Classical spin & gauge models

(the quantum toric code accuracy threshold)

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Quantum computation

Some ideas

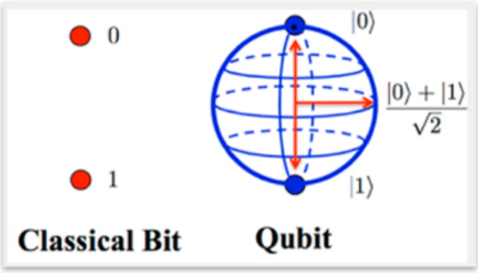
Basic units of information

classical (bit)

$n_i = 0, 1$

quantum (qubit)

$|\psi_i\rangle = a_i|0\rangle + b_i|1\rangle$ with $|a_i|^2 + |b_i|^2 = 1$ - advantage



but problems on the quantum side

Protection of quantum information

against coupling to **environment**

decoherence $\rightarrow$ classical

Action with **quantum gates**

unitary transformations

continuum values $\rightarrow$ dephasing

Quantum encoding

Kitaev's 2D toric quantum code & the stabilizer formalism

Kitaev's 2D toric quantum code

Quantum information is encoded in the **global topological properties** of a macroscopic state and can then be fully **recovered** using **error correcting codes** up to a certain error rate, called **accuracy threshold**

A. Y. Kitaev 97

Quantum error correcting codes could be extensions of the classical repetition ones :

bit 0 copied $0 \mapsto 000$;
transferred with, possibly, one error 010 ;
apply majority rule $010 \mapsto 0$;
original bit recovered

The **stabilizer code** identifies

check operators to detect errors & correction operators to rectify them

D. Gottesman 97

Encoding and error correction

Stabilizer method

A two state **qubit** $|0\rangle$ or $|1\rangle$

Susceptible to **flip**, e.g. $\hat{\sigma}_x|0\rangle = |1\rangle$, and **phase**, e.g. $\hat{\sigma}_z|1\rangle = -|1\rangle$, **errors**

A **code** is a joint state of n qubits, e.g. $\underbrace{|0\rangle \otimes |0\rangle \otimes |1\rangle \otimes |0\rangle \otimes \cdots \otimes |1\rangle}_{n}$

$\overbrace{\hat{\sigma}_x \otimes \hat{\sigma}_z \otimes \hat{\sigma}_x \otimes \cdots \otimes \hat{\sigma}_z}^{n}$ is an example of operator acting on an n qubit state

The **stabilizer operators** $\{\hat{P}_\alpha\}$ form a **group** and leave the basis set of a code $\{|\psi_a\rangle\}$ invariant : $\hat{P}_\alpha|\psi_a\rangle = |\psi_a\rangle$ for all α, a

Quantum information can be stored in $|\psi_a\rangle$ or superpositions $|\psi\rangle$

Take an **error operator** $\hat{E}$ such that $\hat{E}|\psi\rangle = |\psi'\rangle \neq |\psi\rangle$

it anti-commutes with any $\hat{P}_\alpha$, $\quad \hat{E}\hat{P}_\alpha = -\hat{E}\hat{P}_\alpha$, $\quad$ and $\boxed{\hat{P}_\alpha|\psi'\rangle = -|\psi'\rangle}$

Quantum Toric Codes

Definition

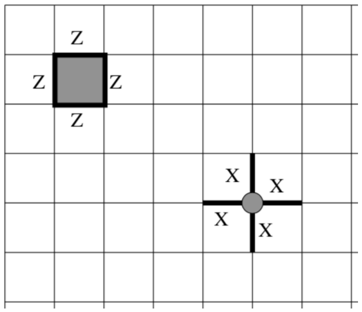
Call $|\psi_i\rangle = a_i|1\rangle + b_i|0\rangle$ the state of the ith **qu-bit**

Place the **qu-bits** on the links of a square lattice defined on a $2d$ surface with non trivial topology, e.g. a **torus**

The state of the $n = L^2$ qu-bits is $|\psi\rangle = \prod_{i=1}^{n} \otimes |\psi_i\rangle$

Local check/stabilizer operators $\hat{P}_\alpha$: tensor product of four Pauli operators acting on the four selected qu-bits on a plaquette or link, $\otimes\, \mathbb{I}$ on all other

$$\hat{O}_p = \prod_{i \in p} \otimes \hat{\sigma}_i^z \qquad \hat{O}_v = \prod_{i \in v} \otimes \hat{\sigma}_i^x$$



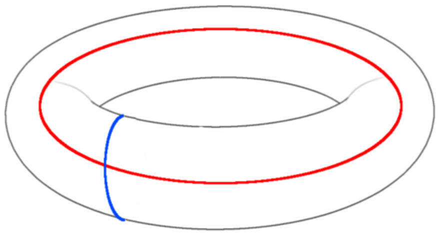
They all commute

can be diagonalized simultaneously

Quantum Toric Codes

Storing information

Hamiltonian



$$\hat{\mathcal{H}} = -J\sum_p \hat{O}_p - J\sum_v \hat{O}_v$$

Model defined on a torus

ground state manifold dim 4

trivial GS $\hat{O}_{p,v}|\mathrm{GS}\rangle = +1|\mathrm{GS}\rangle$ for all p, v

a loop of “reversed” links in one or the other direction

a double loop winding in both directions of this kind

Excitations created by string operators - anyons

Ground states protected against local perturbation

Quantum Toric Codes

Errors

An error occurs (on a link) or a set of errors (on a path)

a **pair of defects** (on the attached vertices) nucleate

To correct it/them

act with operators along a **"recovery chain"** bounded by the two defects

(Like bringing together the two defects and letting them annihilate)

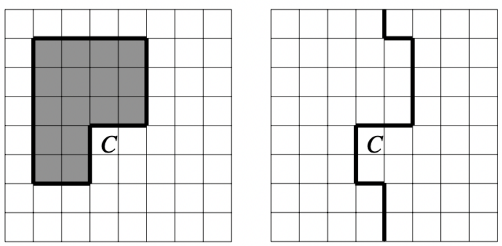
If the closed loop of both the error chain and the recovery chain is

homologically trivial

$\Rightarrow$ correction is possible

homologically nontrivial

$\Rightarrow$ correction is not possible

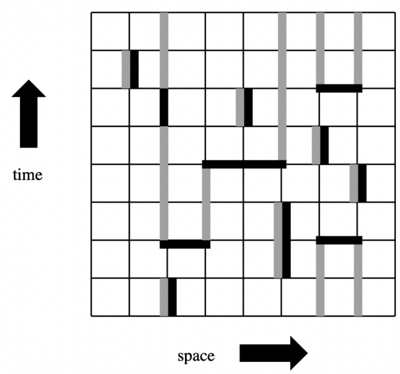


Quantum Toric Codes

Errors assumed to be stochastic and independent

They appear independently on different sites, those created by $\hat{\sigma}_x$ and $\hat{\sigma}_z$ are equally likely and have probability p

The probability that the measured syndrome bit is faulty is q



horizontal (space) :

bit-flip or bit-phase errors p

vertical (time) :

measurement errors q

From the statistics of these lines and the ones used to correct the errors : mapping to **canonical equilibrium of classical statistical physics disordered models**

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Limits to correction

Mapping to classical models

Qubit (flip & phase) error

2D $\pm J$ spin-glass

$$\mathcal{H} = -\frac{1}{2}\sum_{\langle ij\rangle} J_{ij}\sigma_i\sigma_j$$

$$P(J_{ij}) = p\delta_{J_{ij},-J} + (1-p)\delta_{J_{ij},J}$$

Control parameters

βJ & error concentration p

one **Nishimori relation**

$$e^{-2\beta J} = \frac{p}{1-p}$$

Qubit (flip & phase) and measurement errors

sequential measurements $\mapsto$ time

(2+1)D random plaquette gauge model

$$\mathcal{H} = -\sum_{p_h} J_{p_h} \prod_{i\in p_h} \sigma_i - \sum_{p_v} J_{p_v} \prod_{i\in p_v} \sigma_i$$

$$P(J_{p_h}) = p\delta_{J_{p_h},-J} + (1-p)\delta_{J_{p_h},J}$$

$$P(J_{p_v}) = q\delta_{J_{p_v},-K} + (1-q)\delta_{J_{p_v},K}$$

βJ, βK & p (qubit) & q (meas) error prob

two **Nishimori relations**

$$e^{-2\beta J} = \frac{p}{1-p} \qquad e^{-2\beta K} = \frac{q}{1-q}$$

Limits to correction

Accuracy thresholds

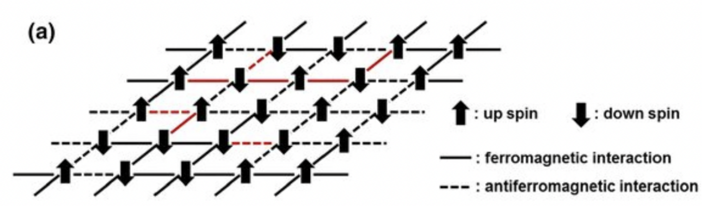
Qubit (flip & phase) error

2D $\pm J$ spin-glass

Nishimori critical point

$(p_N, T_N(p_N))$

between FM and PM phases



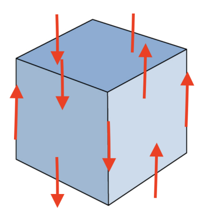
Qubit (flip & phase) and measurement errors

(2+1)D random plaquette gauge model

Disordered **Wegner 71** gauge model

if $p = q \Rightarrow (P_N, T_N(p_N))$

confinement transition



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Plan

Statistical physics perspective

— The $\pm J$ **2D Ising Model**

The equilibrium phase diagram revisited

Nishimori curve & critical behaviour

Out of equilibrium critical dynamics - **Universality ? CFT ?**

Space-time correlations & winding angle, short-time dynamics

— The **pure 3D Gauge Model**

The equilibrium phase diagram revisited

Fortuin-Kasteleyn clusters & geometric loops

Out of equilibrium critical dynamics

— The $\pm J$ and $\pm K$ 3D Gauge Model

The equilibrium phase diagram revisited

Out of equilibrium critical dynamics

$\pm J$ $2d$ Ising Model

Definition

$$\mathcal{H} = -\frac{1}{2} \sum_{\langle ij \rangle} J_{ij} s_i s_j$$

$s_i = \pm 1$

$J_{ij} = \pm J$ **quenched randomness**

$$P(J_{ij}) = p \underbrace{\delta_{J_{ij},-J}}_{\text{AF}} + (1-p) \underbrace{\delta_{J_{ij},J}}_{\text{FM}}$$

$[J_{ij}] = (1-2p)J$

$[J_{ij}^2] = J^2$

p controls the level of **frustration**

$p = 0$ Ferromagnetic Ising Model

$p = 1/2$ (unbiased) Ising Spin-Glass

$p = 1$ Anti-Ferromagnetic Ising Model

symmetry $p \leftrightarrow 1 - p$

Effects of disorder

Results for weak disorder

– **Does disorder kill the ordered phase?** no, but $T_c(p) \searrow$ for $p \nearrow$ expected

– **Effect on the phase transition**

Harris Criterium : the randomness is relevant (irrelevant) if the specific heat exponent α of the pure ($p = 0$) model is positive (negative)

A. B. Harris, J. Phys. C7, 1671 (1974)

but for the $2d$ Ising Model $\alpha = 0$

– **Conformal field theory in $2d$?**

The $n = 0$ Gross-Neveu model (for not too large p, see below)

Vik. S. Dotsenko & Vl. S. Dotsenko, Sov. Phys. JETP Lett. 33, 37 (1981)

– **Do critical exponents change?**

No, close to T_{Is} **Vl. S. Dotsenko, M. Picco & P. Pujol**, Nucl. Phys. 455, 701 (1995)

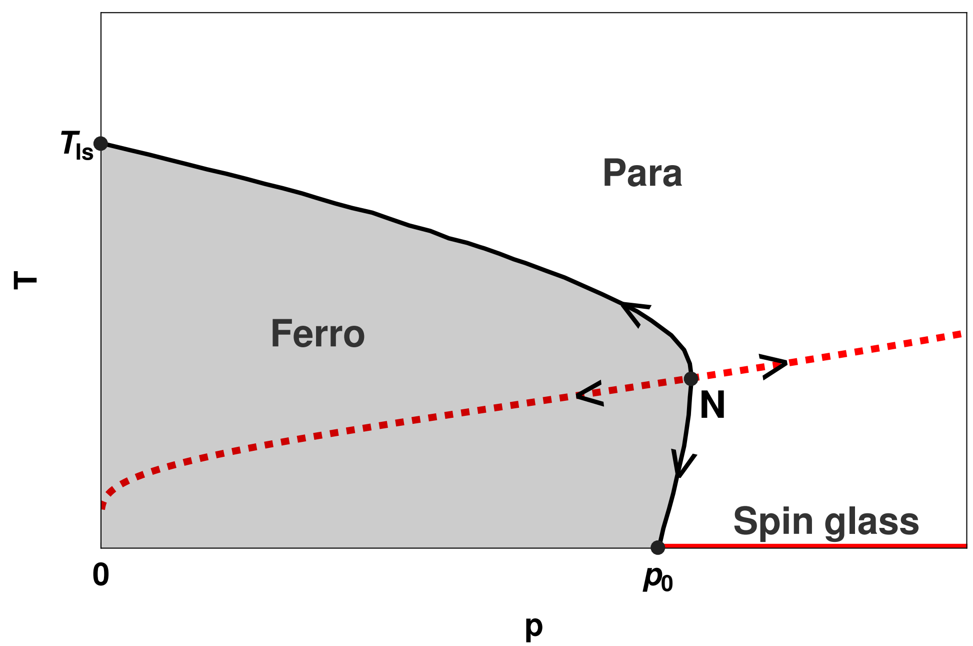
$\pm J$ $2d$ Ising Model

The equilibrium phase diagram ($J = 1$)

Second order phase transition between FM & PM phases

$(T_{\rm Is} = 2.27,\ p = 0)$ **L. Onsager**, Phys. Rev. 65, 117 (1944)

$\downarrow$



Paramagnetic phase

Ferromagnetic phase

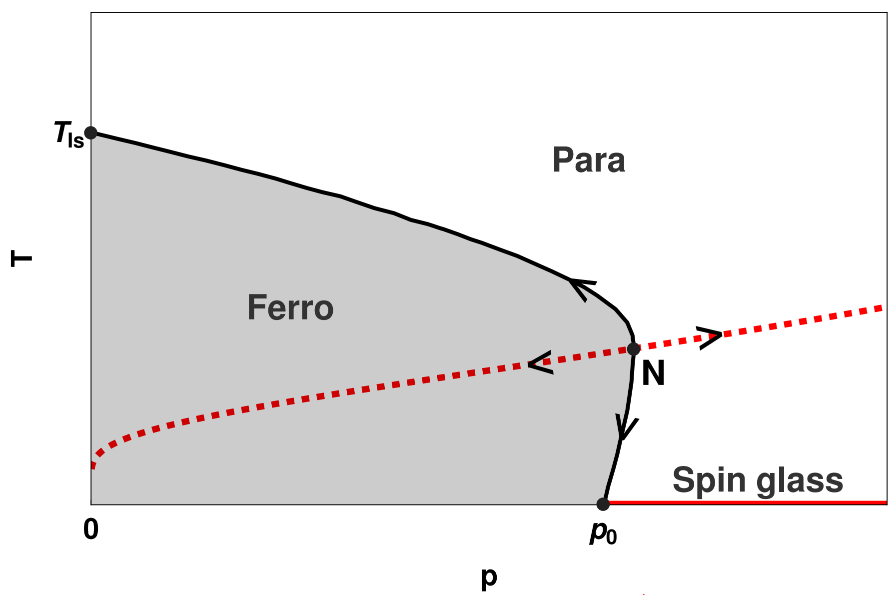
$\pm J$ $2d$ Ising Model

The equilibrium phase diagram

Second order phase transition between FM & PM phases

$(T_{\mathrm{Is}} = 2.27,\ p = 0)$

↓



$$e^{-2\beta J} = \frac{p}{1-p}$$

dotted Nishimori line*

enhanced symmetry properties

$(T_N = 0.95,\ p_N = 0.109)$

*__H. Nishimori__, Prog. Theor. Phys. 66, 1169 (1981)

The Nishimori line

Special features

Local gauge invariance : simultaneous spin and couplings transformation which leave the functional form of $\mathcal{H}$ invariant but change $P(J_{ij})$

On the **Nishimori line** $e^{-2\beta J} = \frac{p}{1-p}$: exact expression for $[\langle \mathcal{H} \rangle](p)$, etc.

The Nishimori line meets the FM-PM transition line at a **tri-critical point** (p_N, T_N)

Phase transition in the Kitaev's **quantum toric code**

A. Yu. Kitaev, Russian Math. Surveys 52, 1191 (1997)

Below p_N encoded information can be protected arbitrarily well

Above p_N it cannot

p is the qu-bit (independent) **error probability**, in the limit of a large code block

E. Dennis, A. Kitaev, A. Landahl & J. Preskill, J. Math. Phys. 43, 4452 (2002)

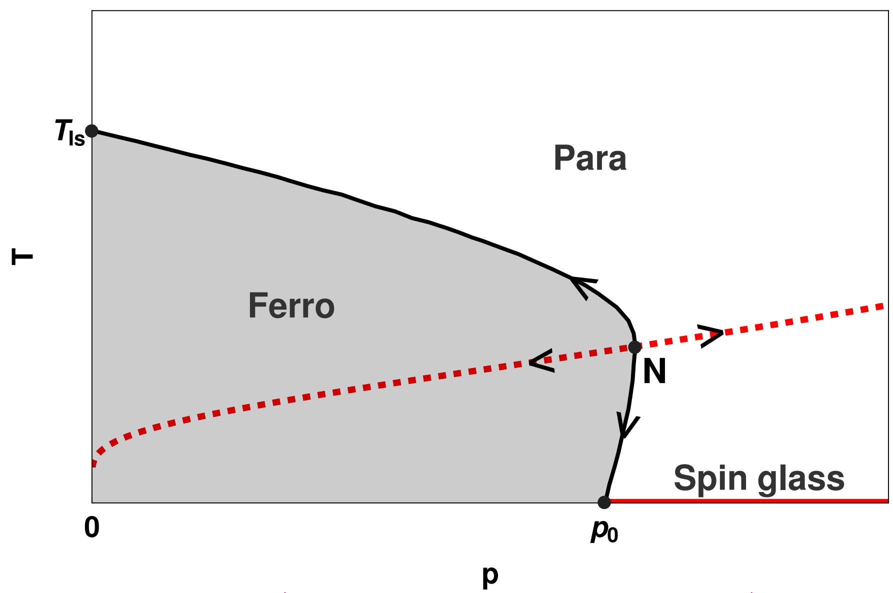
$\pm J$ $2d$ Ising Model

The equilibrium phase diagram

$T_{\rm Is} \geq T > T_N$ Disorder is marginally relevant $\Rightarrow$ $(T_{\rm Is}, p = 0)$ **PM-FM Ising criticality**

Vik. Dotsenko and Vl. Dotsenko, Adv. Phys. 32, 129 (1983)

M. Picco, A. Honecker, and P. Pujol, J. Stat. Mech. P09006 (2006)



$$e^{-2\beta J} = \frac{p}{1-p}$$

dotted Nishimori line*

$(T_0 = 0,\ p_0 = 0.103)$ $\quad$ $(T_N = 0.95,\ p_N = 0.109)$

$0 \leq T < T_N$ **Strong disorder** $\Rightarrow$ $(T_0 = 0, p_0)$ **criticality** and then $T = 0$ **spin-glass**

F. Parisen Toldin, A Pelisetto & E. Vicari, J Stat Phys 135, 1039 (2009)

Critical points

Exponents & equilibrium universality classes

p_c	T_c	ν	η	κ^*	
0	$T_{\text{Is}} = 2.29$	1	0.25	3	FM-PM Ising[1]
$p_N = 0.109$	$T_N = 0.95$	$4/1.5$	0.18	2.22	Bi-critical[2]
$p_0 = 0.103$	$T_0 = 0$	1.5	0.128	1.93	FM-SG[3]
$p_0 < p < 1 - p_0$	$T_{SG} = 0$	∞	0.14	2.1	SG-PM[4]

[1] **L. Onsager**, Phys. Rev. 65, 117 (1944)

[*] **O. Schramm**, Isr. J. Math. 118, 221 (2000) **J. Cardy**, Ann. Phys. 318, 81 (2005)

[2] **W. L. Mc Millan**, PRB 29, 4026 (1984) **M. Hasenbusch** *et al.*, PRE 77, 051115 (2008)

[3] **F. Parisen Toldin, A. Pelissetto & E. Vicari**, J. Stat. Phys 135, 1039 (2009)

[4] **H. Katzgraber, L. W. Lee & I. A. Campbell**, PRB 75, 014412 (2007)
J. Poulter & J. A. Blackman, Phys. Rev. B 72, 104422 (2005).

$2 - 1/\nu = (6-\kappa)/\kappa$ works at $T > T_N$ and also on the SG if $\kappa = 2$

Is there a Conformal Field Theory for the N point ?

At least, what is κ ?

Consider single spin flip stochastic dynamics

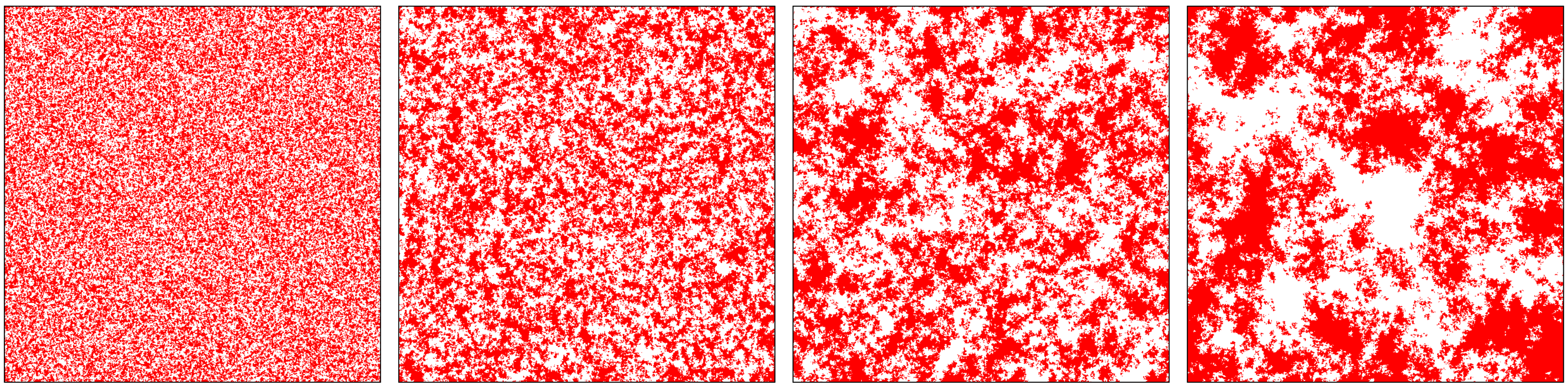
Critical dynamics of the $\pm J$ 2D Ising Model ?

Dynamic Universality ?

$2d$ FM Ising Model

$p = 0$ critical dynamics under single spin MC updates

Instantaneous quench to the Ising FM-PM critical point from $T_i \to \infty$



Progressive growth of critical structures

Typical length scale of critical patches growing algebraically

$$\xi(t) \sim t^{1/z_c}$$

Similar phenomenology expected on the full critical FM-PM line

How to measure z_c ?

Space-time correlations

of simultaneous fluctuations

$$C(r,t) = [\langle s_i(t) s_j(t)\rangle] - [\langle s_i(t)\rangle][\langle s_j(t)\rangle] \quad \text{for} \quad \vec{r}_i - \vec{r}_j = r$$

Scaling for the infinite size $L \to \infty$ system

$$C(r,t) = r^{-\eta} f\left(\frac{r}{\xi(t)}\right)$$

Effective dynamic exponent tends to Dynamic critical exponent

$$\frac{1}{z_{\rm eff}(t)} = \frac{d \ln \xi(t)}{d \ln t} \quad \Rightarrow \quad z_c = \lim_{t\to\infty} z_{\rm eff}(t)$$

$z_c = 2.17$ at the $p = 0$ FM $2d$ case

from Monte Carlo numerical simulations, but also RG, high temperature series expansions, damage spreading, etc

Short-time dynamics

at a critical point

$$m_2(t) = \left[\left\langle \left(\frac{1}{N} \sum_{i=1}^{N} s_i(t) \right)^2 \right\rangle \right]$$

for $R_{\min} \ll \xi(t) \ll \xi_{\rm eq}, L$

Increase right after the quench from $T_i \to \infty$ with (similar to *initial slip* exponent)

$$m_2(t) \sim t^{\zeta} \quad \text{with} \quad \zeta = \frac{1}{z_c}\left(d - \frac{2\beta}{\nu}\right)$$

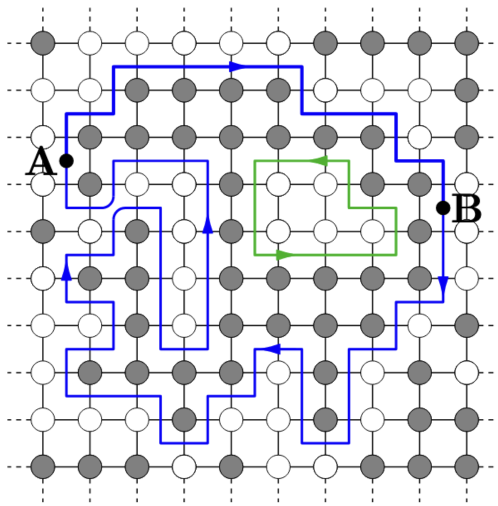
H. Janssen, B. Schaub & B. Schmittmann, Z. Phys. B Cond. Matt. 73, 539 (1989)

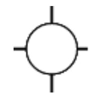
E. V. Albano *et al.*, Rep. Prog. Phys. 74, 026501 (2011)

Winding angle

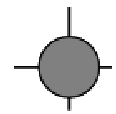
Definition - critical curves

$$[\langle \theta^2(r,t) \rangle]$$





$s_i = 1$



$s_i = -1$

In equilibrium at a critical point

$$[\langle \theta^2(r) \rangle] = c + \frac{4\kappa}{8+\kappa} \ln\left(\frac{r}{a}\right)$$

$d_f = 1 + \kappa/8$ $\kappa = 3$ **Critical Ising** $\kappa = 6$ **Critical percolation**

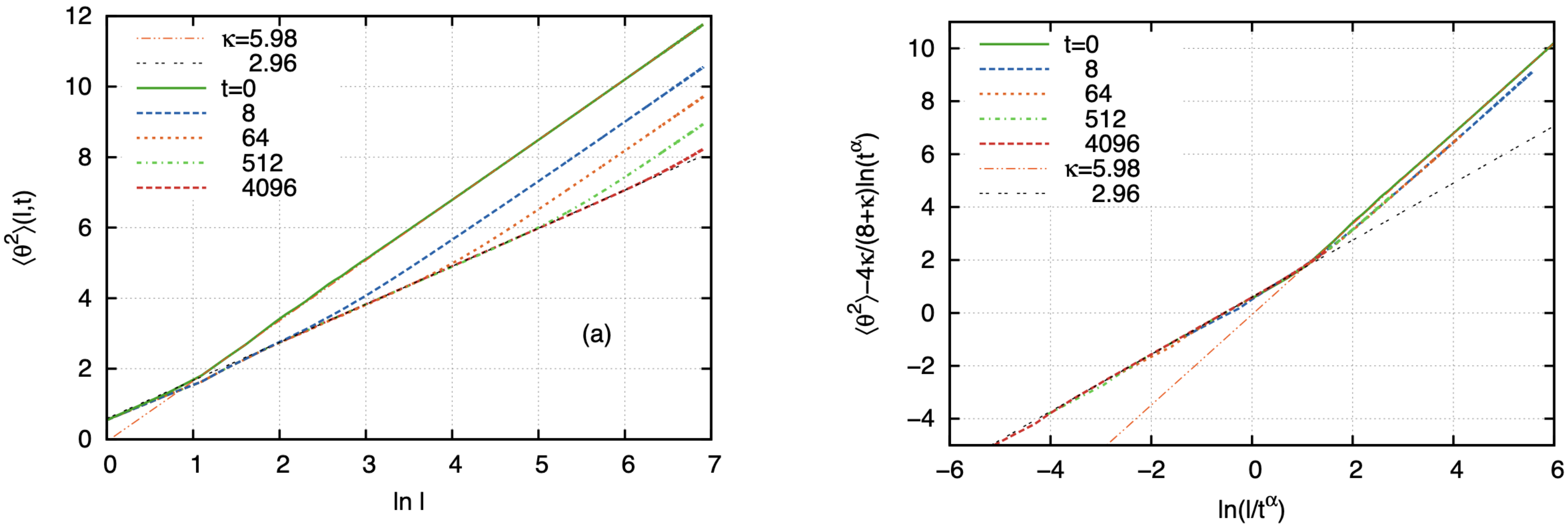
Out of equilibrium

$$[\langle \theta^2(r,t) \rangle] \sim \frac{4\kappa}{8+\kappa} \ln\left(\frac{r}{\xi^{d_f}(t)}\right)$$

Blanchard, LFC, Picco & Tartaglia, 2012-2018

Winding angle

$2d$ FM ($p = 0$) Ising Model quenched from $T_i \to \infty$ to T_{Is}



Out of equilibrium $[\langle\theta^2(r,t)\rangle] \sim \frac{4\kappa}{8+\kappa} \ln\left(\frac{r}{\xi^{d_f}(t)}\right)$ $\alpha = d_f/z_c$

$\kappa = 6$ & $d_f = 7/4$ **Critical percolation** $r > \xi(t) \sim t^{2.17}$ $(t > t_p)$

$\kappa = 3$ & $d_f = 11/8$ **Critical Ising** $r < \xi(t) \sim t^{2.17}$

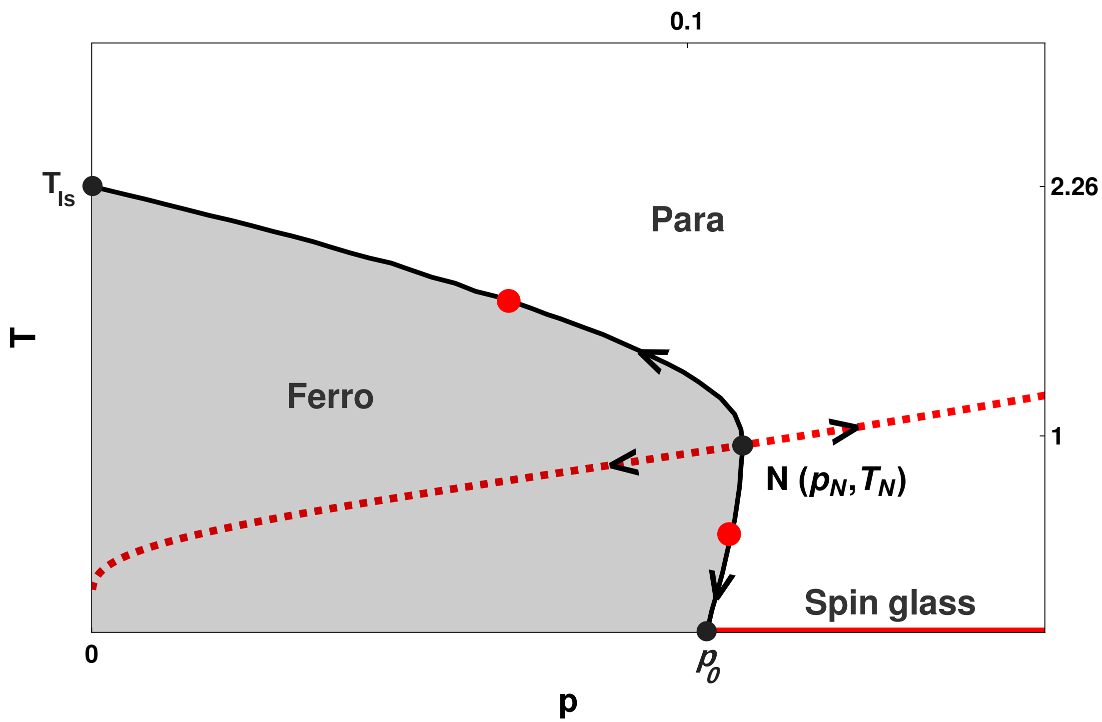
Blanchard, LFC & Picco, J. Stat. Mech. P05026 (2012)

$\pm J\ 2d$ Ising Model

More interesting simulation parameters

Second order phase transition between FM & PM phases

$(T_{\text{Is}} = 2.27,\, p = 0)$



$$e^{-2\beta J} = \frac{p}{1-p}$$

dotted Nishimori line*

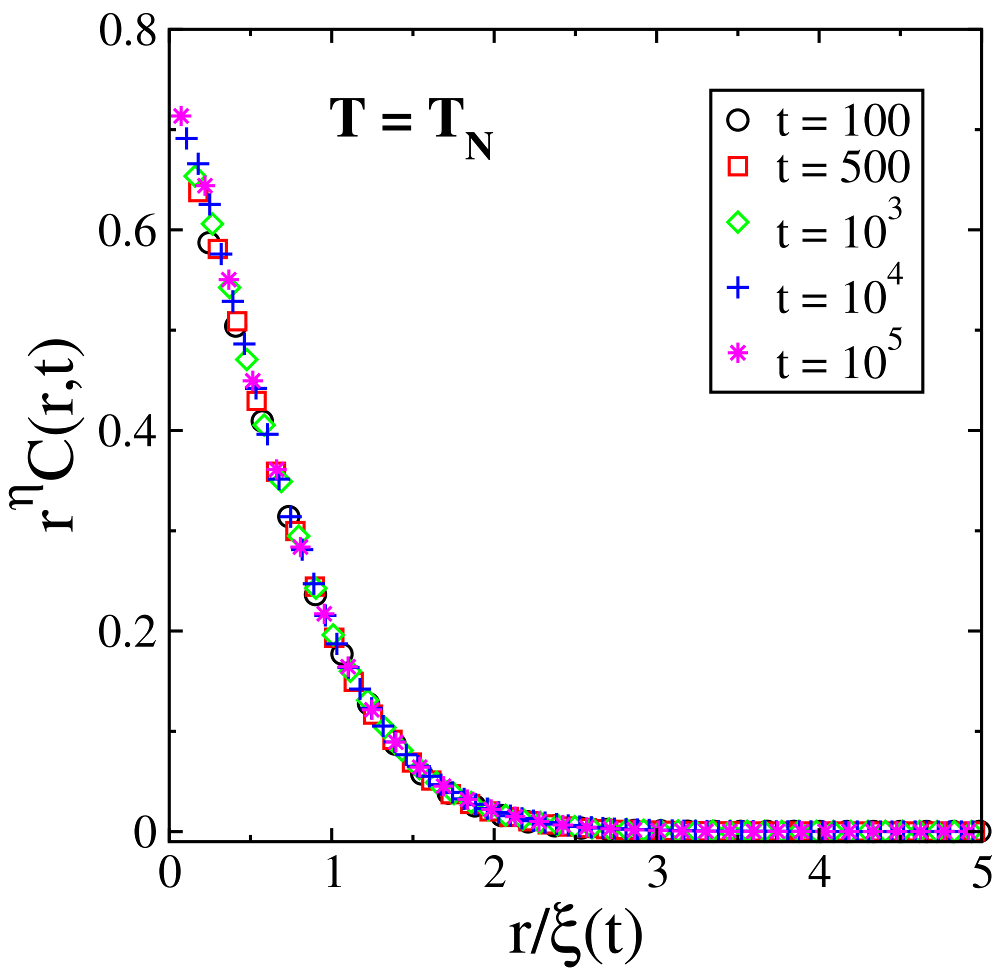
enhanced symmetry properties

$(T_0 = 0,\, p_0 = 0.103)$ $(T_N = 0.95,\, p_N = 0.109)$

Results

Dynamic scaling of the space-time correlation $\Rightarrow \quad \xi(t)?$

$$T = T_N \qquad \eta = 0.18$$



Results

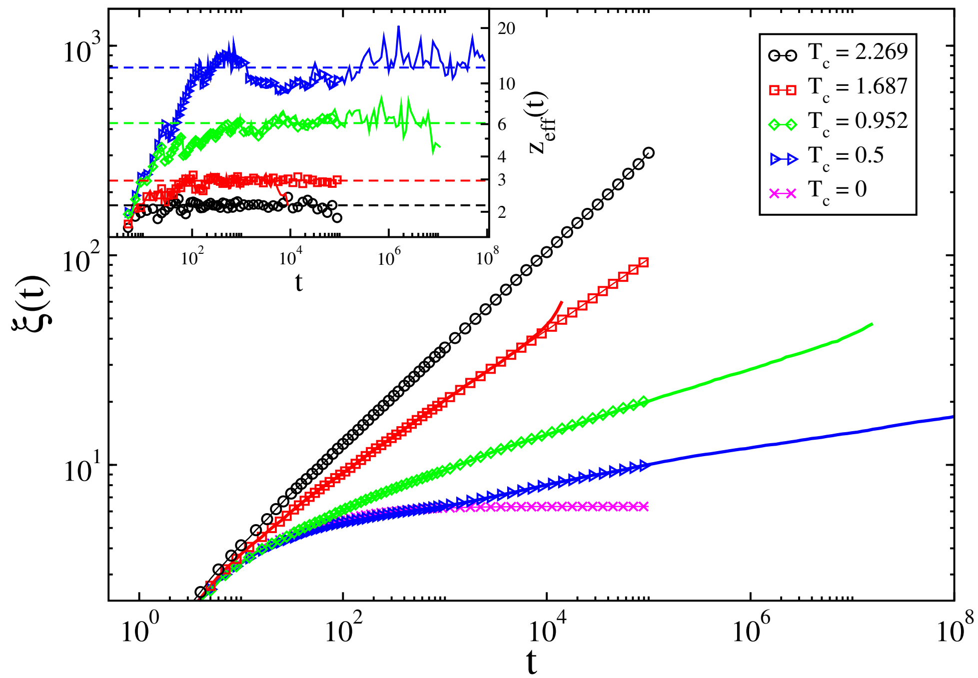
Pre-asymptotic dynamic critical exponent $\Rightarrow z_{\rm eff}^{-1}(t) = d\ln\xi(t)/d\ln t$

$T < T_N$

$T = T_N$

$T > T_N$

$T = T_{\rm Is}$



Data-points

$L = 1024$

Solid lines

$L = 128$

No visible finite size effects

FM-PM Ising critical point $z_c \sim 2.17$ **OK**

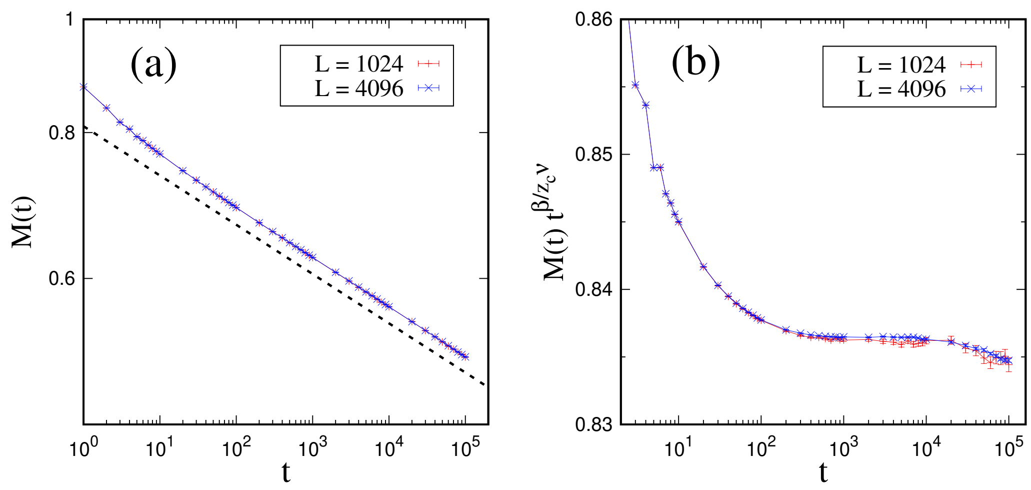
Then, **disorder dependent dynamic critical exponent ?**

Should not be... usually exponents do not vary on critical lines

Results

Decay from a magnetized initial condition $M(t) \sim t^{-\beta/(\nu z_c)}$

$T > T_N$



$\beta/\nu = 0.125$ the Ising critical value and $z_c = 2.96$ from the space-time correlation

Crossover at an L independent time $t_{\text{cross}} \sim 10^4$ presumably fixed by the disorder strength p

very weak drift $z_c \searrow$ after t_{cross}

It was still a pre-asymptotic $z_{\text{eff}}(t)$

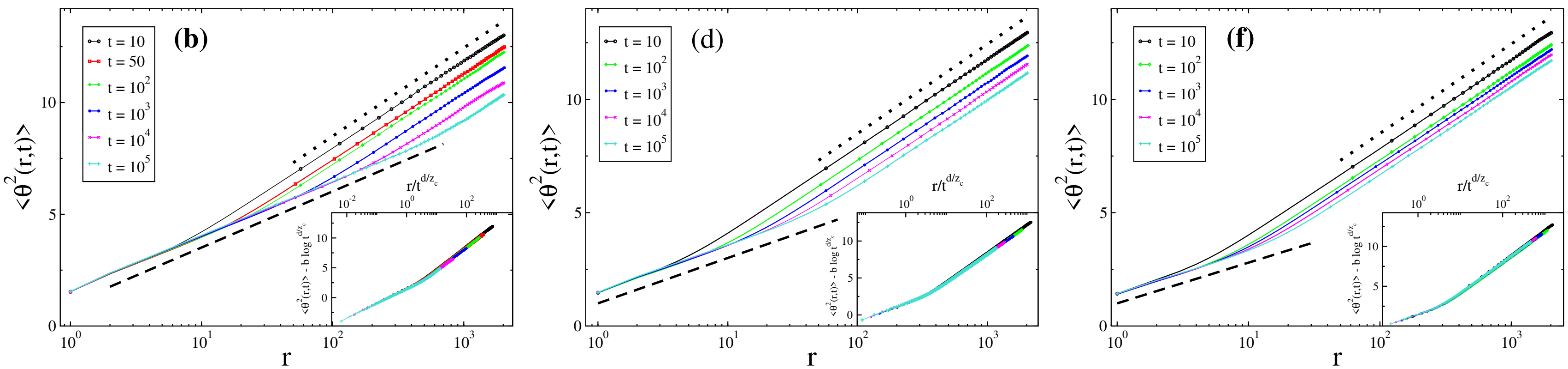
we expect it to converge to $z_c = 2.17$, the critical Ising value

Results

Quenches from $T_i \to \infty$ to T

$\forall T$ $r > \xi(t)$ $\kappa = 6$ **Critical percolation**

. . . . lines



$T = T_{\rm Is}$ $T = T_N$ $T = T_c < T_N$

$r < \xi(t)$ — — — — — — lines

$\kappa = 3$ $\kappa = 2.2$ $\kappa = 1.93$

Critical Ising **CFT ???**

Conclusions 2D $\pm J$ IM

Hard to get strong quantitative results

– $T_N < T \leq T_{\rm Is}$ static universality class of the Ising critical point

Most probably also the same dynamic universality class $z_c \sim 2.17$

– $T = T_N$ new dynamic & static universality classes $z_c \sim 6$ and $\kappa \sim 2.2$

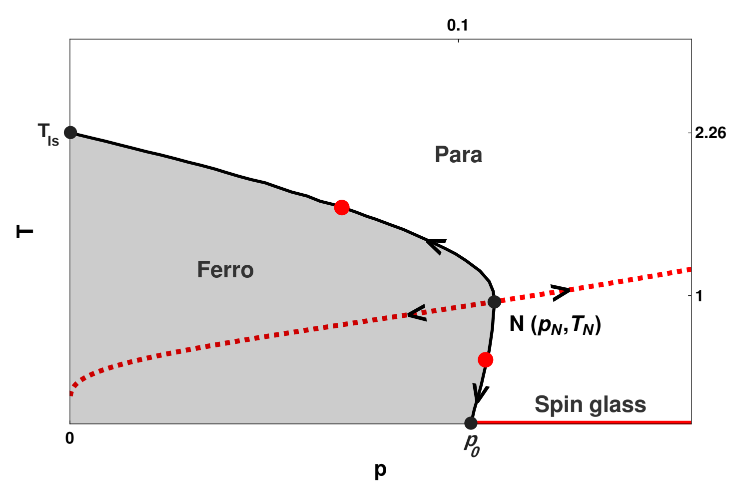
– $T_0 \leq T < T_N$ strong disorder static universality class, $z_{\rm eff}(t) \to \infty$ **but κ?**

The low T dynamics are way too slow to conclude

Agrawal, LFC, Faoro, Ioffe & Picco

Phys. Rev. E 108, 064131 (2023)

Universality with respect to lattice geometry, disorder distribution, etc. : in preparation



Plan

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— The disordered 3D Gauge Model

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Out of equilibrium critical dynamics

Wegner's Gauge Model

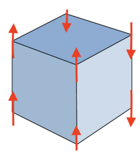
3D Ising Model $\mapsto$ Plaquette Model

Extension of **Kramers-Wannier** $D=2$ duality to $D>2$ $\qquad J=1$

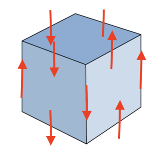
The cubic lattice IM is dual to a $\mathbb{Z}_2$ gauge-invariant IM on a cubic lattice :

the **plaquette model** with Ising spins on the links of a cubic lattice

$$\mathcal{H}_{3\text{DIM}} = -\sum_{\langle ij\rangle} s_i s_j$$



$$\leftrightarrow \quad \mathcal{H}_{\text{plaq}} = -\sum_p \underbrace{\prod_{i\in p}\sigma_i}_{\text{flux } O_p} = -\sum_p O_p$$



The partition functions transform as **Wegner 71**

$$Z_{3\text{DIM}}(\beta) \quad \leftrightarrow \quad Z_{\text{plaq}}(\beta^*)$$

$$(T_c = 4.51) \qquad -\frac{1}{2}\ln\tanh\beta \;=\; \beta^* \qquad (T_c^* = 1.31)$$

high/low $=$ low/high

Spins $\sigma_i \mapsto$ elements in group : **Lattice Gauge Theory** **Kogut 79**

Classical limit of **Kitaev's 03** **Toric Code**

Wegner's Gauge Model

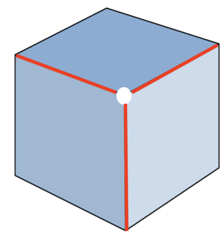
Local gauge invariance

Two obvious $T = 0$ ground states $\sigma_i = 1$ or $\sigma_i = -1$ for all i $\Rightarrow$ $O_p = 1$

Local gauge invariance : reversal of the 6 spins connected to any vertex

$$\sigma_{1\in p}\sigma_{2\in p}\sigma_{3\in p}\sigma_{4\in p} \mapsto$$

$$(-\sigma_{1\in p})(-\sigma_{2\in p})\sigma_{3\in p}\sigma_{4\in p}$$



$\Rightarrow$ **Macroscopic degeneracy of ground states**

Extensive ground state entropy

No local order parameter $\lim_{h\to 0}\langle\sigma_i\rangle_h = 0$ **Wegner 71, Elitzur 75**

Excitations $O_p = -1$ **frustrated plaquettes** magnetic 'charges' created in pairs

disorder the system and eventually lead to a **continuous phase transition**

Wegner's Gauge Model

Local gauge invariance and non-local order parameter

Two obvious $T = 0$ ground states $\sigma_i = 1$ or $\sigma_i = -1$ for all i $\Rightarrow$ $O_p = 1$

Local gauge invariance : reversal of the 6 spins connected to any vertex

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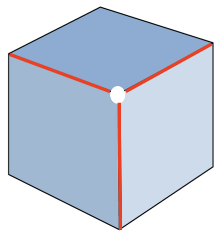
$$(-\sigma_{1\in p})(-\sigma_{2\in p})\sigma_{3\in p}\sigma_{4\in p}$$

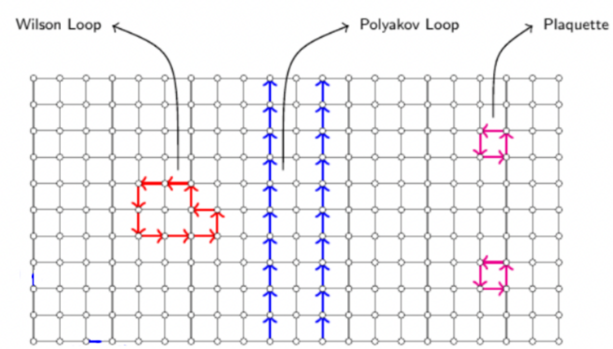


$\Rightarrow$ **Macroscopic degeneracy of ground states**

No local order parameter $\langle \sigma_i \rangle = 0$ but still a **continuous phase transition**

Gauge invariant non-local order parameter : average of the spin product along any **Wilson loop** $W_\ell = \prod_{i\in\ell} \sigma_i$

$$\langle W_{\ell=\partial S} \rangle = \langle \prod_{p \,\mathrm{in}\, \partial S} O_p \rangle \sim \begin{cases} e^{-\alpha(\beta) S} & \text{high } T \\ e^{-\gamma(\beta)\ell} & \text{low } T \end{cases}$$



Wegner's Gauge Model

Local gauge invariance and non-local order parameter

Two obvious $T = 0$ ground states $\sigma_i = 1$ or $\sigma_i = -1$ for all i $\Rightarrow$ $O_p = 1$

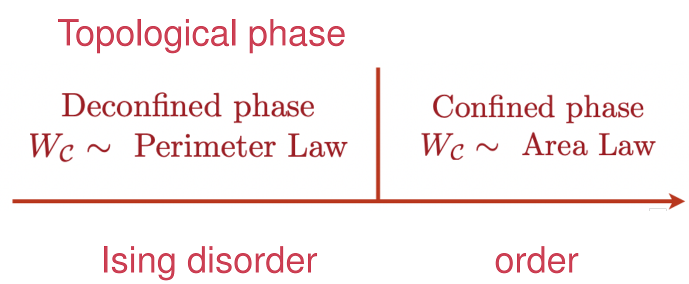
$\Rightarrow$ **Macroscopic degeneracy of ground states**

No local order parameter $\langle \sigma_i \rangle = 0$ but still a **continuous phase transition**

Gauge invariant non-local order parameter : average of the spin product along a **Wilson loop** $W_\ell = \prod_{i \text{ in } \ell} \sigma_i$

Gauge model

$$\langle W_{\ell = \partial S} \rangle = \langle \prod_{p \text{ in } \partial S} O_p \rangle \sim \begin{cases} e^{-\alpha(\beta) S} & \text{high } T \\ e^{-\gamma(\beta)\ell} & \text{low } T \end{cases}$$

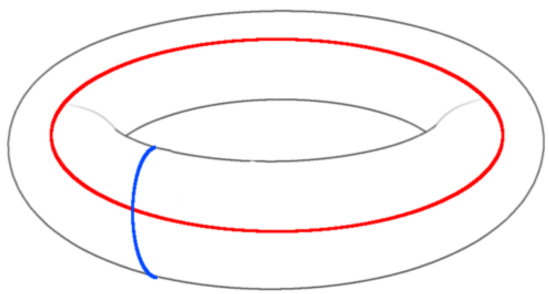


Ising model

3DIM universality class same critical exponents

Wegner's Gauge Model

Non-local order parameter



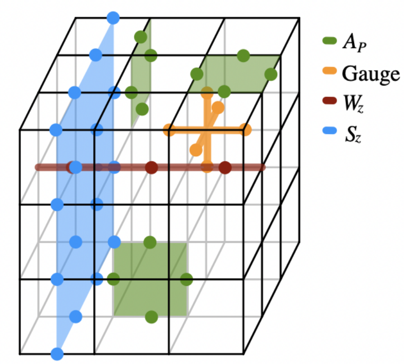
$P_x \equiv \langle \prod_{i\in P_x} \sigma_i \rangle$ on a Polyakov loop

that is a spanning Wilson loop, $\ell \neq \partial S$

Gauge invariant non-local order parameter : the expectation value of a product of spins along a **Polyakov** loop $\ell \neq \partial S$

In $\{\sigma_i'\}$, flip spins on the **plane** only

$$\mathcal{H}_{\rm plaq}(\{\sigma_i\}) = \mathcal{H}_{\rm plaq}(\{\sigma_i'\})$$
$$P_x(\{\sigma_i\}) = -P_x(\{\sigma_i'\})$$



One bit of information per ground state $\{\sigma_i\}$

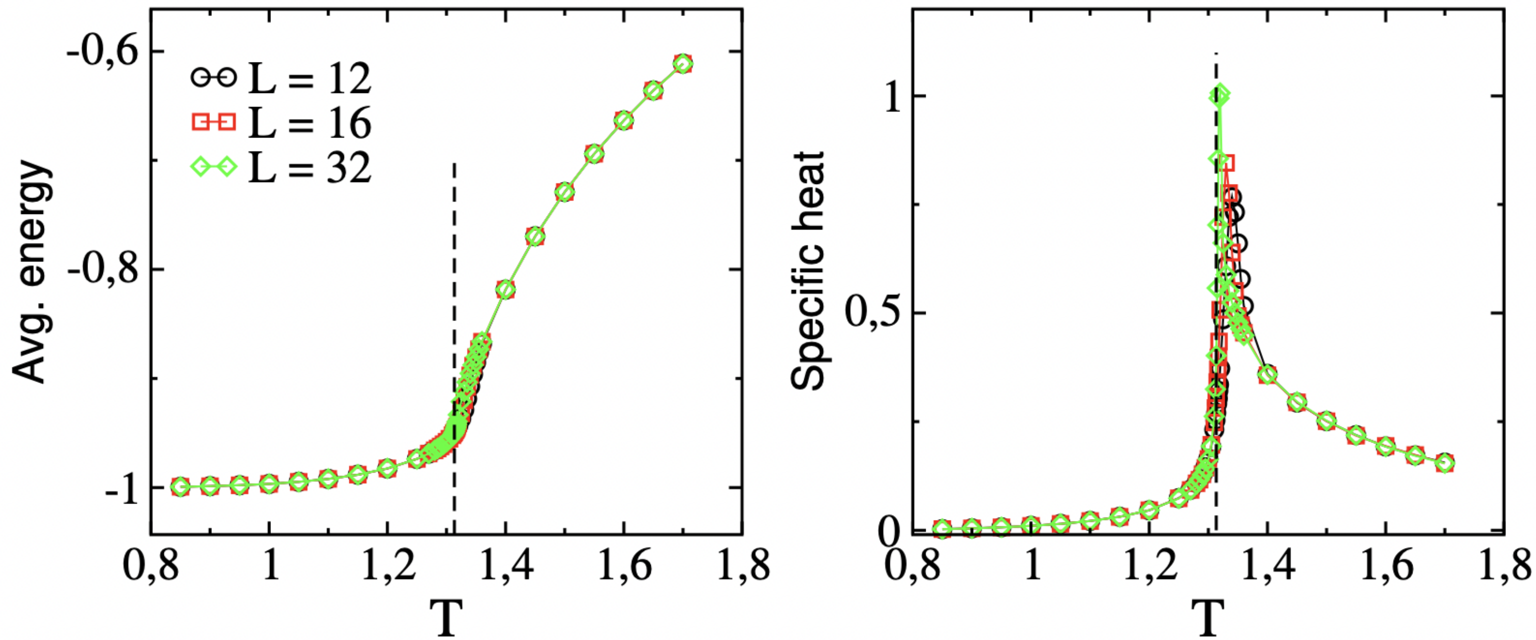
Breaks a global $\mathbb{Z}_2$ symmetry

Robustness against external perturbations

Poulin, Melko & Hastings 18

Phase transition

Standard measurements



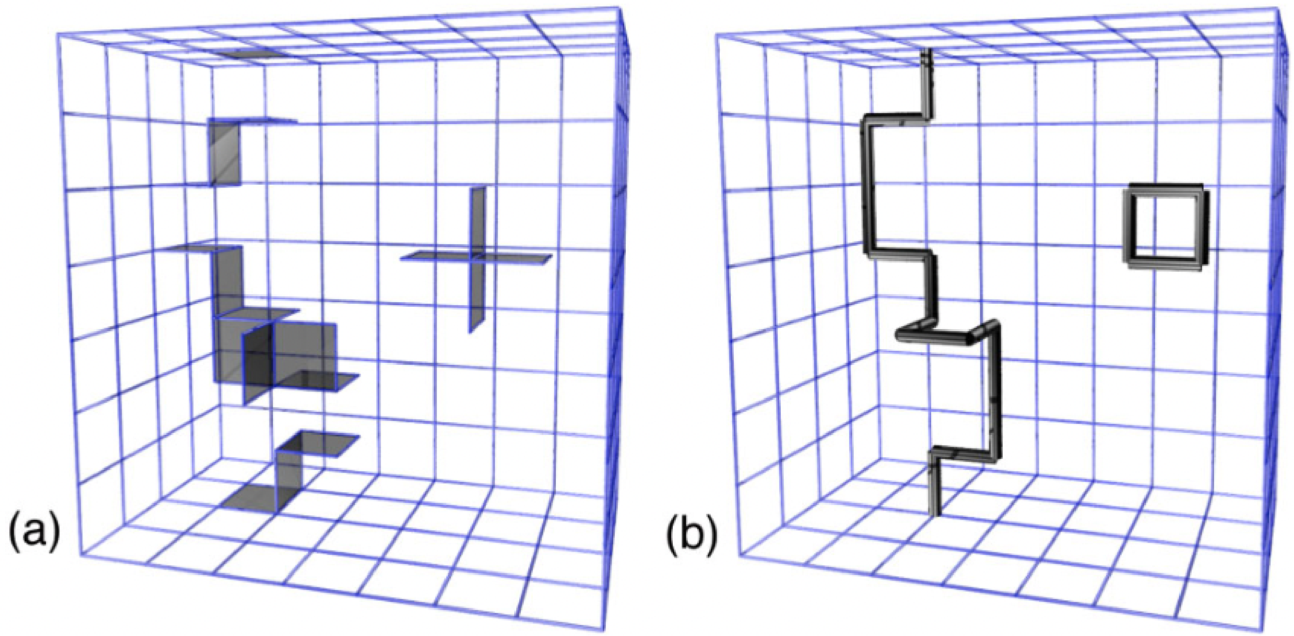
3D Ising Universality class

$\alpha = 0.11,\ \beta = 0.33,\ \gamma = 1.24,\ \delta = 4.79,\ \eta = 0.04,\ \nu = 0.63,\ \omega = 0.83$

Checks **E. Kehl, H. Satz & B. Waltl**, Nucl. Phys. B305 [FS23] (1988)

Excitations

Flux loops



Frustrated plaquettes $O_p = -1$

Threading fluxes - closed loops

A percolating and a finite size loop

Fig. from **Hastings, Watson & Melko 13**

Do defect lines percolate at the thermodynamic critical temperature T_c ?

Yes, Hastings, Watson & Melko 13, Agrawal, LFC, Faoro, Ioffe & Picco 24

If $T_p = T_c$, do the geometric properties of defect loops capture the critical exponents at T_c ?

No, Agrawal, LFC, Faoro, Ioffe & Picco 24

cfr. in the 2D Ising model the geometric clusters (excited droplets of parallel spins) percolate at T_c but they do not have the properties of critical clusters

In the 3D Ising model the geometric clusters percolate at $T_p < T_c$

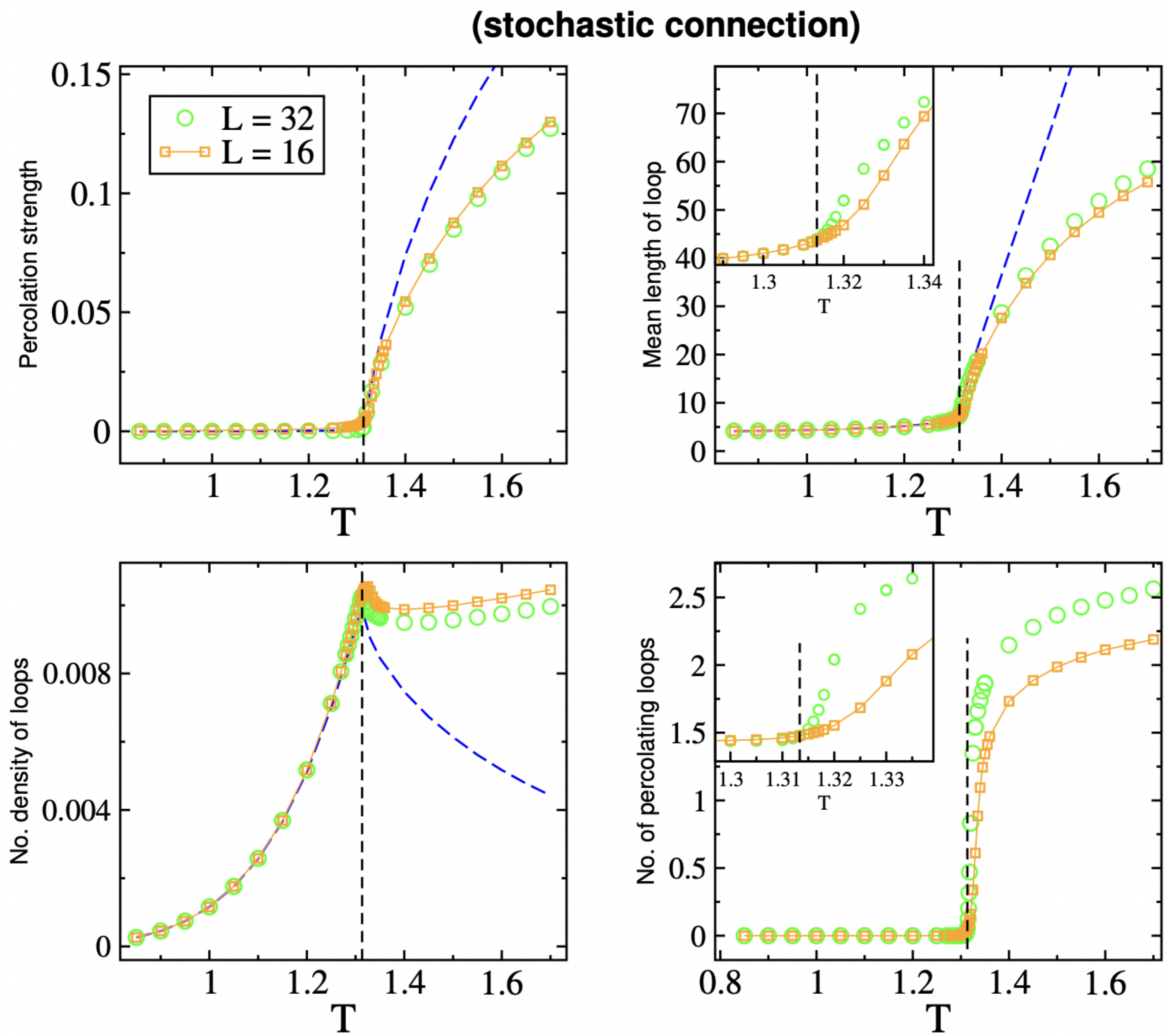
The (smaller) stochastic Fortuin-Kasteleyn clusters percolate at T_c and capture the thermodynamic critical properties of Ising models in all D.

Can Fortuin-Kasteleyn clusters be built and measure critical exponents from them ?

Seems so, Agrawal, LFC, Faoro, Ioffe & Picco, in progress

Phase transition

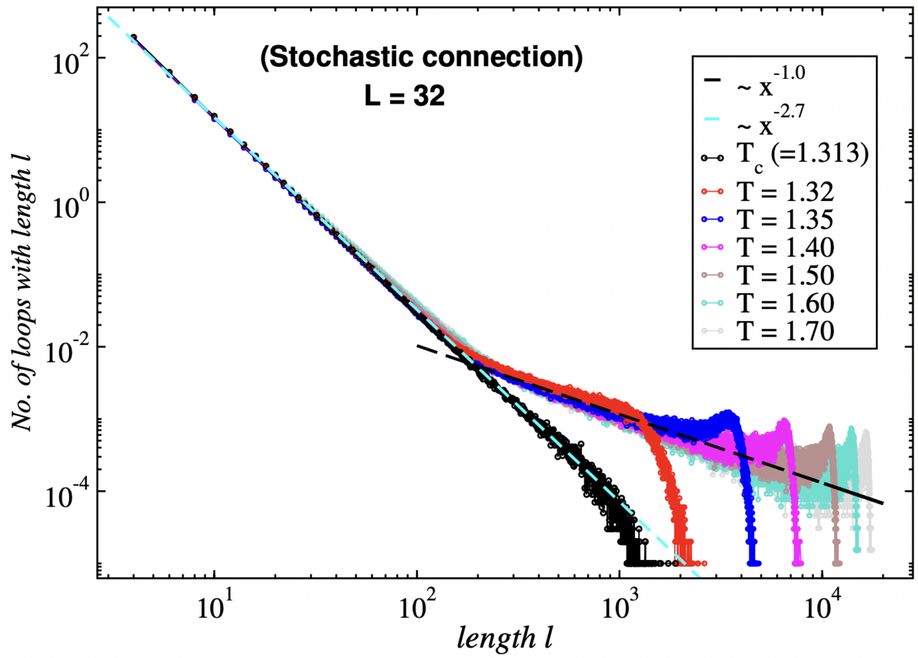
Flux loops



Locate T_c

Phase transition

Flux loops



At $T_c \sim 1.31$

algebraic decay $N^{(S)} \sim l^{-\tau}$

measured $\tau \sim 2.7 \Rightarrow$

fractal dimension

$$D_f = \frac{D}{\tau - 1} = 1.75$$

if $\tau = 5/2 \Rightarrow D_f = 2$

At $T > T_c$ compare to these results

– $l \ll L^2$ Gaussian statistics $l^{-5/2}$

Flory 41, de Gennes 79 (polymers) **Vachaspati & Vilenkin 84** (cosmology)

– $l \gg L^2$ fully-packed loops large-scale statistics l^{-1}

Nahum & Chalker 12 (statistical physics)

U(1) field theory in 3D

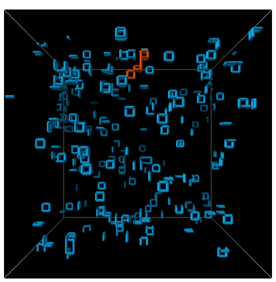
Number of vortex loops in equilibrium (fixed L, varying T)

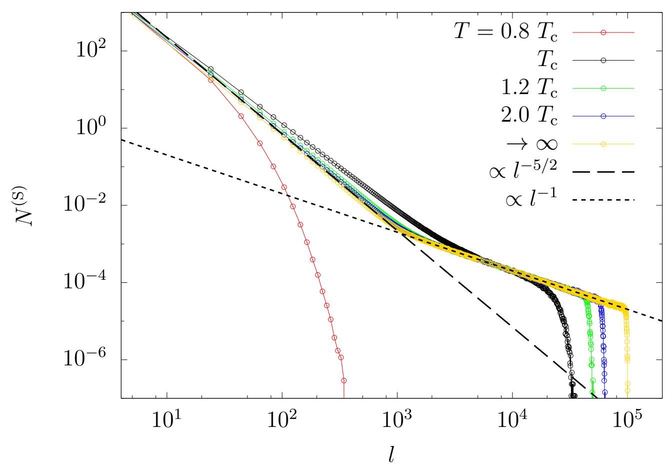
$$\mathcal{L} = c^{-2}|\dot\psi|^2 + i\mu\{\psi^*\dot\psi - \mathrm{cc}\} - |\nabla\psi|^2 + g\rho|\psi|^2 - \frac{g}{2}|\psi|^4$$

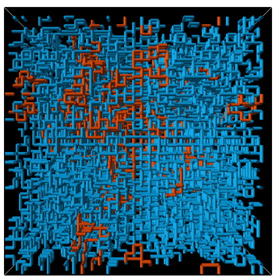
Langevin dynamics $\quad -\gamma\dot\psi$ viscosity, η Gaussian normal noise

time-dependent complex Ginzburg-Landau, stochastic Goldstone $\mu \to 0$ and Gross-Pitaevskii $c \to \infty$ model for BECs close to the Mott insulator transition and in their gaseous phase

Low T







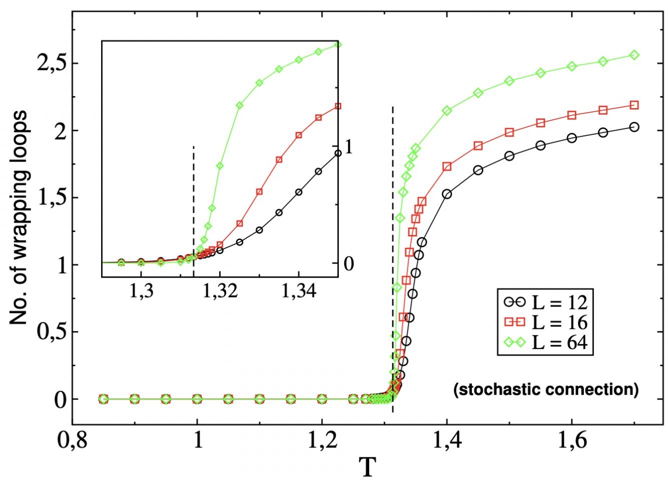
High T

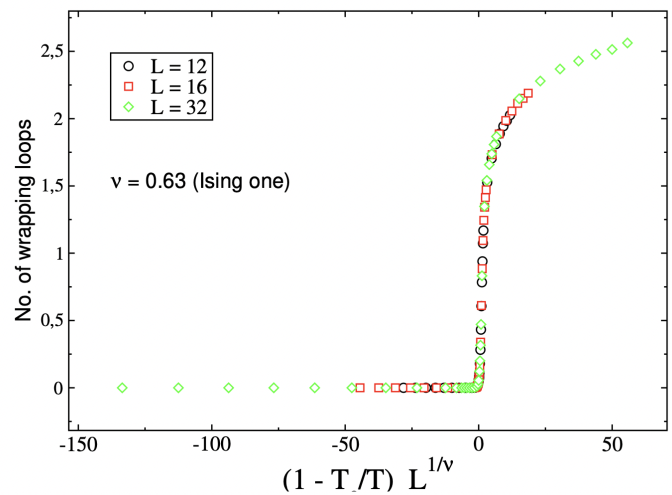
$N^{(S)}(l)$ Number of vortex loops with length l in a system with linear size L and pbc

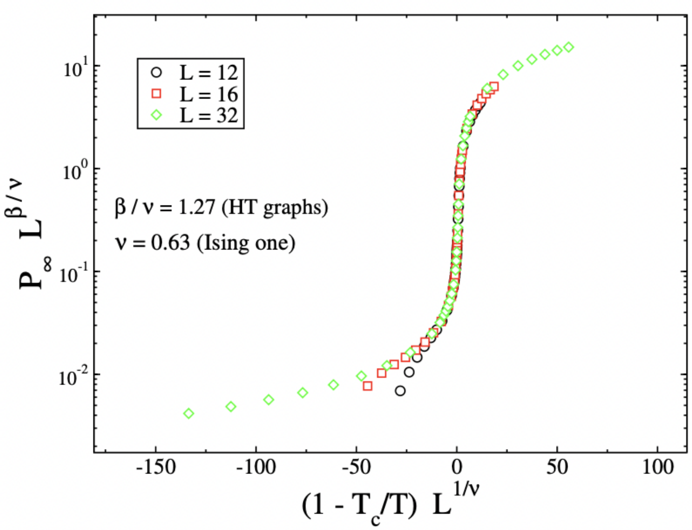
Kobayashi & LFC 16

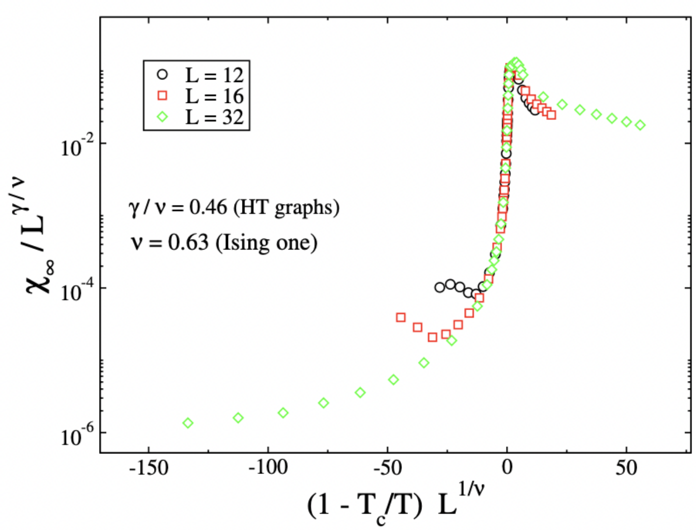
Phase transition

Critical exponents from loop analysis









Crit. Ising

$\nu = 0.63$

Crit. Ising

$\beta/\nu = 0.59$

$\gamma/\nu = 1.96$

do not scale

High-T Ising

$\beta/\nu = 1.27$

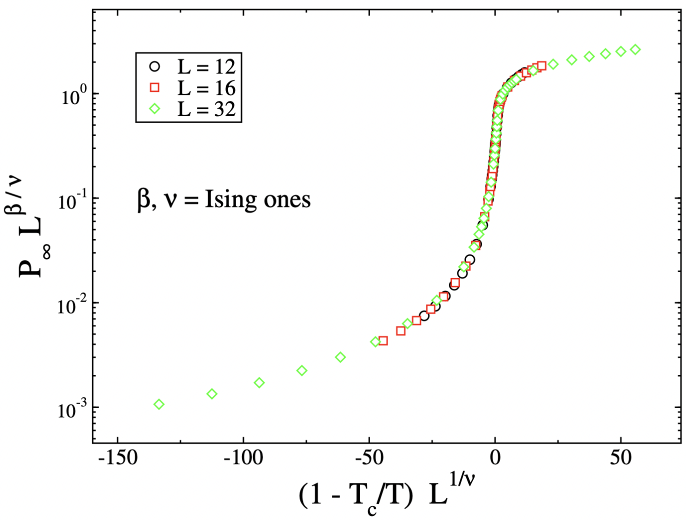
$\gamma/\nu = 0.46$

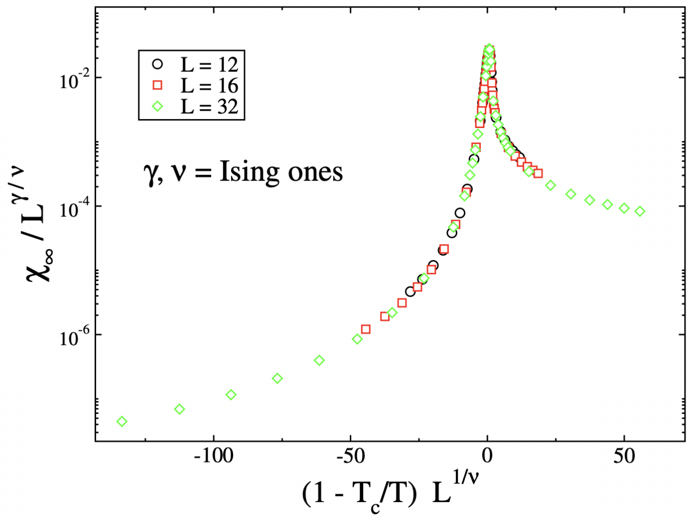
cfr. **Winter, Janke & Schakel** Phys. Rev. E 77, 061108 (2008)

Phase transition

Critical exponents from Fortuin-Kasteleyn analysis

Critical Ising exponents $\nu = 0.63$, $\beta/\nu = 0.59$, $\gamma/\nu = 1.96$ OK





Flux loops are shorten with a probabilistic prescription

In progress, to be checked and improved

cfr. **R. Ben-Av, D. Kandel, E. Katzneison, P. G. Lauwers, and S. Solomon**, J. Stat. Phys. 58, 1/2 (1990)

Thermodynamic mapping between the 3DIM and the plaquette Gauge Model (high/low T) but no obvious relation between their stochastic dynamics.

Value of z_c in the gauge model with single spin flip dynamics?

Energy-energy & Polyakov loop time-delayed correlation functions fitted to decay as $e^{-t/\tau}$ in equilibrium

Ben-Av, Kandel, Katzneison, Lauwers & Solomon J. Stat. Phys. 58, 125 (1989)

$$z_c \sim 2.50 \pm 0.3$$

Kibble-Zurek scaling under annealing

Xu, Castelnovo, Melko, Chamon & Sandvik, PRB 97, 024432 (2018)

$$z_c \sim 2.70$$

Also in progress

Conclusions 3D Gauge Model

– Standard measurements confirm 3D Ising criticality

– Flux loops percolate at T_c but their geometric and statistical properties are non-trivial but do not capture the critical exponents

– FK clusters yield the critical exponents (to be improved)

– Dynamic critical exponent (to be measured)

– Disordered case, especially on Nishimori line (to be studied !)

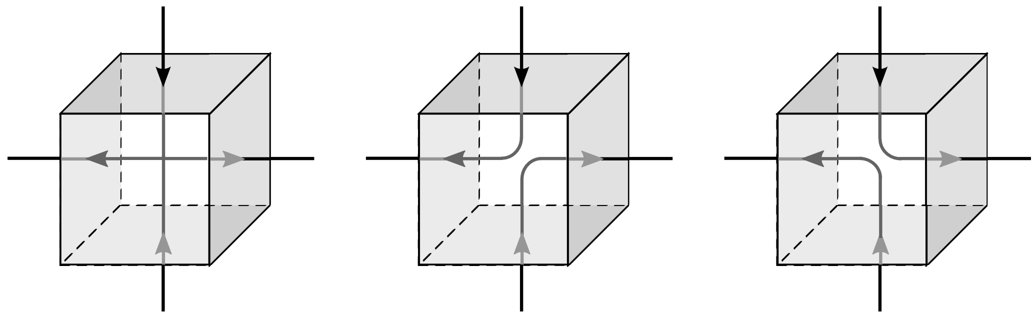
Appendices

Details

U(1) field theory in $3d$

Vorticity & reconnection conventions

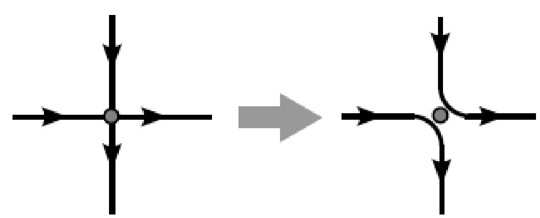
$2\pi v_x = \sum_{\text{plaq}} [\Delta\theta]_{2\pi} = 0, \pm 1, \ldots$ ($\neq 0$ when the field turns around on a plaquette)



One field configuration with two possible line structures

Typical choices : maximal & stochastic reconnection rules

while just one choice in



Kajantie et al. 00, Bittner, Krinner & Janke 05, Kobayashi & LFC 16

Codes

Definition

During the transmission of information, errors may occur

The aim is to minimize their number/strength

Idea, code the message and uncode it at the end

Quantum Toric Codes

Definition

A **qu-bit** is a two-state quantum variable, $|\psi_i\rangle = a_i|\uparrow\rangle + b_i|\downarrow\rangle$

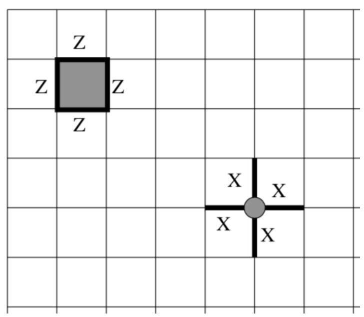
Flip errors, $\hat{\sigma}_x|\uparrow\downarrow\rangle = |\downarrow\uparrow\rangle$ & phase errors, $\hat{\sigma}_z|\uparrow\rangle = \pm|\uparrow\rangle$ occur independently with probability p

Place **qu-bits** on the links of a square lattice defined on a $2d$ surface with non trivial topology, e.g. a torus $\prod_i \otimes|\psi_i\rangle$

Check local operators : plaquette or link operators, tensor product of four Pauli operators acting on the four qu-bits on the links times identities on all other links

Check operators commute

Measurements of check operators yield $+1$ no error, or -1 error.



Quantum Toric Codes

Definition

Stabilizer group G a set of n check operators which applied to a basis state of the quantum error correction code have eigenvalue one, $P_k|\psi_j\rangle = |\psi_j\rangle$ for any kth element in the group and any jth element of the basis. Abelian group

Particular case : product of $\hat{\sigma}_x$ or product of $\hat{\sigma}_z$ operators.

$\prod$ of neighbouring plaquette operators : loop on the lattice.

$\prod$ of neighbouring vertex operators : loop on the dual lattice.

Error operators $E|\psi\rangle = |\psi'\rangle$

String of flip errors on the lattice : vertex operators on the ends yield -1

Correction operators E' such that $E'E \in G$

Another string with the same end points so as to close the loop

Error correction

Optimal toric code decoder threshold

Call p the (independent) probability of a qu-bit error

What is the maximal p such that code can be corrected ?

Probability of a string E' on the lattice that corrects another string of errors E

$$P(E') = (1-p)^N \prod_k \left(\frac{p}{1-p}\right)^{{n'}_k^E} = e^{\beta \sum_{\langle ij \rangle} J_{ij} s_i s_j}$$

$J_{ij} = \pm J$ with probability $1-p, p$ and $p/(1-p) \equiv e^{-2\beta J}$ (Nishimori)

Have to study the sum over all paths E'

$$Z = \sum_{E'/EE' \in G} e^{\beta \sum_{\langle ij \rangle} J_{ij} s_i s_j}$$

Mapping to the classical $\pm J$ $2d$ Ising model on the Nishimori line

p_N is the optimal decoding threshold

Local Gauge invariance

Ising disordered spin models

Transform the Ising spins $s_i = \pm 1$ into new Ising spins $\sigma_i = \eta_i s_i = \pm 1$

Transform the couplings $J_{ij} = \pm J$ into new ones $\overline{J}_{ij} = \eta_i \eta_j J_{ij} = \pm J$

with $\eta_i = \pm 1$ so that $\eta_i^2 = 1$ for all i

The Hamiltonian of the system remains unchanged

$$\mathcal{H}_{\overline{J}_{ij}}[\{\sigma_i\}] = -\sum_{\langle ij \rangle} \overline{J}_{ij} \sigma_i \sigma_j = -\sum_{\langle ij \rangle} J_{ij} s_i s_j = \mathcal{H}_{J_{ij}}[\{s_i\}]$$

but the distribution of couplings may change depending on the η_is

$$P(J_{ij}) \mapsto \overline{P}(\overline{J}_{ij})$$

Valid $\forall$ Ising models with two-body couplings on any lattice/graph

The Nishimori line

Special features

The bimodal distribution of couplings can be rewritten as

$$P(J_{ij}) = (1-p)\,\delta_{J_{ij},J} + p\,\delta_{J_{ij},-J} = \frac{e^{K_p J_{ij}/J}}{2\cosh K_p}$$

with $e^{2K_p} \equiv \dfrac{1-p}{p}$

It transforms according to $$P(J_{ij}) \mapsto \overline{P}(\overline{J}_{ij}) = \eta_i \eta_j \, \frac{e^{K_p \overline{J}_{ij} \eta_i \eta_j / J}}{2\cosh K_p}$$

The **Nishimori line** is defined by $\beta J = K_p = \frac{1}{2}\ln\left(\dfrac{1-p}{p}\right)$

with the limits $p=0, T=0$ and $p=1/2, T\to\infty$

Several exact results can be derived on the Nishimori line

(p_N, T_N) is a **multi-critical point**, different from critical percolation

Wegner's Gauge Model

Gauge invariance and order parameter

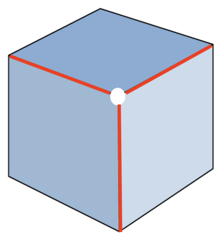
Two obvious $T = 0$ ground states $\sigma_i = 1$ or $\sigma_i = -1$ for all i $\quad\Rightarrow\quad A_P = 1$

Excitations $A_P = -1$ akin to **magnetic fluxes**

Local gauge invariance : reversal of the 6 spins connected to a vertex

$$\sigma_{1\in P}\sigma_{2\in P}\sigma_{3\in P}\sigma_{4\in P} \mapsto$$

$$(-\sigma_{1\in P})(-\sigma_{2\in P})\sigma_{3\in P}\sigma_{4\in P}$$



Macroscopic degeneracy of ground states

No local order parameter $\langle\sigma_i\rangle = 0$ for all i but still a continuous phase transition

Topological transition between **deconfined** (low T) to **confined** (high T) phases

Same universality class as the 3DIM

Random networks

Localization phenomena

Express the partition function as $Z \propto \mathrm{Tr} \prod_k \hat{T}_k$ a product of transfer matrices

All $\hat{T}_k$ are different since disorder-dependent, expressed in terms of $\hat{\sigma}_i^x, \hat{\sigma}_i^z$

Use Jordan-Wigner transformation to introduce fermions, then transform them to Dirac fermions (by doubling the model)

Network tight-binding Hamiltonian for free fermions with random hopping

paramagnet $\equiv$ insulator

Localization problem

ferromagnet $\equiv$ quantum Hall conductor

S. Cho and M. P. A. Fisher, PRB 55, 1025 (1997)

I. Gruzberg, N. Read, and A. Ludwig, PRB 63, 024404 (2001)

F. Merz and J. T. Chalker, PRB 65, 054425 (2002)

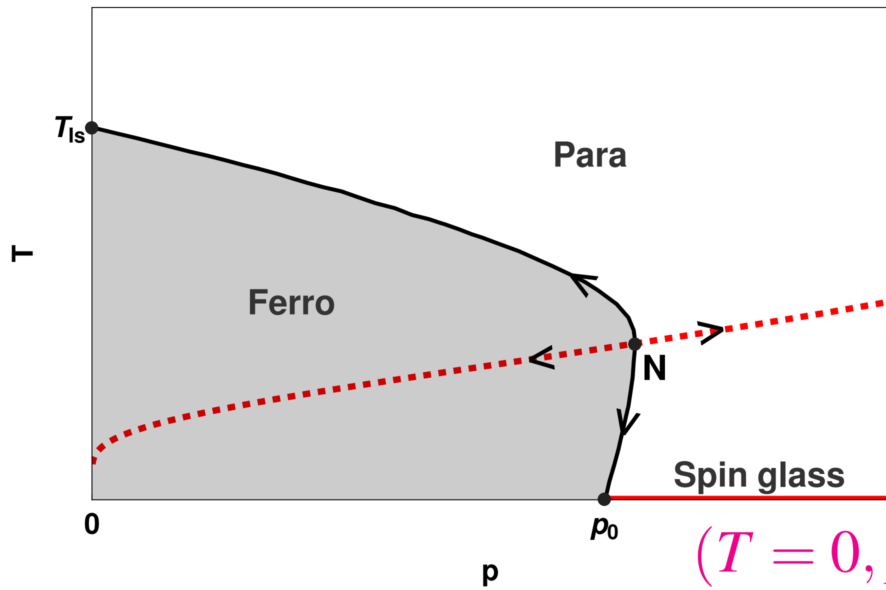
$\pm J$ $2d$ Ising Model

The equilibrium phase diagram

$T_{\rm Is} \geq T > T_N$ Disorder is marginally relevant $\Rightarrow$ $(T_{\rm Is}, p = 0)$ **PM-FM Ising criticality**

A. B. Harris, J. Phys. C : Sol. St. Phys. 7, 1671 (1974)

M. Picco, A. Honecker, and P. Pujol, J. Stat. Mech. P09006 (2006)



$$e^{-2\beta J} = \frac{p}{1-p}$$

dotted Nishimori line*

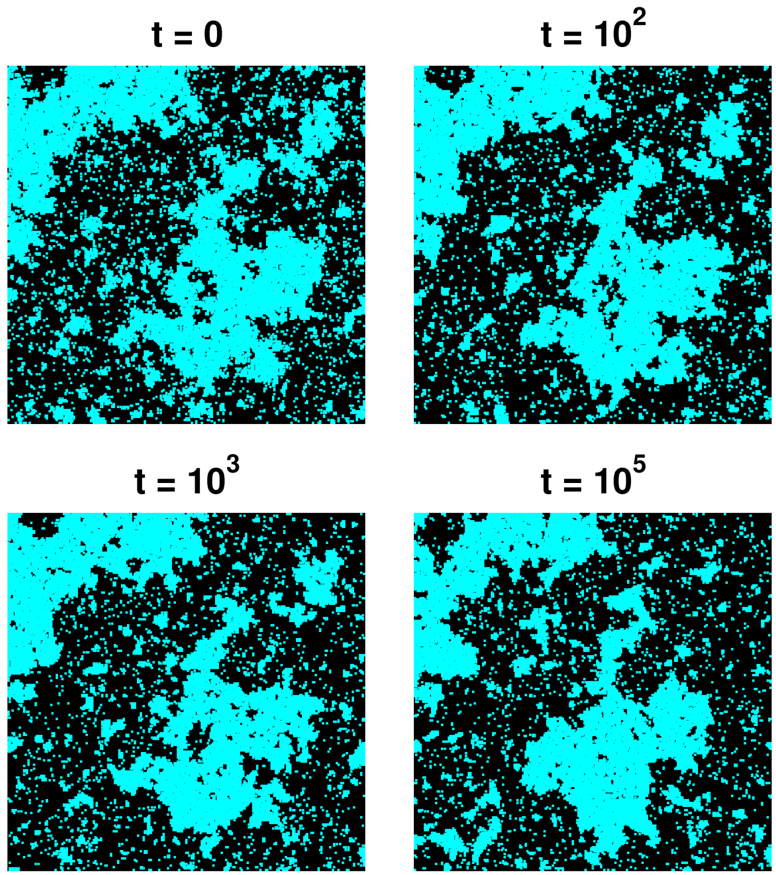
$(T = 0, p_0 < p < 1 - p_0)$ **spin-glass**

T. Jörg, J. Lukic, E. Marinari, and O. C. Martin, Phys. Rev. Lett. 96, 237205 (2006)

F. Parisen Toldin, A Pelisetto, and E. Vicari, Phys. Rev. E 82, 021106 (2010)

Ultra slow dynamics at p_N, T_N

Quench from $T_i = T_{\rm Is}$ to T_N



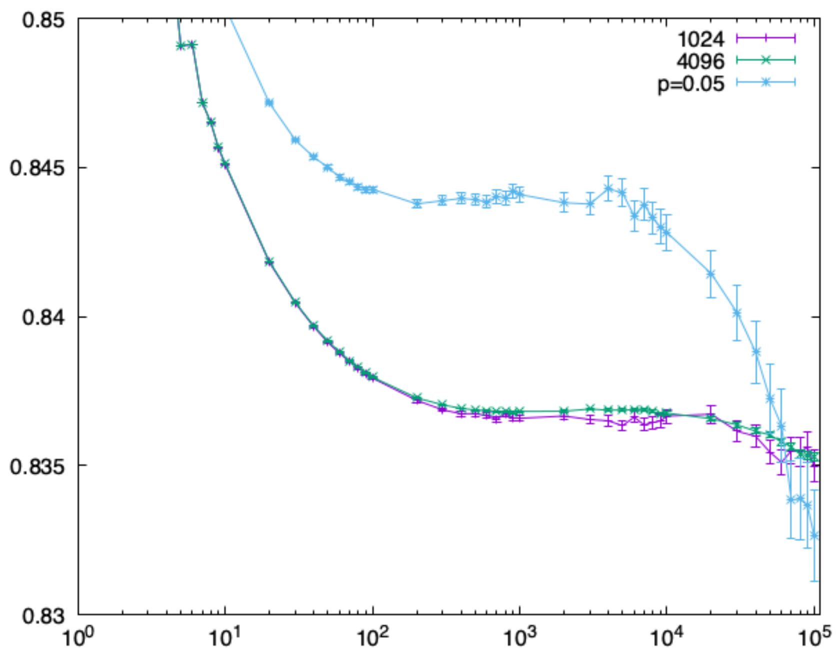
A portion of the system

The overall structure changes very little over a long time span

Results

Decay from a magnetized initial condition $M(t) \sim t^{-\beta/(\nu z_c)}$

$T > T_N$



$z_c = 2.56$ for $p = 0.05$

$z_c = 2.96$ for $p = 0.07$

$\beta/\nu = 0.125$ the Ising critical value and z_c from the space-time correlation

$t_{\text{cross}} \sim 7 \times 10^3$ for $p = 0.05$ $<$ $t_{\text{cross}} \sim 2 \times 10^4$ for $p = 0.07$

L independent t_{cross} (being checked) drift $z_c \searrow$ after t_{cross}

For $p = 0.05$, z_c has already reached 2.2 at $t = 10^5$ (not far from $z_c = 2.17$)

Self-correcting memories

A passive physical **device** that **stores information** robustly at finite T despite fluctuations of its external parameters like magnetic field, pressure, etc

Symmetry broken finite-temperature phase, stable against perturbations

2DIM : stores two bits but they are not stable under a magnetic field

Many equivalent states to store many things

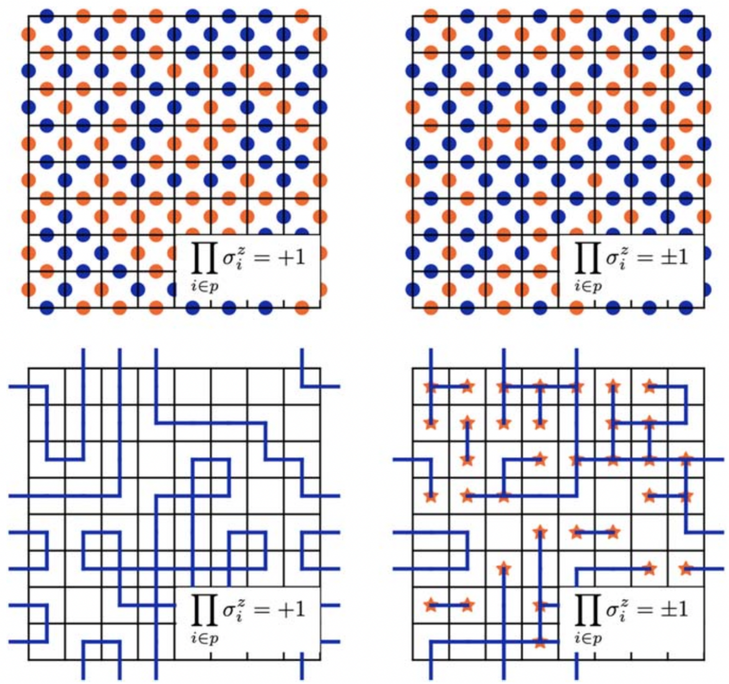
Mixing time grows with the system size

Probability that the system spontaneously transitions from one phase to another, exponentially suppressed with the system size.

The Gibbs phase rule can be avoided in cases with non-local order parameters.

Wegner's Gauge Model

Loop representation : lines along ● spins $\sigma_i = 1$



Spin configuration

Line configuration

$T = 0$ ground state — Closed flux loops

$T \to \infty$ state — $\star$ end points at frustrated plaquettes

Fig. from **Greplova, Valenti, Boschung, Schäfer, Lörch & Huber 20**