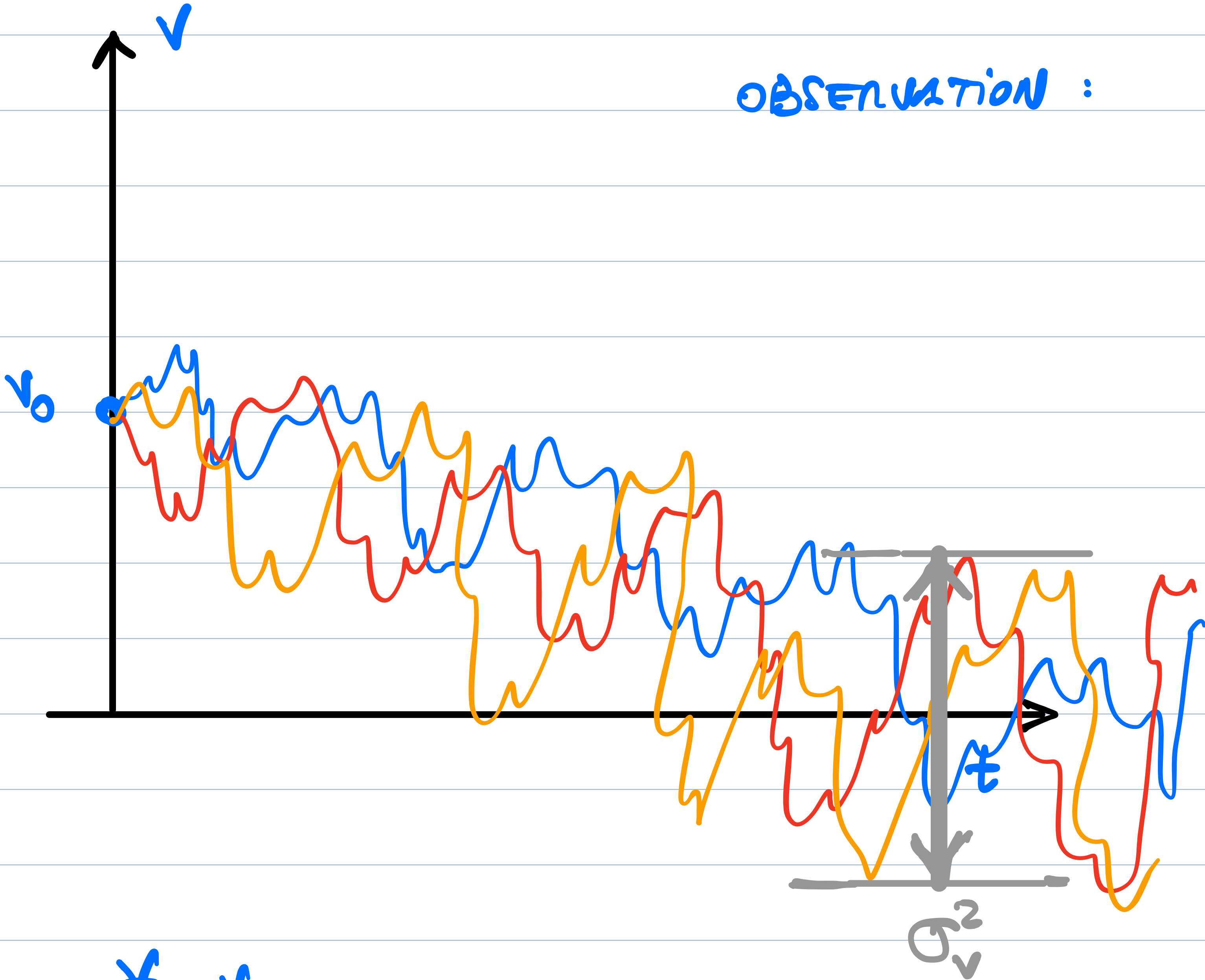
POLLEN

ALIVE OR DEATH

- MANY MANY COLLISIONS BTW WATER MOLECULES & COLLOID IN UNIT TIME (MEASUREMENT CLOCK, eg. 1 SEC)

- EFFECTS : FRICTION & NOISE

OBSERVATION :



$\forall \; v_0$

ALL "TRAJECTORIES" ARE $\neq$
BUT THEY SHARE STATIST.
PROPERTIES

SO THAT

NEWTON'S LAW

$$m\ddot{x} = F_{ENV} \qquad d=1$$

$$= -\gamma\dot{x} + \xi$$

RANDOM FORCE

FRICTION COEFF

WILL SLOW DOWN THE PART

RANDOM FORCE

$\xi(t)$ GAUSSIAN, JUST NEED TO KNOW AVER & CORR. IN TIME

CHOSEN SO THAT:

$\langle v(t) \rangle \longrightarrow 0$ FORGET i.c.

$\frac{m}{2} \langle v^2(t) \rangle \longrightarrow \frac{k_B T}{2}$ EQUIPART. AT AMBIENT TEMP

$\langle \xi(t) \rangle = 0$ $t > t_R = m/\gamma$

$\langle \xi(t)\, \xi(t') \rangle = 2\gamma\, k_B T \;\; \delta(t-t')$

AVER $\langle \dots \rangle$ MEANS OVER REPETITION of EXP.

IMPORTANT:

- SEPARATION of TIME SCALES
  BATH MUCH FASTER THAN SYST
- SEPARATION of SIZES
  BATH MUCH LARGER THAN SYST

WE USE LANGEVIN EQS IN VERY ≠ CASES
WHY?

30'

JUSTIFICATION OF NOISE / DISSIP. IN LANGEVIN EQ.

CLASSICAL LANGEVIN. BROWNIAN MOTION, THEN GENER.

LANGEVIN ~1900

BETTER & MORE GENERAL:

MODEL OF SYST H_{syst} BUT ALSO

MODEL OF BATH & COUPLING

PHONONS, SIMPLEST IDEA

HARMONIC OSCILLATORS & LINEAR COUPLING

$$H_B = \sum_{a=1}^{N_0} \left(\frac{\pi_a^2}{2m_a} + \frac{m_a \omega_a^2}{2} q_a^2 \right)$$

$d=1$

TO SIMPLIFY THE NOTATION

$$H_I = \sum_{a=1}^{N_0} \frac{c_a q_a}{\sqrt{N_0}} \cdot f(x)$$

(COUNTERTERM LATER)

c_a COUPLING CONST. $O(1)$

$\{q_a, \pi_a\}$ OSCILLATOR'S PHASE SPACE VARIABLES

N_0 # OSCILLATORS (WILL GO TO ∞)

SOLVABLE NEWTON / HAM. DYN FOR THE OSC.

LINEAR EQS. FORCED BY THE COUPLING TO x

VIA $f(x)$ USUALLY $f(x) = x$ SIMPLEST CASE

GEN. LATER

SOME VERY SIMPLE CALCULATIONS & USING

THE IMPORTANT ASSUMPTION

EQUILIBRIUM BATH

$\mathcal{P}(\{q_a(0), \pi_a(0)\})$ CANONICAL BOLTZMANN

(COUPLING TO $x(0)$)

TRICKY ISSUE ABOUT A COUNTERTERM TO BE DISCUSSED

TAKING $N_0 \to \infty$ A BATH : ENORMOUS

GAUSSIAN RANDOM INITIAL CONDITIONS $\Longrightarrow$

INDUCE RANDOMNESS / STOCHASTICITY $k_B T$

FRICTION γ

THE DYN NO LONGER CONSERVE ENERGY OR

OTHER CONST MOTION OF THE SYST

$\Longrightarrow$

GENERALIZED LANGEVIN EQS

$$\boxed{m\ddot{x}(t) + \int_0^t dt' \, \Gamma(t-t') \, \dot{x}(t') = F[x(t)] + \xi(t)}$$

$$\xi(t) = -\sum_{a=1}^{N_0} c_a \left[\frac{\pi_a(0)}{m_a \omega_a} \sin \omega_a t + \left(q_a(0) + \frac{c_a \, x(0)}{m_a \omega_a^2} \right) \cos \omega_a t \right]$$

NOISE

$f(x) = x$

STATISTICS OF $\{q_a(0), \pi_a(0)\}$ $\Rightarrow$ THE ONE OF ξ

USUAL EQ CHOICE FOR OSCILLATORS:

$$\langle \xi(t) \rangle = 0 \qquad \langle \xi(t)\, \xi(t') \rangle = k_B T \, \Gamma(t-t')$$

$\langle \cdots \rangle$ MEANS AVERAGE WITH RESPECT TO THE i.c. OF THE OSC.

WITH

$$\Gamma(t-t') = \frac{1}{N_0} \sum_{a=1}^{N_0} \frac{c_a^2}{m_a \omega_a^2} \cos[\omega_a (t-t')]$$

SYMMETRIC FRICTION KERNEL $t \leftrightarrow t'$

FUNCT FORM DEPENDS ON $\{c_a, m_a, \omega_a^2\}$

PROPS OF OSC. BATH

KAWASAKI, ZWANZIG 60s - 70s

USUALLY $S(\omega) = \frac{1}{N_a} \sum_{a=1}^{N_0} \frac{c_a^2}{m_a \omega_a} \delta(\omega - \omega_a)$

$$\Gamma(t-t') = \int_0^\infty d\omega \, \frac{S(\omega)}{\omega} \cos \omega (t-t')$$

THE CHARACTERISTICS OF THE OSCILLATORS

$\Rightarrow$ NOISE, FRICTION PROPS. VIA $\Gamma(t-t')$

FROM CLASSICAL MECHANICS TO STOCHASTIC PROCESSES

REDUCED SYSTEM

NON-MARKOVIAN-MEMORY

SPECTRAL FUNCTION LEADING TO POWER-LAWS

$$\frac{S(\omega)}{\omega} = \frac{2\gamma_0}{\pi} \left(\frac{|\omega|}{\tilde{\omega}}\right)^{\alpha-1} f_c\left(\frac{|\omega|}{\Lambda}\right)$$

USUAL CASES

$$\Gamma(t-t') = \begin{cases} 2\gamma\, \delta(t-t') & a=1 \text{ WHITE OHMIC} \\ \gamma \exp\left(-\frac{|t-t'|}{\tau}\right) & \text{EXP. DECAY} \\ |t-t'|^{-a} & \text{POWER LAW} \end{cases}$$

SUB/SUPER. OHMIC

ALL THE INFO ABOUT BATH IS IN $k_B T$, $\Gamma(t-t')$

OVERDAMPED LIMIT - USUAL IN STAT PHYS.

VALID FOR LARGE FRICTION/MASS $\gamma/m \gg 1$

OFTEN TIME SCALE OF MOMENTUM RELAX. IS VERY SHORT, GAUSSIAN WITH

$$\langle v(t)\rangle \to v_0 \qquad \langle v^2(t)\rangle \to \frac{k_B T}{m}$$

DROP $m\ddot{x}$ FROM EQ. MOTION

SEPARATION OF TIME-SCALES p AND x VERY DIFFERENT

$f(x)$ GENERIC $\qquad H_I = \sum_{a=1}^{N_0} \frac{c_a q_a}{\sqrt{N_0}} f(x)$

$$\boxed{\begin{aligned} m\ddot{x}(t) + f'(x(t)) \int_0^t dt' \, \Gamma(t-t') \, f'(x(t')) \, \dot{x}(t') \\ = F[x(t)] + \xi(t) \, f'(x(t)) \end{aligned}}$$

MULTIPLICATIVE NOISE

FOR THIS GENERIC $+ K \cdot \underset{\smile}{x} \to f(x)$

COUPLING $\longrightarrow$

MENTIONED COUNTERTERM.

TO GET THE EQ. I WROTE, ADD

$$H_c = \frac{1}{2} \sum_{a=1}^{N_0} \frac{c_a^2}{m_a \omega_a^2} x^2$$

AND CHOOSE INITIAL DIST OF $\{ \pi_a(0), q_a(0) \}$ WITH $x(0)$ WITHIN

NEEDED TO AVOID A "RENORM." OF H_{syst}

PHYSICAL CHOICE TO DO IT OR NOT

CONNECTION WITH USUAL
MULT. NOISE WHITE NOISE LANG EQ

$$\dot{x} = \bar{f}(x) + g(x)\,\xi$$ FOUND IN LITERATURE

TAKE $\Gamma(t-t') = 2\gamma_0\,\delta(t-t')$ IN (1)

DROP $m\ddot{x}$: OVERDAMPED LIMIT

SET $\gamma_0 = 1$ BY A REDEF. OF TIME

$$\left(f'(x(t))\right)^2 \dot{x}(t')$$

$$= F[x(t), \lambda(t)] + \xi(t)\, f'(x(t))$$

DIVIDE BY $\left(f'(x(t))\right)^2$ AND REDEFINE THE DETERM FORCE TERM $F/(f')^2 = \bar{f}$

AND

$$g = 1/f'$$

AND THAT'S IT.

MULTI-VARIABLE LANGEVIN EQ.

CAUSALITY

$$m\ddot{x}(t) + f'(x(t))\int_0^{t} dt'\ \Gamma(t-t')\ f'(x(t'))\ \dot{x}(t')$$

$$= F[x(t), \lambda(t)] + \xi(t)\, f'(x(t)) \qquad (1)$$

POSSIBLE TIME-DEP PARAM.

ADD INDICES TO MAKE IT VALID FOR

$i = 1, \dots, N$

$a = 1, \dots, d$

LOOK AT LFC ADV. PHYS. FOR FULL GENERALITY

USUAL ADDITIVE NOISE CASE, ONE INDEPENDENT IDENTICAL BATH PER d.o.f.

$$m\ddot{x}_i^a + \int_0^t dt'\ \Gamma(t-t')\ \dot{x}_i^a =$$

$$= F_i^a[\vec{x}_i(t), \vec{\lambda}_i(t)] + \xi_i^a(t)$$

$$\langle \xi_i^a(t)\ \xi_j^b(t') \rangle = k_B T\ \Gamma(t-t')\ \delta_{ij}\ \delta^{ab}$$

REFS.

DERIVATION & DISCUSSION OF LANG EQ

http://www.lpthe.jussieu.fr/~leticia/SEMINARS/Langevin.pdf

IN ADV. PHYS.

SKETCHY SLIDES

http://www.lpthe.jussieu.fr/~leticia/SEMINARS/ictp-non-Markovian-bis.pdf

APPLICATIONS

60'

INVERSE PROBLEM

OBSERVE THE MOTION OF A TRACER

e.g. BROWNIAN MOTION

CONCLUDE ABOUT THE PROPS OF THE BATH

THE FORCES IT INDUCES ON TRACER

LANGEVIN DID IT THIS WAY, EARLY 1900S

EXPERIMENTS , THEORY

TRACER IN A CELLULAR ENVIRONMENT

HOW TO GO FROM

INFER FROM TRAJECTORIES

P. RONCERAY (MARSEILLE)

MIT GROUP EXPS.

MANY OTHERS

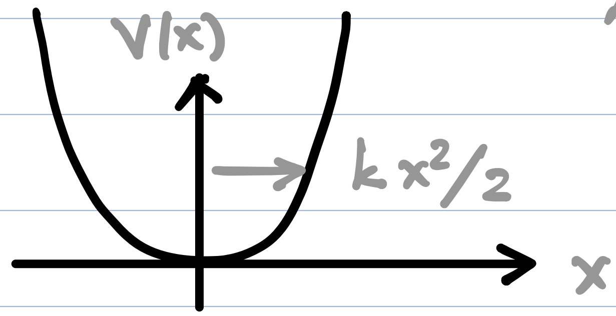
INFERENCE OF BATH - ENVIRONMENT

YANG ET AL 03; MIN, LUO, CHERAYIL, KOU & XIE 05

$$C_x(t,t') = \langle x(t)\, x(t') \rangle$$ MEASURED

$$= \frac{1}{k} E_{\alpha,1}\left(- \frac{k\,|t-t'|^{\zeta}}{g}\right)$$

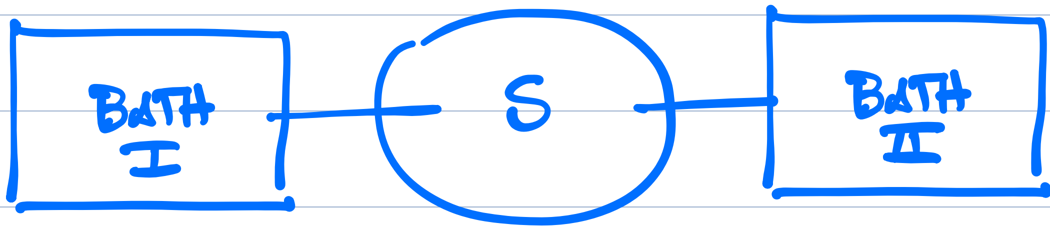
MITTAG LEFFLER - FCT.



$\zeta \simeq 0.5$

BATH?

SYSTEMS SUBMITTED TO TWO BATHS



LANGEVIN EQ FORMULATION

FRICTION $\Gamma_1(t-t') + \Gamma_2(t-t')$

NOISE - NOISE CORR $k_B T_1 \, \Gamma_1(t-t') + k_B T_2 \, \Gamma_2(t-t')$

FOR THE SETTING IN WHICH EACH SYST VARIABLE COUPLED TO BOTH BATHS.

RELEVANT FOR GLASSY PHYSICS → 3rd LECTURE

SYSTEMS FORCED EXTERNALLY BY A TIME-DEPENDENT DRIVE

$$F\,[x(t),\ \lambda(t)\,]$$

SETTING USED IN THE CONTEXT OF

FLUCT. THEOREMS

ENTROPY PRODUCTION

STOCHASTIC THERMODYNAMICS

ACTIVE ORNSTEIN-UHLENBECK PARTICLES

cf F. van Wijland's talks

also my seminar

OVERDAMPED LIMIT

$$\gamma \dot{x} = F + \eta$$

$$\dot{\eta} = -\frac{1}{\tau}\eta + \xi$$

$$\langle \xi(t) \rangle = 0$$

$$\langle \xi(t)\, \xi(t') \rangle = k_B T \;\; \delta(t-t')$$

INTEGRATING η:

$$\eta(t) = \underbrace{\eta(0)}_{0}\, e^{-t/\tau} + \int_0^t dt'\, e^{-(t-t')/\tau}\, \xi(t')$$

$$\langle \eta(t) \rangle = 0$$

$$\langle \eta(t)\, \eta(t') \rangle \longrightarrow k_B T\, \tau\, e^{-(t-t')/\tau}$$

$$\langle \eta(t)\,\eta(t')\rangle = \int_0^t dt'' \int_0^{t'} dt''' \, e^{-\frac{(t-t'')}{\tau}-\frac{(t'-t''')}{\tau}} \underbrace{\langle \xi(t'')\,\xi(t''')\rangle}_{2 k_B T\, \delta(t''-t''')}$$

$t > t'$

$$= 2 k_B T \int_0^{t'} dt'' \; e^{-(t-t''+t'-t'')/\tau}$$

$$= 2 k_B T \; e^{-(t+t')/\tau} \int_0^{t'} dt'' \; e^{2t''/\tau}$$

$$= \cancel{2} k_B T \; \frac{\tau}{\cancel{2}} \left[e^{-(t-t')/\tau} - e^{-(t+t')/\tau} \right]$$

$$\xrightarrow[t' \gg \tau]{} k_B T \; \tau \; e^{-(t-t')/\tau}$$

EXPONENTIAL CORR.
IN TIME OF NOISE η
PERSISTENCE TIME τ

$$\gamma_0 \frac{[x]}{[t]} = [F] = \frac{[E]}{[x]} \Longrightarrow \gamma_0 [x^2] = [k_B T]\,[t]$$

FROM EQ. ON x (1)

$$\frac{[\eta]}{[t]} = \frac{[\eta]}{[\tau]} = [\xi] \Longrightarrow [\eta] = [\xi]\,[t]$$

$$\Downarrow$$

$$[t] = [\tau] \checkmark$$

FROM EQ. ON η (2)

$$[\xi^2] = \frac{[k_B T]}{[t]}$$

FROM NOISE ξ CORR, IN (2):

$$[\eta] = \frac{[k_B T]^{1/2}}{[t]^{1/2}} \cdot [t] = [k_B T]^{1/2}\,[t]^{1/2} \quad (3)$$

ALSO $[\eta] = [f] = \frac{[E]}{[x]} = \frac{[k_B T]}{[x]}$ FROM EQ. ON x

USING (1) $$[\eta] = \frac{[k_B T]^{1/2}}{[t]^{1/2}} [\gamma_0]^{1/2} \quad (4)$$

(3) & (4) $\Longrightarrow$ $[t] = [\gamma_0]^{1/2}$

THE NOISE TERM η AND THE FRICTION TERM $\gamma \dot{x}$ ARE NOT BALANCED

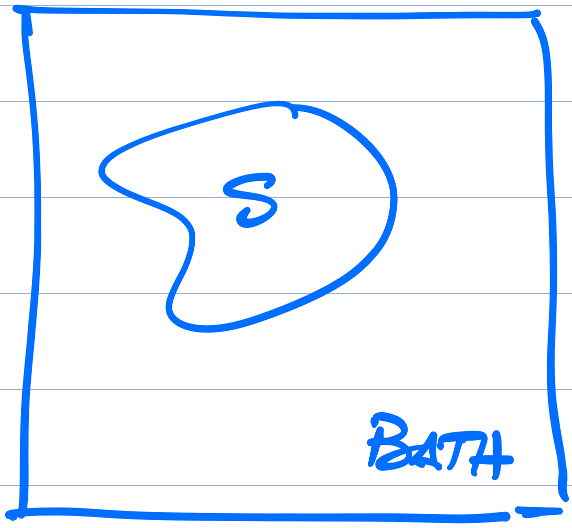
THE LANG EQ IS NOT OF THE FORM DERIVED ABOVE FOR EQUIL BATHS.

$\Rightarrow$ OUT OF EQUIL. ENVIRONMENT

BACK TO FORMALISM

1h 1/2

SUMMARY 1st LECTURE



INTEGRATE AWAY BATH
if IN EQUIL.

$\Longrightarrow$ GENERALIZED LANG EQ

w/ $m\ddot{x}$, RETARDED FRICTION,
COLOURED NOISE POSSIBLY MULTIP.

SIMPLIFICATIONS

OVERDAMPED $m\ddot{x} \to$ DROP
MARKOV $\Gamma(t-t') \to 2\gamma\,\delta(t-t')$
ADDITIVE NOISE

TODAY : OTHER WAYS OF DEALING W/ THESE SITUATIONS

IN THEIR MOST GRAL. FORM OR APPROX.

1- FOKKER - PLANCK / KRAMERS
& LINDBLAD FOR QUANTUM

MARKOV

2- GEN. FUNCT

MSR - JD

SCHWINGER - DYSON (CTP)

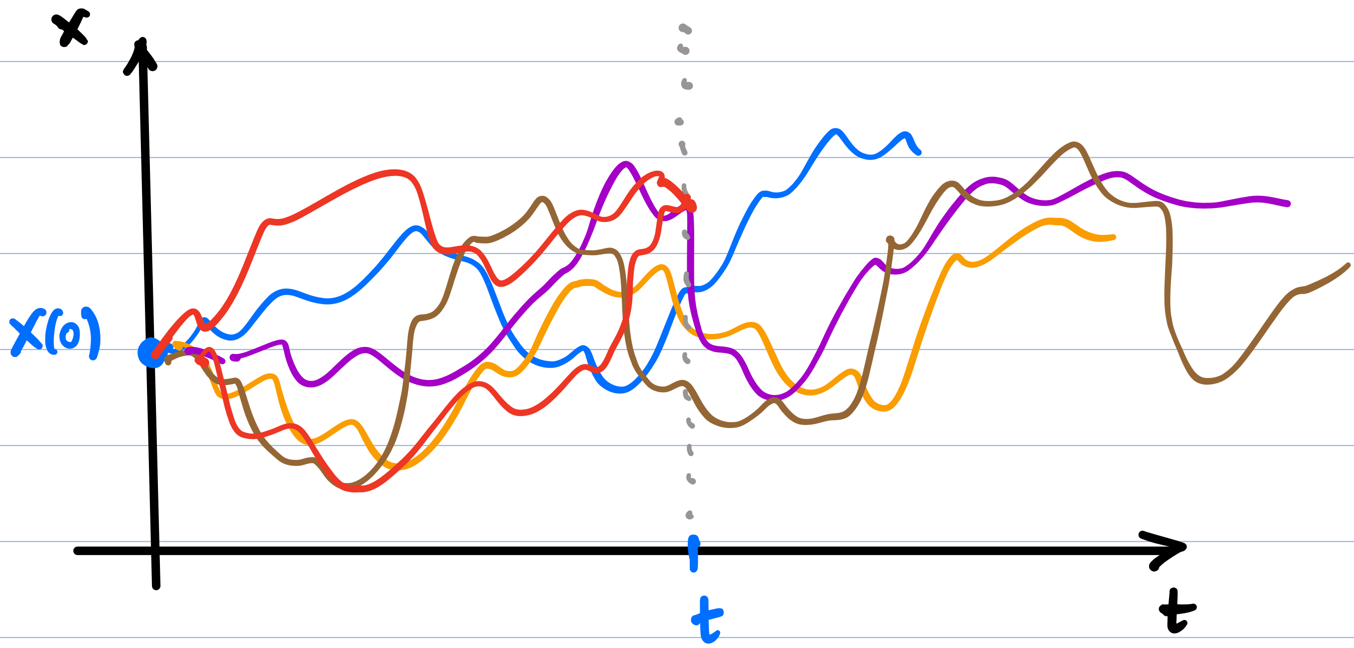
FEYNMAN VERNON

GENERIC

KRAMERS / FOKKER-PLANCK EQUATION

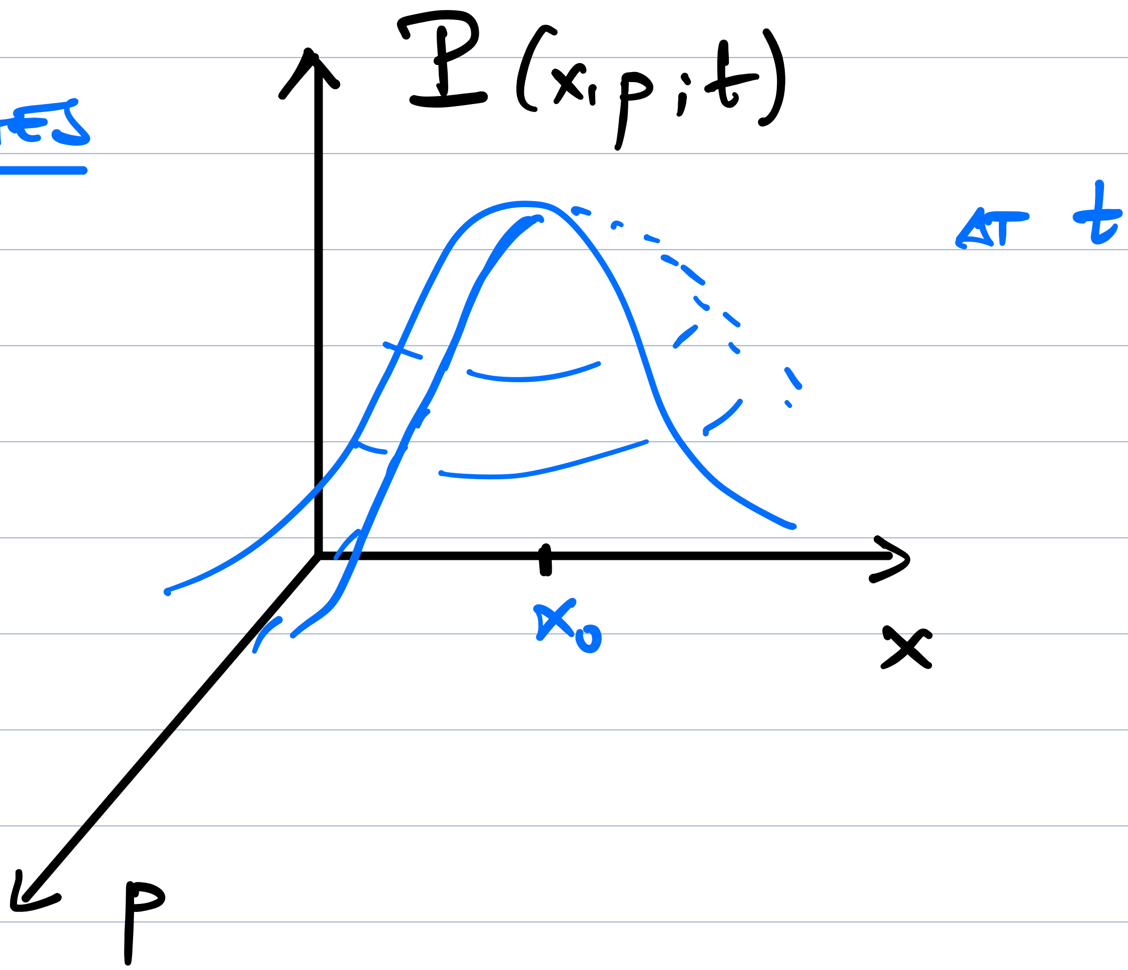
$\mathcal{P}(x, p; t)$ PROB. OF PARTICLES HAVING $\{x, p\}$ AT TIME t KNOWING INITIAL PROB.

eg. JUST x (OVERDAMPED) $d=1$ $N=1$

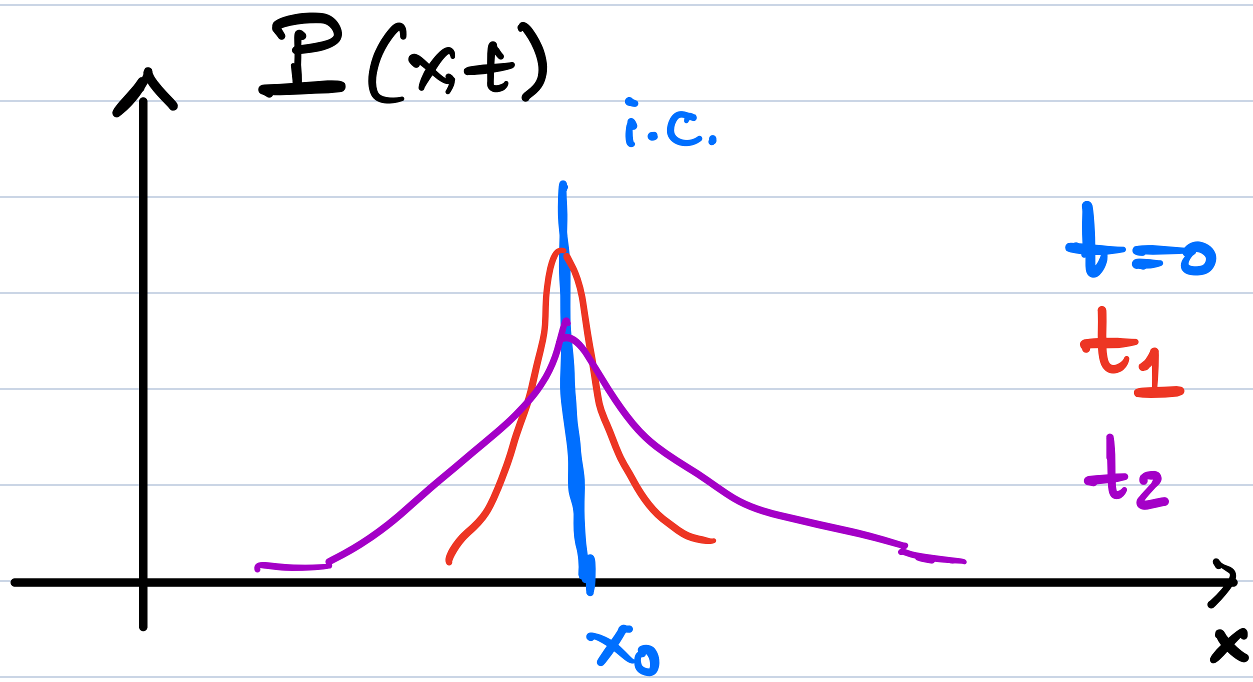


READ $\mathcal{P}(x; t)$ HERE

SKETCHES



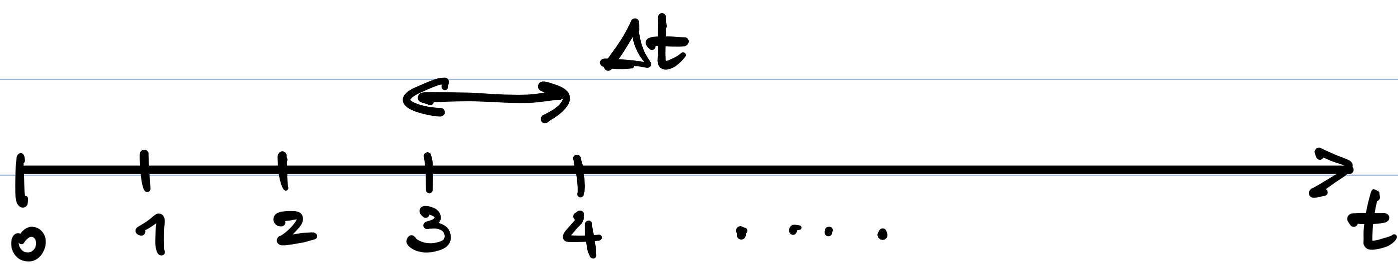
EASIER TO VISUALIZE
OVER-DAMPED LIMIT $t \gg m/\gamma_0$



HOW DO WE DERIVE AN EQ. THAT RULES THE t- EVOL OF

$$\mathcal{P}(p,x;t) \quad ?$$

THINK ABOUT THE DISCRETIZED IN TIME PROCESS



$t_k = k \cdot \Delta t$ $\qquad$ $k = 0, 1, 2 \ldots$

FOR A GENERIC NON-MARKOV PROCESS

$$\mathcal{P}(p, x; t+\Delta t) =$$

$$= \int dx' \int dp' \, \mathcal{P}(p, x; t+\Delta t \,|\, p', x', t; \text{ WHOLE HISTORY BEFORE})$$

$$\mathcal{P}(p', x'; t; \text{ WHOLE HISTORY BEFORE})$$

BUT CAN'T DO MUCH WITH THIS

MARKOV ASSUMPTION

ONLY FOR δ-CORR NOISE !

$$\mathcal{P}(p, x; t+\Delta t) =$$

$$\int dp_0 \int dx_0 \, \mathcal{P}(p, x; t+\Delta t \,|\, p_0, x_0; t) \, \mathcal{P}(p_0, x_0; t)$$

IT'S LIKE AN I.C.

THAT'S WHY I CALL IT x_0 FROM NOW ON

TRANSITION PROB.

$$\mathcal{P}(p, x; t+\Delta t \mid p_0, x_0; t) =$$

$$= \left\langle \delta(p - p_\xi(t+\Delta t))\, \delta(x - x_\xi(t+\Delta t)) \right\rangle$$

IMPOSING

$p = p_\xi(t+\Delta t)$ $\qquad$ $x = x_\xi(t+\Delta t)$

GIVEN BY A TRAJECTORY

WITH THE i.c. $\begin{cases} p(t) = p_0 \\ x(t) = x_0 \end{cases}$

AND THE EVOLUTION GIVEN BY THE LANG EQ. WITH NOISE $\xi(t)$ AT TIME t

FOCUS ON ADDITIVE NOISE CASE.

GENER. TO MULT NOISE IS POSSIBLE

TAKE OVER DAMPED LIMIT (EASIER)

GEN TO UNDER-DAMPED ALSO OK

THE UPDATING EQ FOR $\mathcal{P}(x,t)$ IS:

$$\mathcal{P}(x;t+\Delta t) = \int dx_0\, \mathcal{P}(x,t+\Delta t \mid x_0,t)\, \mathcal{P}(x_0,t)$$

WITH

$$\mathcal{P}(x;t+\Delta t \mid x_0;t) = \langle \delta(x - x_{\xi}(t+\Delta t)) \rangle$$

LET'S LOOK AT THE RHS

USE THE LANGEVIN EQ.

$$\gamma \dot{x}(t) = F[x(t)] + \xi(t)$$

AND DISCRETIZE IT:

$$x(t+\Delta t) = x(t) + \Delta x = \underbrace{x_0}_{IC} + \Delta x$$

WITH (ITÔ DISCRETIZATION)

$$\Delta x = \frac{1}{\gamma} F[x_0]\, \Delta t + \frac{\Delta t}{\gamma} \xi(t)$$

TAYLOR EXPAND THE δ:

$$\delta\big(x - x(t+\Delta t)\big) = \delta(x - x_0 - \Delta x)$$

$$\simeq \delta(x-x_0) - \delta'(x-x_0)\,\Delta x$$

$$+ \frac{1}{2} \delta''(x-x_0) \underbrace{\Delta x^2}_{\text{IMPORTANT !}} + O(\Delta x^3)$$

AND TAKE THE AVER. OVER NOISE

$$\langle \delta(x - x(t+\Delta t)) \rangle \simeq$$

$$\simeq \delta(x-x_0) - \delta'(x-x_0) \langle \Delta x \rangle + \frac{1}{2} \delta''(x-x_0) \langle \Delta x^2 \rangle$$

WITH

$$\langle \Delta x \rangle = \frac{1}{\gamma} F[x_0] \Delta t + \frac{\Delta t}{\gamma} \cancelto{0}{\langle \xi(t) \rangle}$$

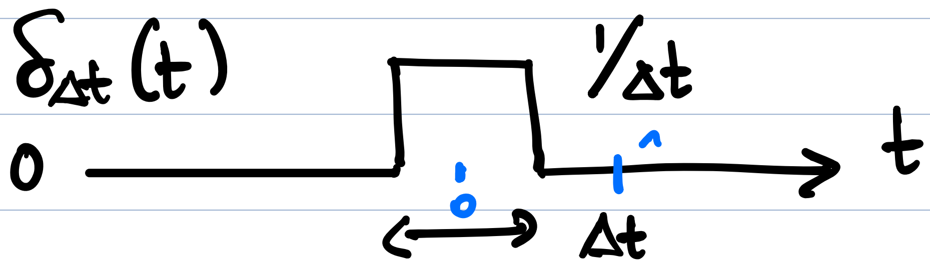
$$\langle \Delta x^2 \rangle = \frac{1}{\gamma^2} F^2[x_0] \boxed{\Delta t^2} + \frac{2}{\gamma^2} F[x_0] \Delta t^2 \cancelto{0}{\langle \xi(t) \rangle}$$

$$O(\Delta t) \Longleftarrow + \frac{\Delta t^{\cancel{2}}}{\gamma^2} \underbrace{\langle \xi^2(t) \rangle}_{\frac{2\gamma k_B T}{\cancel{\Delta t}}}$$

WHY $\langle \xi^2(t) \rangle \propto 1/\Delta t$?

DISCRETIZATION OF

$$\langle \xi(t)\,\xi(t') \rangle = 2\gamma k_B T\, \delta(t-t')$$



LIKE THIS $\quad \delta_{\Delta t}(t) \propto \dfrac{1}{\Delta t} \xrightarrow[\Delta t \to 0]{} \infty$

$$\int_{-\infty}^{\infty} dt\, \delta_{\Delta t}(t) = \frac{1}{\Delta t} \cdot \Delta t = 1$$

PROPS OF δ : OK

ALL IN ALL

$$\langle \Delta x^2 \rangle = \overbrace{\frac{1}{\gamma^2} F[x_0]\, \Delta t^2}^{\text{NEGLIGIBLE}} + \frac{\Delta t^{\cancel{2}}}{\gamma^2}\, 2\gamma k_B T \frac{1}{\cancel{\Delta t}} + \dots$$

ALSO

$$= 2 \Delta t \frac{k_B T}{\gamma} + \Theta(\Delta t^2)$$

NOW, BACK IN THE TRANSITION PROB

$$P(x_i, t+\Delta t \mid x_0 ; t) = \langle \delta(x - x_{\xi}(t+\Delta t)) \rangle$$

$$= \delta(x - x_0) - \delta'(x - x_0) \langle \Delta x \rangle + \frac{1}{2} \delta''(x - x_0) \langle \Delta x^2 \rangle$$

$$= \delta(x-x_0) - \delta'(x-x_0)\,\frac{1}{\gamma}\,F[x_0]\,\Delta t$$

$$+ \frac{1}{\cancel{2}}\,\delta''(x-x_0)\,\cancel{2}\,\Delta t\,\frac{k_B T}{\gamma} + O(\Delta t^2)$$

AND INSERTING IN THE UPDATING EQ FOR P (KEEPING $O(\Delta t)$)

$$P(x,t+\Delta t) = \int dx_0\, \underbrace{P(x,t+\Delta t|x_0,t)}_{\text{REPLACE}}\, P(x_0,t)$$

$$= \int dx_0 \Big[\delta(x-x_0) - \delta'(x-x_0)\,\frac{F(x_0)}{\gamma}\,\Delta t$$

$$+ \delta''(x-x_0)\,\frac{k_B T}{\gamma}\,\Delta t \Big] P(x_0,t)$$

INTEGRATE

$$= P(x,t) - \frac{1}{\gamma}\,\frac{\partial}{\partial x}\Big(F(x)\,P(x,t)\Big)\,\Delta t$$

$$+ k_B T/\gamma\,\frac{\partial^2 P(x,t)}{\partial x^2}\,\Delta t$$

ARRANGE ON LEFT-HAND-SIDE

$$\boxed{\gamma \frac{P(x,t+\Delta t) - P(x,t)}{\Delta t} \rightarrow \gamma \frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left(F(x)\, P(x,t) \right) + k_B T \frac{\partial^2 P(x,t)}{\partial x^2}}$$

FOKKER-PLANCK EQ.

IN PHASE SPACE

KRAMERS EQ

BOTH MOMENTUM & POSITION

UNDERDAMPED

$$\frac{\partial P}{\partial t} = \underbrace{- \frac{p}{m} \frac{\partial P}{\partial x} + \frac{\partial}{\partial p}\left(-F P\right)}_{\text{HAMILT. FLOW}}$$

$$\underbrace{+ \gamma \frac{\partial}{\partial p}(p P) + \gamma k_B T \frac{\partial^2 P}{\partial p^2}}_{\text{DISSIPATIVE PART}}$$

BOTH CASES

ONLY FOR MARKOV!

IF COLOURED NOISE $\Longrightarrow$ NO LOCAL - IN TIME DIFF EQ LIKE THESE CAN BE DERIVED.

STILL USEFUL SINCE MANY CASES HAVE WHITE - UNCORRELATED NOISE NO MEMORY

EXTENSION TO MULT. NOISE $\Longrightarrow$ OK.
BUT DISCRETIZATION HAS TO BE DONE CAREFULLY

WHAT IS IT USEFUL FOR ?
SOME CALC. ARE EASIER &

45'

PROOFS OF APPROACH TO EQUIL

GIBBS – BOLTZMANN

$$F = -V'(x) \Longrightarrow$$

$$P(x,p) = \frac{e^{-\beta H_{syst}(\{x,p\})}}{Z(\beta)}$$

$$H_{syst} = K(p) + V(x)$$

IS A STATIONARY SOLUTION $\frac{\partial P(x,p;t)}{\partial t} = 0$

BUT IF WE DON'T HAVE KRAMERS / FP ?

IT'S HARDER TO PROVE THAT THE LANGEVIN PROCESS TAKES TO GIBBS-BOLTZ.

IF CONFINING POT $V(x)$ AND NOISE IN EQUIL $\Longrightarrow$ IT DOES, EVEN FOR COLOURED NOISE

IF MULT. WHITE NOISE $\Longrightarrow$ SOME ISSUES.

t_{EQ} FINITE BUT COULD BE $\boxed{t_{EQ}(N)}$

LINDBLAD EQ — QUANTUM

TAKE A QUANTUM SYSTEM IN A MIXED STATE

$$\hat{\rho} = \sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|$$

BASIS OF HILBERT SPACE

$$\sum_\alpha p_\alpha = 1$$

IT'S AN OPERATOR $\Rightarrow$ FOLLOWS HEISENBERG

$$\frac{d\hat{\rho}(t)}{dt} = -\frac{i}{\hbar}\left[\hat{H}, \hat{\rho}(t)\right]$$

IF SYST IS ISOLATED FROM REST OF THE WORLD

TAKE SYST + ENV + COUPLING

$$\hat{H} = \hat{H}_{syst} + \hat{H}_{BATH} + \hat{H}_{COUPLING}$$

COUPLES SYST'S OPERATOR, SAY $\hat{x}$ TO

BATH OPERATORS, SAY $\{\hat{b}_a\}$

INTEGRATE AWAY THE $\{\hat{b}_a\}$S $\Longrightarrow$ REDUCE

DEFINE $\hat{\rho}^{RED}_{SYST} = \mathrm{Tr}_{\substack{BATH \\ \{\hat{b}_a\}}} \hat{\rho}$

INTEGRATE AWAY OPERATORS FROM BATH.

$\Longrightarrow$ UNDER SIMILAR ASSUMPTIONS TO

THE ONES OF LANGEVIN EQ'S DERIV

- ASSUME COUPLING SWITCHED ON AT $t=0$

- ASSUME BATH MUCH FASTER

  AND MUCH LARGER $N_b \gg 1$ THAN SYST

- MARKOV APPROX.

DERIVE LINDBLAD'S EQ:

$$\frac{d\hat{\rho}^{RED}_{SYST}}{dt} = -\frac{i}{\hbar}\left[\hat{H}_{SYST}, \hat{\rho}^{RED}_{SYST}\right]$$

$$+ \underbrace{\sum_{mn} h_{mn} \left(\hat{L}_n \hat{\rho}^{RED}_{SYST} \hat{L}^{\dagger}_m - \frac{1}{2}\left\{ \hat{L}^{\dagger}_m \hat{L}_n, \hat{\rho}^{RED}_{SYST} \right\} \right)}_{\text{FROM THE COUPLING TO BATH.}}$$

IN PRACTICE: CHOOSE $\{\hat{L}_m\}$

WEISS 2000, BAUER PETRUCCIANI

SCHIRÒ & FAZIO SCIPOST 2025

J. MARINO ET AL 2025

PROBLEMS

- IT DOES NOT HAVE $\frac{e^{-\beta \hat{H}_{syst}}}{Z}$ AS ASYMPT. STAT. SOLUTION !

- IT'S LOCAL IN TIME ⟹ NO MEMORY

SIMILAR TO KRAMERS / FOKKER_PLANCK

SOL. STILL ⟹ OK FOR Q OPTICS, ETC.

AND CAN GIVE SOME IDEAS

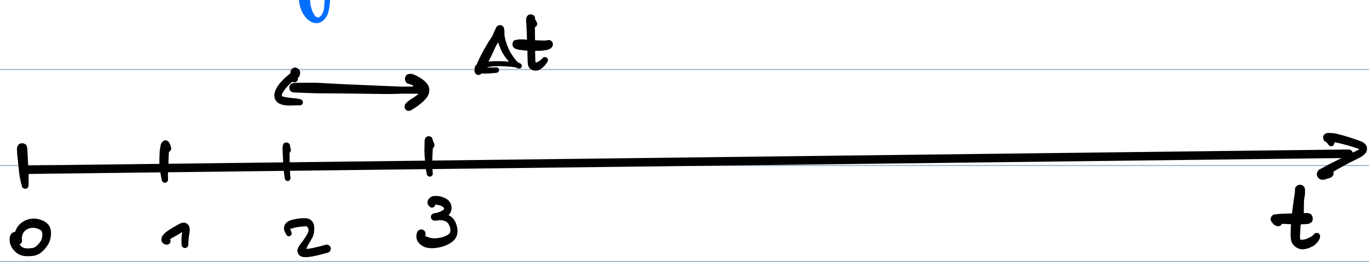
⟹ MY SEMINAR ON QUANTUM ACTIVE

FUNCTIONAL FORMALISM

MSR - JD CLASSICAL

$\langle A(x(t)) \rangle_\xi$ WE WANT TO CALC. SUCH AVER. W/ $A(x)$ A FUNCTION - OBSERV. OF x

SIMPLIFY AND USE OVERDAMPED



$$t_\ell = \ell \cdot \Delta t$$

$x_\xi(t)$ SOLUTION TO LANG. EQ. STRESS ξ DEP.

SO $\langle A(x_\xi(t)) \rangle_\xi$

SKETCH OF CONSTRUCTION

$$\langle A(x_\xi(t)) \rangle_\xi =$$

$$= \underbrace{\int \mathcal{D}\xi \, \mathbb{P}(\{\xi\})}_{} \, A(x_\xi(t))$$

AVERAGE OVER ALL NOISE REALIZATIONS WITH THEIR WEIGHT

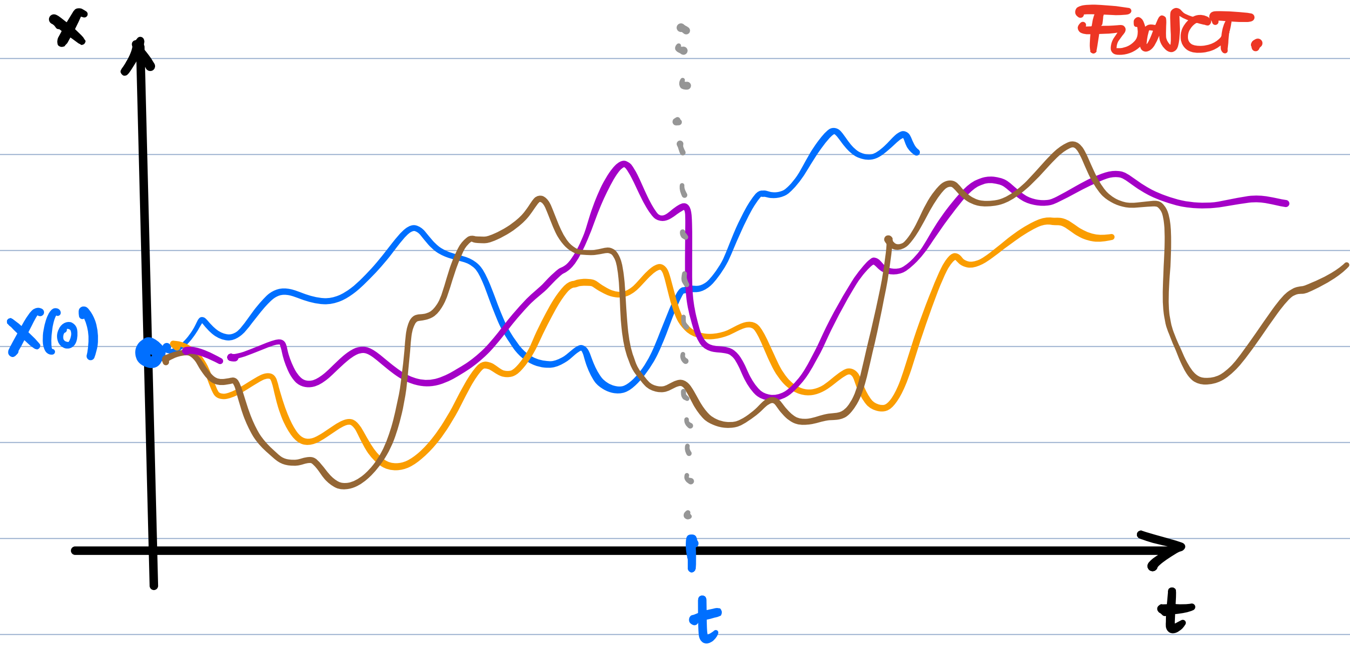
$$\int \underbrace{\prod_{l=0} d\xi(t_l)} \, \underbrace{e^{-\frac{1}{k_B T} \sum_{ll'} \xi_l \Gamma^{-1}_{ll'} \xi_{l'}}}$$

DISCRETIZED INT MEASURE

DISCRETIZED GAUSSIAN WEIGHT

FOR COLOURED NOISE IT MIXES ξ_l AT DIFF TIMES THROUGH KERNEL

FOR GENERATING FUNCT.



$$Z_{dyn}(\lambda) = \left\langle e^{\int_0^t dt' \, \lambda(t') \, x_\xi(t')} \right\rangle_\xi$$

MEANS A SUM OVER ALL REALI'Z OF ξ

WITH THEIR

$$P[\{\xi(t)\}]$$

AND $x_\xi(t)$ SOL. OF LANG EQ.

THE TRICK

WHICH I'LL NOT EXPLAIN FOR LACK OF TIME

TRANSFORM THIS PATH INTEGRAL

INTO ONE WHERE ONE INTEGRATES OVER

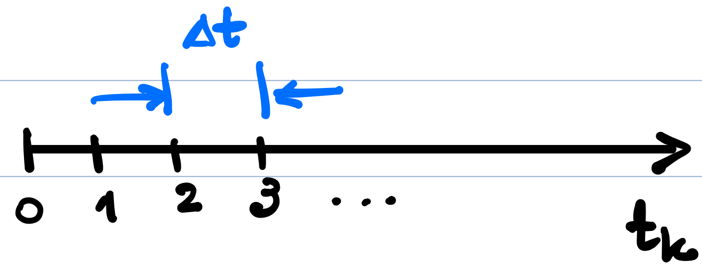
$$\underbrace{x(t)}_{\text{ORIGINAL VAR.}} \quad \text{AND} \quad \underbrace{i\hat{x}(t)}_{\text{AUXIL. VAR.}}$$

AND OBTAIN A GENERATING FUNCT.

$$Z_{dyn}[\lambda] = \int \mathcal{D}x \int \mathcal{D}i\hat{x}\; e^{-S[x,i\hat{x}]}\, e^{\int dt\, \underbrace{\lambda(t)}_{\text{SOURCE}} A(x(t))}$$

HINTS ON HOW TO OBTAIN THE ACTION S

- DISCRETIZE TIME



- DISCRETIZE LANGEVIN EQ.

- IF INERTIA PRESENT OR COLOURED NOISE
  $\neq$ PRESCRIPTIONS ARE EQUIVALENT FOR
  $\Delta t \to 0$

- IF OVERDAMPED & WHITE NOISE :
  CAREFUL !
  SPECIALLY WHEN CONSTRUCTING THE GEN. FUNCT.

- IMPOSE THE DISC. LANG. EQ W/ δ

- EXPON. δ ~ FOURIER TRANSF.

- INTEGRATE OVER NOISE

ARNOULX DE PIREY, LFC, LECOMTE, VAN WIJLAND 22

THE INFLUENCE FUNCTIONAL

THE ZWANZIG-KAWASAKI WITHIN PATH INT.

$$Z_{dyn}[\lambda] = \int \prod_a dq_a \prod_a d\pi_a \int dx \int dp \; e^{\int_0^t dt' \lambda(t') x(t')}$$

$$\exp\left\{- S\left[\{x, p ; q_a, \pi_a\}\right]\right\}$$

λ SOURCE

INTEGRATE AWAY THE $\{\pi_a, q_a\}$

$$Z_{dyn}[0] = \int dx \int dp \; \exp\left\{- S\left[\{x, p\} ; k_B T, \Gamma\right]\right\}$$

$$S = S_{syst}\left[\{x, p\}\right] + \underbrace{S_{INF}\left[\{x, p\} ; k_B T, \Gamma\right]}$$

BATH EFFECT $\Rightarrow$ INFLUENCE FUNCT.

THIS IS A FORMALISM WHICH CAN BE GENERALIZED QUANTUM MECHANICALLY.

FEYNMAN - VERNON 60s

FOR EXAMPLE, FOR A SINGLE VARIABLE BILINEARLY COUPLED TO AN OSCILLATOR BATH

CLASSICALLY

$$S[x, i\hat{x}] = \underbrace{\ln P_0(x(0), \dot{x}(0))}_{\text{INITIAL COND.}}$$

$$- \int_0^t dt' \, i\hat{x}(t') \underbrace{\left\{ m\ddot{x}(t') + \int_0^{t'} dt'' \, \Gamma(t'-t'') \, \dot{x}(t'') - F[x(t')] \right\}}_{\text{LANG EQ. (APART FROM NOISE)}}$$

$$\underbrace{+ \frac{k_B T}{2} \int_0^t dt' \int_0^t dt'' \, i\hat{x}(t') \, \Gamma(t'-t'') \, i\hat{x}(t'')}_{\text{INTEGRATION OVER NOISE}}$$

$$+ \underbrace{\ln J[x]}$$

JACOBIAN - HAS TO DO WITH DISCRETIZATION USED TO DEFINE THE STOCH PROCESS AND BUILD THE PATH INTEGRAL

WHY IS THIS FORMALISM USEFUL? EXACT NO APPROX.!

- PERTURB. TH.
- SADDLE-POINT / MEAN-FIELD
- SYMMETRIES → WARD IDENTITIES, e.g. FLUCT THS.

CORRELATIONS

$$\langle x(t)\, x(t')\rangle = \int \mathcal{D}x\, \mathcal{D}i\hat{x}\; e^{-S[\;]}\, x(t)\, x(t')$$

LINEAR RESPONSE

$$\left.\frac{\delta \langle x(t)\rangle_h}{\delta h(t')}\right|_{h=0} = \frac{\delta}{\delta h(t')} \int \mathcal{D}x\, \mathcal{D}i\hat{x}\; e^{-S_h[\;]}\, x(t)\Big|_{h=0}$$

IN:

$$F_h[x(t)] = F[x(t)] + h(t)\, x(t)$$

RECALL $\quad e^{\int_0^t dt''\, i\hat{x}(t'')\, F_h[x(t'')]}\quad$ IN PATH INTEG.

$$\Longrightarrow \left.\frac{\delta \langle x(t)\rangle_h}{\delta h(t')}\right|_{h=0} = \int \mathcal{D}x\, \mathcal{D}i\hat{x}\; e^{-S[\;]}\, x(t)\, i\hat{x}(t')$$

$$= \langle x(t)\, i\hat{x}(t')\rangle$$

CAUSALITY, $\neq 0$ $t \geq t'$

RESPONSE FIELD

SYMMETRIES

PROVE FDT IF

$P_0(x(0), i\dot{x}(0))$ MAXWELL-BOLTZM.

$F[x] = -V'[x]$ POTENTIAL FORCE WITH THE SAME $V[x]$ USED TO DRAW i.c.

$\Gamma(t-t')$ THE SAME IN FRICTION AND NOISE CORR.

PROOF. IT'S BASED ON THE INVAR. OF ACTION UNDER SIMULT TRANSF OF $\{x(t), i\hat{x}(t)\}$

$$S[x(t), i\hat{x}(t)] = S[\underbrace{x(-t)}_{\uparrow}, \underbrace{i\hat{x}(-t)}_{\uparrow} + \underbrace{\beta \frac{d}{dt} x(-t)}_{\text{TRANSL.}}]$$

TIME REVERSAL

ALSO THE MEASURE $\mathcal{D}x\, \mathcal{D}i\hat{x}$ IS INVARIANT & INTEGRATION INTERVAL TOO

CONSEQUENCE ON CORRELATIONS & LINEAR RESP.²

NEWTON INFLUENCE

NOTE THAT S_{syst} & S_{Diss} ARE SEPARATELY
+ JAC IN VARIANT

$$\langle x(-t)\, i\hat{x}(-t') \rangle =$$

ALL THE $\langle \cdots \rangle$ OPERATION IS INV. SINCE S & MEASURE ARE

$$\left\langle x(t) \left[i\hat{x}(t') - \beta \frac{d}{dt'} x(t') \right] \right\rangle \Longrightarrow$$

$$\langle x(t)\, i\hat{x}(t') \rangle = \langle x(-t)\, i\hat{x}(-t') \rangle + \beta \langle x(t)\, \dot{x}(t') \rangle$$

FDT

$$R(t,t') = R(-t,-t') + \beta \frac{\partial}{\partial t'} C(t,t')$$

THEREFORE, IF FDT $\Longrightarrow$ EQUIL. À LA MAXWELL BOLTZMANN

THEN THESE PROCESSES DO TAKE SYST TO EQUIL. UNDER ABOVE CONDITIONS

QUANTUM

VERY SIMILAR !

COMPLEX PLANE

FEYNMAN'S

PATH INT → STAT MECH.

$e^{-\beta\hat{H}}/\hat{Z} = \hat{\rho}_{in}$

$-it/\hbar \to -\beta$

$t = -i\tau$

$\tau = \beta\hbar$

IN ORDER TO CALCULATE CORR. LIKE

$$\langle \hat{A}(t)\hat{B}(t') + \hat{B}(t')\hat{A}(t) \rangle$$

NO TIME ORDERED, TRICK GO FORTH & BACK.

$$\frac{x^+(t) + x^-(t)}{2} \xrightarrow[\hbar \to 0]{} x(t)$$

$$\frac{x^+(t) - x^-(t)}{\hbar} \xrightarrow[\hbar \to 0]{} i\hat{x}(t)$$

BATH:

$$\tilde{\Gamma}(t-t') = \int_0^\infty d\omega\ S(\omega)\ \coth\left(\frac{\beta\hbar\omega}{2}\right) \cos\omega(t-t')$$

$$\coth x : \lim_{x\to 0} \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{2}{\cancel{1}+x-\cancel{1}+x} = \frac{1}{x}$$

$$\lim_{\beta\hbar\to 0} \tilde{\Gamma} = \int_0^\infty d\omega\ S(\omega)\ \frac{2}{\beta\hbar\omega} \cos\omega(t-t')$$

$$= \frac{k_B T}{\hbar} \underbrace{\int_{-\infty}^{\infty} d\omega\ \frac{S(\omega)}{\omega} \cos\omega(t-t')}_{\Gamma(t-t')}$$

THIS KERNEL APPEARS AS A TERM IN $e^{\frac{i}{\hbar} S_{diss}}$

NOISE

$$\frac{iS_{diss}}{\hbar} = \frac{(-1)}{\hbar}\int_0^t dt' \int_0^t dt'' \underbrace{\left(x^+(t') - x^-(t')\right)}_{i\hat{x}(t')\hbar} \tilde{\Gamma}(t'-t'') \underbrace{\left(x^+(t'') - x^-(t'')\right)}_{i\hat{x}(t'')\hbar}$$

CLASSICAL

$$\xrightarrow[\hbar\to 0]{} -\frac{\cancel{\hbar^2}}{\cancel{\hbar}} \frac{k_B T}{\cancel{\hbar}} \int_0^t dt' \int_0^t dt''\ i\hat{x}(t)\ \Gamma(t-t')\ i\hat{x}(t'')$$

THERE IS ANOTHER TERM THAT DOES NOT DEP. ON TEMP. AND PLAYS THE RÔLE OF THE RETARDED FRICTION (RESPONSE OF BATH) RELATED BY QUANTUM FDT TO THIS ONE

$$\gamma'(\omega) = -\,\Theta(t-t') \int_0^\infty d\omega\, S(\omega)\, \sin\omega\,(t-t')$$

$$= \frac{1}{4\hbar} \lim_{\varepsilon\to 0^+} \int \frac{d\omega'}{\pi} \frac{1}{\omega-\omega'+i\varepsilon}\, th\, \frac{\beta\hbar\omega'}{2}\, 2\hbar\, \tilde{F}(\omega')$$

APPLICATIONS

— NOISE-INDUCED PH. TRANS

HOT TOPIC IN QUANTUM

SCHWINGER - KELDYSH

\+ FEYNMAN VERNON

MSR-JD

COMPLETE FORMALISM THAT LET'S US TREAT ANY KIND OF BATH - NOT ONLY OHMIC

SYMMETRIES

TRANSFORMATION OF PATHS IN THE COMPLEX PLANE

C. ARON, G. BIROLI & LFC 18

Summary 1st & 2nd lectures

- I justified the generalized Langevin eq. beyond Brownian motion
  generic memory kernel & noise-corr related by FDT if eq. bath

- sketched the classical gen. funct. construction

- discussed its time-rev. symm. $\Rightarrow$ FDT & then equil. I haven't done it see notes if inter.

- sketched the construct of quantum gen fct - Schwinger-Keldysh w/ generic bath - Feynman-Vernon
  not done either

- the kernels for a quantum osc baths can be calculated as well

IMPORTANT

ALL THESE FORMALISMS SHOULD ADMIT (IF BATH IN EQ.) $F = -V'$ AND

TO TAKE SYST FOR $t \gg t_{eq}$ (PARAM.) TO CANON. EQUIL

$$\frac{e^{-\beta H_{syst}}}{Z(\beta)}$$

NOT EASY IN GENERAL

→ OK FOR FOKKER PLANCK / KRAMERS

→ SYMM. IN Zdyn FOR COLOURED NOISE

LINDBLAD ... UHM ...

t_{eq} ?? COULD BE VERY LONG AND SYST SIZE-DEPENDENT

⇒ LONG OUT OF EQ. RELAXATION

STILL FOR EQ. BATH & $F = -V'$

3rd LECTURE

LET'S DISCUSS SOME PHYSICAL PROBLEMS NOW.

1- SEP. OF TIME SCALES p VS. x $\Rightarrow$ EX.

2- DROP INERTIA, JUST x. SEPARATION OF TIME SCALES IN ITS OWN BEHAVIOUR.
NOT SO SURPRISING. THINK ABOUT RW AGAIN, FOR

$t < t_{eq}^{(p)}$ $\Rightarrow$ BALLISTIC $x(t) \simeq \frac{p}{m} t$

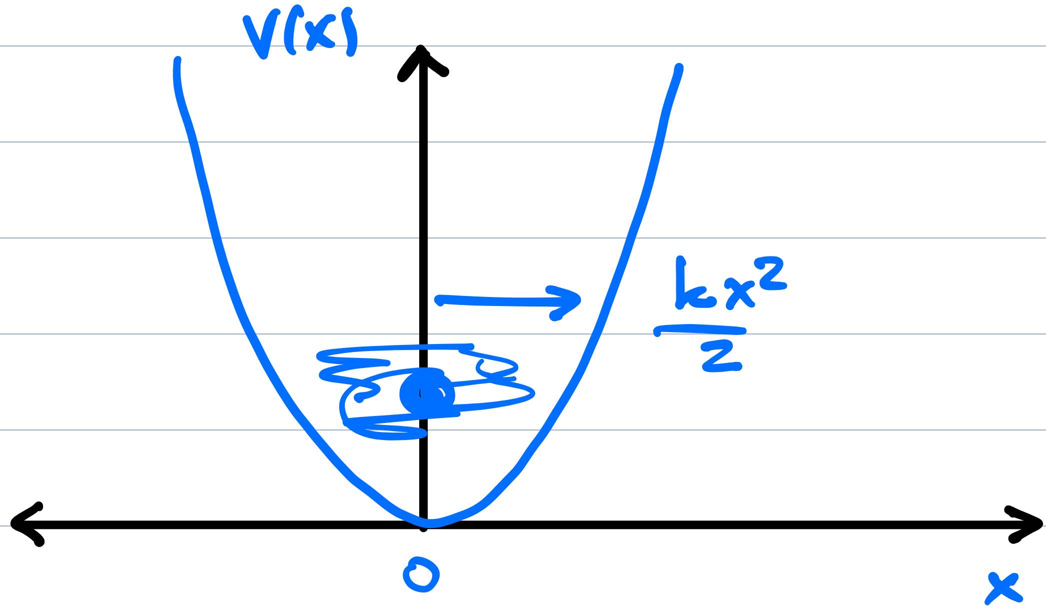
$t > t_{eq}^{(p)}$ $\Rightarrow$ DIFFUSIVE WHEN NOISE CONTROLS IF $V=0$

LET'S LOOK AT ANOTHER SIMPLE EX IN THE OVERDAMPED LIMIT

$$\gamma \dot{x} = -kx + \xi + h$$

TO MEASURE LINEAR RESP.

A PARTICLE IN A HARM WELL UNDER ADD. WHITE NOISE



$t_{eq}^{(x)} = \frac{\gamma}{k}$

$x_0 = 0$

$$x(t) = \cancel{x_0}^{\,0}\, e^{-k/\gamma\, t} + \frac{1}{\gamma}\int_0^t dt''\, e^{-k/\gamma\,(t-t'')} \left[\xi(t'') + h(t'')\right]$$

$$\langle x(t)\rangle_h = \frac{1}{\gamma}\int_0^t dt''\, e^{-\frac{k}{\gamma}(t-t'')}\, h(t'')$$

LINEAR RESPONSE TO AN INSTANT KICK

$$R(t,t') = \left.\frac{\delta\langle x(t)\rangle_h}{\delta h(t')}\right|_{h=0} = \frac{1}{\gamma}\, e^{-\frac{k}{\gamma}(t-t')}\, \underbrace{\theta(t-t')}_{\text{CAUSAL}}$$

WE NOTE THAT

$$R(t,t') = \bar{R}(t-t')$$

ABUSE OF NOTATION $R(t-t')$

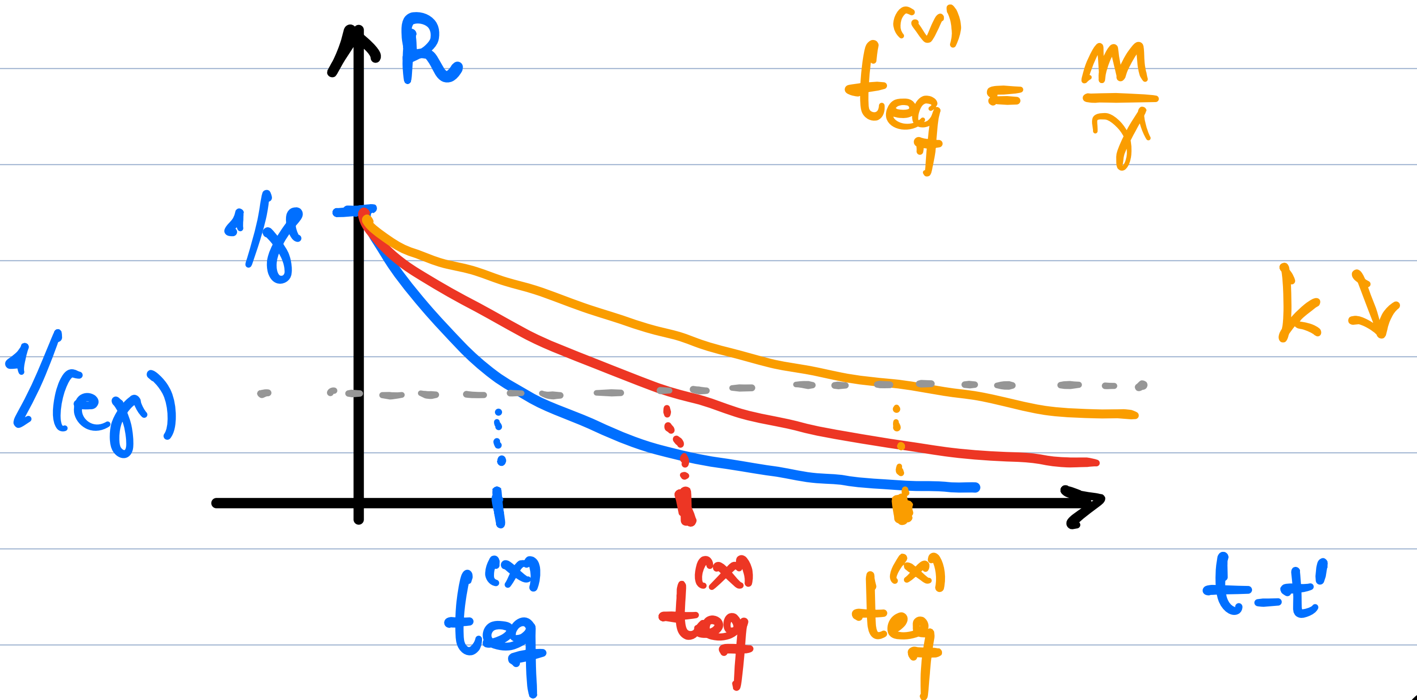
STATIONARY TIME-TRANSL. INVARIANT

$$t \to t+\Delta \qquad t' \to t'+\Delta$$

$$\boxed{t_{eq}^{(x)} = \frac{\gamma}{k}}$$ TIME SCALE FOR RELAX.

NB $t_{eq}^{(x)} \xrightarrow[k\to 0]{} \infty$

$$t_{eq}^{(v)} = \frac{m}{\gamma}$$



FOR DECREASING k THE $t_{eq}^{(x)}$ INCREASES

$$R(t-t') \xrightarrow[t \gg t_{eq}^{(x)}]{} 0$$

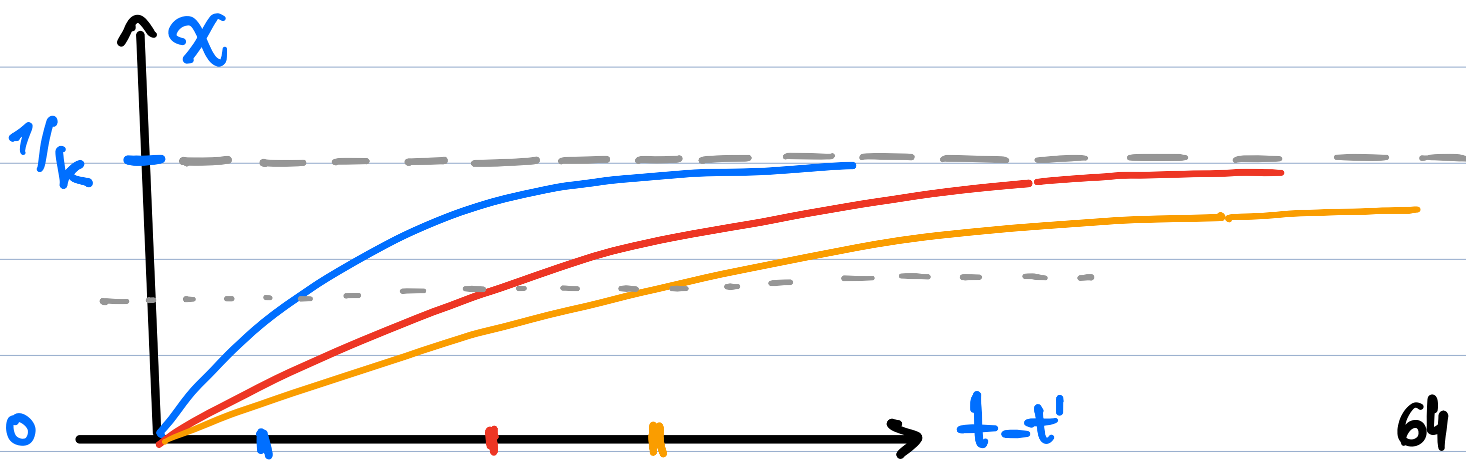
INTEGRATED LINEAR RESPONSE $\chi(t,t)=0$

$$\chi(t,t') = \int_{t'}^{t} dt'' \, R(t,t'')$$

$$= \int_{t'}^{t} dt'' \, \frac{1}{\gamma} e^{-\frac{k}{\gamma}(t-t'')}$$

$$= \frac{1}{\cancel{\gamma}} \frac{\cancel{\gamma}}{k} \left(1 - e^{-\frac{k}{\gamma}(t-t')}\right)$$

$$\xrightarrow[t-t' \gg t_{eq}^{(x)}]{} 1/k$$



SELF-CORRELATION

$$\langle x(t)\, x(t') \rangle_{h=0} = \frac{k_B T}{k}\left[e^{-\frac{k}{\gamma}|t-t'|} - \underbrace{e^{-\frac{k}{\gamma}(t+t')}}_{\to 0} \right]$$

STATIONARY

SAME $t^{(x)}_{eq} = \frac{\gamma_0}{k}$

$t+t' \to 0$
iff $k \neq 0$

$$C(t,t') \xrightarrow[t' \gg \tau]{} \frac{k_B T}{k}\, e^{-\frac{k}{\gamma}|t-t'|}$$

$$= C(t-t')$$

STATIONARY

TAKE A t'-DERIV.

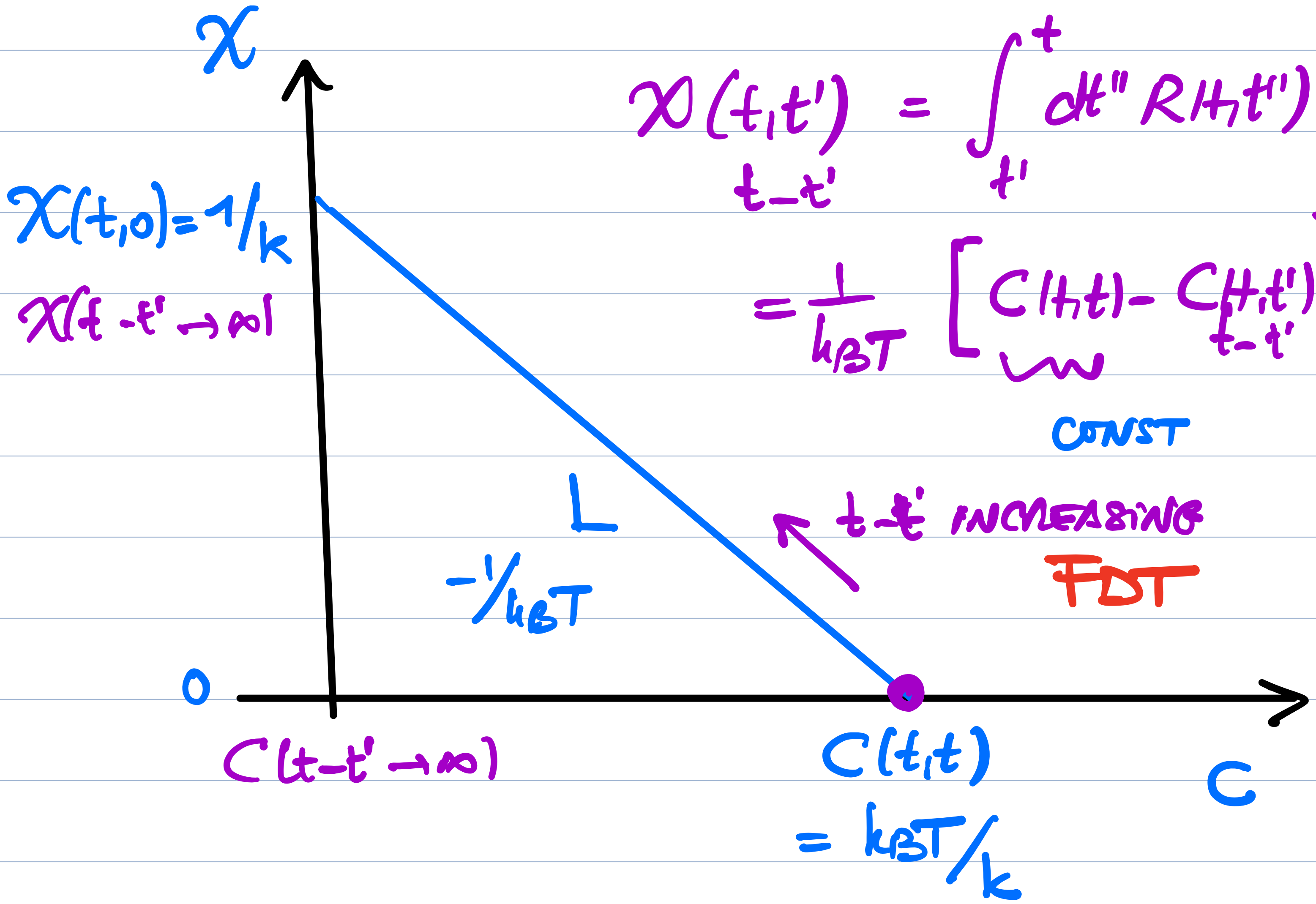
$$\frac{\partial \langle x(t)\, x(t') \rangle}{\partial t'} \xrightarrow[t>t']{} \frac{k_B T}{\cancel{k}} \frac{\cancel{k}}{\gamma}\, e^{-\frac{k}{\gamma}(t-t')} \qquad t>t'$$

$$\partial_{t'} C(t,t') = \frac{k_B T}{\gamma}\, e^{-\frac{k}{\gamma}(t-t')}$$

cfr
$$\boxed{R(t,t') = \frac{1}{k_BT}\,\partial_{t'} C(t,t') \qquad \text{FDT}} \qquad \forall\, t>t'$$

$$\underbrace{\chi(t,t')}_{t-t'} = \int_{t'}^{t} dt''\, R(t,t'')$$

$$= \frac{1}{k_BT}\Big[\underbrace{C(t,t)}_{\text{CONST}} - \underbrace{C(t,t')}_{t-t'}\Big]$$



FDT is ACTUALLY A <u>MODEL INDEP</u> RESULT, VALID ∀ SYST EVOLVING IN EQUIL. FOR ANY COLOURED NOISE

EXERCISE

STUDY WHAT HAPPENS IF YOU TAKE

$k \to 0$

NO CONFINING POTENTIAL

RANDOM WALK $\to$ DIFFUSION

$$C(t,t) = \langle x^2(t) \rangle \longrightarrow 2Dt$$

IS NOT A CONSTANT

$$\Rightarrow \quad C(t,t) \neq C(t-t) = C(0)$$

NO STAT., NO FDT $\Rightarrow$ NO EQUIL

$$\boxed{\text{FLAT DIRECTIONS IN THE POTENTIAL MAKE THE EQUIL. TIME DIVERGE}}$$

WHAT HAPPENS IF YOU COUPLE THE SAME PART IN A HARM WELL TO TWO $\neq$ BATH ?

CHOICE

$$\Gamma_f(t-t') = 2\gamma_f\,\delta(t-t')$$

$$\Gamma_s(t-t') = \frac{\gamma_s}{\tau_s}\,e^{-|t-t'|/\tau_s}$$

$$\tau_f \ll \tau_s$$

WE HAD $t_{eq}^{(x)} = \frac{\gamma}{k}$ BUT NOW IT'S CLEAR THAT IT'LL BE MORE COMPLEX

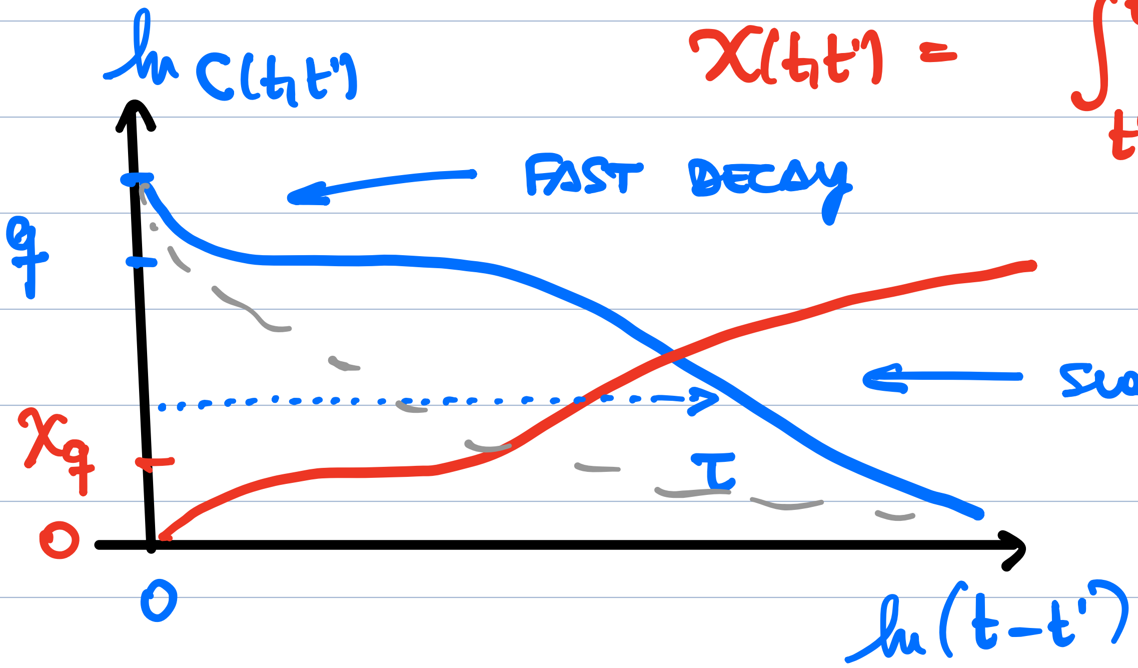
$$\int_0^t dt' \left[\Gamma_s(t-t') + 2\gamma_f\,\delta(t-t')\right]\dot{x}(t') =$$

$$= -k\,x(t) + \xi(t) + \eta(t) + h(t)$$

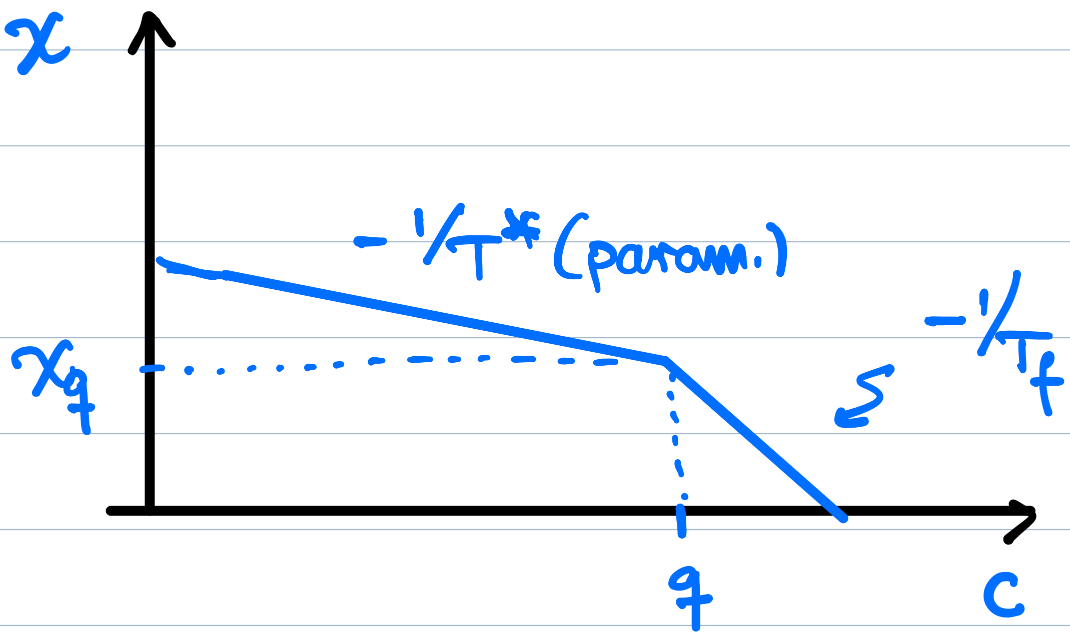
STILL A LINEAR 1st ORDER DIFF EQ.
SOLVABLE ANALYTICALLY.

CHOOSE PARAM TO GET SEP. OF TIME SCALES

$\chi(t,t') = \int_{t'}^{t} dt'' \, R(t,t'')$

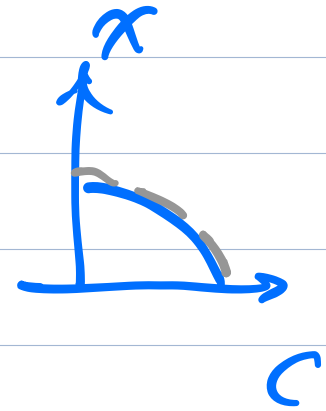


TWO STEP RELAX.



WITH TWO TEMP IN FDT MEASUREM.

IMAGINE A SYST SUCH THAT



$$\lim_{\substack{t, t' \gg t^* \\ C(t,t') = C}} X(t,t') = X(C)$$

ONE CAN DEFINE THE

EFFECTIVE TEMPERATURE

AS:

$$\frac{-1}{k_B T_{eff}(C)} = X'(C)$$

- IN THE EQUIL. CASE $X(C) = \frac{1}{k_B T}\left(C(0) - C\right)$

$$T_{eff}(C) = T$$

- IN THE TWO SCALE CASE $X(C)$ PIECE WISE

$$T_{eff}(C) = \begin{cases} T_f & C > q \\ T_s & C < q \end{cases}$$

FINALLY COMPLEX SYSTEMS, BEYOND SINGLE VARIABLE.

DMFM WHAT IS IT?

METHODS TO DEAL WITH DYNAMICS IN MEAN-FIELD.

GROSSO MODO. SINGLE OUT ONE VARIABLE OF YOUR WHOLE SYSTEM AND IDENTIFY SELF-CONSISTENTLY WHICH IS THE FRICTION & NOISE-NOISE KERNEL THAT SHOULD GO IN THE GEN. LANG. EQ.

CALL x THE CHOSEN VARIABLE

$$m\ddot{x} + \underbrace{\gamma \dot{x}}_{\text{EXT BATH}} = \underbrace{\int_0^t dt' \, \Sigma(t,t')\, x(t')}_{\text{INTERNAL EFFECT}} + \underbrace{\xi + \eta}_{\text{EXT BATH}}$$

$$\langle \xi(t)\, \xi(t') \rangle = 2\gamma k_B T\, \delta(t-t')$$

$$\langle \eta(t)\, \eta(t') \rangle = D(t,t')$$

NB Σ is like $\dfrac{d\Gamma(t-t')}{dt'}$ a "RESPONSE"

D is like $\Gamma(t-t')$ a "CORR"

WHO ARE Σ AND D ?

IN "MEAN-FIELD" MODELS THEY ARE FCTS OF $\{C, R\}$

$C = \langle x(t)\, x(t') \rangle$ SELF-CORR

$R = \left. \dfrac{\delta \langle x(t) \rangle}{\delta h(t')} \right|_{h=0}$ LINEAR RESP.

OF THE SYST ITSELF

eg. P-SPIN DISORDER MODEL $p \geq 3$

$$H_{syst} = -\sum_{i_1 < \dots < i_p} J_{i_1 \dots i_p} x_{i_1} \cdots x_{i_p}$$

$$\sum_{i=1}^{N} x_i^2 = N \quad \text{SPH. CONST}$$

$J_{i_1 \dots i_p}$ QUENCHED RANDOM COUPLINGS

GAUSSIAN DIST CONVENIENTLY NORMALIZED

$$D(t,t') = J^2 \frac{p}{2} C^{p-1}(t,t')$$

$$\Sigma(t,t') = J^2 \frac{p(p-1)}{2} C^{p-2}(t,t') R(t,t')$$

cfr J. KENT-DOBIAS TALK

eg F. VAN WIJLAND'S LECTURES

OF COURSE, THIS IS A MUCH HARDER PROBLEM TO SOLVE!

- SIMILAR STRUCT. CLASSICALLY & QUANTUM

- CAN BE PUT IN A COMPUTER

- CAN BE USED TO WRITE CLOSED EQS.
  FOR $\{C, R\}$ $\longrightarrow$ SCHWINGER - DYSON EQS.
  BBGKY HIERARCHY CLOSED FOR MF MODELS

PHYSICS ? GLASSY MODELS.

THESE MODELS / EQS. HAVE A

DYN. PHASE TRANS.

- $T > T_d$

RELAX. AS <u>IN THERMAL EQ.</u>

$t > t_{eq}$ $(N \to \infty)$: FINITE

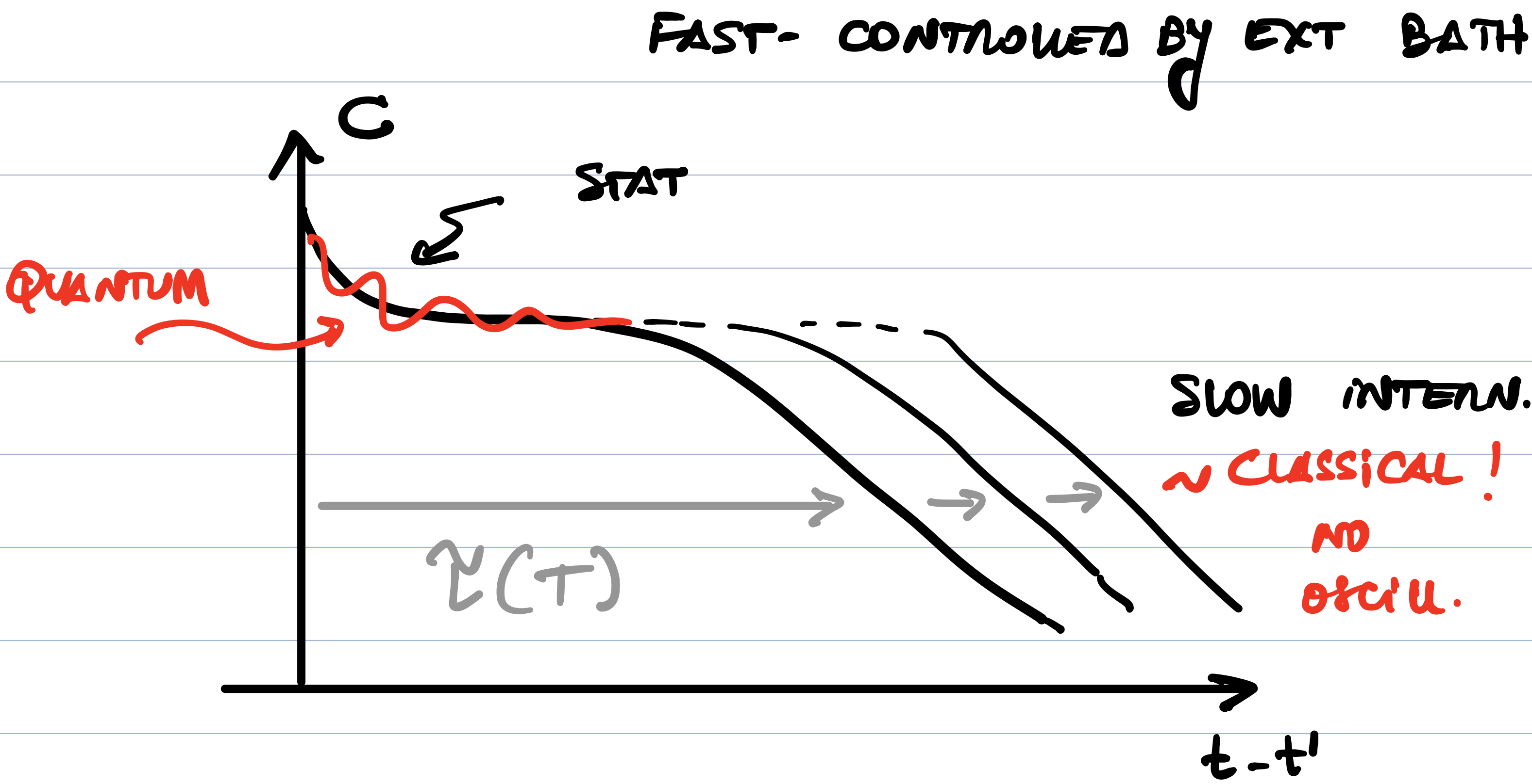
STILL, SLOW RELAX. FOR $T \gtrsim T_d^+$

$\sim$ LIKE CRITICAL SLOWING DOWN
IN USUAL PH. TRANS.

FDT HOLDS.

TWO SCALE RELAX. $\Longrightarrow$ SELF INDUCED.

$T > T_d$



$\tau(T) \uparrow$ FOR $T \searrow$ TOWARDS T_d^+

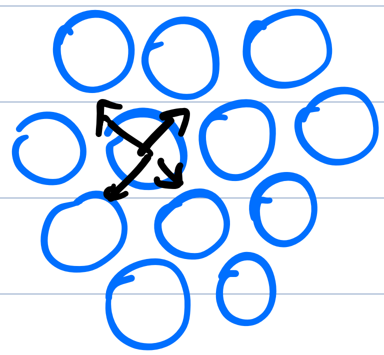
DIFFERENT CURVES FOR $\neq T$

$$\chi(C) = -\frac{1}{k_B T}\left(C(0) - C\right) \qquad \text{FDT}$$

QUANTUM VERSION HOLDS TOO.

HOW DOES ONE UNDERSTAND THE SEP. OF TIME SCALES ?

NB C is A GLOBAL CORR.



SHORT TIME DIFF.

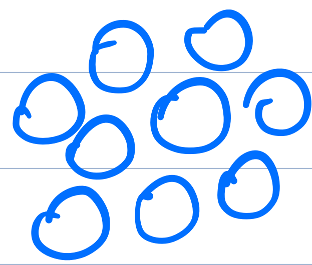
$t-t' < \tau(T)$

ONLY FLUCT. WITHIN CAGES

LONG TIME - DIFF

$t-t' > \tau(T)$

STRUCT RELAX



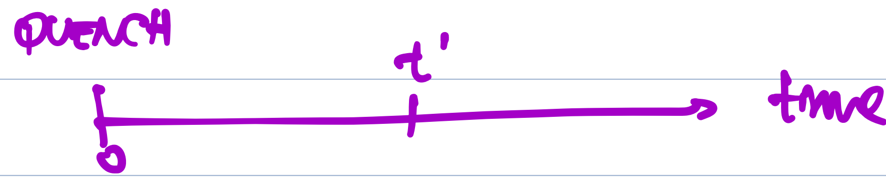
DIFF ORGAN. OF PART.

BUT ALL TAKES PLACE IN EQUIL.

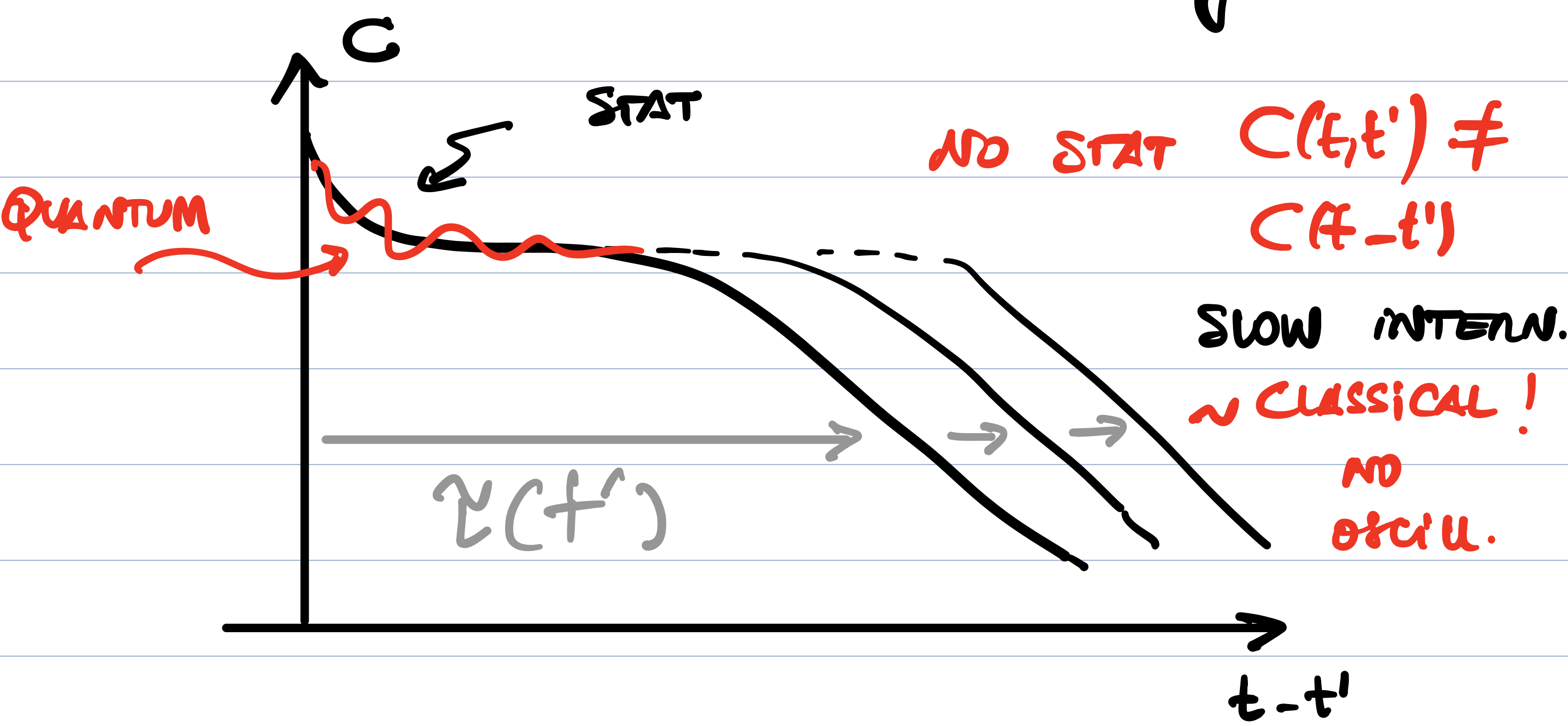
- $T < T_d$

RELAX. OUT OF EQUILIBRIUM

t' WAITING TIME

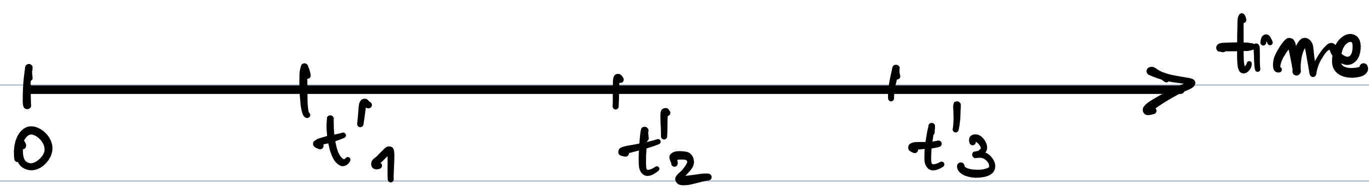


FAST- CONTROLLED BY EXT BATH



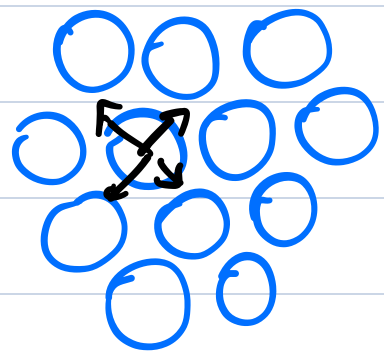
$\tau(t') \uparrow$ FOR $t' \uparrow$

AGING



HOW DOES ONE UNDERSTAND THE SEP. OF TIME SCALES ?

NB C is A GLOBAL CORR.



SHORT TIME DIFF.

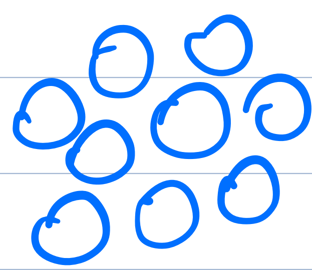
$t - t' < \tau(t')$

ONLY FLUCT. WITHIN CAGES

LONG TIME - DIFF

$t - t' > \tau(t')$

STRUCT RELAX

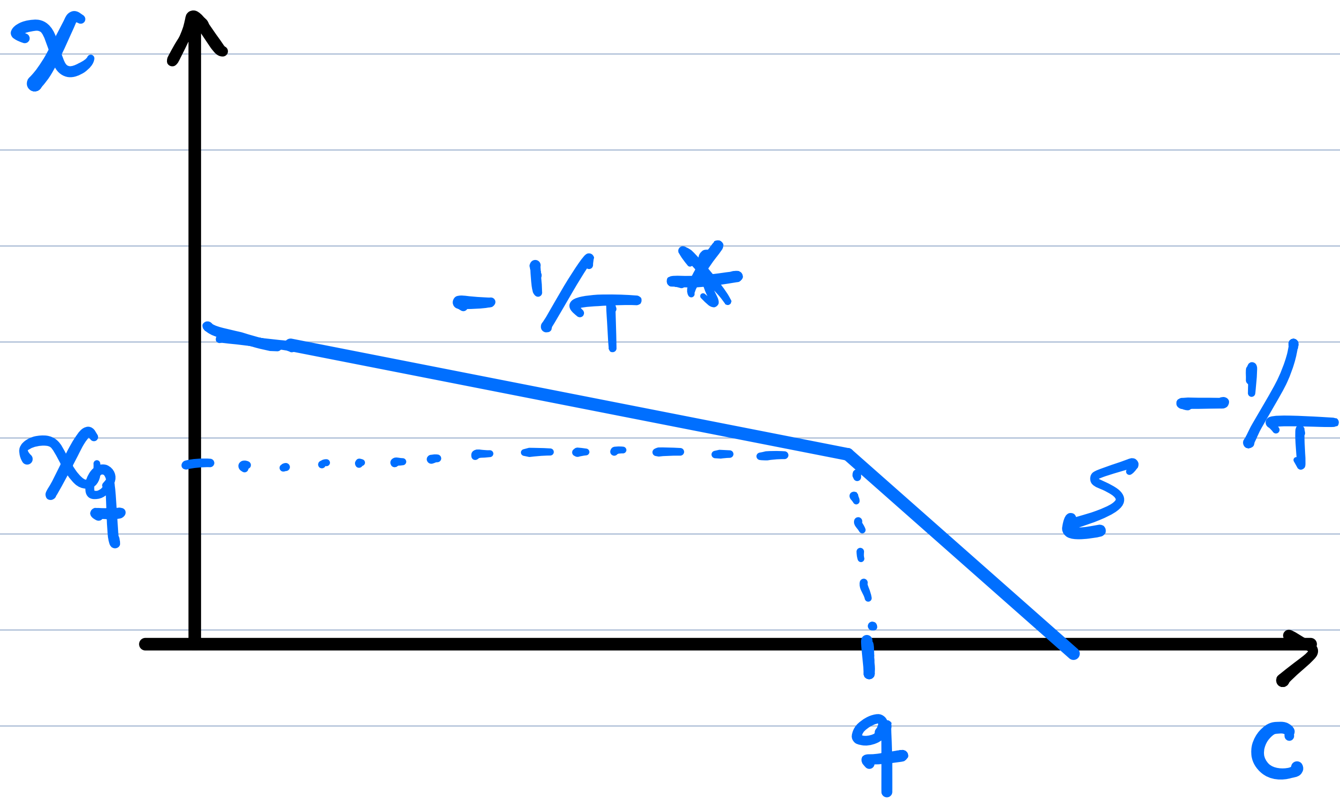


DIFF ORGAN. OF PART.

SECOND RELAX. OCCURS OUT OF EQ.

FLUCT-DISS. RELATION

INTEGRATED RESP. VS. CORR. FCT.



WITH TWO
TEMP
IN FDT
MEASUREM.

CLASSICAL

$$-\frac{1}{T_{eff}(c)} = \frac{dx(c)}{dc} \begin{cases} c > q & -\frac{1}{T} \\ c < q & -\frac{1}{T^*} \end{cases}$$

THERE IS AN <u>EFFECTIVE TEMP</u>

$$T^* > T$$

SELF- GENERATED IN THE SYST

WHY $T^* > T$?

THIS IS A QUENCH FROM $T_i > T_d$

KIND OF "MEMORY" OF THE INITIAL CONDITION

HEATING FROM $T_i < T_d$

ONE OBSERVES $T^* < T$

CONSISTENTLY

WITH THE IDEA OF "MEMORY"

FROM i.c.

RELATION TO PHASE SPACE STRUCT.

THERE IS !

CONF ENTROPY

COMPLEXITY

$$\frac{1}{k_B T^*} = \frac{\partial \Sigma_{conf}}{\partial f^{TAP}} \Bigg|_{f^{TAP}_{threshold}}$$

IN p-spin MODELS

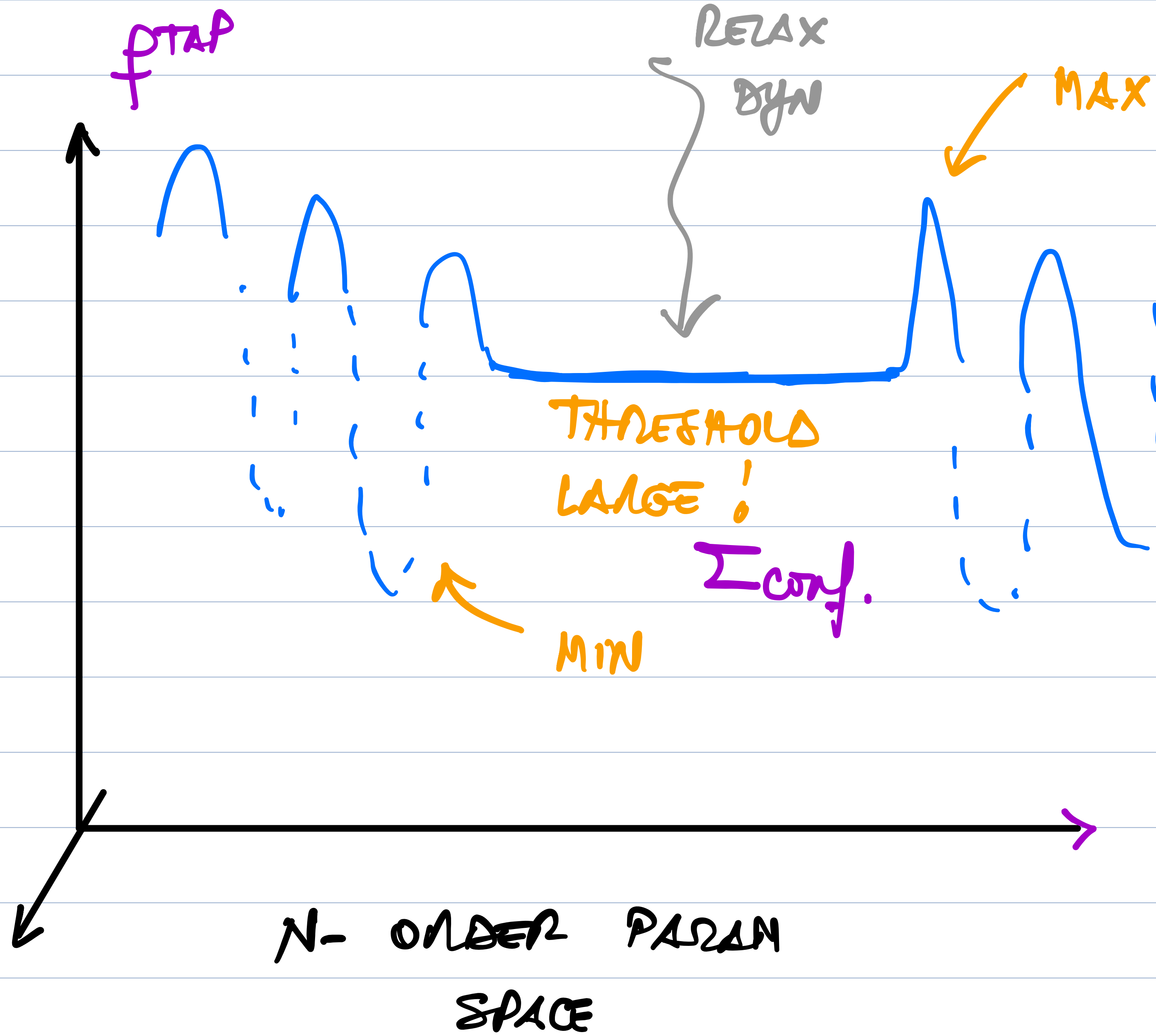
MICROCAN. TEMP

cfr JARON'S TALK

BRUNO'S LECTURE

FREE-ENERGY LANDSCAPE PERSPECTIVE

$f^{TAP} \approx$ GINZBURG-LANDAU FREE-ENERGY DENS. $T < T_d$



NOT JUST 1 !

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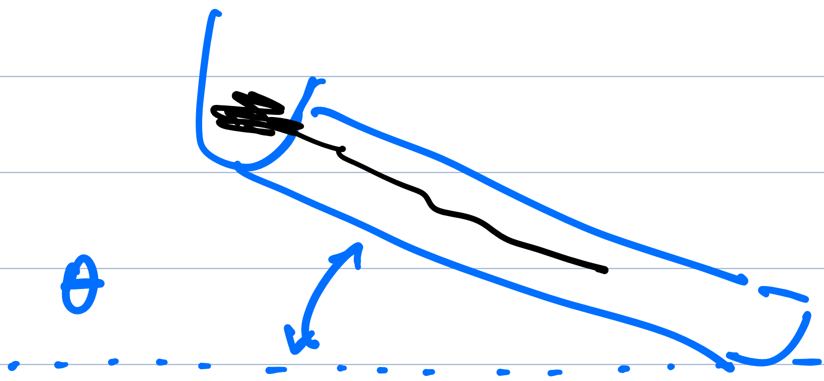
ONE HAS TO IMAGINE THE THRESHOLD AS A NETWORK OF SLIGHTLY TILTED CHANNELS

FAST RELAX.
AS IN A "HARMONIC WELL"



SLOW DIFF.
AS IN AN ALMOST FLAT POT.

VERY SMALL θ

FREE-ENERGY LANDSCAPE VIEW

QUANTUM

QUENCHES FROM DISORDERED PHASE DOWN TO WITHIN THE ORDERED ONE

$c > \frac{1}{2}$ $\Rightarrow$ QUANTUM FOT AT T

$c < \frac{1}{2}$ $\Rightarrow$ CLASSICAL FOT AT T^*

A FREE-ENERGY LANDSCAPE DESCRIPTION IS ALSO POSSIBLE

SUMMARY

1- INTRO TO STOCH PROCESSES

LANGEVIN
FOKKER-PLANCK / KRAMERS
PATH INTEGRALS

2- A BIT OF PHENOM. OF GLASSY DYNAMICS.

SOME REFS

LFC ADV PHYS. 1st PART 2024

LCF ANNUAL REV. COND MATT 2024
LES HOUCHES 2002

ALSO NOTES IN MY WEB PAGE