A Short Review of Classical Active Matter

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Special thanks to

S. Mossa & D. Loi (Grenoble) – 2008 - 2011

G. Gonnella and his group in Bari 2014 – present

D Levis and his group in Barcelona 2018 – present

M. Schirò (College de France) & R. Fazio (ICTP) – present

ICTP, Trieste, 2026

Definitions and realisations

Active Matter

Definition - Biological inspiration

Active matter is composed of large numbers of active "agents", which consume energy and thus move or exert mechanical forces

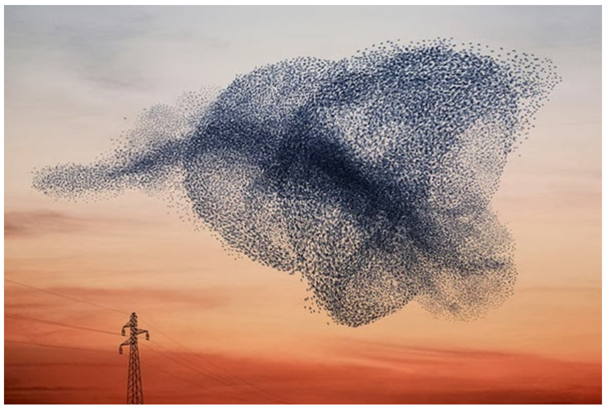
Due to the energy consumption, these systems are intrinsically out of thermal equilibrium

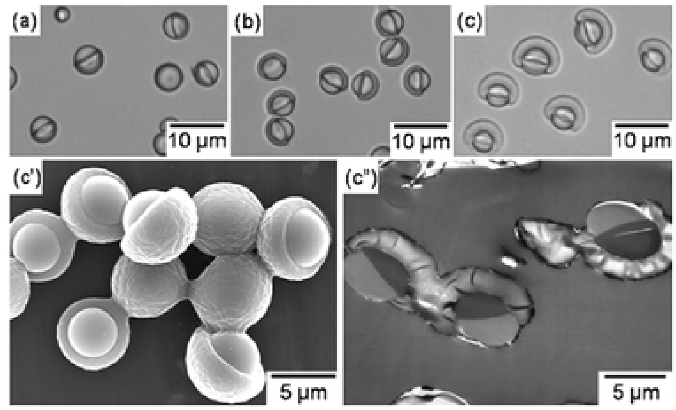
Homogeneous energy injection (not from the borders, *cfr.* shear)

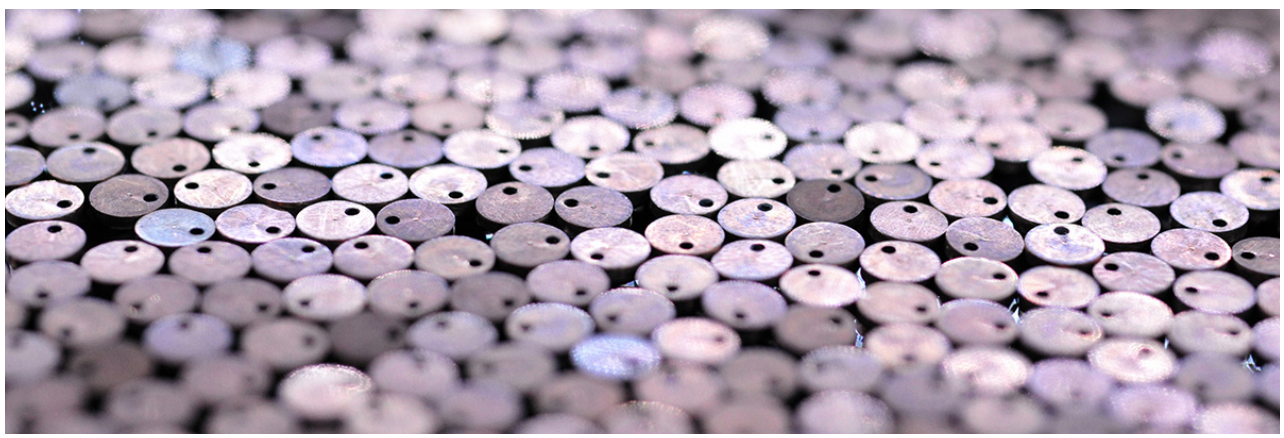
Coupling to the environment (bath) allows for dissipation and the access of some asymptotic state

Active Matter

Natural & artificial systems







Experiments & observations **Bartolo et al.** Lyon, **Bocquet et al.** Paris, **Cavagna et al.** Roma, **di Leonardo et al.** Roma, **Dauchot et al.** Paris, just to mention some Europeans

Active matter

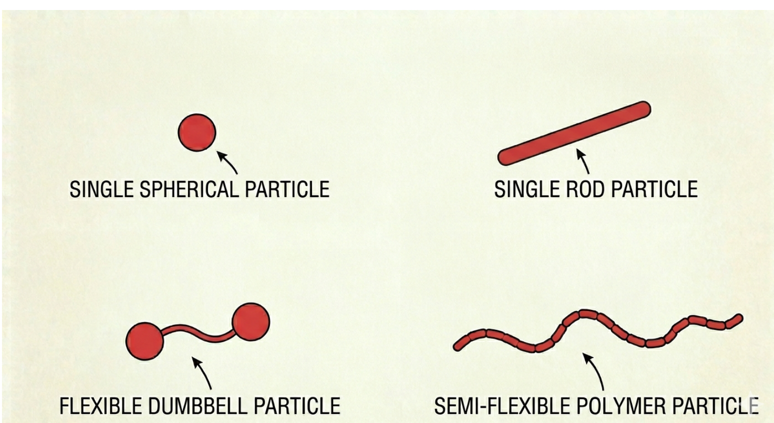
Realisations & modelling

- Wide range of scales: macroscopic to microscopic

  Natural examples are birds, fish, cells, bacteria.

  Artificial realisations are Janus particles, asymmetric grains, toys, etc.

- Embedding spaces in $3d$, $\boxed{2d}$ and $1d$.
- Particle shapes



- Self-propulsion mechanism

  Microscopic - details often ignored

- Energy injection mechanism

  Details ignored - models proposed

  Non-conservative forces

Active Matter

Global goal - from our "community"

To understand the **collective** behaviour of **active matter**

from the **statistical physics** viewpoint

with the help of extensive **numerical simulations**

and **theoretical arguments/analytic calculations**

Statistical Physics approach

One has to start from the **single particle/hydro/field theory** modelling

Single Particle Models

Brownian Motion

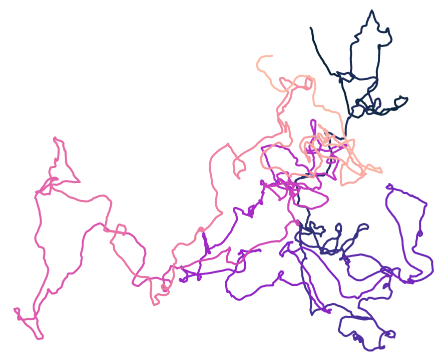
Langevin description

A spherical particle immersed in a liquid

Force on the particle due to the coupling to the environment

No external force

$$\overbrace{m\ddot{\mathbf{r}} = \mathbf{F}_{\rm env}}^{\text{Newton}} = \underbrace{-\gamma\dot{\mathbf{r}}}_{\text{friction}} + \underbrace{\boldsymbol{\xi}}_{\text{noise}}$$



$\boldsymbol{\xi}$ Gaussian white noise with $\langle \xi_a(t) \rangle = 0$ and $\langle \xi_a(t)\xi_b(t') \rangle = 2\gamma k_B T \delta_{ab}\delta(t-t')$

$a, b = 1, \ldots, d$

Averaged position $\langle \mathbf{r}(t) \rangle = \mathbf{r}_0$ the initial position

Balance between noise and dissipation ensures $\lim_{t \gg m/\gamma} \langle v_a^2(t) \rangle = {}^{k_B T}/_{m}$

Brownian Motion

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Force on the particle due to the coupling to the environment

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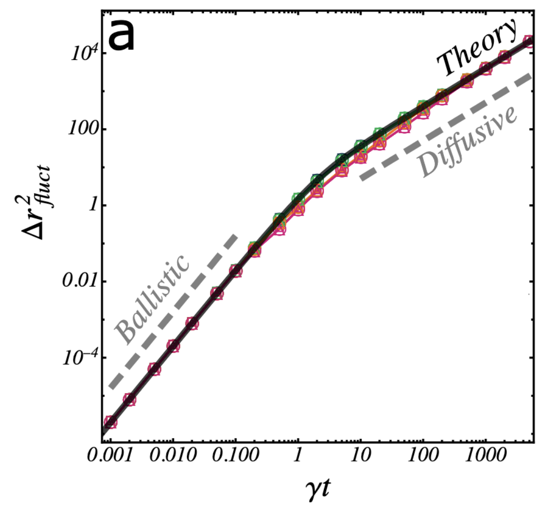


Mean-square displacement $\Delta^2(t,0) = \langle(\mathbf{r}(t) - \mathbf{r}_0)^2\rangle$

At $t < {}^{m}/_{\gamma}$ *ballistic* $\Delta^2 \propto t^2$

At $t > {}^{m}/_{\gamma}$ *diffusive* $\Delta^2 \propto D_T\, t$

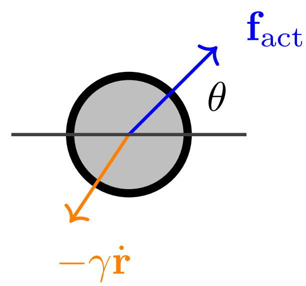
Diffusion coefficient $D_T = {}^{k_B T}/_{\gamma}$

Active Brownian Particle

Langevin equations

Active force $f_{\rm act}$ along $\hat{\mathbf{n}} = (\cos\theta, \sin\theta)$ a direction following a random walk

$$
\begin{aligned}
m\ddot{\mathbf{r}} &= f_{\rm act}\hat{\mathbf{n}} - \gamma\dot{\mathbf{r}} + \boldsymbol{\xi} \\
\dot{\theta} &= \eta
\end{aligned}
$$



$\boldsymbol{\xi}$ and η Gaussian white noises

$\langle \xi_a(t)\rangle = \langle \eta(t)\rangle = 0$, $\langle \xi_a(t)\,\xi_b(t')\rangle = 2\gamma k_B T \delta_{ab}\delta(t-t')$ and $\langle \eta(t)\,\eta(t')\rangle = 2D_\theta\delta(t-t')$

Persistence time $\tau_p = 1/D_\theta$ and length $\ell_p = v_0\,\tau_p = (f_{\rm act}/\gamma)\,\tau_p$

Péclet number Pe = $f_{\rm act}\sigma/(k_B T)$ measures the activity

Active Brownian Particle

Langevin equations – trajectories

Active force $f_{\rm act}$ along $\hat{\mathbf{n}} = (\cos\theta, \sin\theta)$ a direction following a random walk

$$
\begin{aligned}
m\ddot{\mathbf{r}} &= f_{\rm act}\hat{\mathbf{n}} - \gamma\dot{\mathbf{r}} + \boldsymbol{\xi} \\
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Active Ornstein Uhlenbeck

Over-damped Langevin equations

Long times, beyond the inertia time-scale $t \gg {}^{m}\!/\!_{\gamma}$

$$\dot{\mathbf{r}} = \mathbf{v} \qquad\qquad \tau\,\dot{\mathbf{v}} = -\mathbf{v} + \eta$$

Gaussian white noise with $\langle \eta_a(t) \rangle = 0$ and $\langle \eta_a(t)\eta_b(t') \rangle = 2D\,\delta_{ab}\,\delta(t-t')$

The random velocity $\mathbf{v}$ is correlated as

$$\langle v_a(t) v_b(t') \rangle = \delta_{ab}\,\frac{D}{\tau}\,e^{-|t-t'|/\tau}$$

Unbalanced correlated noise and memoryless friction $\Longrightarrow$

out-of-equilibrium environment

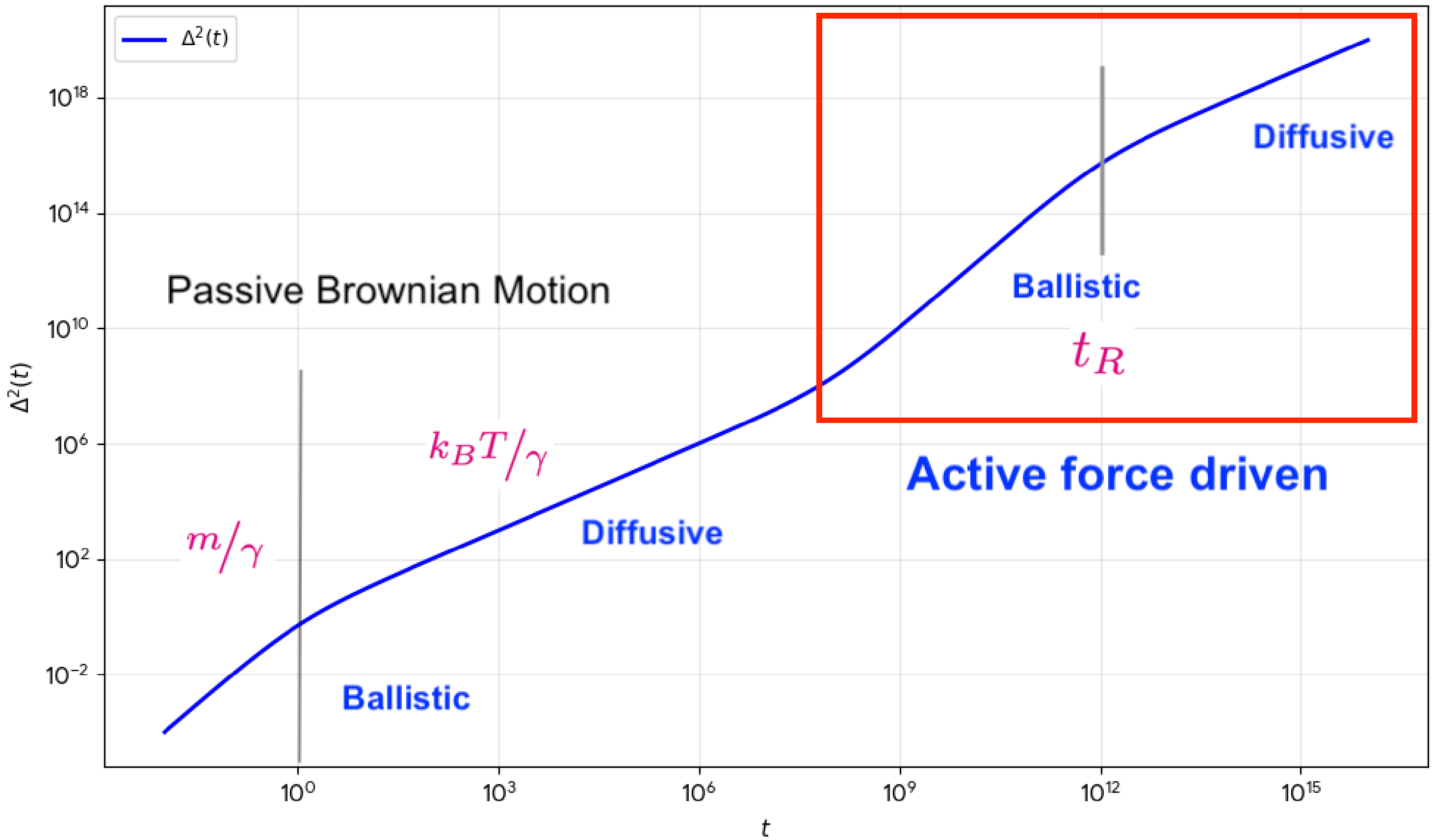
Similar **phenomenology** as the one of the ABPs but simpler to deal with,

no additional degree of freedom **Martin et al 2021**

ABP & AOUP

Mean-square displacement

$$\Delta^2(t) \equiv \langle (\mathbf{r}(t) - \mathbf{r}(0))^2 \rangle = \langle (\mathbf{r}(t) - \mathbf{r}_0)^2 \rangle = \langle (\mathbf{r}(t) - \langle \mathbf{r}(t) \rangle)^2 \rangle \equiv \mathrm{Var}(\mathbf{r}(t))$$



Passive Brownian Motion τ^* Active Driven Motion

Active Brownian Particle

Mean-square displacement

Active force $f_{\rm act}$ along $\hat{\mathbf{n}} = (\cos\theta, \sin\theta)$ a direction following a random walk

$$m\ddot{\mathbf{r}} = f_{\rm act}\hat{\mathbf{n}} - \gamma\dot{\mathbf{r}} + \boldsymbol{\xi} \qquad\qquad \dot{\theta} = \eta$$

Mean-square displacement $\Delta^2(t) = \langle(\mathbf{r}(t) - \mathbf{r}(0))^2\rangle$

Time	Regime	$\Delta^2 \propto$	Physical Interpretation
$t < {}^{m}/_{\gamma}$	ballistic	${}^{k_B T}/_{m}\; t^2$	Inertial thermal velocity dominates
${}^{m}/_{\gamma} < t < \tau^*$	diffusive	$D_T\; t$	Damping $\mapsto$ thermal Brownian motion
$\tau^* < t < t_R$	ballistic	${}^{f_{\rm act}^2}/_{\gamma^2}\; t^2$	Self-prop speed $v_0 = {}^{f_{\rm act}}/_{\gamma}$ dominates
$t_R < t$	diffusive	$D_A\; t$	Rotational diff randomizes $\mathbf{v}_0$

Diffusion coefficients $\underbrace{D_T = {}^{k_B T}/_{\gamma}}_{\text{thermal}}$ $\quad$ $\underbrace{D_A = D_T + c\,\mathrm{Pe}^2}_{\text{active}}$

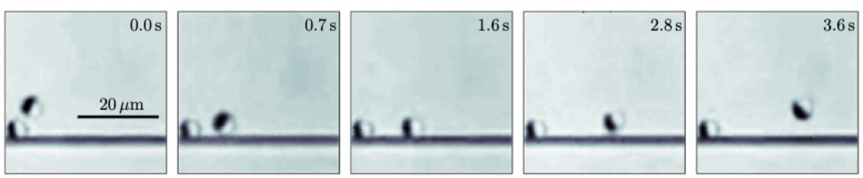
Active Brownian Particle

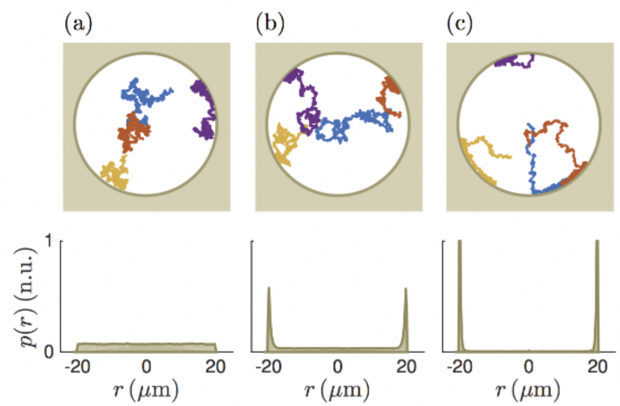
Aggregation close to walls

Active force $f_{\rm act}$ along $\hat{\mathbf{n}} = (\cos\theta, \sin\theta)$ a direction following a random walk

$$m\ddot{\mathbf{r}} = f_{\rm act}\hat{\mathbf{n}} - \gamma\dot{\mathbf{r}} + \boldsymbol{\xi} \qquad \dot{\theta} = \eta$$

Motion of Janus colloids & ABPs





close to a wall **in a pore**

Many Particle Models

Vicsek model

Minimal (cellular automata) model for flocking

Flocking due to any kind of self-propulsion and alignment with neighbours

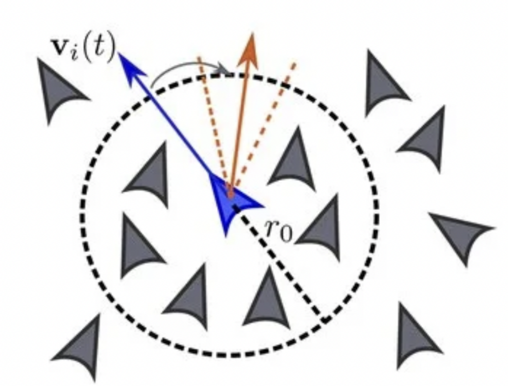
The individual's direction is updated according to the mean $\overline{\ldots}$ over its neighbours

$$\hat{\boldsymbol{v}}_i(t+\delta t) = \overline{\hat{\boldsymbol{v}}_j(t)}_{|\boldsymbol{r}_i-\boldsymbol{r}_j|<r_0} + \boldsymbol{\eta}_i(t)$$

plus noise & normalization

moves at constant speed v_0 in the new direction

$$\boldsymbol{r}_i(t+\delta t) = \boldsymbol{r}_i(t) + v_0 \hat{\boldsymbol{v}}_i(t+\delta t)\,\delta t$$



Control parameters :

noise temperature & density

$\hat{\boldsymbol{v}}_i$ similar to $\boldsymbol{s}_i$ in the Heisenberg (or XY) ferro models – though moving in real space

Spontaneous symmetry breaking of polar order $\boldsymbol{p}(t) = \frac{1}{N}\sum_{i=1}^{N} \hat{\boldsymbol{v}}_i(t) \neq 0$ even in $2d$

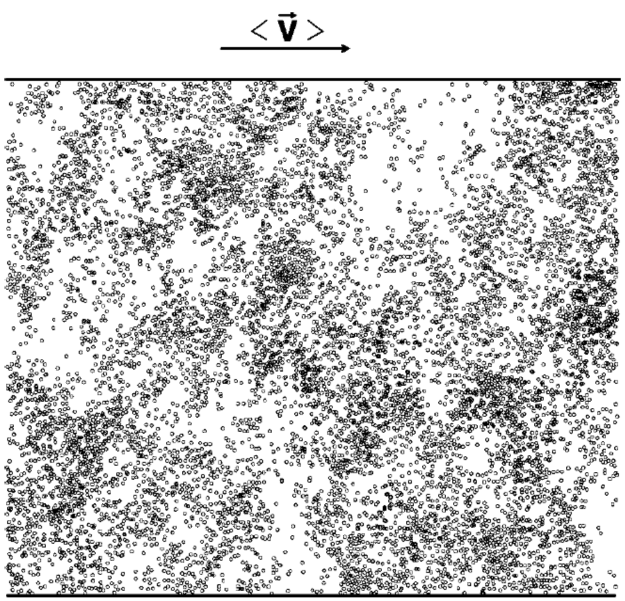
Vicsek, Czirók, Ben-Jacob, Cohen & Shochet 95

Toner-Tu Model

Continuum model for dry active matter

Order parameter

center-of-mass $\langle \boldsymbol{v}_{\mathrm{c.o.m.}} \rangle \neq 0$



continuous rotational symmetry spont broken

Possible in low d out of equilibrium

Breakdown of linearized theory

$\Longrightarrow$ large fluctuations

Argument

Improved transport suppresses the very fluctuations that give rise to it, leading to long-range order in $d = 2$

Giant density fluctuations

Toner & Tu 95 & 98

Toner-Tu Model

Continuum model for the flock velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$

$$\partial_t \boldsymbol{v} + \underbrace{\lambda_1(\boldsymbol{\nabla}\cdot\boldsymbol{v})\boldsymbol{v} + \lambda_2(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} + \lambda_3\boldsymbol{\nabla}v^2}_{\text{Navier-Stokes w/no Galilean invariance}} =$$

$$\underbrace{\alpha_1\boldsymbol{v} - \alpha_2 v^2\boldsymbol{v}}_{\text{“Potential force” imposing } v^2 = \alpha_1/\alpha_2} \quad \underbrace{-\boldsymbol{\nabla}P}_{\text{“Pressure variation”}}$$

$$\underbrace{+D_B\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v}) + D_T\nabla^2\boldsymbol{v} + D_2(\boldsymbol{v}\cdot\boldsymbol{\nabla})^2\boldsymbol{v}}_{\text{Dissipative terms}} \underbrace{+\boldsymbol{\eta}}_{\text{Noise}}$$

$$P = \sum_{n=1}^{\infty}\sigma_n(\rho - \rho_0)^n$$

Pressure imposing $\sim$ uniform density, $\rho - \rho_0$ small

$$\partial_t\rho + \boldsymbol{\nabla}\cdot(\boldsymbol{v}\rho) = 0$$

Toner & Tu, PRE 98

$\alpha_1 < 0$ ($\alpha_1 > 0$) in the homogenous (flocking) phase

Dry active matter e.g. herd of wildebeest, flock of birds, bacteria on agar plate

Momentum is dissipated into the substrate

Flocking in dry active matter

Vicsek & Toner-Tu Models, self-alignment

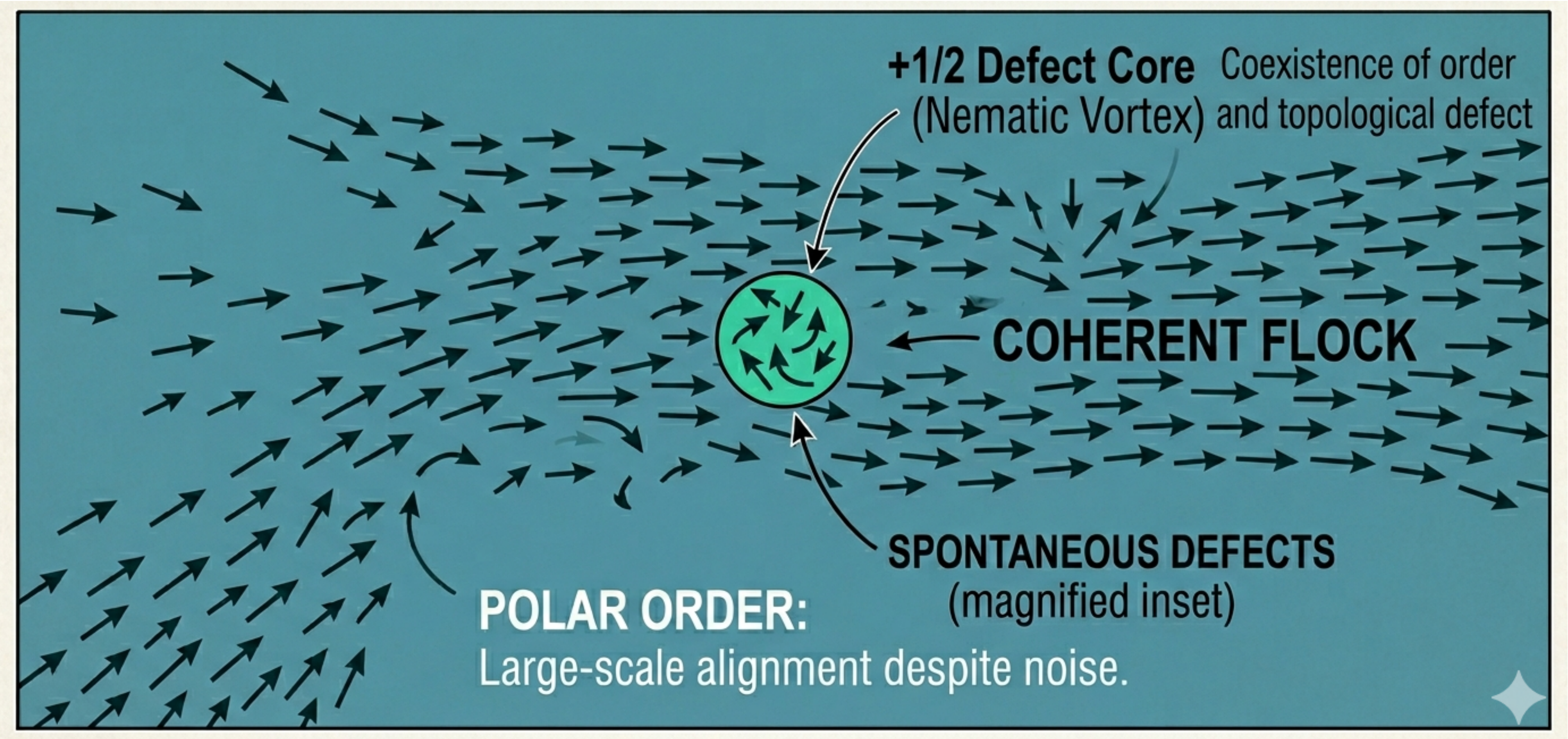


Image generated by Gemini

Flocking also in a **Hamiltonian fluid with velocity-spin coupling**

Casiulis, Tarzia, LFC & Dauchot 20

Fluctuating Hydrodynamics

Wet nematic active matter

Molecule density field c, diffusion tensor $\mathbb{D}$

Traceless & symm tensor field $\mathbb{Q}$ (head-tail symm) for orient order, γ rot friction

Fluid: velocity field $\boldsymbol{u}$, density field ρ, passive shear viscosity η, pressure P

Stress tensor $\mathbb{S} = \mathbb{S}_{\text{passive}} + \mathbb{S}_{\text{active}}$

with $\mathbb{S}_{\text{active}} = \alpha\mathbb{Q}$, and $\alpha > 0$ for pushers and $\alpha < 0$ for pullers

$$(\partial_t + \boldsymbol{u}\cdot\boldsymbol{\nabla})\, c = \boldsymbol{\nabla}\cdot(\mathbb{D}\boldsymbol{c}) \qquad \text{mass conservation}$$

$$\rho\,(\partial_t + \boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u} = \eta\nabla^2\boldsymbol{u} - \boldsymbol{\nabla}P + \boldsymbol{\nabla}\mathbb{S} \qquad \text{momentum conservation}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0 \qquad \text{incompressible fluid}$$

$$(\partial_t + \boldsymbol{u}\cdot\boldsymbol{\nabla})\,\mathbb{Q} = -\frac{1}{\gamma_{\mathbb{Q}}}\frac{\delta F}{\delta\mathbb{Q}} + \mathbb{F}(\boldsymbol{\nabla}\boldsymbol{u},\mathbb{Q}) + \boldsymbol{\eta}_{\mathbb{Q}} \qquad \text{fluid-molecule coupling}$$

e.g., suspension of cytoskeletal filaments or elongated swimming bacteria, and **active turbulence**

Marchetti, Joanny, Ramaswamy, Liverpool, Prost, Rao & Simha 13

Wet Polar/Nematic Active Matter

Fluctuating Hydrodynamics

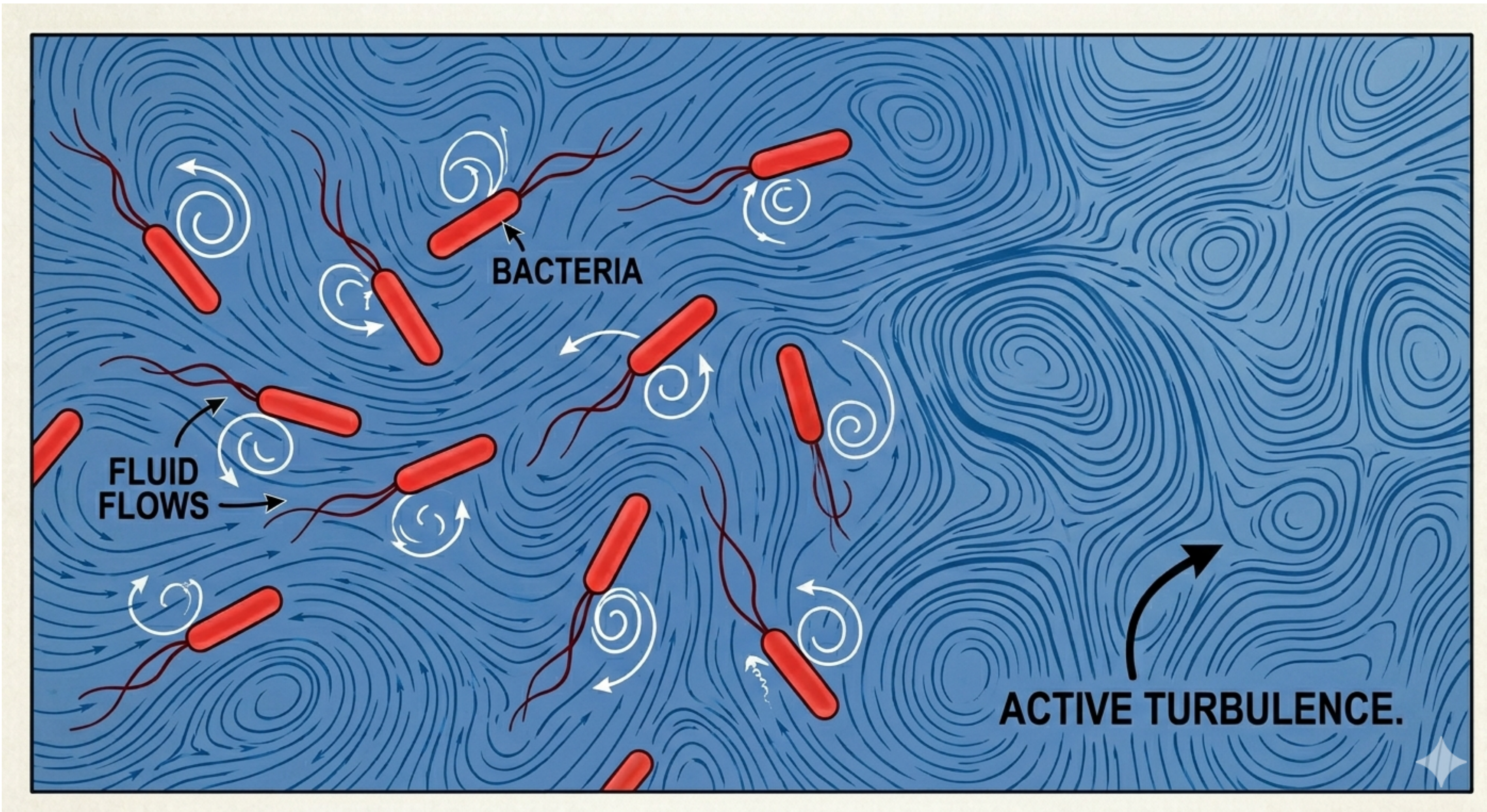
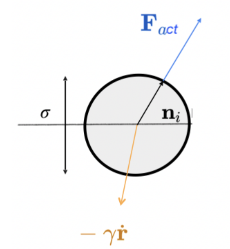


Image generated by Gemini

Active Brownian Particles

(Overdamped) Langevin equations for, e.g. Janus particles



Active force $f_{\rm act}$ along $\mathbf{n}_i = (\cos\theta_i, \sin\theta_i)$

$$m\ddot{\mathbf{r}}_i + \gamma\dot{\mathbf{r}}_i = f_{\rm act}\mathbf{n}_i - \nabla_i \sum_{j(\neq i)} U_{\rm Mie}(r_{ij}) + \boldsymbol{\xi}_i \qquad \dot{\theta}_i = \eta_i$$

$\mathbf{r}_i$ position of the ith particle & $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$ inter-part distance,

$U_{\rm Mie}$ **short-range strongly repulsive** Mie potential

No alignment mechanism

Péclet number Pe = $f_{\rm act}\sigma/(k_BT)$

Over-damped dynamics $m/\gamma = 0.1$

$\boldsymbol{\xi}$ and η Gaussian white noises

$\langle\xi_a(t)\rangle = \langle\eta(t)\rangle = 0$, $\langle\xi_a(t)\,\xi_b(t')\rangle = 2\gamma k_BT\delta_{ab}\delta(t-t')$ and $\langle\eta(t)\,\eta(t')\rangle = 2D_\theta\delta(t-t')$

Persistence time $\tau_p = 1/D_\theta$ and length $\ell_p = v_0\,\tau_p = (f_{\rm act}/\gamma)\,\tau_p$

$\phi = \pi\sigma^2 N/(4S)$ the packing friction

Romanczuk, Bär, Ebeling, Lindner & Schimansky-Geier 12

Active Models B & H

à la Ginzburg-Landau, also for, e.g., Janus particles

Scalar order parameter locally conserved

Cahn-Hilliard – Model B for phase separation

Continuity equation $\partial_t \phi(\boldsymbol{x}, t) = \boldsymbol{\nabla} \cdot \boldsymbol{J}(\boldsymbol{x}, t)$ with current

Active Model B

$$\boldsymbol{J}(\boldsymbol{x}, t) = \underbrace{-\boldsymbol{\nabla} \frac{\delta F[\phi]}{\delta \phi(\boldsymbol{x}, t)}}_{\text{equilibrium forcing}} \underbrace{-\boldsymbol{\nabla} \lambda [\boldsymbol{\nabla} \phi(\boldsymbol{x}, t)]^2}_{\text{non-eq } \neq -\boldsymbol{\nabla} \delta F / \delta \phi} + \underbrace{\boldsymbol{\eta}(\boldsymbol{x}, t)}_{\text{noise}}$$

Active Model B+

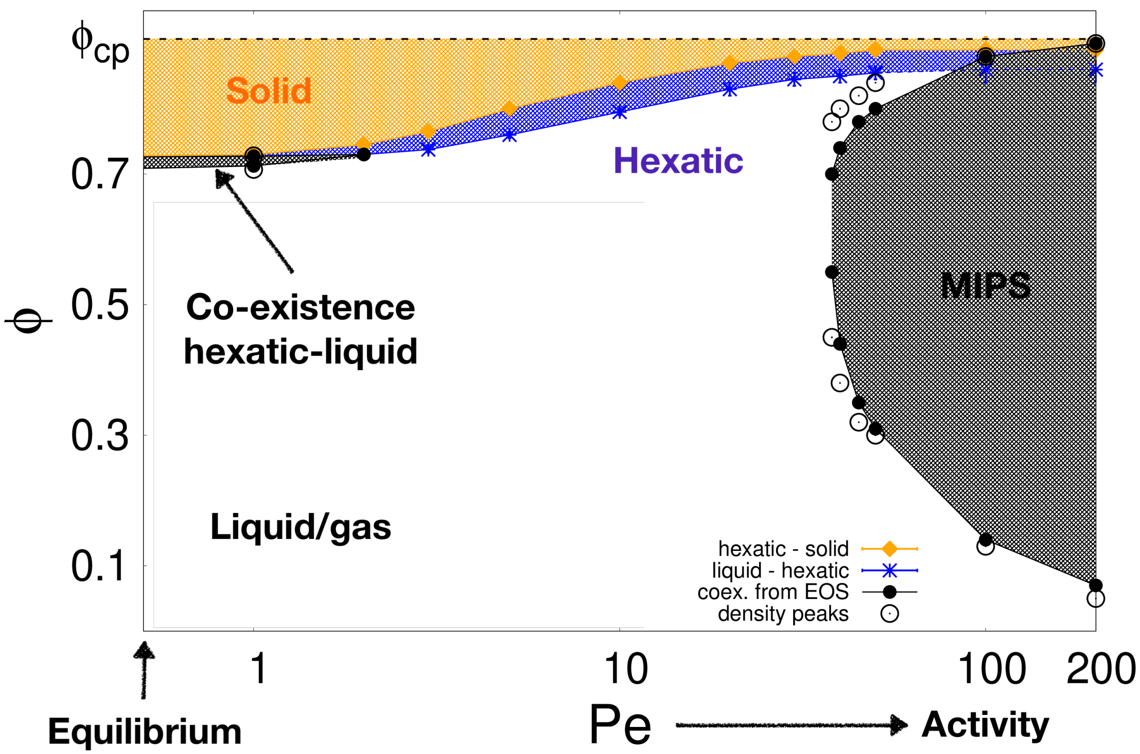
$$\boldsymbol{J} \mapsto \boldsymbol{J} + \underbrace{\zeta (\nabla^2 \phi) \boldsymbol{\nabla} \phi - \frac{1}{2} \boldsymbol{\nabla} (\boldsymbol{\nabla} \phi)^2}_{\text{high-order terms order } \nabla^3 \text{ and } \phi^2}$$

Active Model H with hydrodynamics as well

Cates & collaborators 14-17

Active Brownian Particles

Motility Induced Phase Separation – Collective behaviour



Gray zone at high Pe

Motility induced

phase separation (MIPS)

gas & dense

Cates & Tailleur 15

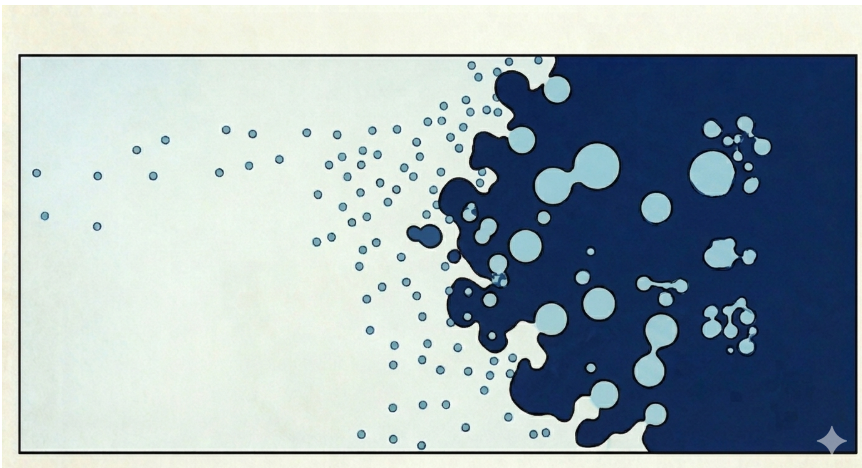
Farage, Krinninger & Brader 15

Pressure $P(\phi, \text{Pe})$ (EOS), correlations $G_T(r)$, $G_6(r)$, and distributions of ϕ_i, $|\psi_{6i}|$

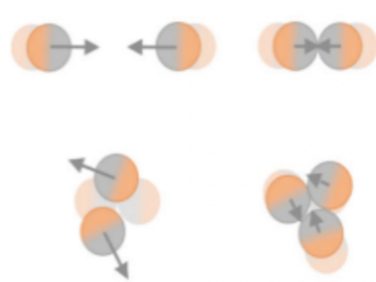
Digregorio, Levis, Suma, LFC, Gonnella & Pagonabarraga 18

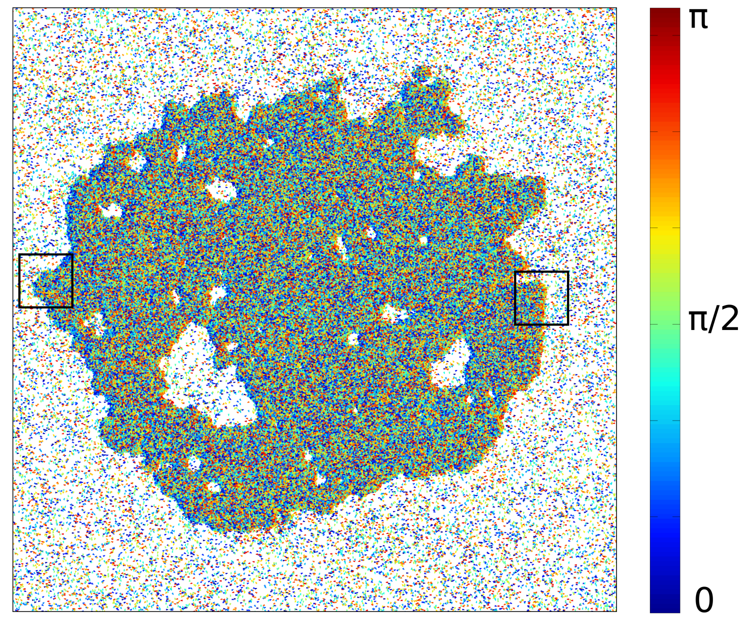
MIPS

Active Brownian Particles & Model B+



Sketch generated by Gemini





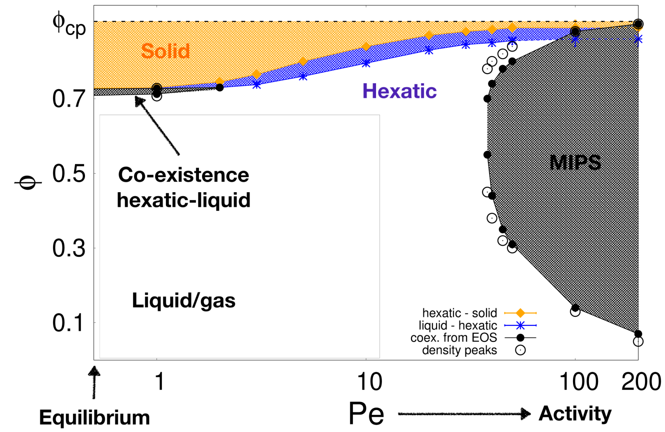
Particles collide heads on and aggregate even under repulsive interactions

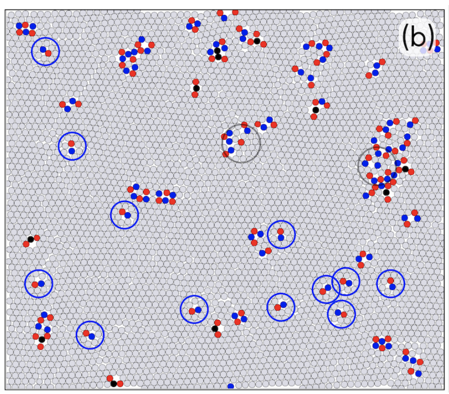
Digregorio et al 18

Plenty of Interesting Features

Full phase diagram of ABPs

solid, **hexatic**, **liquid** & **MIPS**





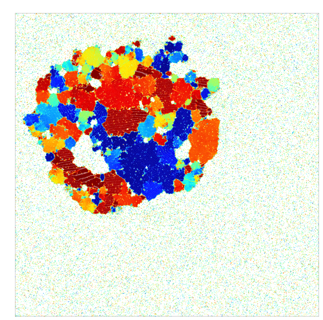
Point-like **dislocations** & **disclinations**

and **clustered** defects in

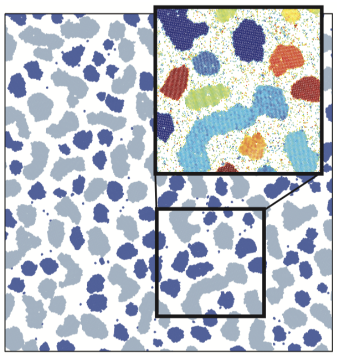
passive & active $2d$ models.

In **MIPS**

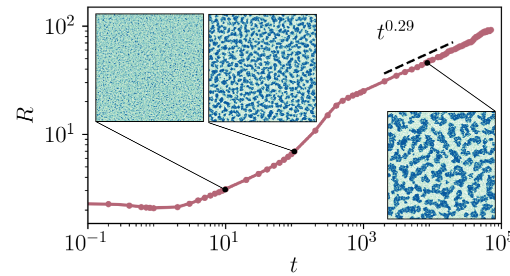
Micro *vs.* macro: hexatic patches & bubbles



Plenty of Interesting Features

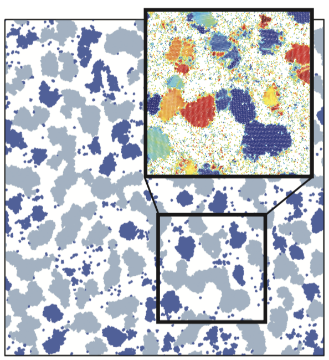


Difference between



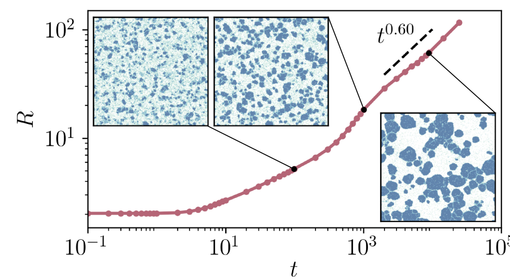
Passive

ABPs



Active

growth



ABDs

Ostwald ripening & cluster-cluster diffusive aggregation in active case
cluster-cluster aggregation almost not present in passive

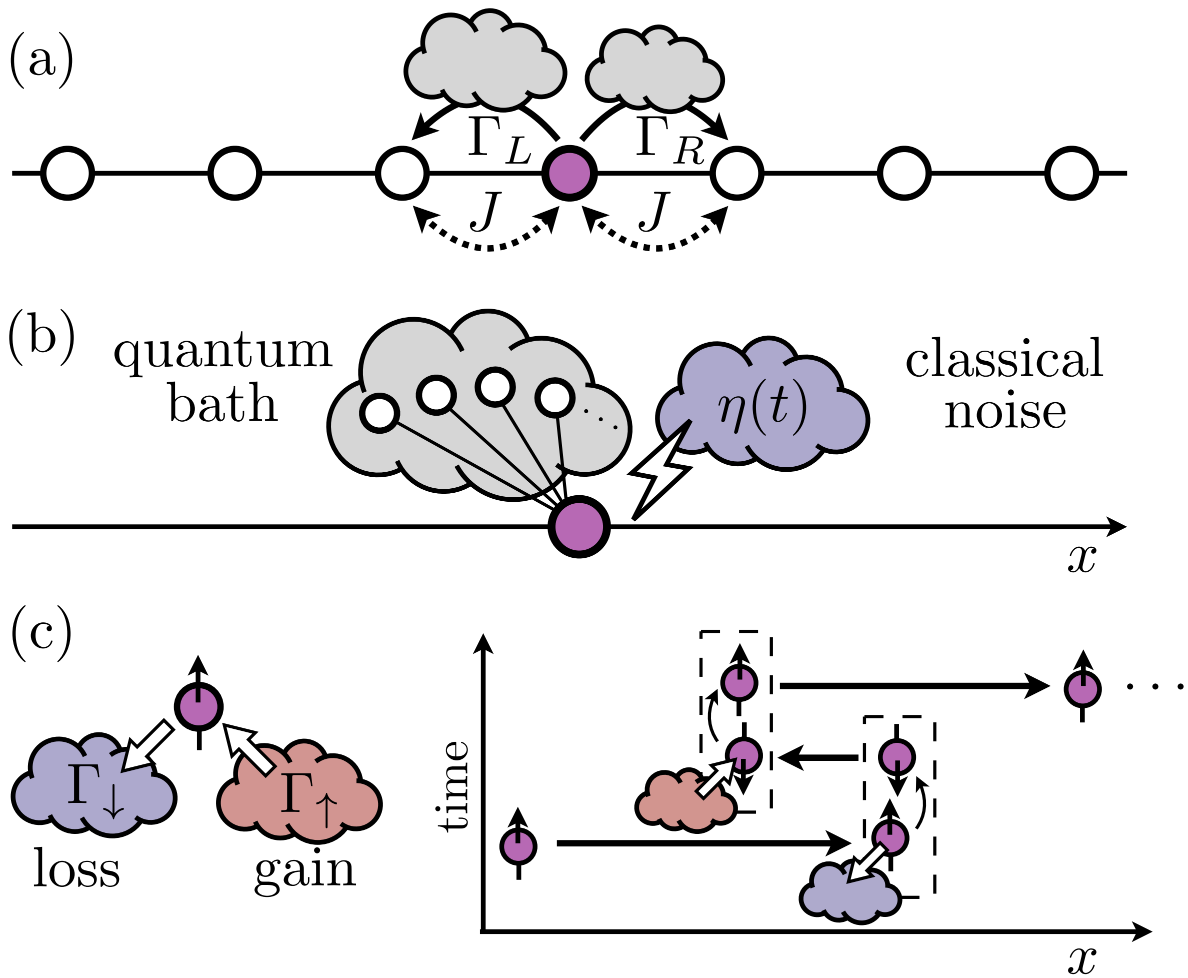
Co-existence of regular and fractal clusters

Heterogeneous orientational order in large active clusters

Quantum Active Particles

Quantum Active Particles

Sketches of three models - playing with the environments



Gipouloux, Brunelli, LFC, Fazio & Schirò 26