Dynamic Mean-Field Theory

aging, weak long-term memory & effective temperatures

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Introductory talk

Plan

— Many-body systems in interaction

some examples

— Collective dynamics

e.g. domain growth coarsening & the growing length

— Spontaneous and perturbed global relaxation

self-correlation and linear response

— Non-equilibrium Complex Dynamics

coarsening & glassy dynamics

separation of time scales & effective temperatures

effective temperatures

Many-body Systems in Interaction

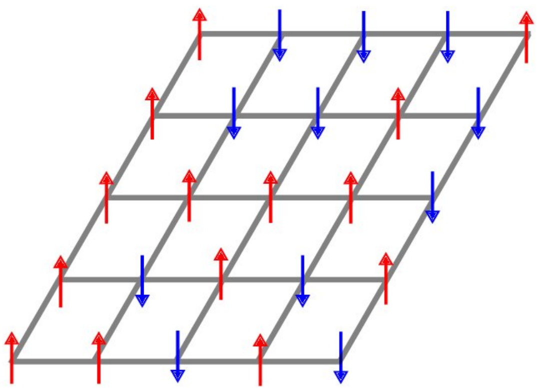
Some examples

Many-body systems

Some examples

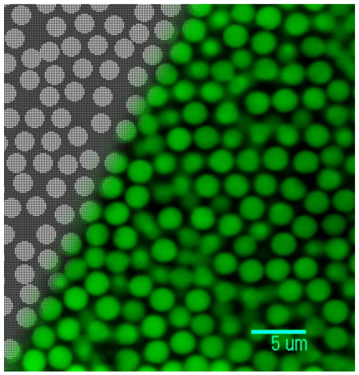
Ferromagnetic Ising Model

$$\mathcal{V} = -J \sum_{\langle ij \rangle} s_i s_j$$



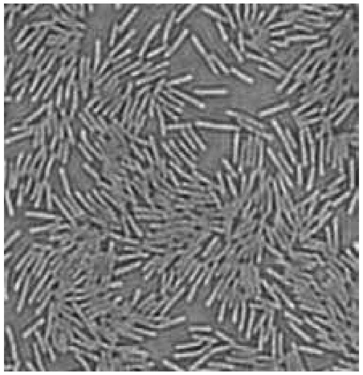
Particles in Interaction

$$\mathcal{V} = \sum_{i \neq j} V(r_{ij})$$



Active Matter

$$\vec{\mathcal{F}}_i \neq -\vec{\nabla}_i \mathcal{V}$$



In physical systems the action-reaction principle is respected, in other examples it is not

Also many examples beyond physics, like **ecosystems, markets**, etc. $\vec{\mathcal{F}}_{i \to j} \neq \vec{\mathcal{F}}_{j \to i}$

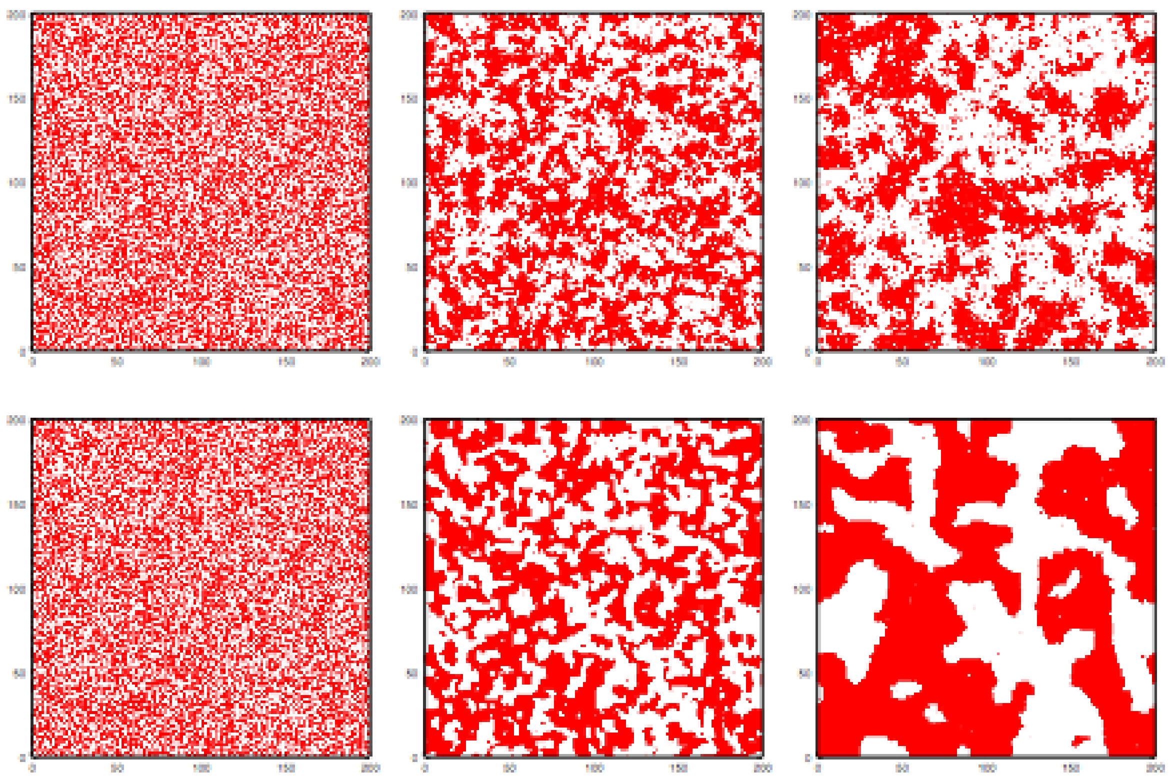
Collective dynamics

the simplest example, coarsening

$2d$ Ising model

Snapshots after an instantaneous quench from $T_0 \to \infty$ to $T \leq T_c$

$t = 0$ t_1 t_2



$T = T_c$

$T < T_c$

At $T = T_c$ **critical dynamics** At $T < T_c$ **coarsening**

A certain number of **interfaces** or **domain walls** in the last snapshots.

Phenomenon

In both cases one sees the growth of ‘red and white’ patches and **interfaces** surrounding such geometric domains.

Spatial regions of **local equilibrium** (with vanishing, at T_c, or non-vanishing, at $T < T_c$, order parameter) grow in time and

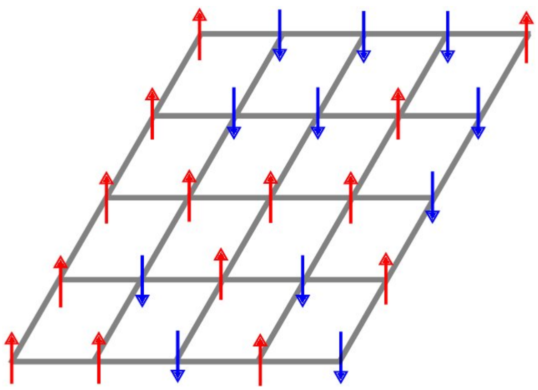
a single **growing length** $\mathcal{R}(t, T/J)$ can be identified and it is at the heart of *dynamic scaling*.

Many-body systems

Some examples

Ferromagnetic Ising Model

$$\mathcal{V} = -J \sum_{\langle ij \rangle} s_i s_j$$



Particles in Interaction

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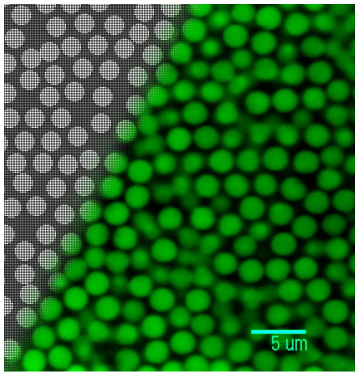

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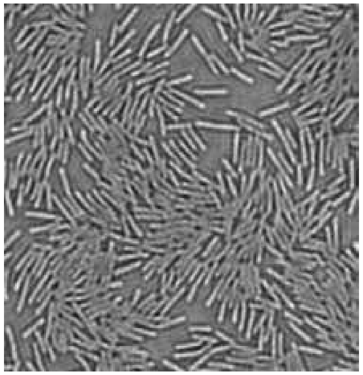

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Collective dynamics

if there is no obvious length ?

Global observables

Two-time correlation and linear responses

Two-time dependencies

Self-dislacement and linear response

The two-time displacement and integrated linear response

$$\Delta^2(t,t_w) \equiv \frac{1}{N}\sum_i [\langle (x_i(t)-x_i(t_w))^2\rangle]$$

$$\chi(t,t_w) \equiv \frac{1}{N}\sum_i \int_{t_w}^{t} dt'\, R(t,t') = \frac{1}{N}\sum_i \int_{t_w}^{t} dt' \; [\left.\frac{\delta\langle x_i(t)\rangle_h}{\delta h_i(t')}\right|_{h=0}]$$

Extend the notion of **order parameter**

They are not related by FDT out of equilibrium

The averages are thermal (and over initial conditions) $\langle \ldots \rangle$

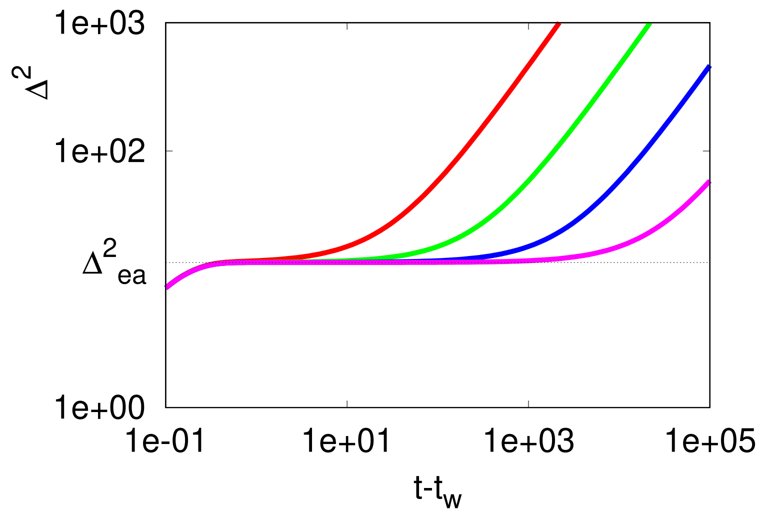
and over quenched randomness $[\ldots]$ (if present)

t_w waiting-time and t measuring time

Mean-square displacement

Relevant to follow single particle motion

$$\Delta^2(t,t_w) \equiv \frac{1}{N}\sum_i \left[\langle (x_i(t)-x_i(t_w))^2\rangle\right] \qquad T < T_c$$



$t_{w1} < t_{w2} < t_{w3} < t_{w4} <$

Two scales $\Delta^2_{\text{eq}}(t-t_w) + \Delta^2_{\text{ag}}(t,t_w)$

$$\Delta^2_{\text{eq}}(t-t_w) \sim f_{\text{eq}}\left(\frac{e^{-t/t_{\text{eq}}}}{e^{-t_w/t_{\text{eq}}}}\right) \qquad \Delta^2_{\text{ag}}(t,t_w) \sim f_{\text{ag}}\left(\frac{\mathcal{R}(t)}{\mathcal{R}(t_w)}\right)$$

In glassy systems, for which there is no clear visualization of $\mathcal{R}$

Physical aging

Older systems (more time elapsed after the quench)

relax **more slowly** than younger ones

Breakdown of stationarity of the self-correlation

$$\Delta^2(t, t_w) \neq \Delta^2(t - t_w)$$

In each regime, equilibrium and aging, scaling*

$$\Delta^2(t, t_w) = \Delta^2\left(\frac{\mathcal{R}(t)}{\mathcal{R}(t_w)}\right)$$

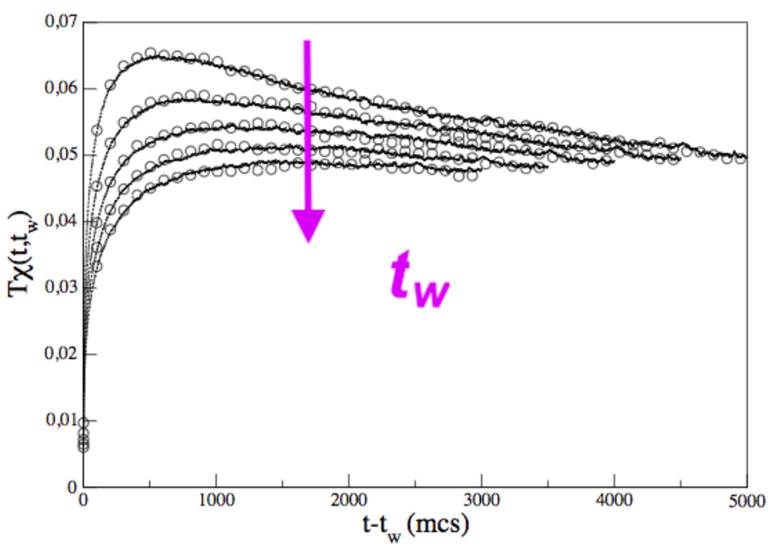
*the scaling form can be proven from general properties of temporal correlation functions

No obvious interpretation of $\mathcal{R}(t)$ in aging **glassy** systems

Two-time linear response

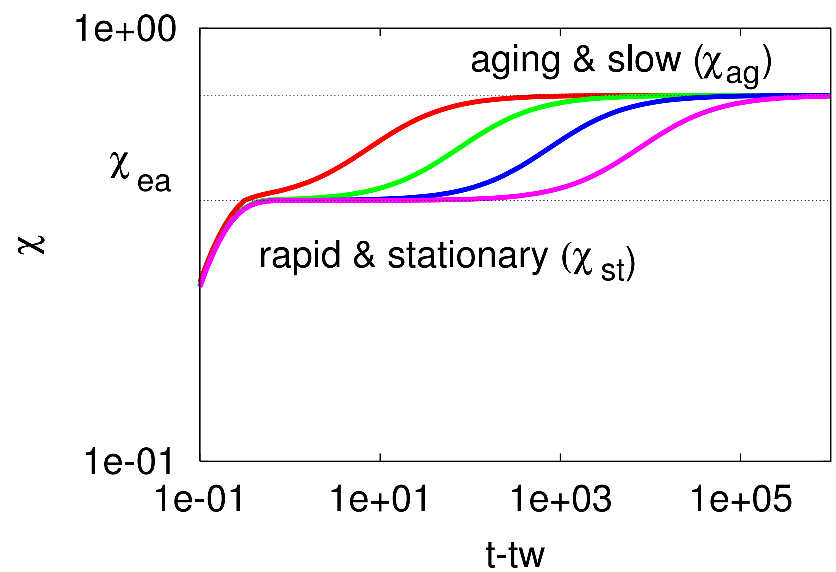
An important difference

Coarsening



Lippiello, Corberi & Zannetti 05

Glassy



Sketch **Chamon & LFC 07**

Weak long-term memory in the glassy but not in the coarsening problem.
In the latter, just the stationary part survives asymptotically, contrary to the sketch on the right valid for glasses & spin-glasses.

Memory

Older systems (more time elapsed after the quench)

relax **more slowly** than younger ones

Breakdown of stationarity of the integrated linear response

$$\chi(t,t_w) \neq \chi(t-t_w)$$

In the aging regime, difference between coarsening & glassy

$$\chi(t,t_w) = t^{-a}\,\chi\left(\frac{\mathcal{R}(t)}{\mathcal{R}(t_w)}\right) \quad \text{or} \quad \chi(t,t_w) = \chi\left(\frac{\mathcal{R}(t)}{\mathcal{R}(t_w)}\right)$$

Coarsening Glassy

(but no obvious interpretation of $\mathcal{R}(t)$ in aging **glassy** systems)

Mean-Field Modelling

Usual Curie-Weiss for PM-FM

More unusual for glasses

Glassy mean-field models

Classical p-spin spherical

Potential energy

$$\mathcal{V} = -\sum_{i_1 \neq \ldots \neq i_p} J_{i_1 \ldots i_p} x_{i_1} \ldots x_{i_p} \qquad p \text{ integer}$$

quenched random couplings $J_{i_1 \ldots i_p}$ drawn from a Gaussian $P[\{J_{i_1 \ldots i_p}\}]$

(over-damped) **Langevin dynamics** for continuous spins $x_i \in \mathbb{R}$

coupled to a white bath $\langle \xi(t) \rangle = 0$ and $\langle \xi(t) \xi(t') \rangle = 2\gamma k_B T \delta(t - t')$

$$\gamma \frac{dx_i}{dt} = -\frac{\delta \mathcal{V}}{\delta s_i} + z_t x_i + \xi_i$$

z_t is a Lagrange multiplier that fixes the spherical constraint $\sum_{i=1}^{N} x_i^2 = N$

$p = 2$ mean-field **domain growth**

$p \geq 3$ RFOT modelling of **fragile glasses**

One (surprising) Prediction

from coarsening & glassy mean-field models

and its further development

Fluctuation-dissipation

Linear relation between χ and Δ^2 in equilibrium

$$P(\{x_i\}, t_w) \to P_{\rm eq}(\{x_i\})$$

- The dynamics are stationary

$$\Delta^2_{AB}(t,t_w) = \langle [A(t) - B(t_w)]^2 \rangle = [C_{AA}(0) + C_{BB}(0) - 2C_{AB}(t-t_w)]$$

$$\to \Delta^2_{AB}(t-t_w)$$

- The fluctuation-dissipation theorem between spontaneous (Δ^2_{AB}) and induced (R_{AB}) fluctuations

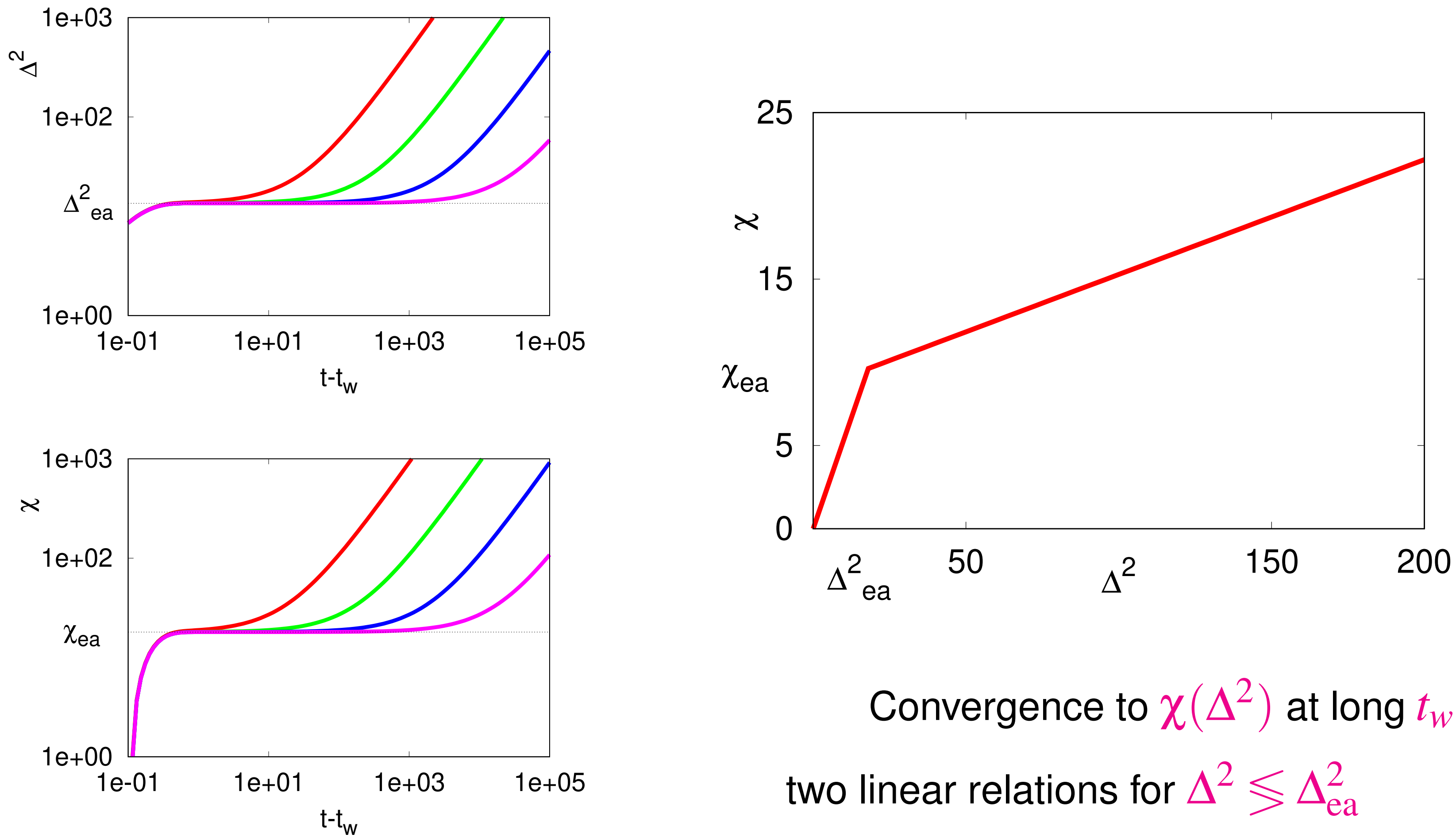
$$R_{AB}(t-t_w) = \frac{1}{2k_B T} \frac{\partial \Delta^2_{AB}(t-t_w)}{\partial t} \, \theta(t-t_w)$$

holds and implies

$$\chi_{AB}(t-t_w) \equiv \int_{t_w}^{t} dt' \, R_{AB}(t,t') = \frac{1}{2k_B T} [\Delta^2_{AB}(t-t_w) - \Delta^2_{AB}(0)]$$

Glassy non-equilibrium dynamics

Fluctuation-dissipation relation : parametric plot



Convergence to $\chi(\Delta^2)$ at long t_w

two linear relations for $\Delta^2 \lessgtr \Delta^2_{ea}$

Analytic solution to the p-spin model **LFC & J. Kurchan 93**

& effective temperature interpretation **LFC, Kurchan & Peliti 97**

Fluctuation-dissipation

Linear relation between χ and Δ^2 out of equilibrium ?

$$P(\{x_i\}, t_w) \neq P_{\text{eq}}(\{x_i\})$$

- The dynamics are not stationary

$$\Delta^2_{AB}(t,t_w) = \langle [A(t) - B(t_w)]^2 \rangle \not\to \Delta^2_{AB}(t-t_w)$$

- The fluctuation-dissipation theorem between spontaneous (Δ^2_{AB}) and induced (R_{AB}) fluctuations

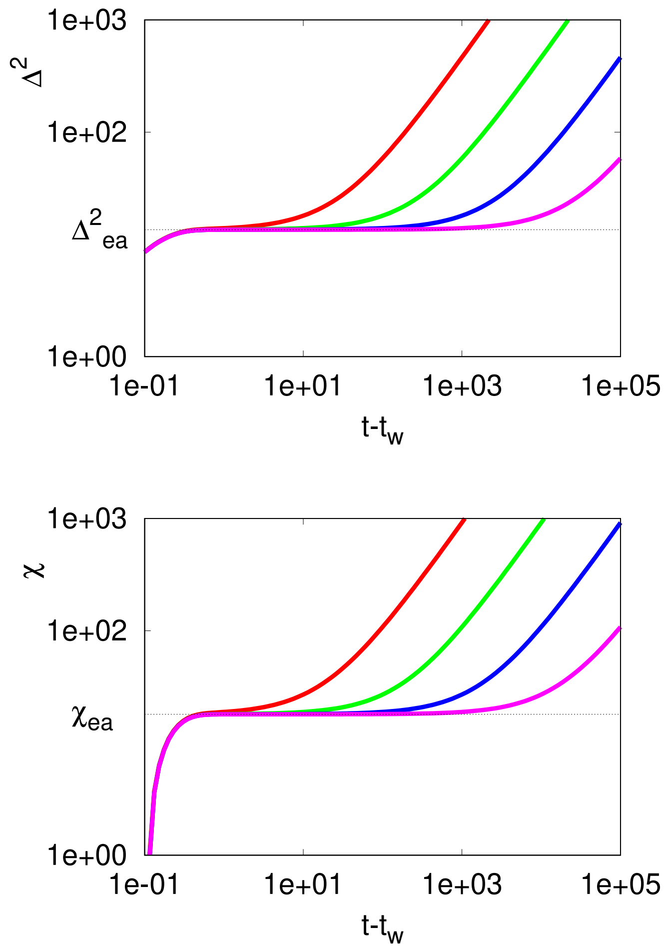
$$R_{AB}(t,t_w) \neq \frac{1}{2k_B T} \frac{\partial \Delta^2_{AB}(t,t_w)}{\partial t} \, \theta(t-t_w)$$

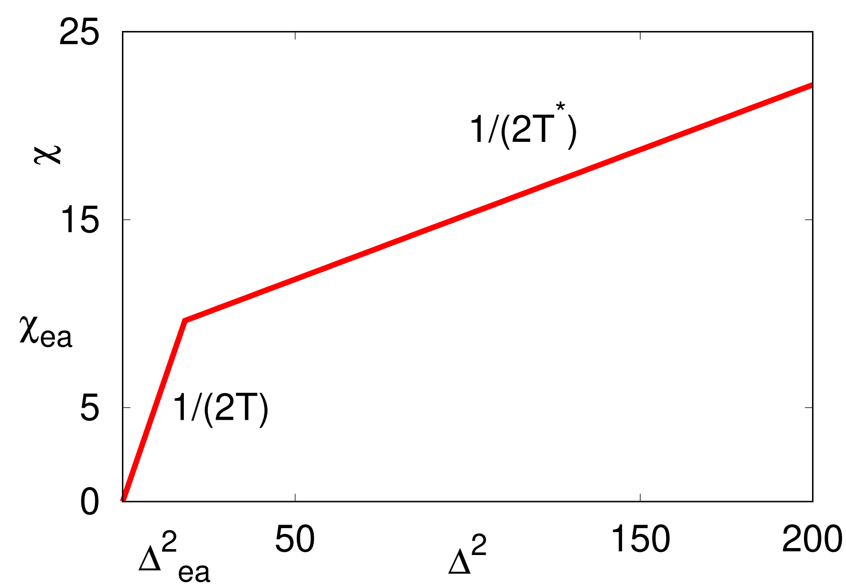
does not hold but one can propose

$$\chi_{AB}(t,t_w) \equiv \int_{t_w}^{t} dt' \, R_{AB}(t,t') = \frac{[\Delta^2_{AB}(t,t_w) - \Delta^2_{AB}(t,t)]}{2k_B T_{\text{eff}}(t,t_w)}$$

Glassy non-equilibrium dynamics

Fluctuation-dissipation relation : parametric plot





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Glassy non-equilibrium dynamics

Interpretation

- **Short-scale** re-arrangements follow the equilibrium bath rules

  The FDT is the equilibrium one with the temperature of the bath T

- **Large-scale** re-arrangements do not follow the equilibrium bath rules

  but the systems' own internal slow dynamics.

  The equilibrium FDT does not hold, it is modified in a rather simple

  way, as if it was applying but with another temperature value T^*

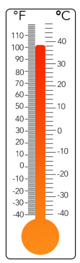
Is this interpretation correct ?

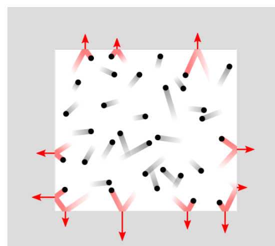
Statistical physics

Accomplishments

- Microscopic definition & derivation of thermodynamic concepts

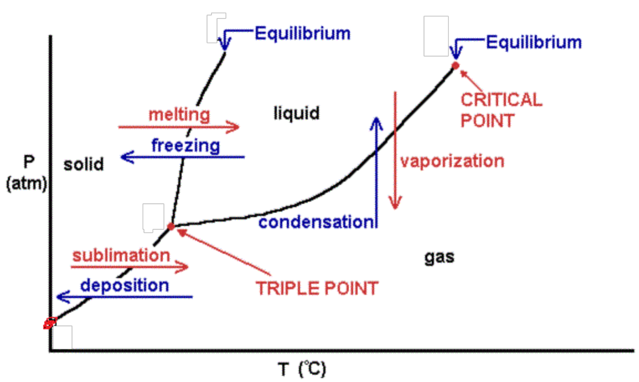
  (**temperature** , pressure, *etc.*) and laws (equations of state, *etc.*)





$$PV = nRT$$

- Theoretical understanding of collective effects $\Rightarrow$ phase diagrams



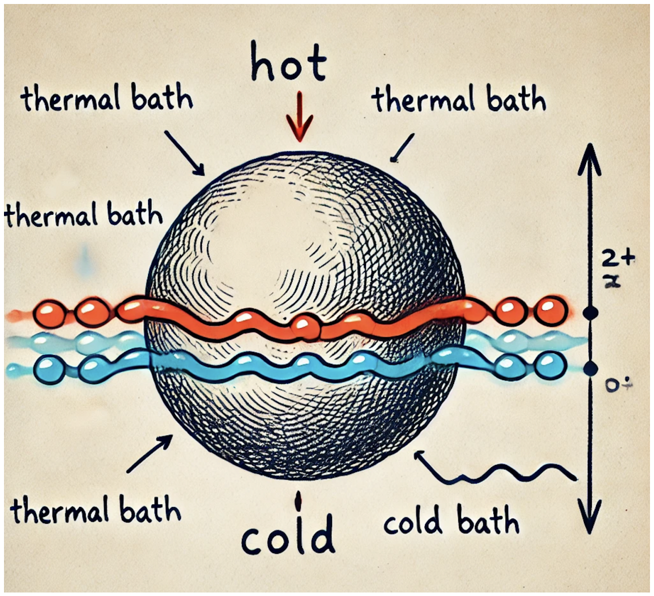
Phase transitions : sharp changes in the macroscopic behavior when an external (*e.g.* the temperature of the environment) or an internal (*e.g.* the interaction potential) parameter is changed

- Calculations can be difficult but the theoretical frame is set beyond doubt

FDT & effective temperatures

Can one interpret the slope as a temperature ?

Diffusion in a complex bath



Sketch created by ChatGPT

$$\Gamma = \Gamma_{\rm cold} + \Gamma_{\rm hot}$$

$$\Gamma_{\rm cold}(t-t') = 2\gamma\delta(t-t')$$

and temperature T

$$\Gamma_{\rm hot}(t-t') = \gamma_{\rm hot}\, e^{-(t-t')/\tau}$$

and temperature T^*

FDT & effective temperatures

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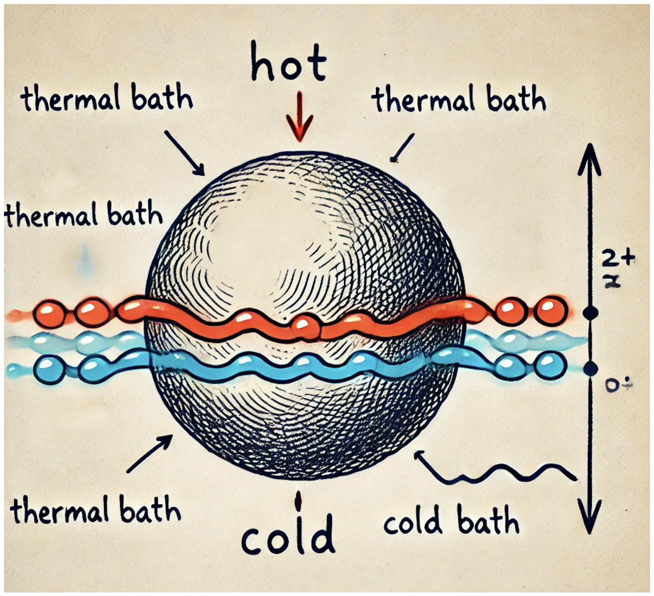

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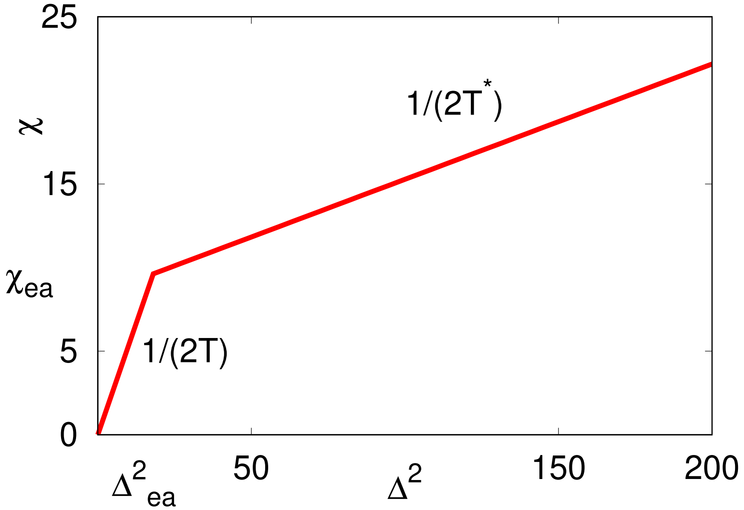

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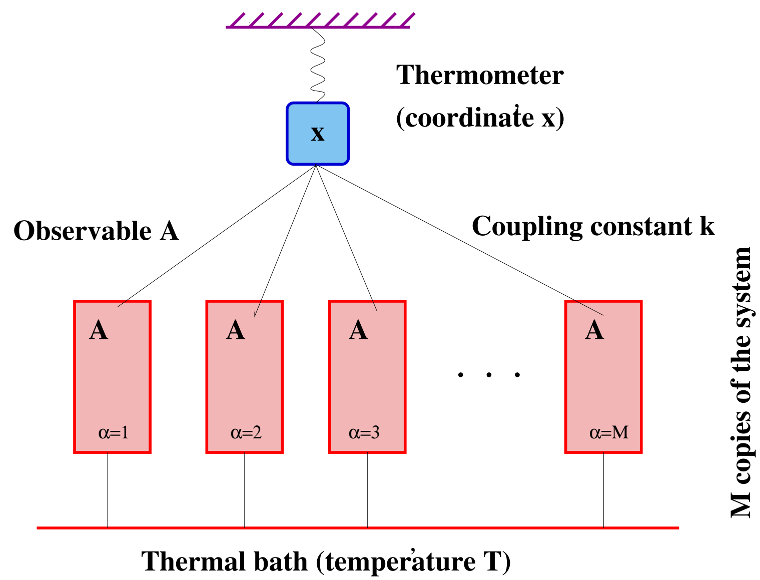


(1) Measurement with a **thermometer** with

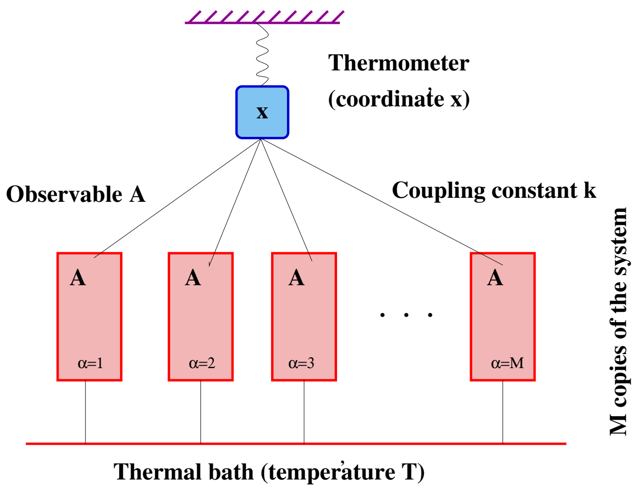
- Short internal time scale τ_0, fast dynamics is tested and T is recorded.
- Long internal time scale τ_0, slow dynamics is tested and T^* is recorded.

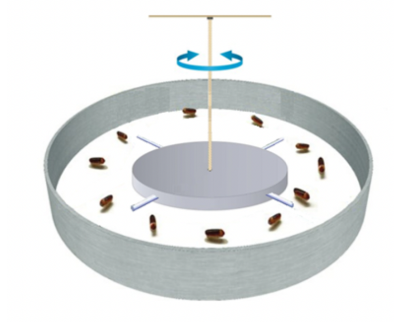
(2) **Partial equilibration** (3) **Direction of heat-flow**

LFC, Kurchan & Peliti 97

FDT & effective temperatures

Can one interpret the slope as a temperature ?





Grigera & Israeloff 99 - glassy

D'Anna, Mayor, Barrat, Loreto & Nori 03 - granular

Boudet, Jagielka, Guerin, Barois, Pistolesi & Kellay 24

artificial active matter - robots

Measurement with a **thermometer** with

- Short internal time scale τ_0, fast dynamics is tested and T is recorded.
- Long internal time scale τ_0, slow dynamics is tested and T^* is recorded.

Therm Uncertainty Relations

FDT violations & entropy production

Langevin process - Kramers equation for $P(x,v;t)$

Function $\mathcal{H}(t) = \int dxdv\, P(x,v;t)\, [T \ln P(x,v;t) + H(x,v)]$

such that $\dot{\mathcal{H}} \leq 0$ and $\dot{\mathcal{H}} = 0$ for $P(x,v,t) = P_{\rm eq}(x,v)$

Like an "out of equilibrium free-energy"

Kubo, Toda & Hashitume 65

The FDT violation $|2T\chi(t,t_w) - \Delta^2(t,t_w)|$ of a relaxing system is bounded by

$$\boxed{|2T\chi(t,t_w) - \Delta^2(t,t_w)| \leq \langle x^2(t)\rangle \int_{t_w}^{t} ds \left(-\frac{d\mathcal{H}(s)}{ds}\right)^{1/2}}$$

LFC, Dean & Kurchan 97

FDT & Fluctuation Theorems

Take a glass *out of equilibrium* and take it into a

driven steady glassy state

with a perturbing force.

For which entropy production rate does a fluctuation theorem hold ?

Since there is no meaning to T but there is to T_{eff} the proposal is to replace

$$\int_{-\tau/2}^{\tau/2} dt\, \frac{W(t)}{T} \quad \to \quad \int_{-\tau/2}^{\tau/2} dt\, \frac{W(t)}{T_{\text{eff}}(t)}$$

with $T_{\text{eff}}(t)$ the effective temperature as measured from

the fluctuation-dissipation relation of the *unperturbed* relaxing system

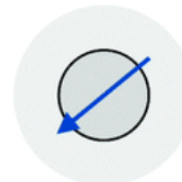
with, e.g., its two values T and T^*

Zamponi, Bonetto, LFC & Kurchan 05

Active Brownian particles

The standard model – ABPs

Spherical particles with diameter σ_d



Environment $\Longrightarrow$ Langevin dynamics

Scales $\Longrightarrow$ over-damped motion

Self-propulsion $\Longrightarrow$ active force F_{act} along $n_i = (\cos\theta_i(t), \sin\theta_i(t))$

$$\underbrace{\gamma \dot{r}_i}_{\text{friction}} = \underbrace{F_{\text{act}} n_i}_{\text{propulsion}} \underbrace{- \nabla_i \sum_{j(\neq i)} U(r_{ij})}_{\text{inter-particle repulsion}} + \underbrace{\xi_i}_{\text{translational white noise}} \qquad \underbrace{\dot{\theta}_i = \eta_i}_{\text{rotational white noise}}$$

$2d$ packing fraction $\boxed{\phi = \pi\sigma_d^2 N/(4S)}$ Péclet number $\boxed{\text{Pe} = F_{\text{act}}\sigma_{\text{d}}/(k_B T)}$

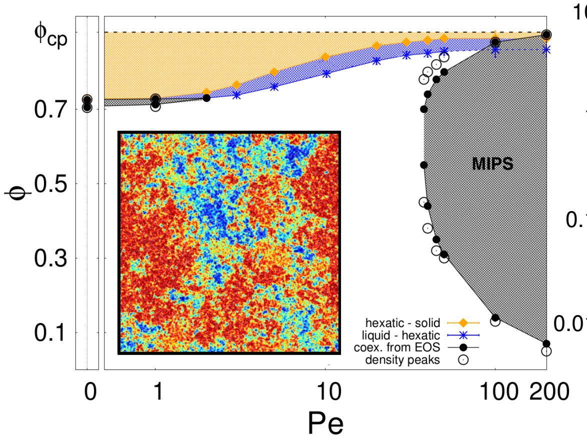
Bialké, Speck & Löwen 12, Fily & Marchetti 12

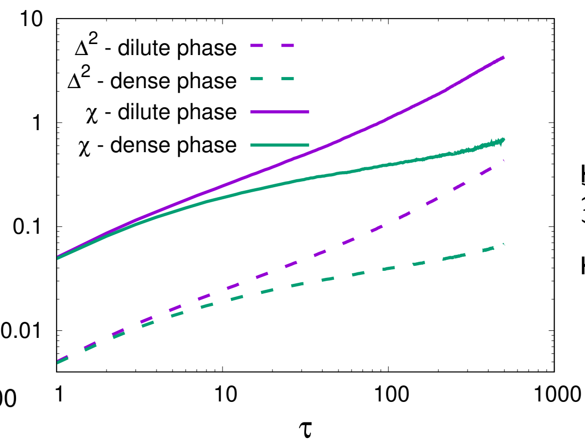
Teff = T

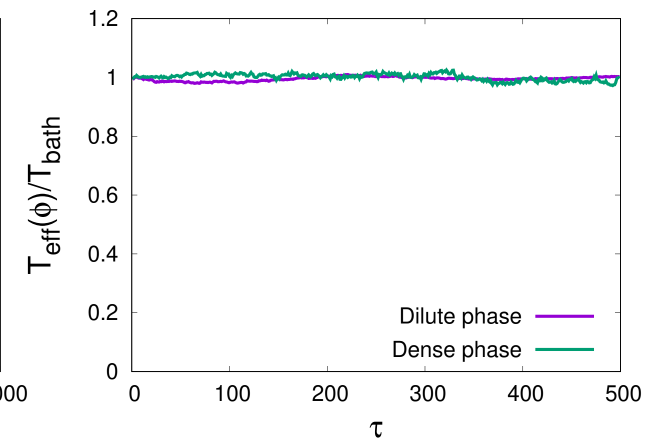
Co-existence in equilibrium

Pe = 0 $\quad \phi = 0.710$

Integrated linear response & mean-square displacement : their ratio (FDT) $\tau = t - t_w$







Method : linear response computed with Malliavin weights (no perturbation applied) as proposed by **Szamel** for active matter systems.

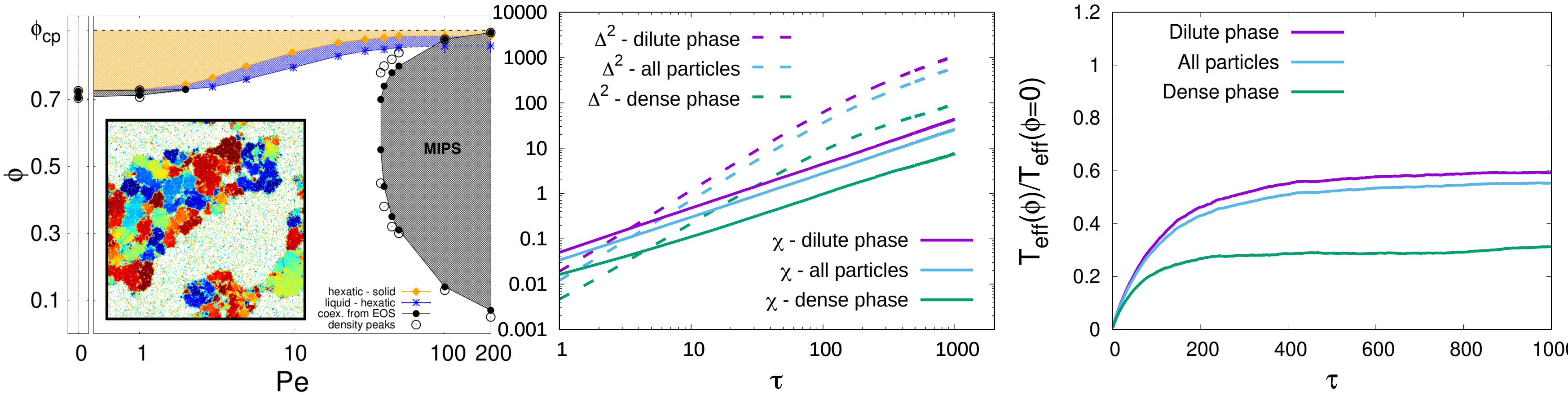
Petrelli, LFC, Gonnella & Suma 20

Teff $\neq$ T

Co-existence in MIPS

Pe = 50 $\phi = 0.5$

Integrated linear response & mean-square displacement : their ratio (FDR) $\tau = t - t_w$



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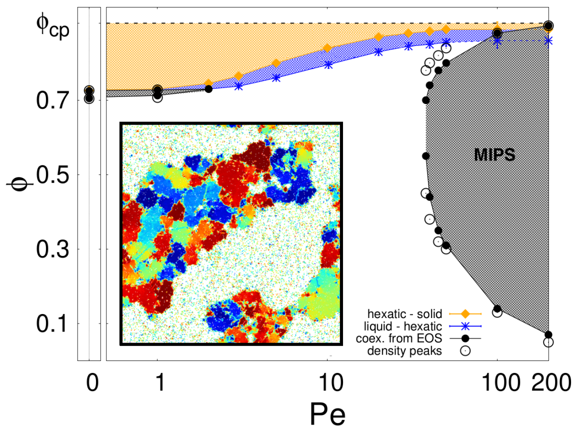
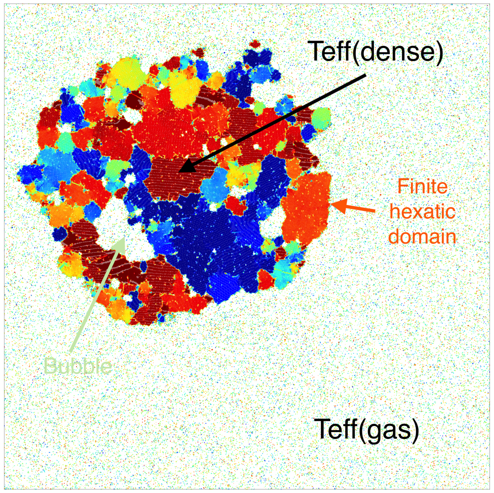
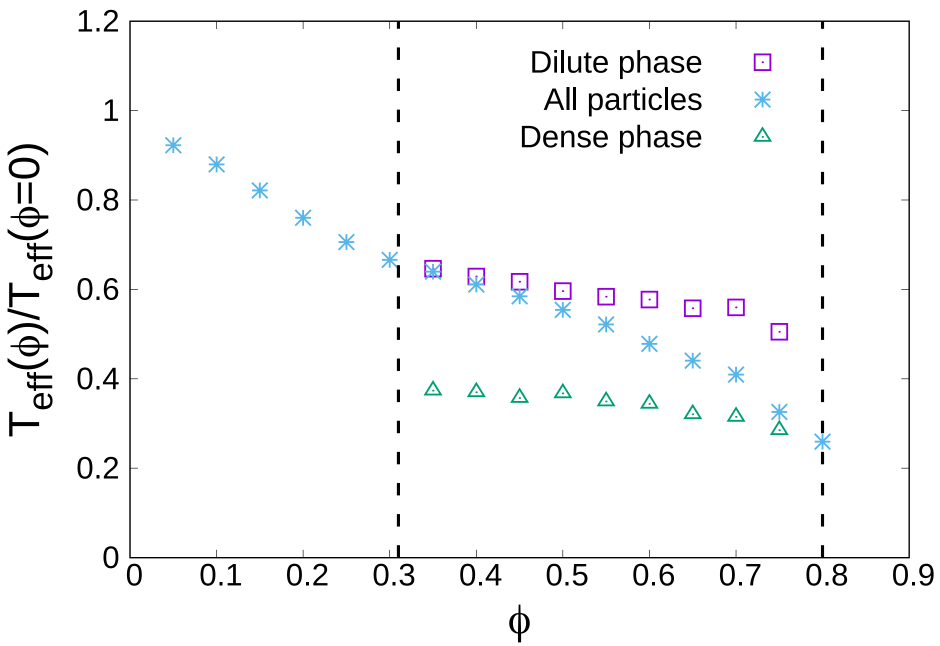
Teff in MIPS



ϕ_{cp}
0.7
0.5
0.3
0.1
ϕ
MIPS
hexatic - solid
liquid - hexatic
coex. from EOS
density peaks
0
1
10
100
200
Pe



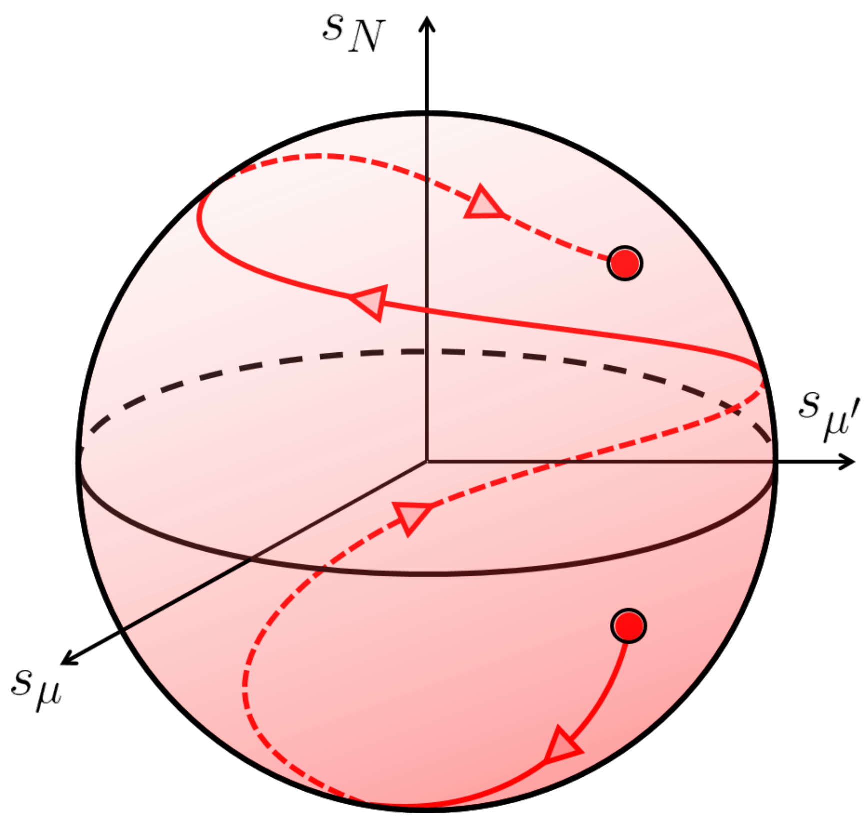
Teff(dense)
Finite hexatic domain
Bubble
Teff(gas)



$T_{eff}(\phi)/T_{eff}(\phi=0)$
1.2
1
0.8
0.6
0.4
0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
ϕ
Dilute phase
All particles
Dense phase

Classical dynamics

A particle on the sphere under anisotropic harmonic potentials



Integrable system

Neumann 1850, Uhlenbeck 80s

$I_\mu(\{x_\nu, p_\nu\})$ for $\mu = 1, \ldots, N$ known

Constraints

$\phi: \ \sum_\mu s_\mu^2 - N = 0$

$\phi': \ \sum_\mu s_\mu p_\mu = 0$

Correlation and response

Fluctuation-dissipation theorem in Boltzmann equilibrium

$$C(t,t_w) = \frac{1}{N}\sum_{\mu=1}^{N} \langle s_\mu(t) s_\mu(t_w)\rangle_{i.c.}$$ self correlation

$$R(t,t_w) = \frac{1}{N}\sum_{\mu=1}^{N} \left.\frac{\delta\langle s_\mu(t)\rangle_{i.c.}}{\delta h_\mu(t_w)}\right|_{h=0}$$ linear response

Stationary limit $C(t,t_w) \mapsto C_{\mathrm{st}}(t-t_w)$ and $R(t,t_w) \mapsto R_{\mathrm{st}}(t-t_w)$

Fourier transforms

$$\hat{C}(\omega) = \text{F.T. } C_{\mathrm{st}}(t-t_w)$$

$$\hat{R}(\omega) = \text{F.T. } R_{\mathrm{st}}(t-t_w)$$

Fluctuation-dissipation thm

$$-\frac{\mathrm{Im}\hat{R}(\omega)}{\omega\hat{C}(\omega)} = \beta$$

Correlation and response

Fluctuation-dissipation theorem in Boltzmann equilibrium

$$C(t,t_w) = \frac{1}{N}\sum_{\mu=1}^{N} \langle s_\mu(t) s_\mu(t_w)\rangle_{i.c.}$$ self correlation

$$R(t,t_w) = \frac{1}{N}\sum_{\mu=1}^{N} \left.\frac{\delta\langle s_\mu(t)\rangle_{i.c.}}{\delta h_\mu(t_w)}\right|_{h=0}$$ linear response

Stationary limit $C(t,t_w) \mapsto C_{\text{st}}(t-t_w)$ and $R(t,t_w) \mapsto R_{\text{st}}(t-t_w)$

Fourier transforms

$$\hat{C}(\omega) = \text{F.T. } C_{\text{st}}(t-t_w)$$
$$\hat{R}(\omega) = \text{F.T. } R_{\text{st}}(t-t_w)$$

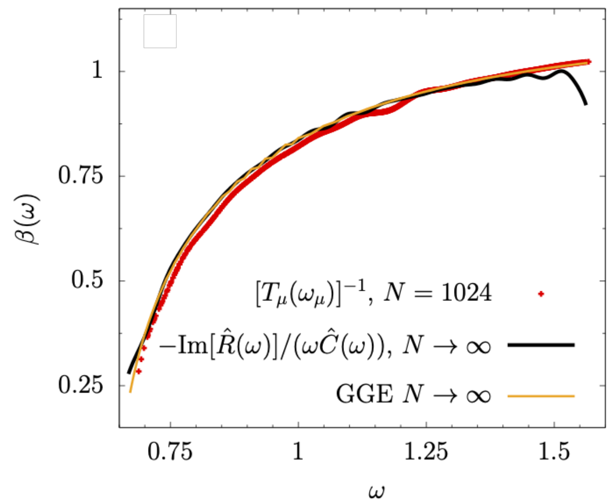
Fluctuation-dissipation thm

$$-\frac{\text{Im}\hat{R}(\omega)}{\omega\hat{C}(\omega)} = \beta_{\text{eff}}(\omega)$$

Read $\beta_{\text{eff}}(\omega)$

Frequency domain FDR

The T_μs from the FDR at $\omega_\mu = [(z_f - \lambda_\mu)/m]^{1/2}$ in Phase I



A way to measure the mode temperatures with a single measurement

$$\beta_{\mathrm{eff}}(\omega_\mu) = -\mathrm{Im}\hat{R}(\omega_\mu)/(\omega_\mu \hat{C}(\omega_\mu)) = \beta_\mu$$

No "partial equilibration" contradiction from the effective temperature perspective. The modes are uncoupled, they do not exchange energy, and can then have different T_μs

Idea in **LFC, de Nardis, Foini, Gambassi, Konik & Panfil 17** for **quantum**

Barbier, LFC, Lozano, Nessi 22

Final remarks

Some other applications/extensions of DMFT

– Large d approach to glassiness

Agoritsas, Charbonneau, Kurchan, Maimbourg, Parisi, Urbani & Zamponi, ...

– Ecological models

Altieri, Biroli, Bunin, Cammarotta & Roy, ...

– Neural networks & non-reciprocal interactions

Crisanti & Sompolinsky 80s, Brunel et al., etc.

LFC, Kurchan, Le Doussal & Peliti 90s, Berthier, Barrat & Kurchan 00s

Biroli, Mignacco, Urbani, Zdeborová, ...

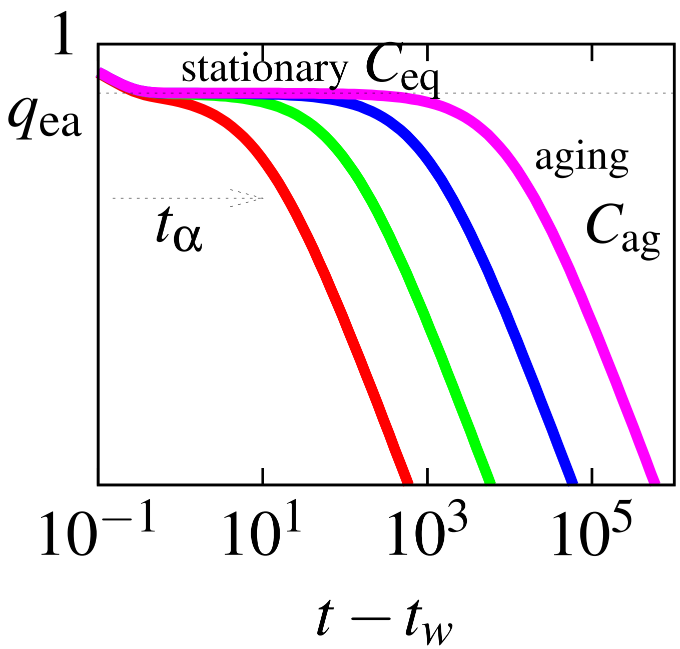
Final remarks

Time reparametrization invariance

Separation of time-scales

In the long t_w limit

Fast $t - t_w \ll t_w$



The aging part is slow

Slow $\mathcal{R}(t)/\mathcal{R}(t_w) = O(1)$

$$C_{ag}(t, t_w) \sim f_{ag}\left(\frac{\mathcal{R}(t)}{\mathcal{R}(t_w)}\right)$$

$$\partial_t C_{ag}(t, t_w) \propto \frac{\dot{\mathcal{R}}(t)}{\mathcal{R}(t)} \xrightarrow[t\to\infty]{} 0$$

$$\boxed{\partial_t C_{ag}(t, t_w) \ll C_{ag}(t, t_w)}$$

Eqs. for the slow relaxation $C_{ag} < q_{ea}$:

Approx. asymptotic time-reparametization invariance $\boxed{t \to h(t)}$

Time reparametrization

Example : the equation $(\partial_t - z_t)R(t,t_w) = \int dt' \, \Sigma(t,t')R(t',t_w)$

- Focus on times such that $z_t \to z_\infty$, $C \sim C_{\rm ag}$ and $R \sim R_{\rm ag}$

- Separation of time-scales (drop $\partial_t R$ and approximate the integral) :

$$-z_\infty R_{\rm ag}(t,t_w) \sim \int dt' \, D'[C_{\rm ag}(t,t')] R_{\rm ag}(t,t') \, R_{\rm ag}(t',t_w) \qquad (1)$$

- The transformation

$$t \to h_t \equiv h(t) \qquad \left\{ \begin{array}{l} C_{\rm ag}(t,t_w) \to C_{\rm ag}(h_t,h_{t_w}) \\ R_{\rm ag}(t,t_w) \to \frac{dh_{t_w}}{dt_w} R_{\rm ag}(h_t,h_{t_w}) \end{array} \right.$$

with h_t positive and monotonic leaves eq. (1) **invariant** :

$$-z_\infty R_{\rm ag}(h_t,h_{t_w}) \sim \int dh_{t'} \, D'[C_{\rm ag}(h_t,h_{t'})] R_{\rm ag}(h_t,h_{t'}) \, R_{\rm ag}(h_{t'},h_{t_w})$$

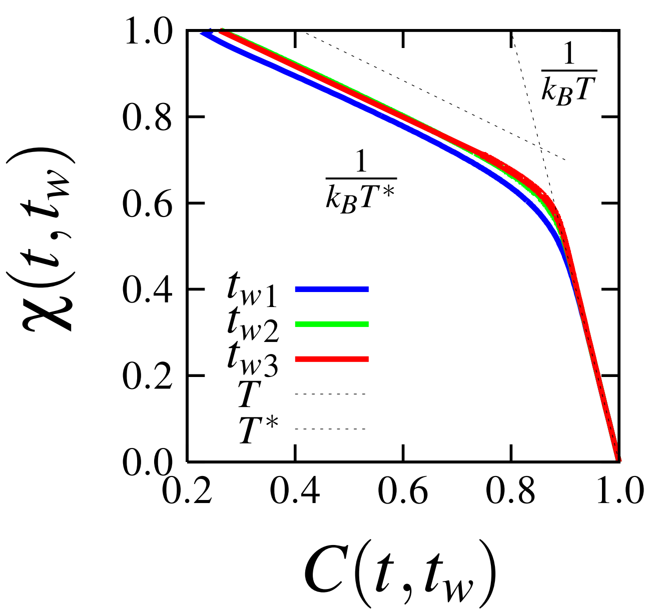
Time reparametrization

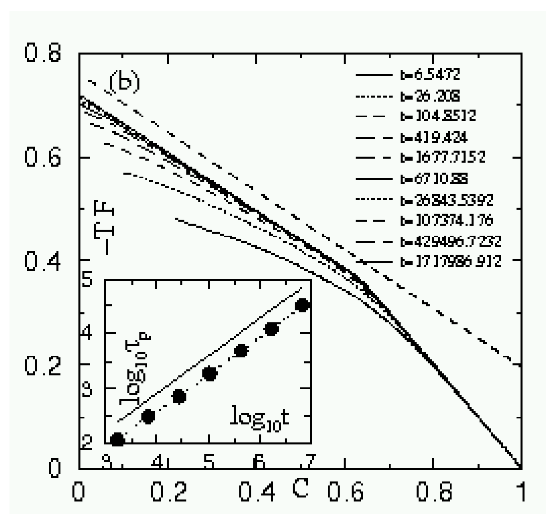
One can compute analytically $f_{\rm ag}$ and $\chi_{\rm ag}(C_{\rm ag})$

for times t and t_w such that $C_{\rm ag}(t,t_w) \sim f_{\rm ag}\left(\frac{\mathcal{R}(t)}{\mathcal{R}(t_w)}\right)$, *e.g.*

$$\chi_{\rm ag}(t,t_w) \sim \frac{1-q_{\rm ea}}{T} + \frac{1}{T^*}\left[q_{\rm ea} - C_{\rm ag}(t,t_w)\right]$$

but not the 'clock' $\mathcal{R}(t)$





Kim & Latz 00 very precise numerical solution

Implications on Fluctuations

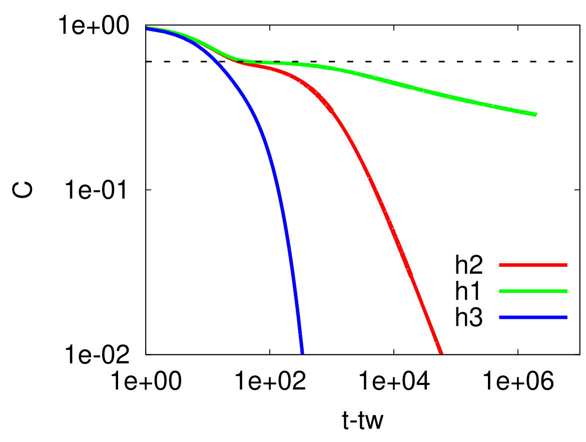
Leading fluctuations

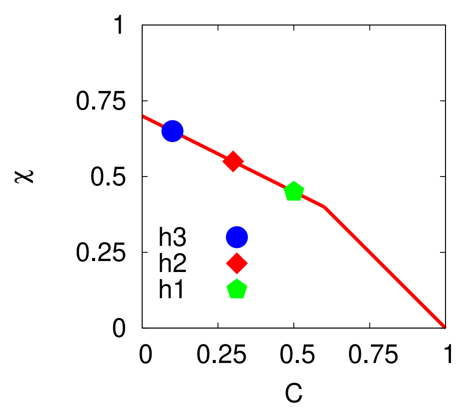
Global to local correlations & linear responses

$$C_{\rm ag}(t,t_w) \approx f_{\rm ag}\left(\frac{\mathcal{R}(t)}{\mathcal{R}(t_w)}\right)$$ **global correlation**

Global time-reparametrization invariance $\Rightarrow C^{\rm ag}_{\vec r}(t,t_w) \sim f_{\rm ag}\left(\frac{h_{\vec r}(t)}{h_{\vec r}(t_w)}\right)$

Ex. $h_{\vec r_1} = \frac{t}{t_0}$, $h_{\vec r_2} = \ln\left(\frac{t}{t_0}\right)$, $h_{\vec r_3} = e^{\ln^{a>1}\left(\frac{t}{t_0}\right)}$ in different spatial regions





Castillo, Chamon, LFC, Iguain & Kennett 02, 03

Chamon, Charbonneau, LFC, Reichman & Sellitto 04

Jaubert, Chamon, LFC & Picco 07

Conclusions on Fluctuations

Fluctuations

(Annoying) global time-reparametrization invariance $t \to h(t)$ in models in which

- $C_{ag}(t, t_w) \gg \partial_t C_{ag}(t, t_w)$ (slow dynamics)
- $\chi_{ag}(t, t_w) \gg \partial_t \chi_{ag}(t, t_w)$ (weak long-term memory)

and finite effective temperature $T_{\rm eff} < +\infty$ **Chamon, LFC & Yoshino 06**

Reason for the large dynamic fluctuations (heterogeneities) $h(\vec{r}, t)$

Effective action for $\varphi(\vec{r}, t)$ in $h(\vec{r}, t) = e^{-\varphi(\vec{r}, t)}$ **Chamon & LFC & Yoshino 07**

Quantum : the rapid equilibrium regime is modified but the slow aging one is classical controlled by a $T_{\rm eff} > 0 \Rightarrow$ the same applies

LFC & Lozano 98, 99 ; Kennett & Chamon 00, 01

Each problem

with its own peculiarities

& much more to say !

Dynamic equations

Conservative dynamics - closed classical systems

In the $N \to \infty$ limit exact causal Schwinger-Dyson equations

$$(m\partial_t^2 - z_t)R(t,t_w) = \int dt'\, \Sigma(t,t')R(t',t_w) + \delta(t-t_w)$$

$$(m\partial_t^2 - z_t)C(t,t_w) = \int dt'\, \left[\Sigma(t,t')C(t',t_w) + D(t,t')R(t_w,t')\right]$$

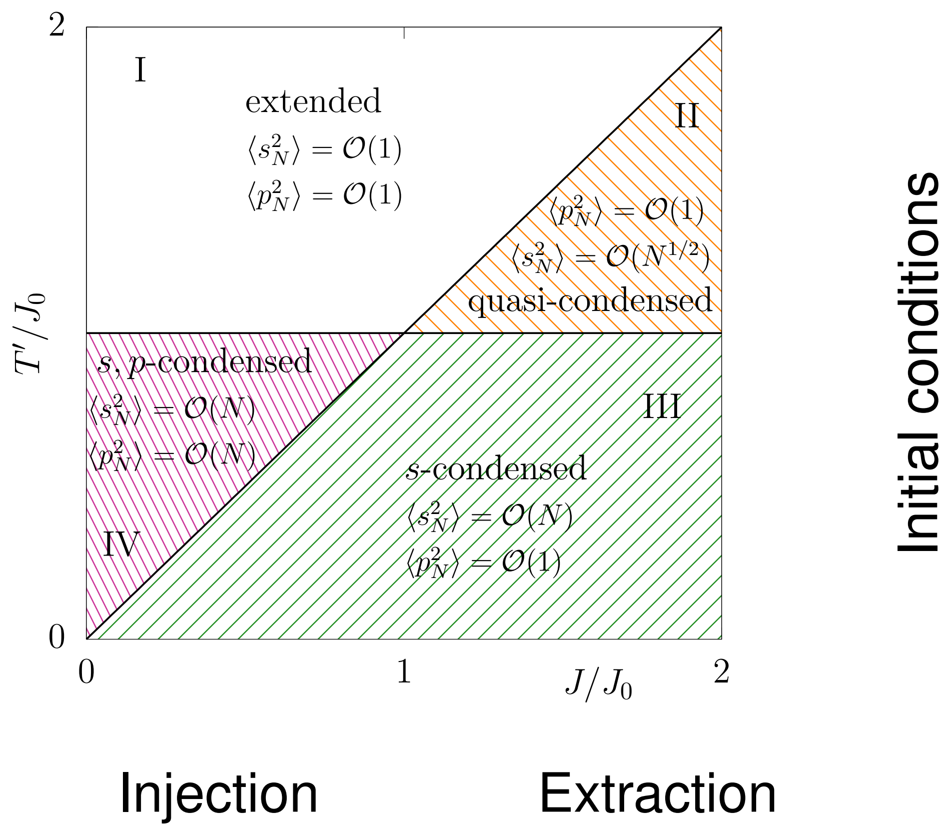
$$+\frac{\beta_0 J_0}{J}\sum_{a=1}^{n} D_a(t,0)C_a(t_w,0)$$

$$(m\partial_t^2 - z_t)C_a(t,0) = \int dt'\, \Sigma(t,t')C_a(t',0) + \frac{\beta_0 J_0}{J}\sum_{a=1}^{n} D_b(t,0)Q_{ab}$$

$a = 1,\ldots,n \to 0$, replica method to deal with $e^{-\beta_0 \mathcal{H}_0}$ and fix Q_{ab}

The $p = 2$ integrable model

The phase diagram



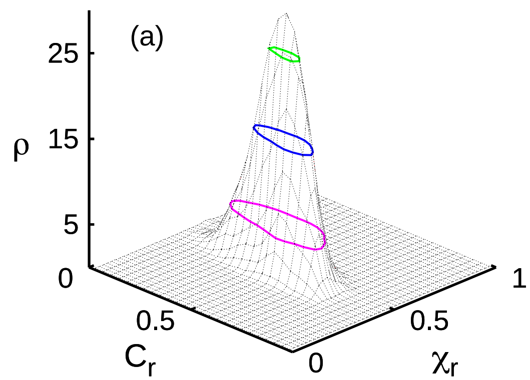
For all parameters $\lim_{t \gg t_{st}} \lim_{N \to \infty} \overline{\langle s_\mu^2(t) \rangle_{i.c.}} = \langle s_\mu^2 \rangle_{\rm GGE}$ etc.

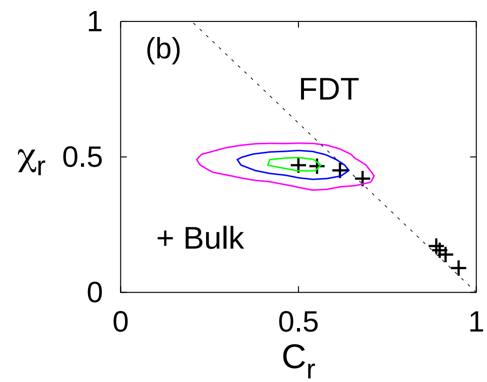
Barbier, LFC, Lozano, Nessi, Picco & Tartaglia 18-22

Local correlations & responses

$3d$ Edwards-Anderson spin-glass

$$C_{\vec{r}}(t,t_w) \equiv \frac{1}{V_{\vec{r}}} \sum_{i \in V_{\vec{r}}} s_i(t) s_i(t_w) \;, \quad \chi_{\vec{r}}(t,t_w) \equiv \frac{1}{V_{\vec{r}}} \sum_{i \in V_{\vec{r}}} \int_{t_w}^{t} dt' \; \left. \frac{\delta s_i(t)}{\delta h_i(t')} \right|_{h=0}$$





+ Bulk : Parametric plot $\chi(t,t_w)$ vs $C(t,t_w)$ for t_w fixed and 7 $t\ (> t_w)$

ρ corresponds to the maximum t yielding the smallest C (left-most +)

Castillo, Chamon, LFC, Iguain, Kennett 02

Kinetically constrained models + Charbonneau, Reichman & Sellitto 04

Sigma Model

Conditions & expression

$$h(\vec{r},t) = e^{-\varphi(\vec{r},t)} \qquad\qquad C_{ag}(\vec{r},t,t_w) = f_{ag}\left(e^{-\int_{t_w}^{t} dt'\, \partial_{t'}\varphi(\vec{r},t')}\right)$$

i. The action must be invariant under a global time reparametrization $t \to h(t)$.

ii. If our interest is in short-ranged problems, the action must be written using local terms. The action can thus contain products evaluated at a single time and point in space of terms such as $\varphi(\vec{r},t)$, $\partial_t\varphi(\vec{r},t)$, $\nabla\varphi(\vec{r},t)$, $\nabla\partial_t\varphi(\vec{r},t)$, and similar derivatives.

iii. The scaling form in eq. (29) is invariant under $\varphi(\vec{r},t) \to \varphi(\vec{r},t) + \Phi(\vec{r})$, with $\Phi(\vec{r})$ independent of time. Thus, the action must also have this symmetry.

iv. The action must be positive definite.

These requirements largely restrict the possible actions. The one with the smallest number of spatial derivatives (most relevant terms) is

$$\mathcal{S}[\varphi] = \int d^d r \int dt \left[K \, \frac{(\nabla\partial_t\varphi(\vec{r},t))^2}{\partial_t\varphi(\vec{r},t)} \right] , \qquad (30)$$

Chamon & LFC 07

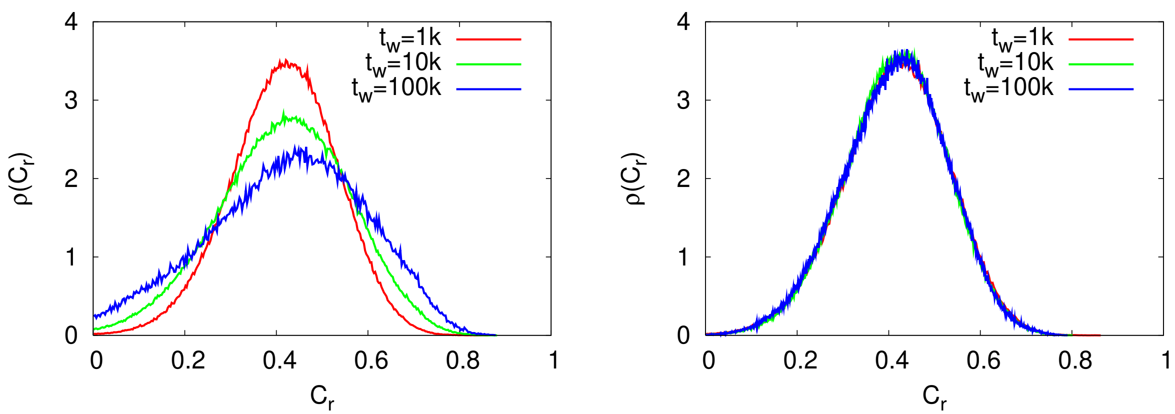
Sigma Model

Some consequences - $3d$ Edwards Anderson model

$$h(\vec{r},t) = e^{-\varphi(\vec{r},t)} \qquad\qquad C_{ag}(\vec{r},t,t_w) = f_{ag}(e^{-\int_{t_w}^{t} dt'\, \partial_{t'} \varphi(\vec{r},t')})$$

Distribution of local correlations depends on times t, t_w only through C, ξ

$$\rho(C_{\vec{r}};\ t, t_w, \ell, \xi(t,t_w)) \to \rho(C_{\vec{r}};\ C_{ag}(t,t_w), \ell/\xi(t,t_w))$$



t, t_w such that $C_{ag}(t,t_w) = C$ $\qquad$ ℓ such that $\ell/\xi = \text{cst}$ $\qquad$ **Jaubert, Chamon, LFC, Picco 07**

predictions on the form of ρ derived from $S[\varphi]$ too

Tests in Lennard-Jones systems **Avila, Castillo, Mavimbela, Parsaeian 06-12**

How general is this ?

Coarsening & domain growth

e.g. the d-dimensional $O(N)$ model in the large N limit (continuous space limit of the Heisenberg ferro with $N \to \infty$)

N component field $\vec{\phi} = (\phi_1, \ldots, \phi_N)$ with Langevin dynamics

$$\partial_t \phi_\alpha(\vec{r},t) = \nabla^2 \phi_\alpha(\vec{r},t) + \lambda |N^{-1} \phi^2(\vec{r},t) - 1| \, \phi_\alpha(\vec{r},t) + \xi_\alpha(\vec{r},t)$$

$\phi_\alpha(\vec{k},0)$ Gaussian distributed with variance Δ^2

Time reparametrization invariance is reduced to time rescalings

$$t \to h(t) \qquad \Rightarrow \qquad t \to \lambda t$$

Same in the $p = 2$ spherical model

Chamon, LFC, Yoshino 06

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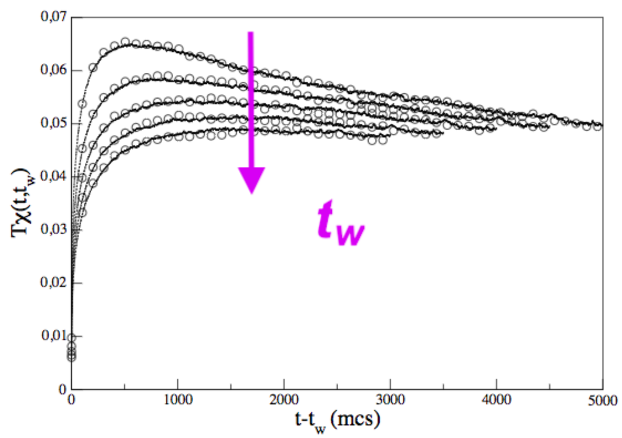
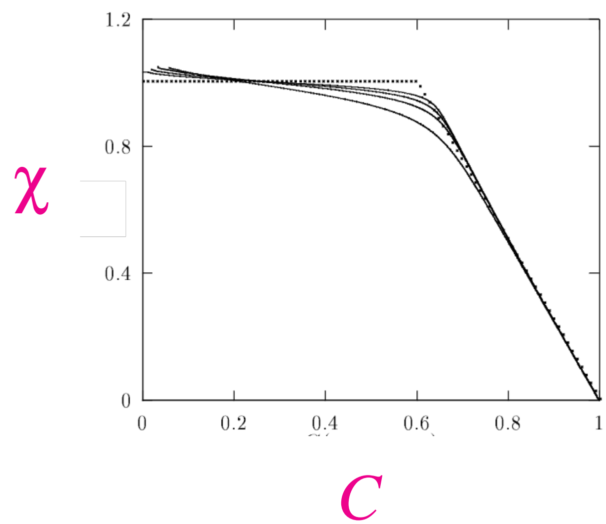
Coarsening & domain growth

Time reparametrization invariance is reduced to time rescalings

$$t \to h(t) \qquad \Rightarrow \qquad t \to \lambda t$$





Ising FM, $O(N)$ field theory, or $p = 2$ spherical model

Related to $T^* \to \infty$ and simplicity of free-energy landscape

Triangular relations

Scaling of the aging global correlation

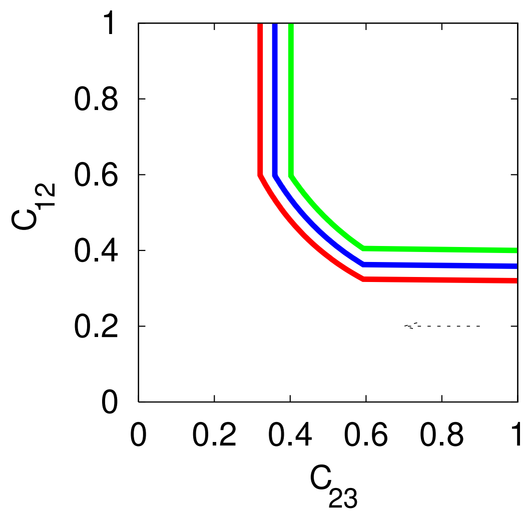
Take three times $t_1 \geq t_2 \geq t_3$ and compute the three global correlations

$$C(t_1, t_2),\ C(t_2, t_3),\ C(t_1, t_3)$$

If, in the aging regime $C_{ag}^{ij} \equiv C_{ag}(t_i, t_j) = f_{ag}\left(\frac{h(t_i)}{h(t_j)}\right)$ with $t_i \geq t_j \Rightarrow$

$$C_{ag}^{12} = f_{ag}\left(\frac{h(t_1)}{h(t_3)}\frac{h(t_3)}{h(t_2)}\right) = f_{ag}\left(\frac{f_{ag}^{-1}(C_{ag}^{13})}{f_{ag}^{-1}(C_{ag}^{23})}\right)$$

q_{ea}



q_{ea}

choose t_3 and t_1 so that $C^{13} = 0.3$

the arrow shows the t_2 'flow' from t_3 to t_1

e.g. $C^{12} = q_{ea} C^{13} / C^{23}$

Triangular relations

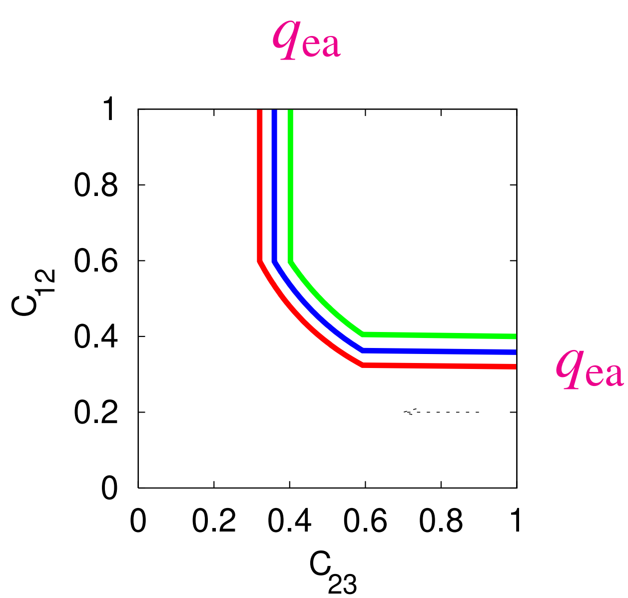
Scaling of the slow part of the global correlation

Take three times $t_1 \geq t_2 \geq t_3$ and compute the three local correlations

$$C_{\vec r}(t_1,t_2),\ C_{\vec r}(t_2,t_3),\ C_{\vec r}(t_1,t_3)$$

If, in the aging regime $C^{ij}_{\vec r} \equiv C_{\vec r}(t_i,t_j) = f_{\rm ag}\left(\frac{h_{\vec r}(t_i)}{h_{\vec r}(t_j)}\right)$ with $t_i \geq t_j \Rightarrow$

$$C^{12}_{\vec r} = f_{\rm ag}\left(\frac{f^{-1}_{\rm ag}(C^{13}_{\vec r})}{f^{-1}_{\rm ag}(C^{23}_{\vec r})}\right)$$



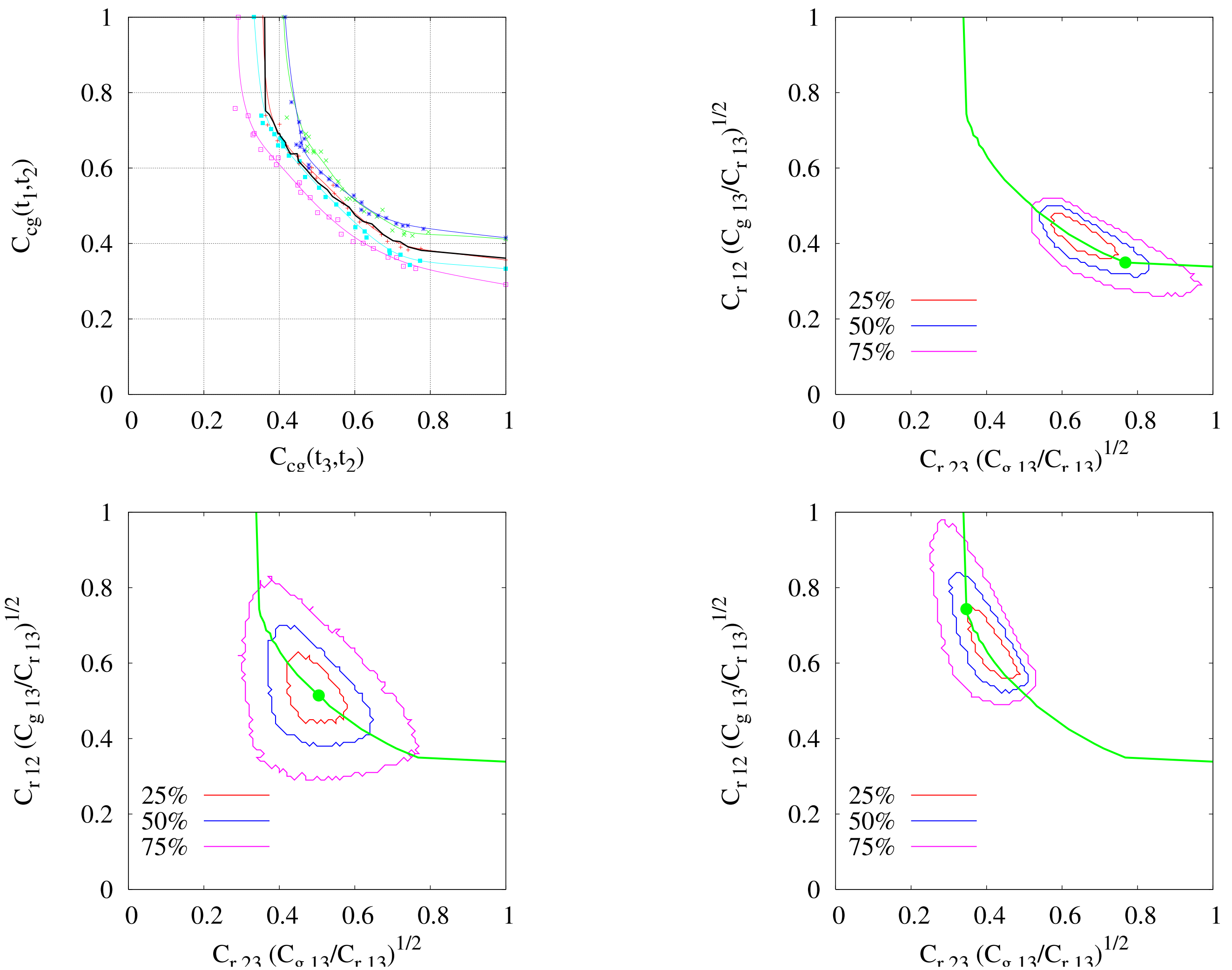
choose t_3 and t_1 so that $C^{13} = 0.3$

the arrow shows the t_2 ‘flow’ from t_3 to t_1

e.g. $C^{12}_{\vec r} = q_{\rm ea}\, C^{13}_{\vec r}/C^{23}_{\vec r}$.

Triangular relations

$3d$ Edwards-Anderson model



Jaubert, Chamon, LFC & Picco 07