Phase ordering kinetics and percolation in two dimensions

Leticia F. Cugliandolo

Sorbonne Université

Laboratoire de Physique Théorique et Hautes Energies

Institut Universitaire de France

leticia@lpthe.jussieu.fr

www.lpthe.jussieu.fr/~leticia/seminars

In collaboration with

Jeferson Arenzon, Thibault Blanchard, Alan Bray, Federico Corberi, Ingo Dierking, Marco Esposito, Michikazu Kobayashi, Ferdinando Insalata, Marcos-Paulo Loureiro, Marco Picco, Hugo Ricateau, Yoann Sarrazin, Alberto Sicilia and Alessandro Tartaglia.

Bangkok, January 2019

Quenches in statistical physics models

passing by critical percolation !

What is it about ?

Classical open systems

Statistical physics framework

Stochastic dissipative dynamics

Out of equilibrium

coarsening – phase ordering kinetics

percolation – fractality (but static uncorrelated problem ! So ?)

Phenomenon

The talk focuses on a very well-known example

Dynamics following a change of a control parameter

- If there is an equilibrium phase transition, the **equilibrium phases** are known on both sides of the transition.

  i.e. the asymptotic state is known.

- For a purely dynamic problem, the **absorbing states** are known.

- The **dynamic mechanism** towards equilibrium is understood

  the systems try to order locally in one of the few competing states.

Interests and goals

Practical & fundamental interest, *e.g.*

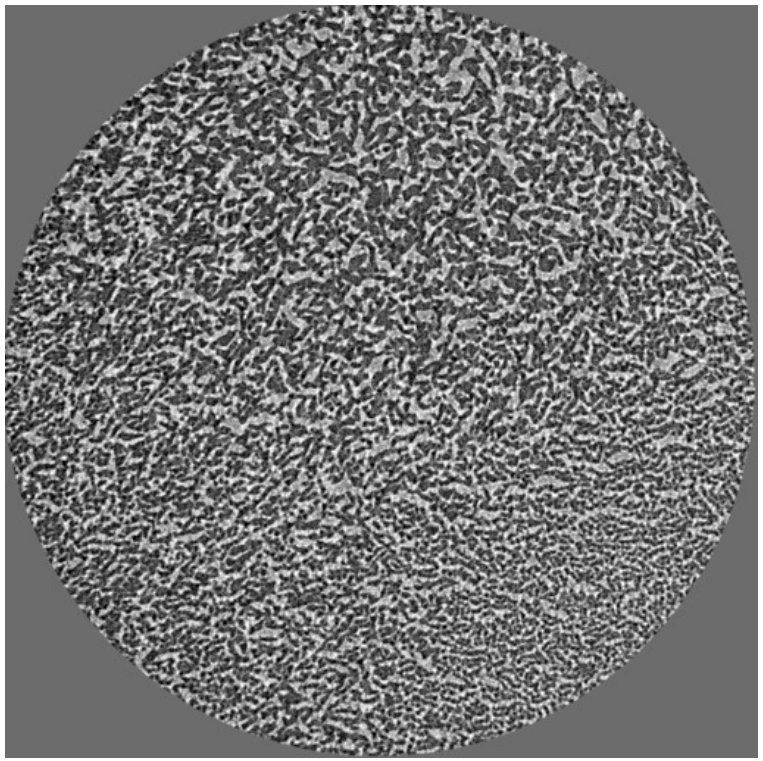
- Mesoscopic structure effects on the opto-mechanical properties of phase separating glasses
- Cooling rate effects on the density of topological defects in cosmology and condensed matter

Some issues

- The role played by the initial conditions & short-time dynamics
- Full geometric characterisation of the structure
- When does the usual dynamic scaling regime set in ?
- The role played by the cooling rate

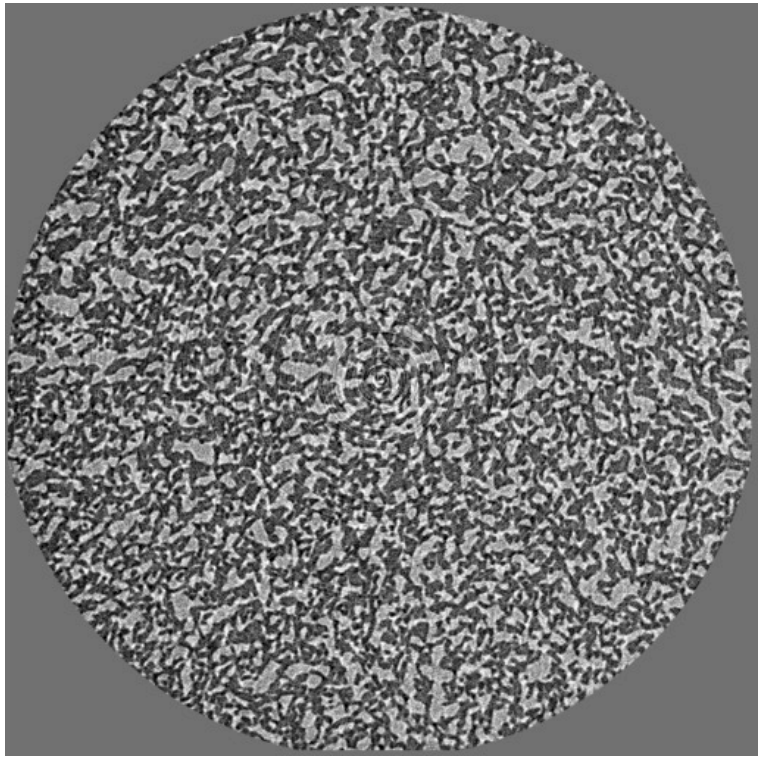
that are related to each other.

Phase separation in glasses



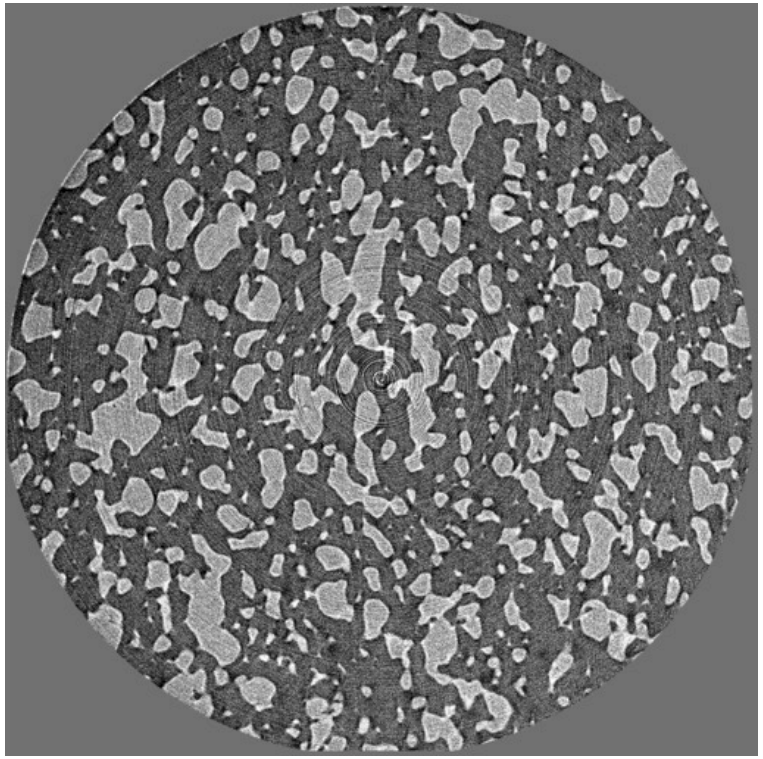
$t = 1$ min

Phase separation in glasses



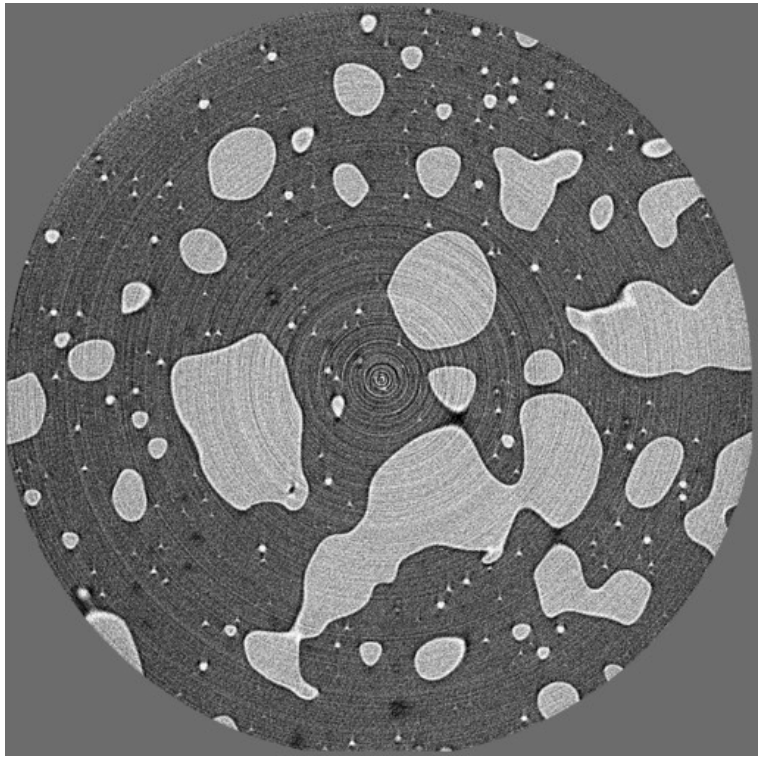
$t = 4$ min

Phase separation in glasses



$t = 16$ min

Phase separation in glasses



$t = 64$ min

Framework

The talk is on a very well-known problem

The stochastic dynamics of the $2d$ Ising model after an instantaneous quench from high to low temperature with non-conserved order parameter dynamics

- There is a 2nd order phase transition.

  The **equilibrium phases** are: the paramagnet at high T

  the (degenerate) ferromagnet at low T

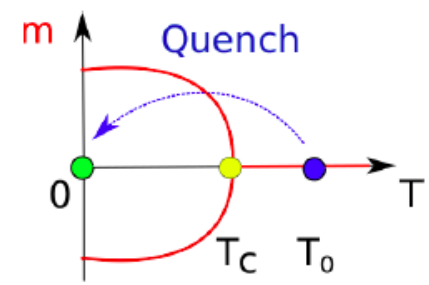
- Standard knowledge :

  The **dynamic mechanism** is curvature-driven domain growth

$2d$ Ising Model (IM)

Archetypical example for classical magnetic systems

$$H = -J \sum_{\langle ij \rangle} s_i s_j$$



$s_i = \pm 1$ Ising spins.

$\langle ij \rangle$ sum over nearest-neighbours

$J > 0$ ferromagnetic coupling constant.

critical temperature $T_c > 0$ for $d > 1$

Monte Carlo rule $s_i \to -s_i$ accepted with

$p = 1$ if $\Delta E < 0$

$p = e^{-\beta \Delta E}$ if $\Delta E > 0$

$p = 1/2$ if $\Delta E = 0$

Non-conserved order parameter dynamics [$\uparrow\downarrow$ towards $\uparrow\uparrow$] etc. allowed.

[$m = 0$ to $m = 2$]

Phenomenon

Similar questions can be asked in very well-known problems in math, *e.g.*

Dynamics of a voter model starting from

a random initial condition

- Purely dynamic, violation of detailed balance, no phase transition
- Two absorbing states
- The **dynamic mechanism** towards absorption is understood

  domain growth is driven by interfacial noise

$2d$ Voter Model (VM)

Archetypical example of opinion dynamics

H does not exist - kinetic model

$s_i = \pm 1$ Ising spins that

sit on the vertices of a lattice.

Voter update rule

choose a spin at random, say s_i

choose one of its $2d$ neighbours at random, say s_j

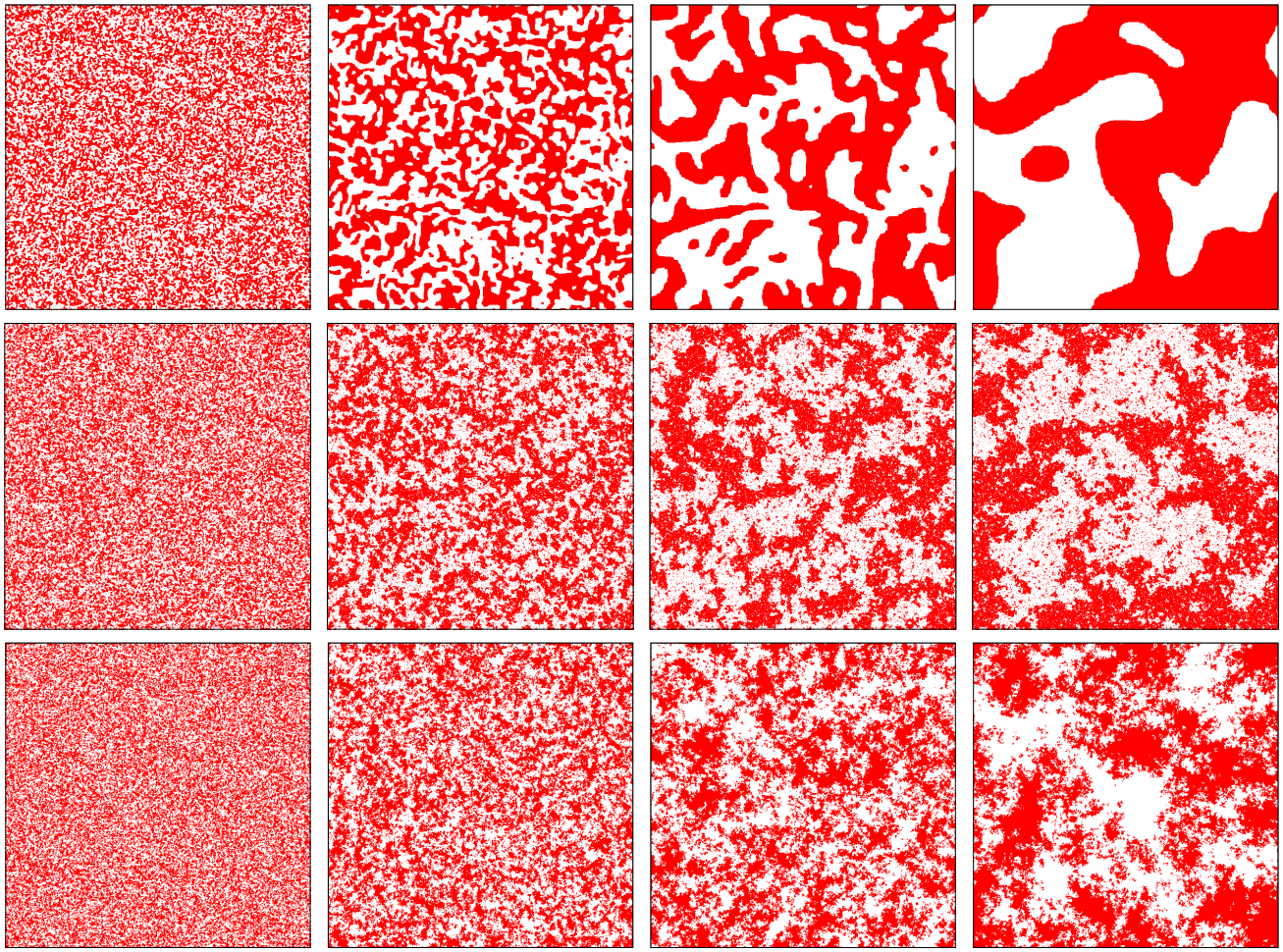
set $s_i = s_j$

In two dimensions full consensus, *i.e.* $m = L^{-d} \sum_{i=1}^{L^d} s_i = \pm 1$ is reached in a timescale $t_C \simeq L^2$ (with $\ln L$ corrections)

Clifford & Sudbury 73, Holley & Liggett 75, Cox & Griffeaths 86

Phase ordering kinetics

$s_i = \pm 1$ **at** $t = 0$ **MCs, snapshots at** $t = 4,\ 64,\ 512, 4096$ **MCs**



Ising

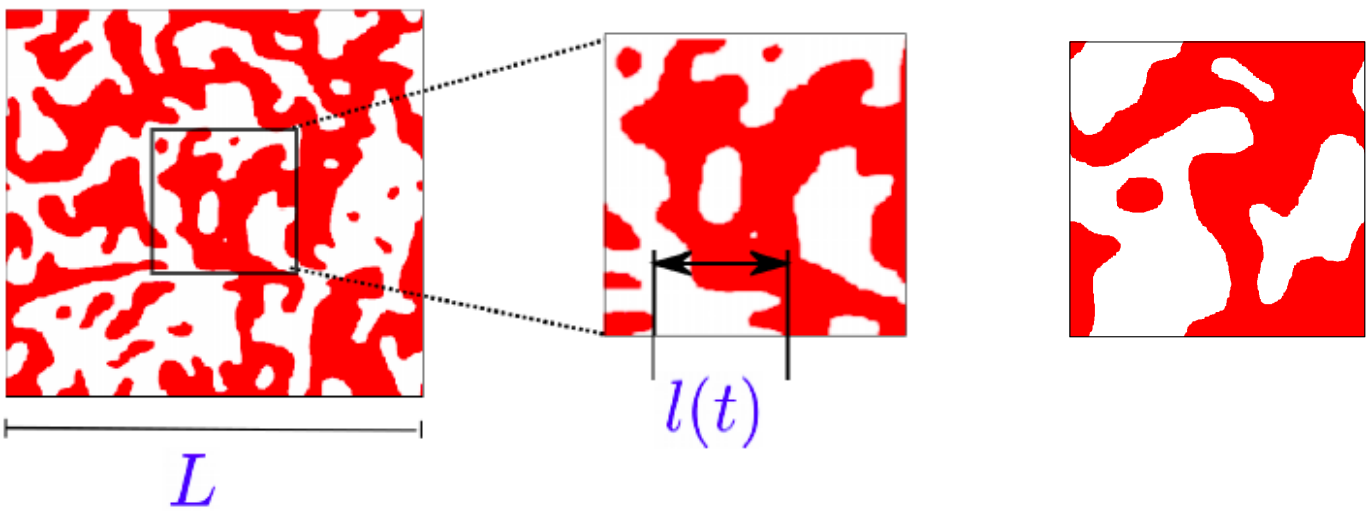
$T = 0$

T_c

Voter

Dynamic scaling

in phase ordering kinetics



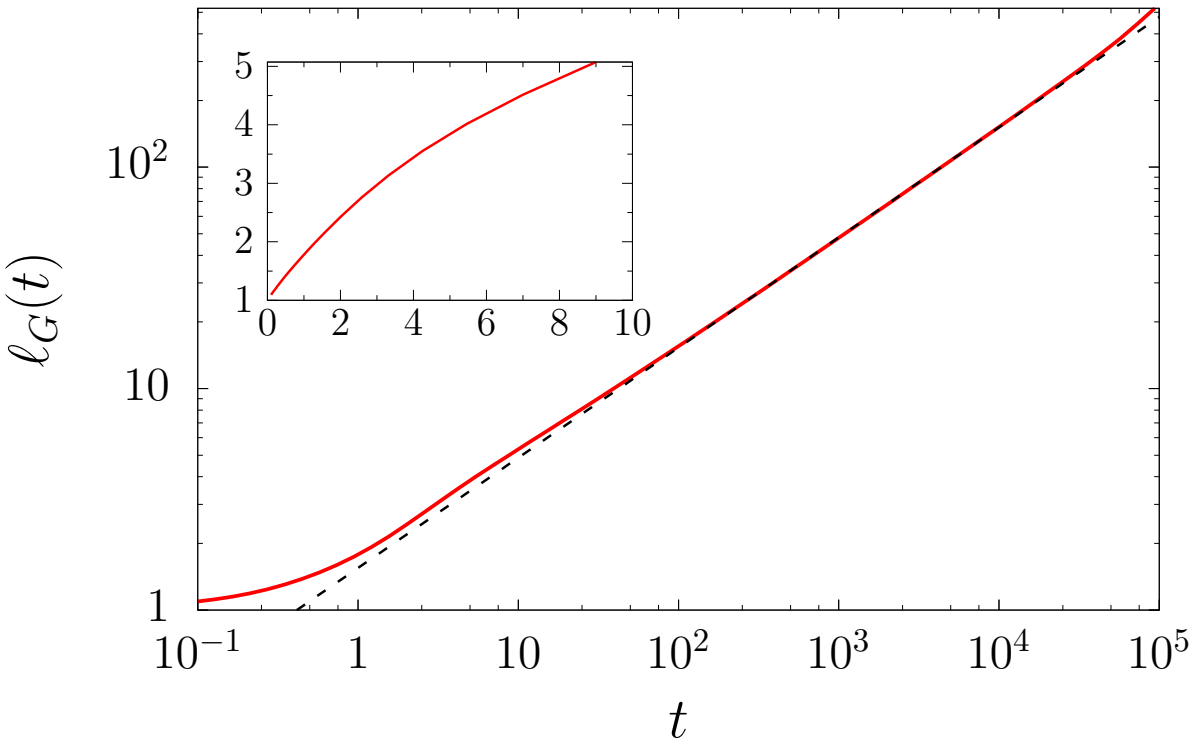
Growing length $l(t)$

Typically $l(t) \simeq t^{1/z_d}$

Excess energy w.r.t. the equilibrium one stored in the domain walls

Growing length

From excess energy



$$\ell_G(t) = \frac{E_{\rm eq}(T)}{E_{\rm eq}(T) - E(t,T)} \simeq t^{1/z_d}$$ and $$z_d = 2$$

Dynamic scaling

At late times there is a single length-scale, the typical radius of the domains $l(t)$, such that the domain structure is (in statistical sense) independent of time when lengths are scaled by $l(t)$, *e.g.*

$$C(r,t) \equiv \langle\, s_i(t) s_j(t)\,\rangle|_{|\vec{x}_i - \vec{x}_j| = r} \sim \langle\phi\rangle_{eq}^2\; f\left(\frac{r}{l(t)}\right)$$

$$C(t,t_w) \equiv \langle\, s_i(t) s_i(t_w)\,\rangle \sim \langle\phi\rangle_{eq}^2\; f_c\left(\frac{l(t)}{l(t_w)}\right)$$

etc. when $r \gg \xi_{eq}$, times such that $t, t_w \gg t_0$ and $C < \langle\phi\rangle_{eq}^2$.

Review Bray 94

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etc. when $r \gg \xi_{eq}$, times such that $t, t_w \gg t_0$ and $C < \langle\phi\rangle_{eq}^2$.

Is this really all there is ?

Interests and goals

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- Cooling rate effects on the density of topological defects in cosmology and condensed matter

Some issues

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- Full geometric characterisation of the structure
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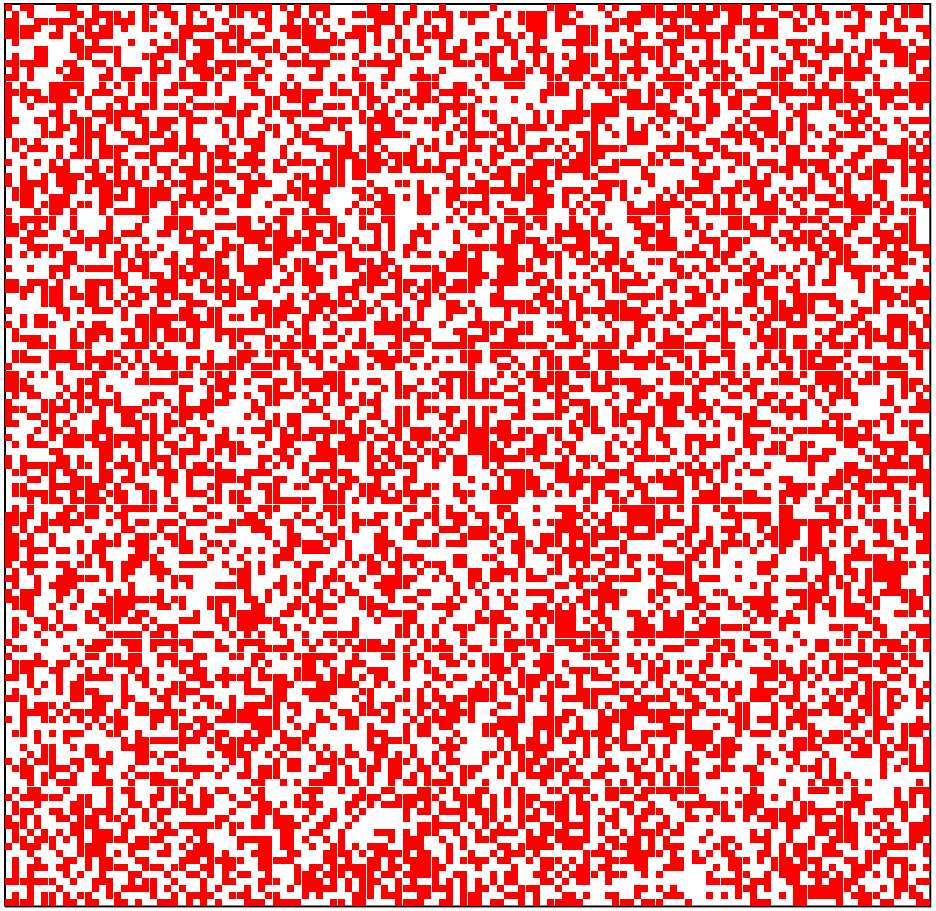
that are related to each other.

2d square IM at T=0



t=0.0

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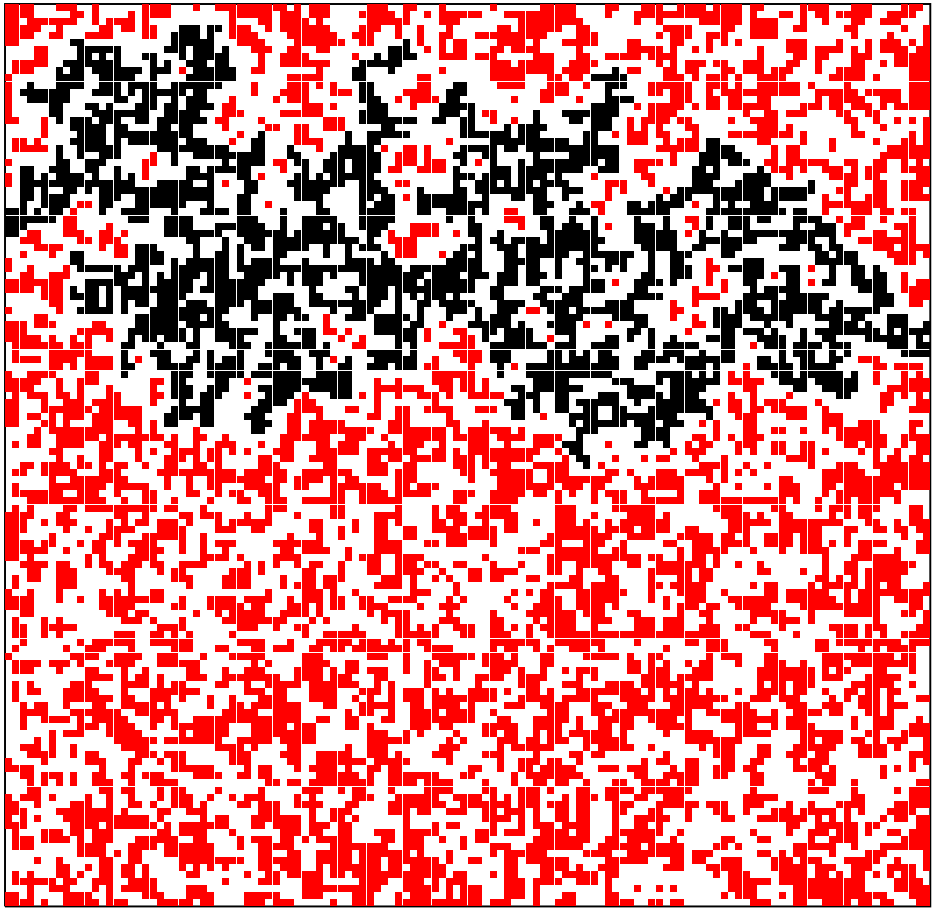
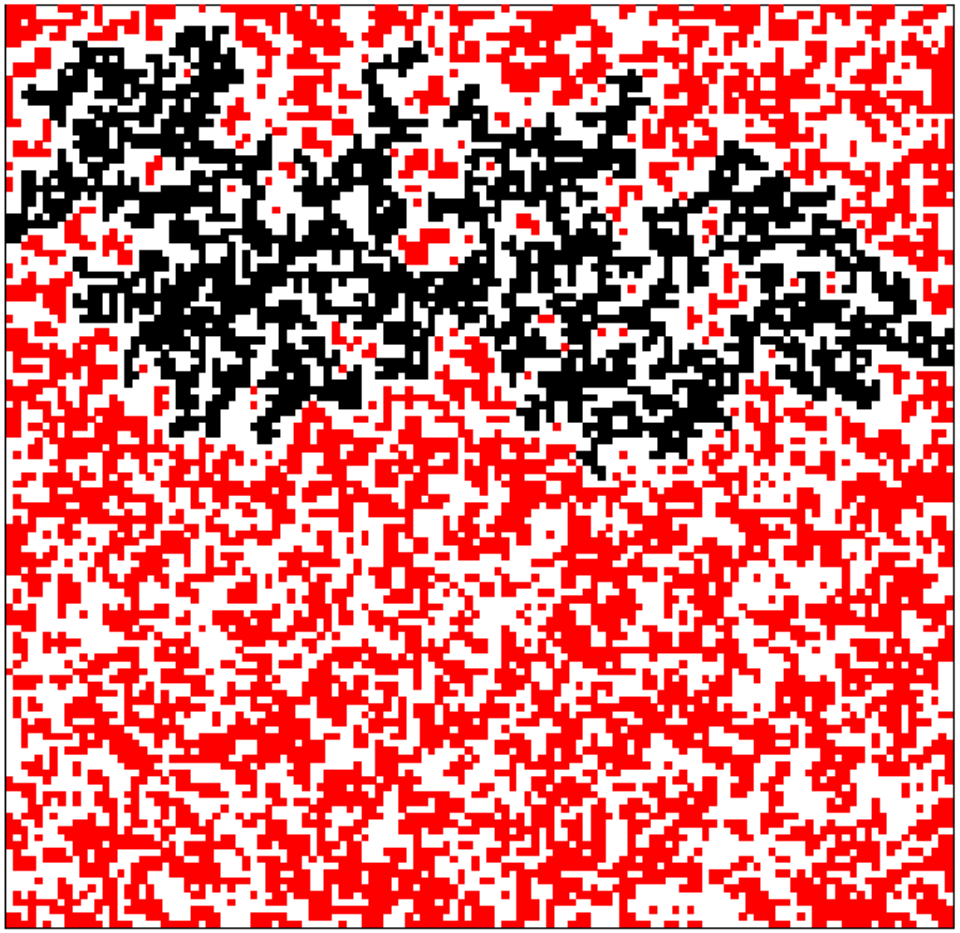


t=0.57533

Spanning cluster

Has this cluster something to do with (critical) percolation ?



Percolation

Purely geometric problem

Take a lattice Λ in d spatial dimensions.

Define a site occupation variable $n_i = 1, 0$ with probability $p, 1-p$

In the limit $L \to \infty$ there is a continuous phase transition at p_c such that the probability of there being a cluster of occupied nearest-neighbour sites that crosses a sample from one end to another in at least one Cartesian direction

$$\lim_{L\to\infty} P(p, L) \begin{cases} = 0 & \text{if } p \leq p_c \\ > 0 & \text{if } p > p_c \end{cases}$$

p_c depends on Λ and d.

At p_c the spanning cluster has fractal properties that are well characterised

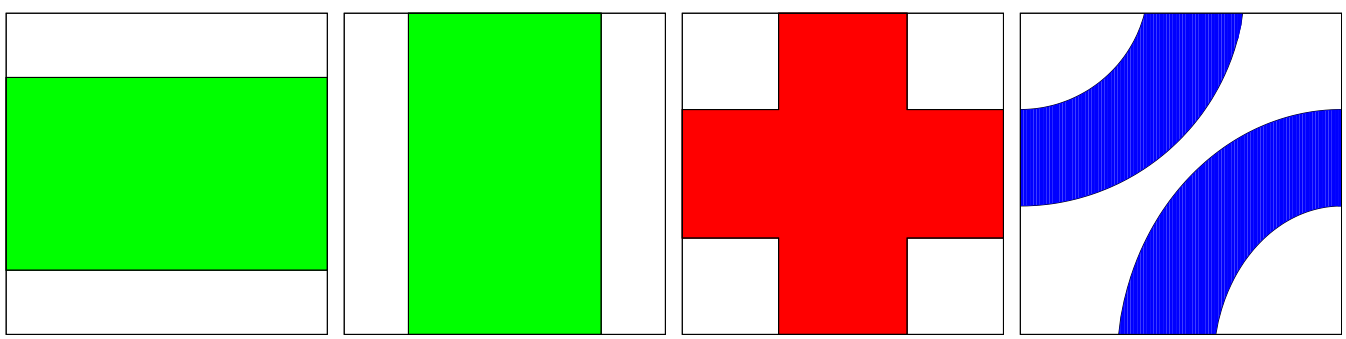
Percolation

Purely geometric problem

Probability of percolation along the horizontal *or* vertical directions, $\pi_{h,v}$

Probability of percolation along the horizontal *and* vertical directions, π_{hv}

(On a torus) Probability of percolation along the diagonal direction, π_d



Known exactly from SLE & CFT calculations.

Percolation

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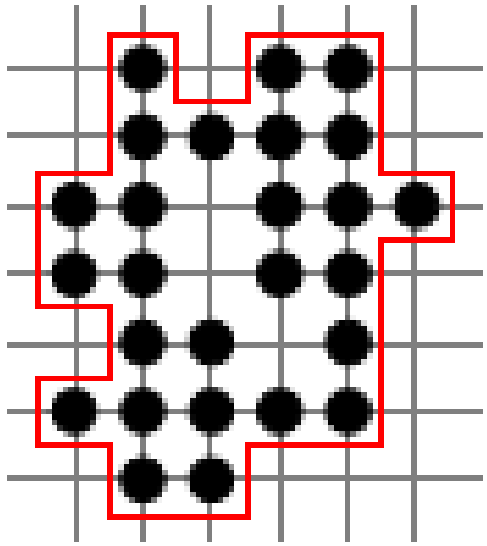
p_c depends on Λ and d.

The distribution of finite size clusters is algebraic at p_c

Percolation

Purely geometric problem



Domain area: sum of filled dots $\mathcal{N}_d, \tau_d, D_A$

External boundary or hull: red broken line $\mathcal{N}_\ell, D_\ell$

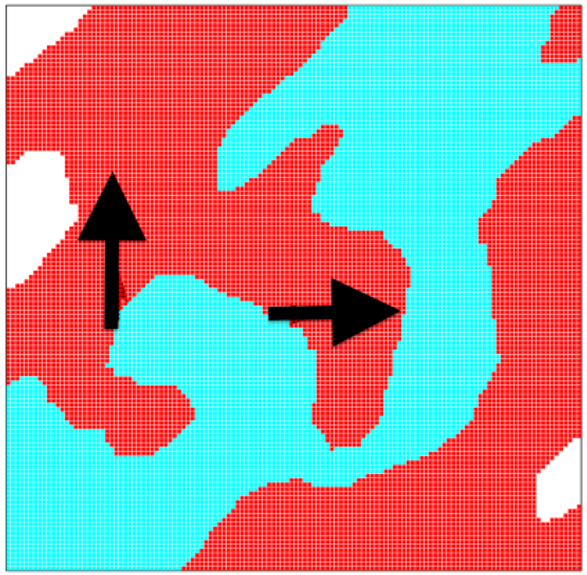
Hull-enclosed area: sum of lattice sites within the red boundary (including the two empty sites) $\mathcal{N}_h, \tau, D$

Percolation

Purely geometric problem

Winding angle *vs.* curvilinear length of the wall

$$\langle \theta^2(x) \rangle = \text{ct} + \frac{4\kappa}{8+\kappa} \ln x \qquad \text{with} \qquad \kappa = 6$$



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Spanning cluster

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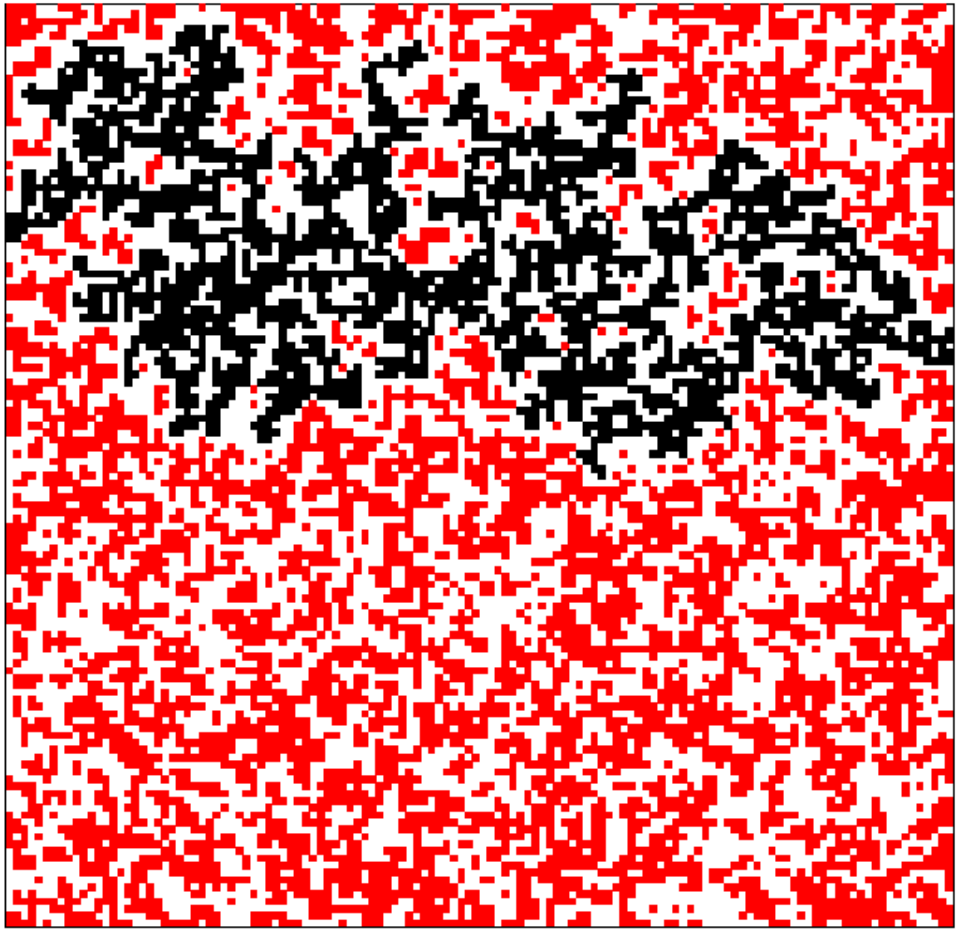


Back to the dynamic problem

2dIM : various lattices

Initial conditions for an instantaneous quench

Equilibrium at infinite temperature, $T_0 \to \infty$, initial condition.

The spins take ± 1 values with probability $1/2$.

Site occupation variable $n_i = (s_i + 1)/2 = 1, 0$ with $p = 1/2$

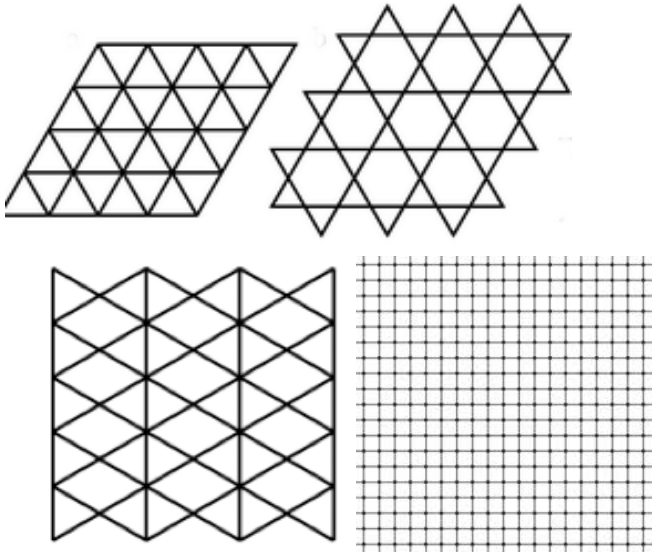
From a site percolation perspective:

$p_c = 0.65$ Kagome lattice.

$p_c = 0.59$ Square lattice.

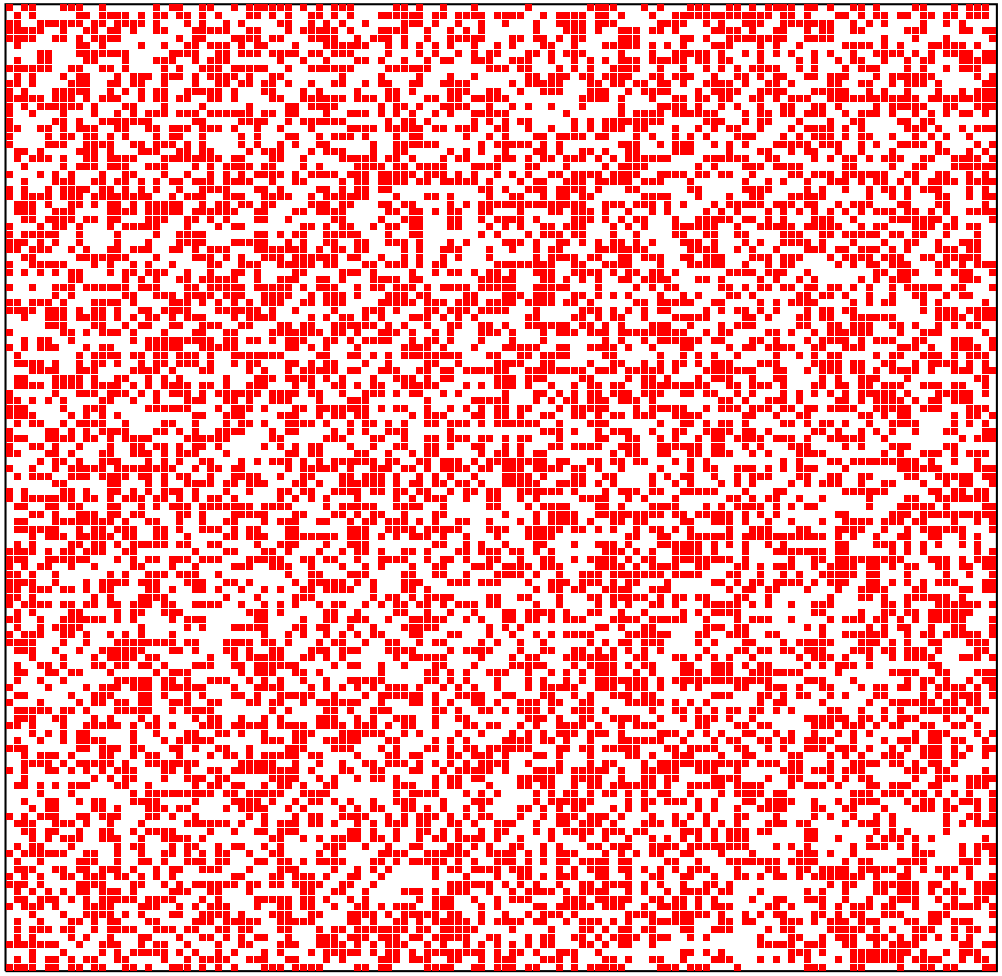
$p_c = 0.55$ Bow-tie lattice.

$p_c = 0.5$ Triangular lattice.



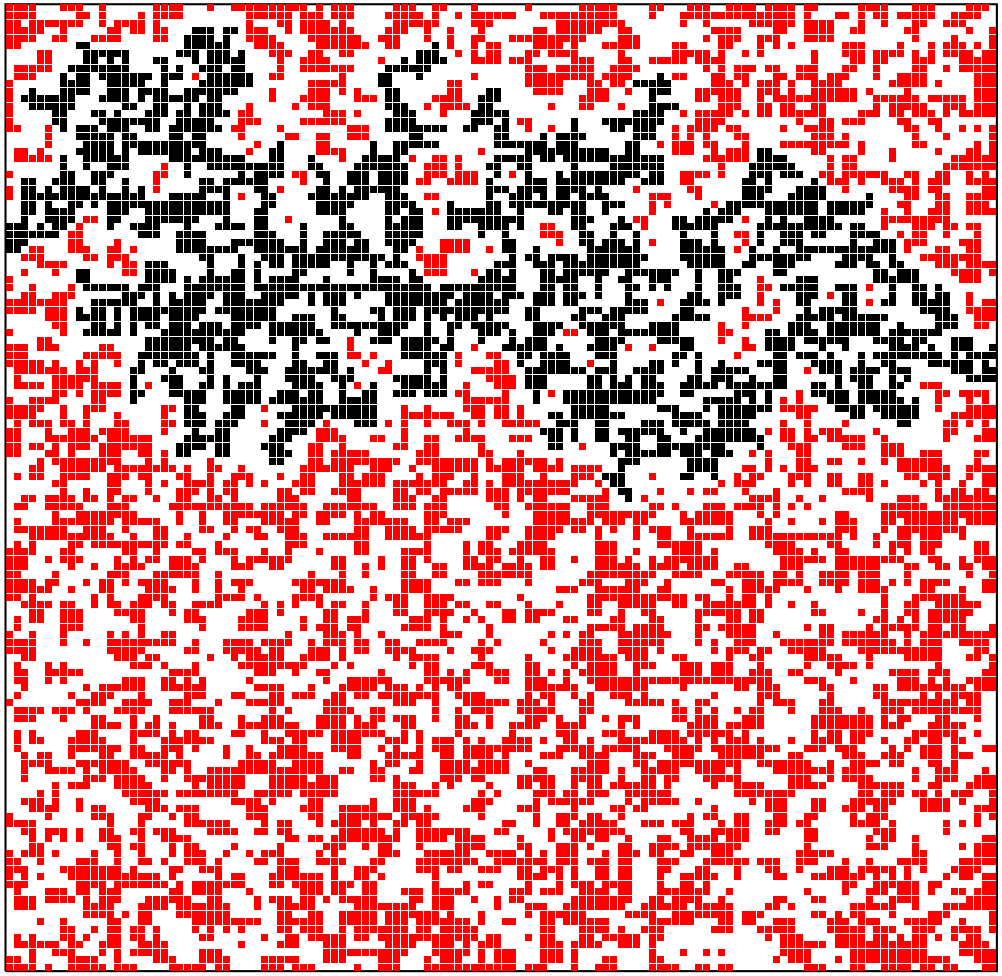
Initial condition at $p = 0.5$, below p_c

2d square IM at T=0



t=0.0

2d square IM at T=0



t=0.57533

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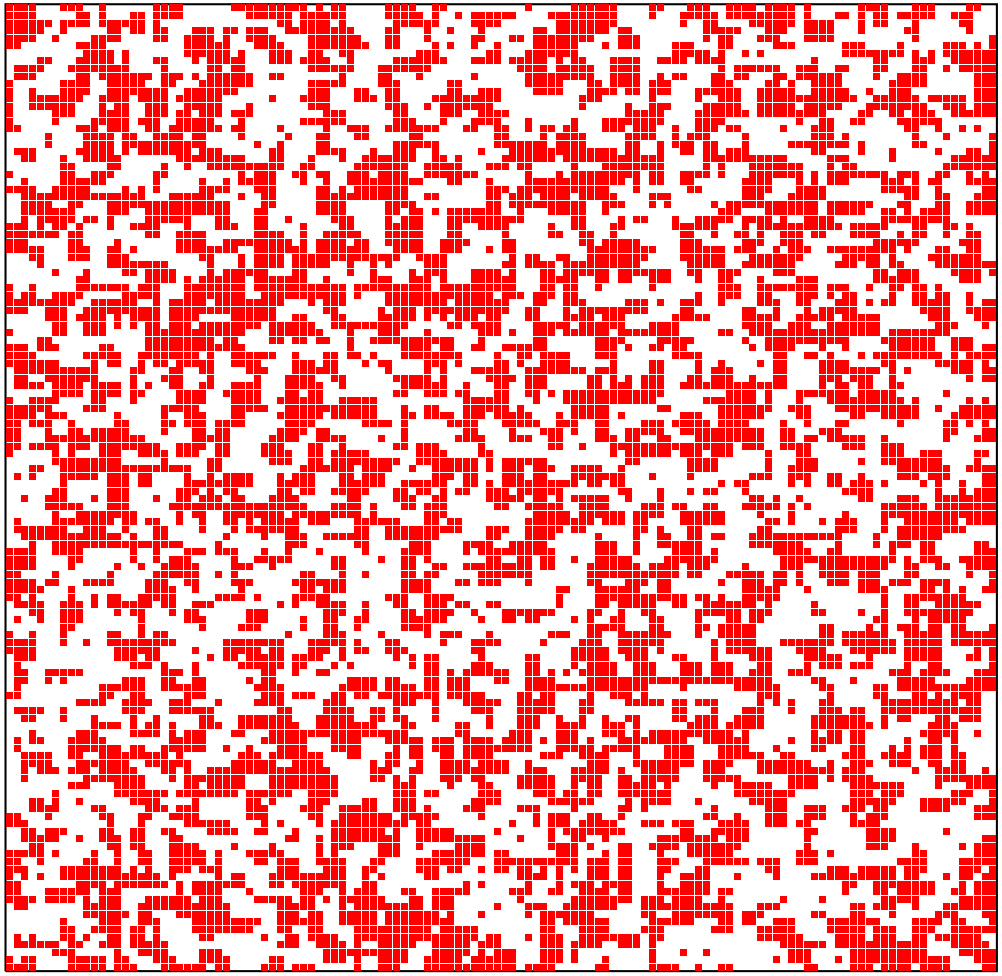
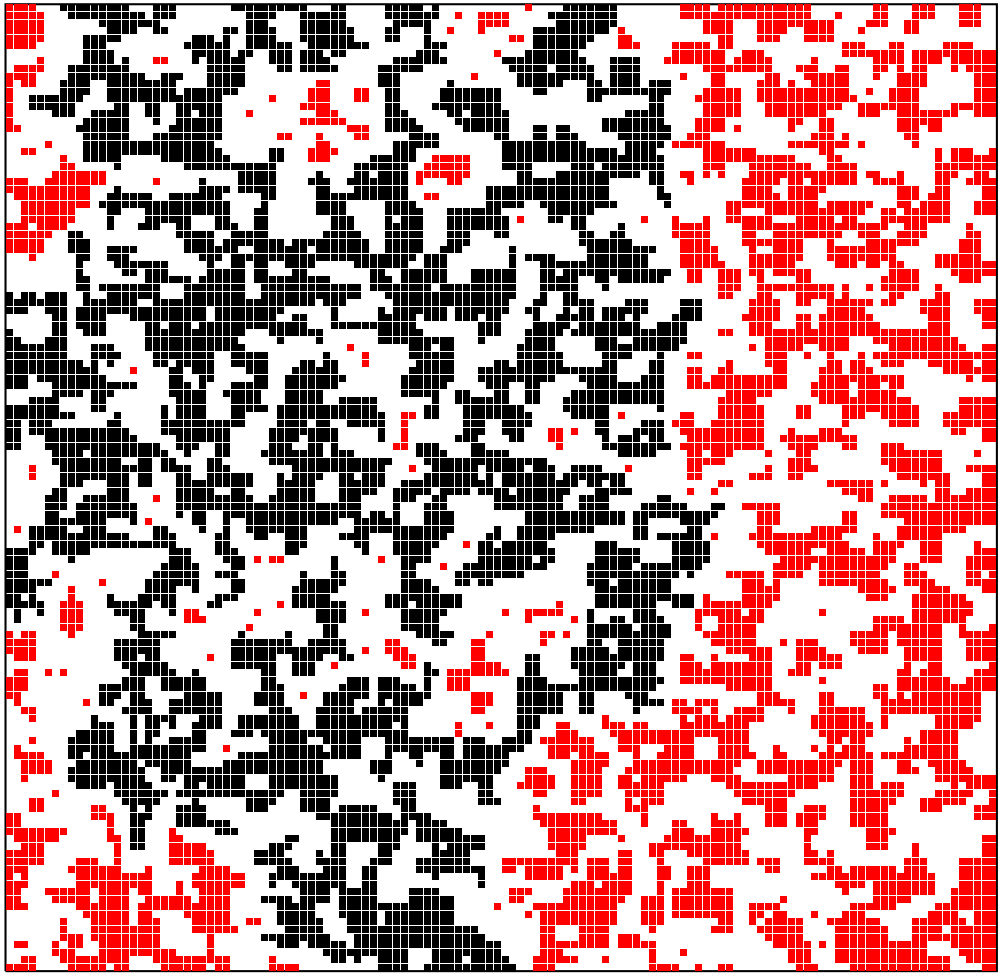


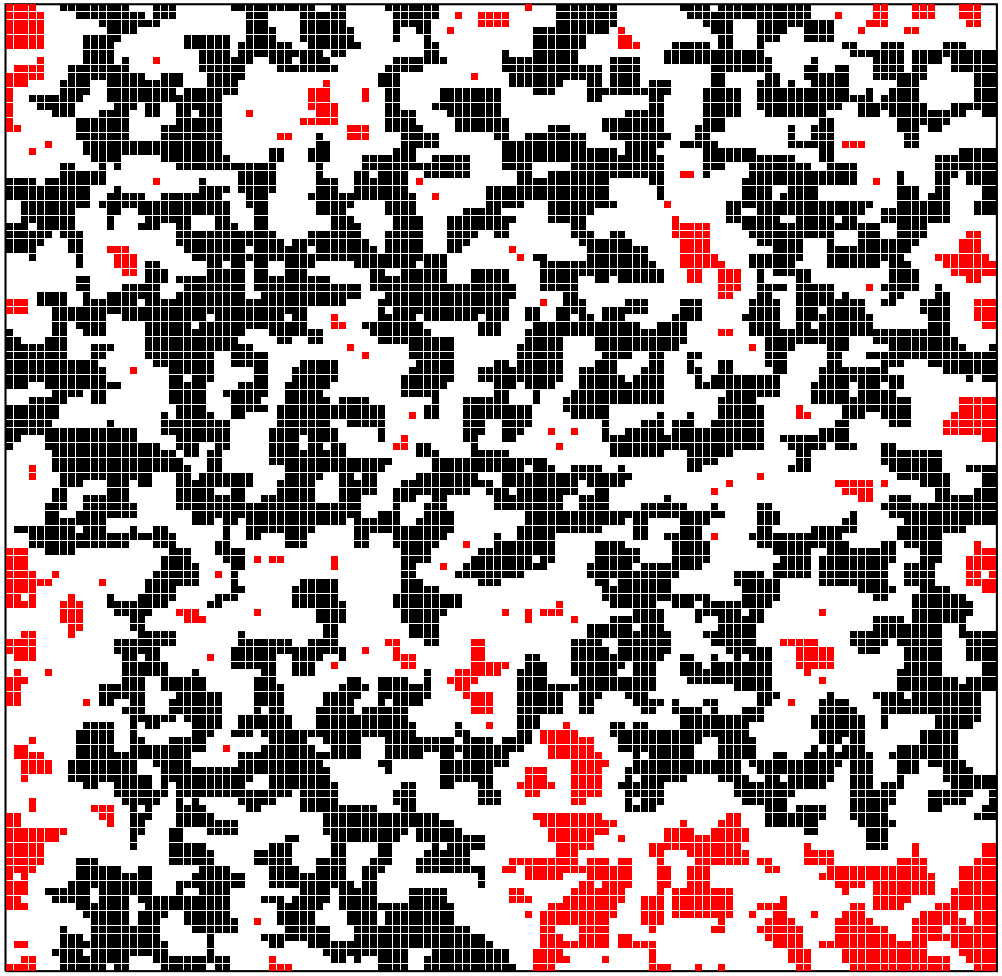
t=0.94844

2d square IM at T=0



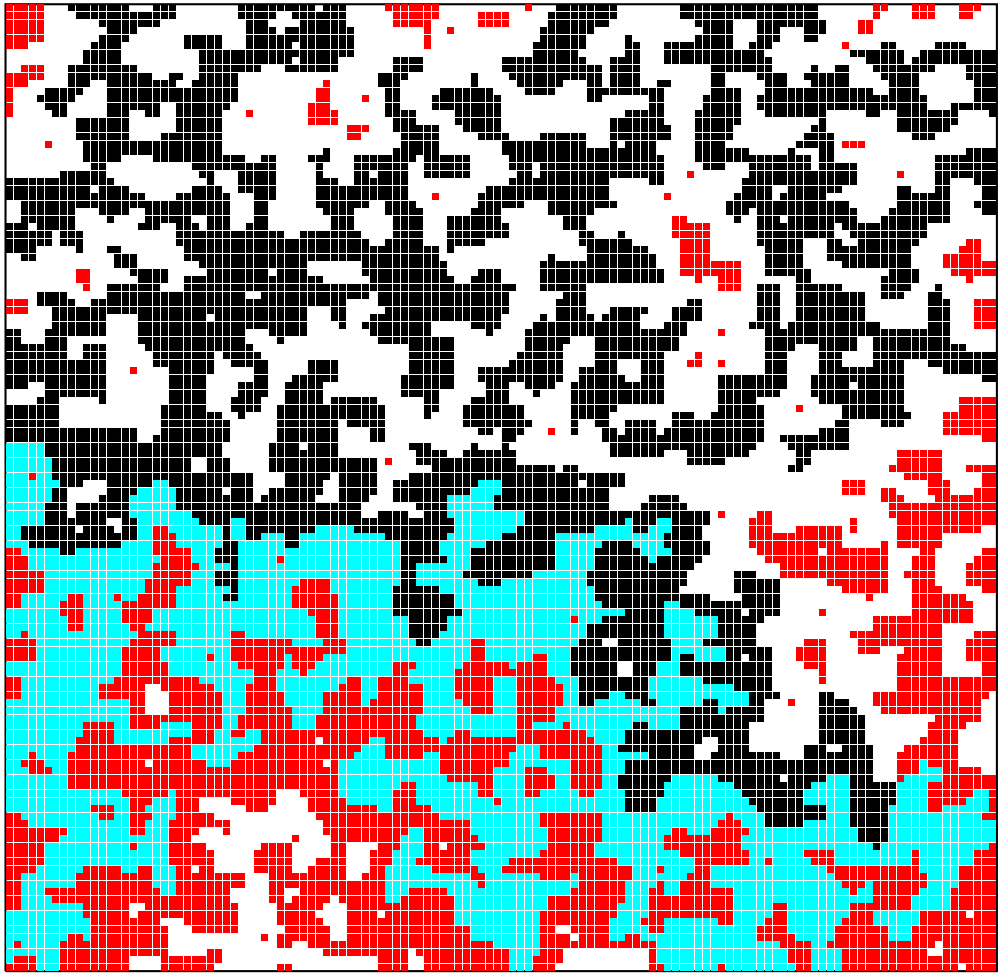
t=2.00847

2d square IM at T=0



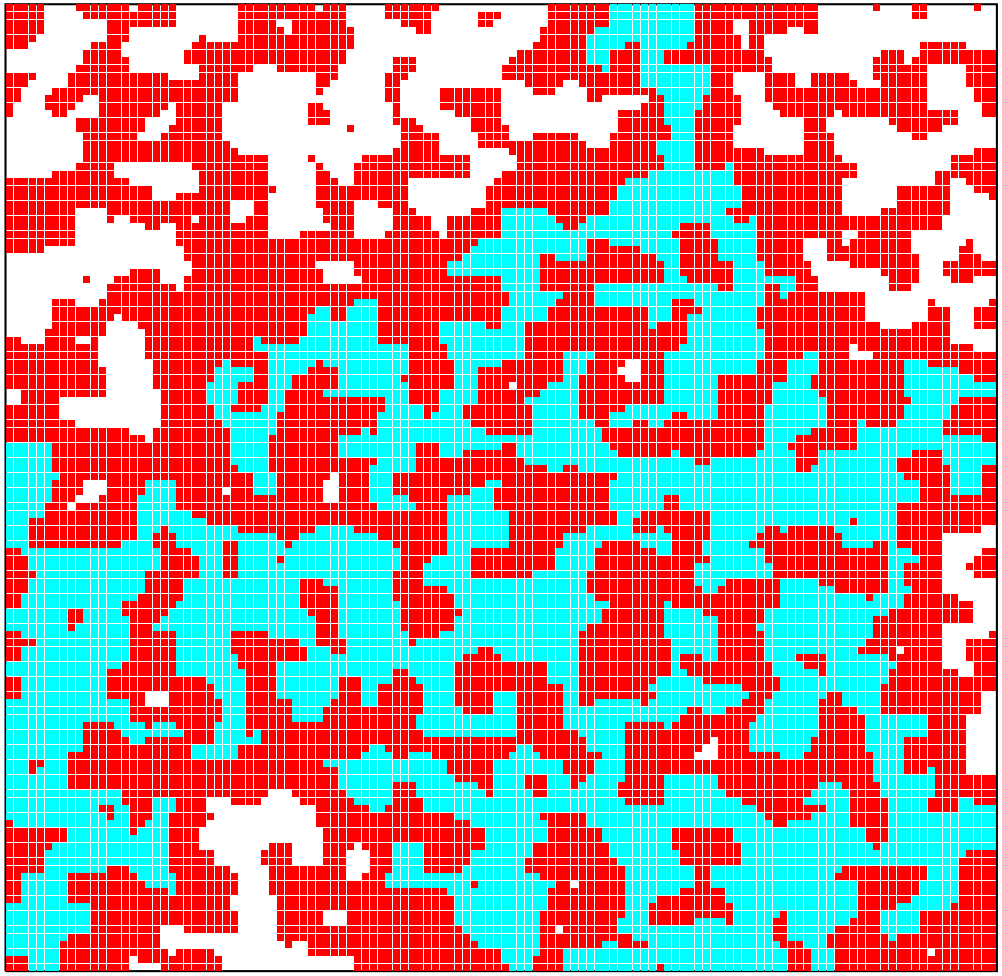
t=2.57898

2d square IM at T=0



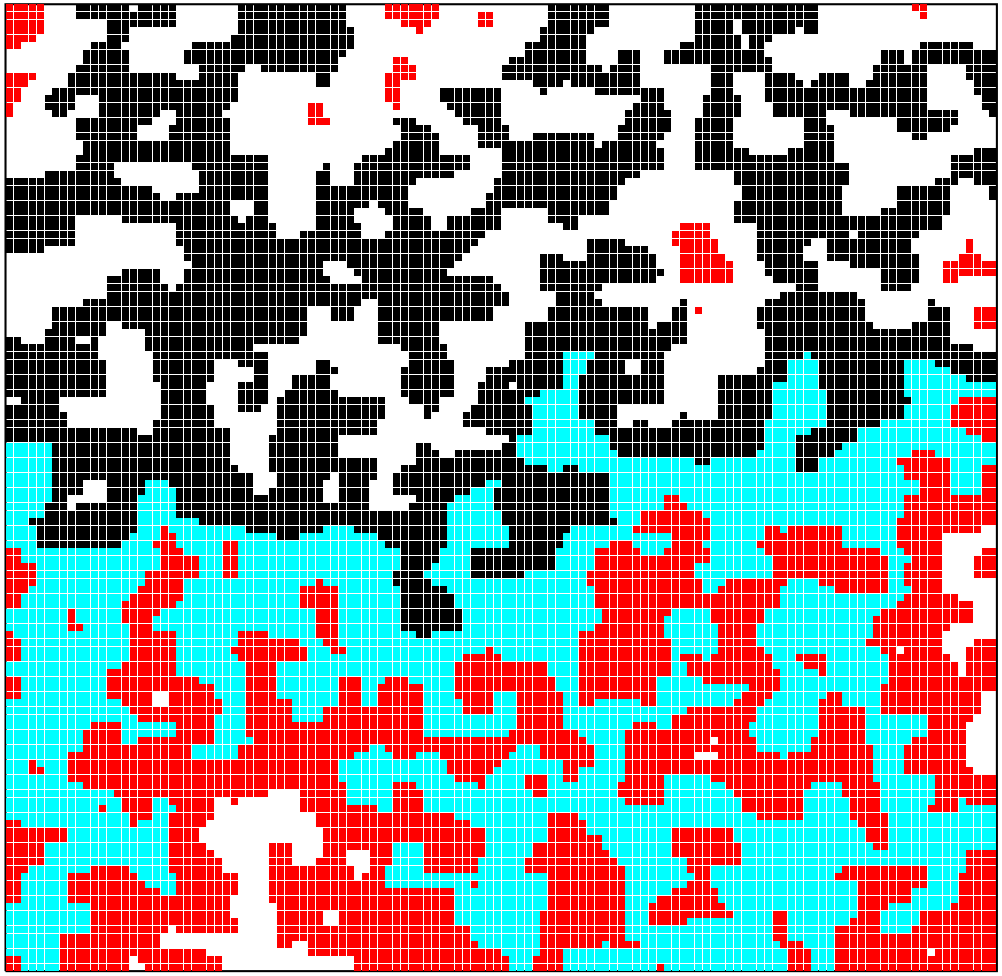
t=3.99211

2d square IM at T=0



t=6.58423

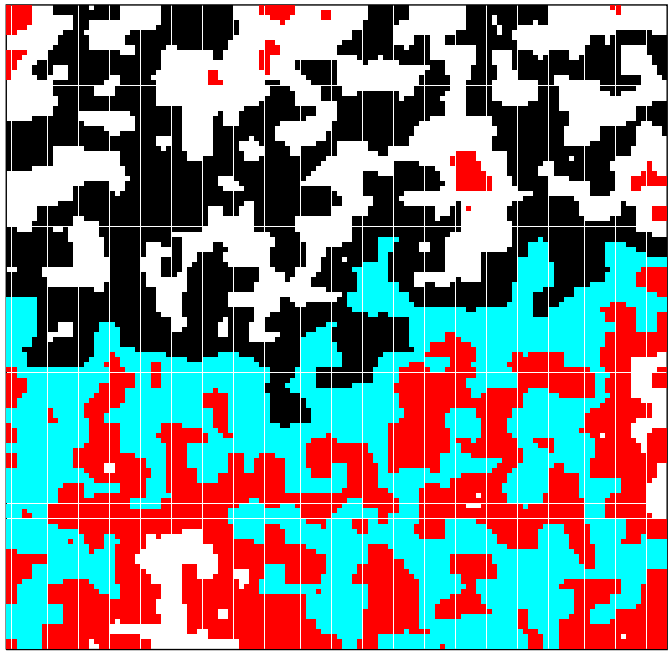
2d square IM at T=0



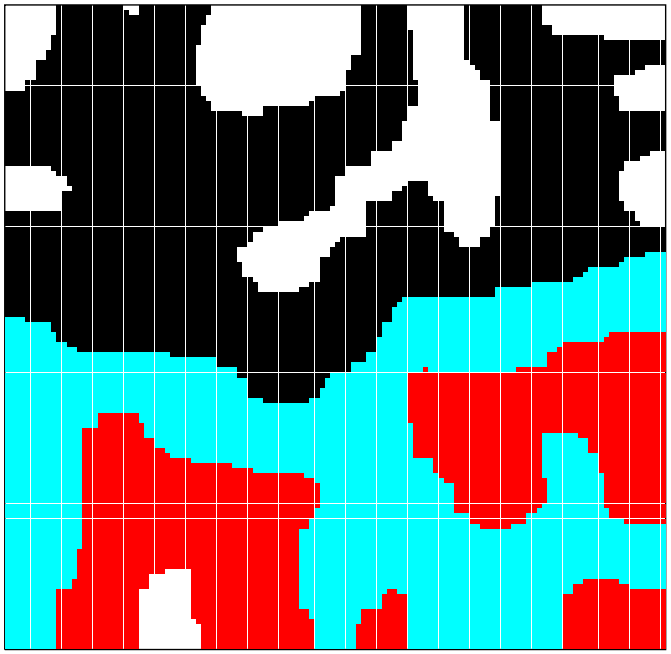
t=7.46144

2d square IM at T=0

The percolating structure was decided at $t_p \simeq 8$ MCs



t=7.46144

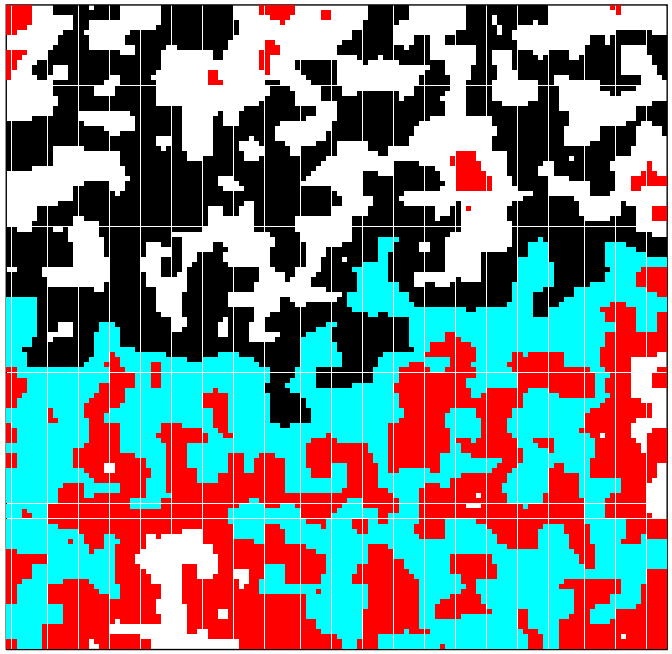


t=128.0

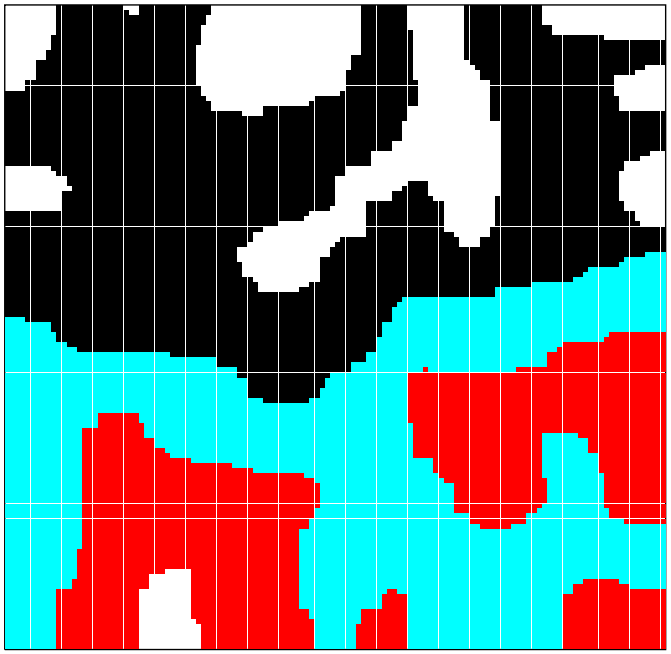
Arenzon, Bray, LFC & Sicilia 07 Blanchard, Corberi, LFC & Picco 14

2d square IM at T=0

The final configuration will be one with two horizontal stripes



t=7.46144

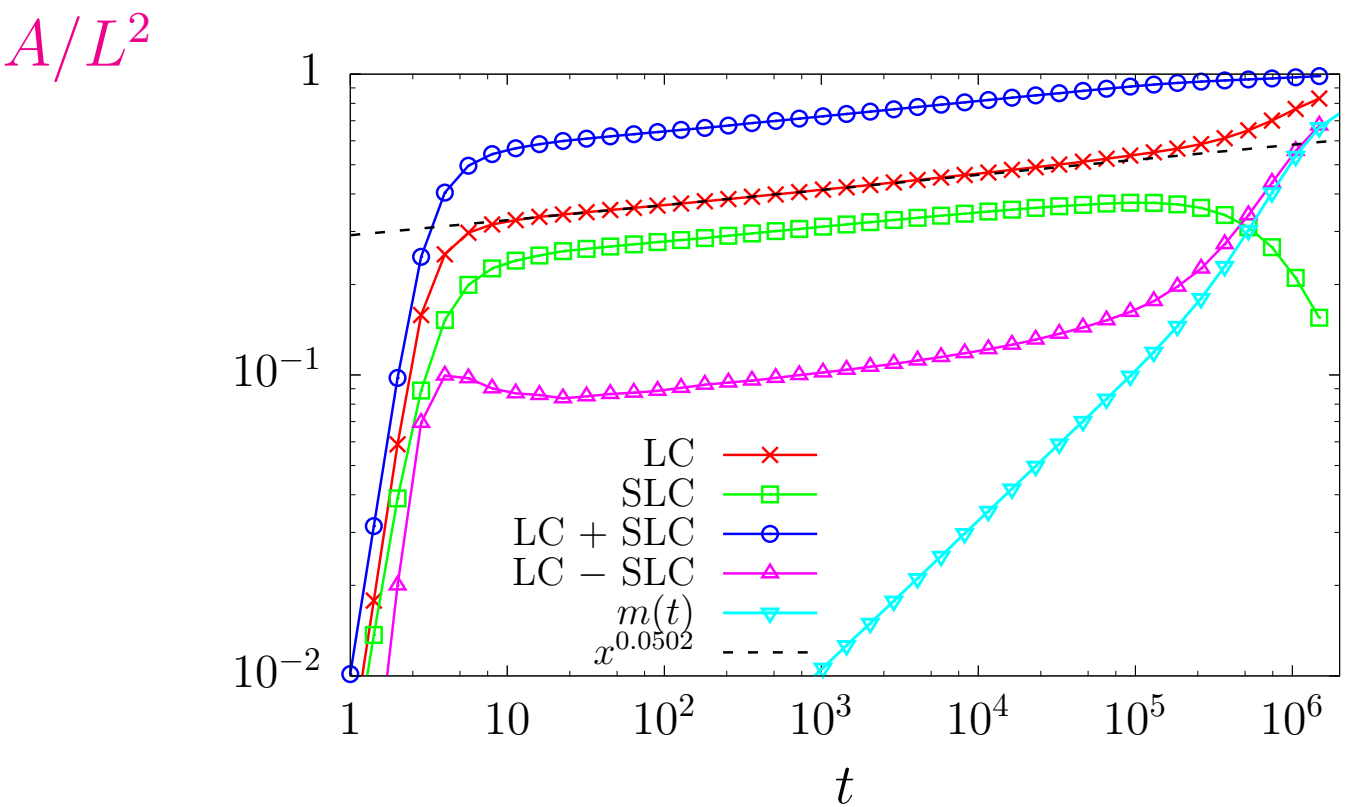


t=128.0

Olejarz, Krapivsky & Redner 12, Blanchard & Picco 13

What is going on ?

Rapid growth of the two largest clusters in $t \simeq 10$ MCs

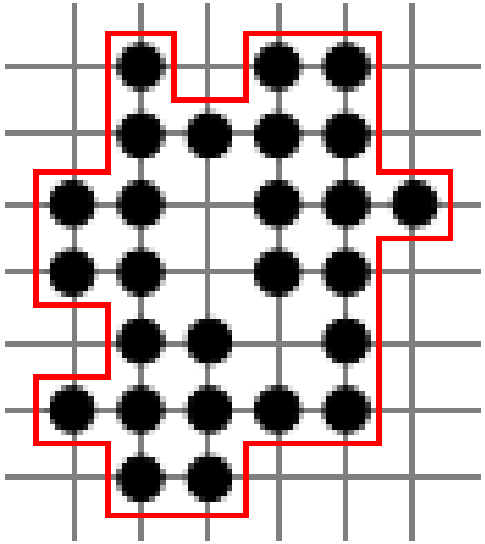


Further algebraic growth after $t \simeq 10$ MCs

Data averaged over many runs. I'll come back to these data later.

Cluster analysis

Counting and measuring



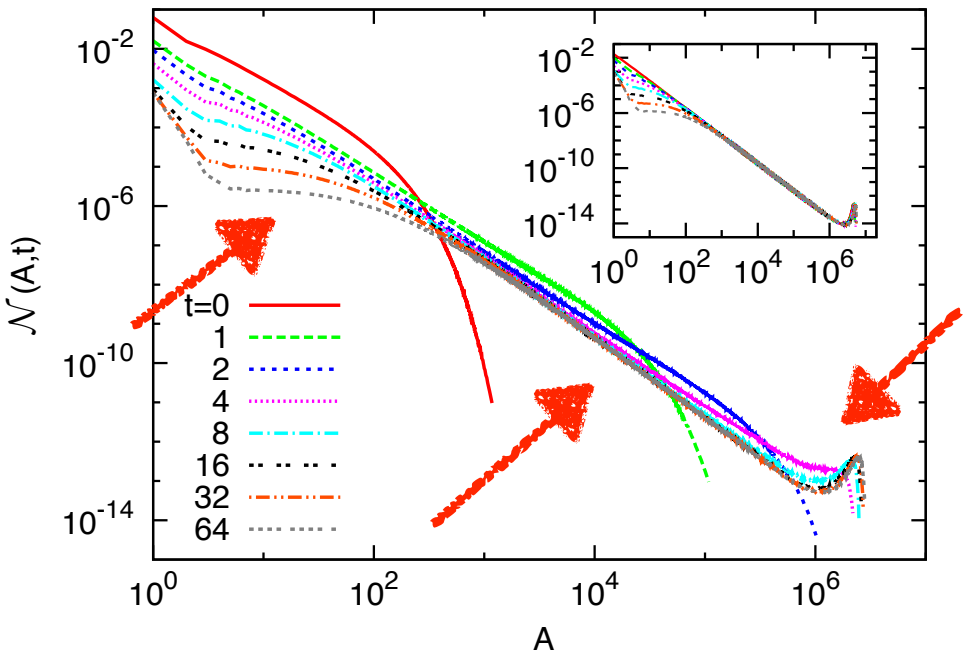
Domain area: sum of filled dots

External boundary or hull: red broken line

Hull-enclosed area: sum of lattice sites within the red boundary (including the two empty sites) $\mathcal{N}_h(A,t)$

Is it critical percolation ?

Full distribution of areas after a quench from $T_0 \to \infty$



Three dynamic regimes.

– small areas $A \leq l^2(t)$: flat

– large but finite areas $l^2(t) \leq A \leq L^2$: power law decay

– even larger areas $A = \mathcal{O}(L^{D_A})$: a bump

Is it critical percolation ?

Tail and finite size scaling of the bump

Take A to be the hull-enclosed area or the domain area.

At critical percolation, finite size scaling of the number density of areas

$$\boxed{\mathcal{N}(A,L) = 2c\, A^{-\tau_A} + N_p(A/L^{D_A})}$$

with $D_A = d/(\tau_A - 1)$ the fractal dimension of the percolating clusters.

Stauffer & Aharony 94

For hull-enclosed areas $\tau_A = 2$ and $D_A = 2$

For domain areas $\tau_A = 187/91 \simeq 2.05$ and $D_A \simeq 1.9$

NB the corresponding exponents for critical Ising conditions are different but take values close to these.

The constants $2c$ are known, *e.g.* $2c_h \approx (16\pi\sqrt{3})^{-1}$ and $2c_d \approx 0.06$

Cardy & Ziff 03, Sicilia, Arenzon, Bray & LFC 07

Statistics of finite areas

Assuming percolation established at $t = t_p$

$$n_h(A,t) \equiv \frac{(2)c_h}{(A+\lambda t)^2} \qquad n_d(A,t) \approx \frac{(2)c_d\,(\lambda_d t)^{\tau_A - 2}}{(A+\lambda_d t)^{\tau_A}}$$

in the long time limit $t \gg t_p$. Change in notation $n = \mathcal{N}$

We derived the expected scaling forms, as $l(t) = (\lambda t)^{1/2}$:

$$l^4(t) n_h(A,t) = f_h\left(\frac{A}{l^2(t)}\right) \qquad l^4(t) n_d(A,t) \approx f_d\left(\frac{A}{l^2(t)}\right)$$

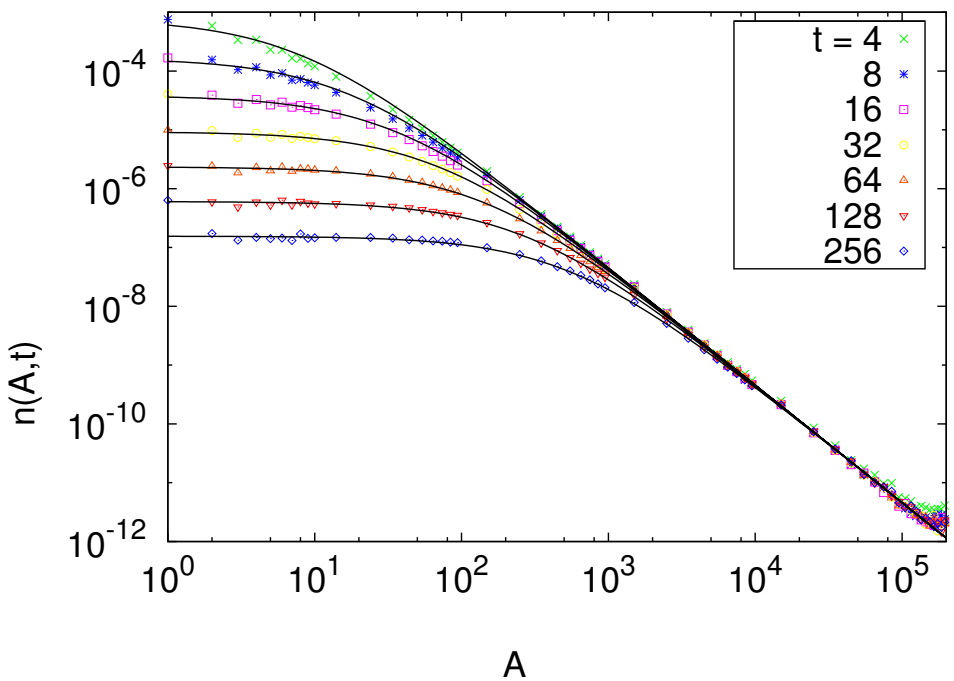
The new parameters are $c_d = c_h + O(c_h^2)$ and $\lambda_d = \lambda + O(c_h)$. Moreover, the sum rules, $N_h(t) = N_d(t)$ and $\int dA\; A\, n_d(A,t) = 1$ relate c_h to τ (or τ') !

Arenzon, Bray, LFC & Sicilia 07

Simulations vs. theory

Number density of (finite) hull-enclosed areas per unit area

$T_0 \to \infty$ **and** $T = 0$



Solid lines

analytical prediction :

$$n_h(A,t) \equiv \frac{(2)c_h}{(A+\lambda t)^2}$$

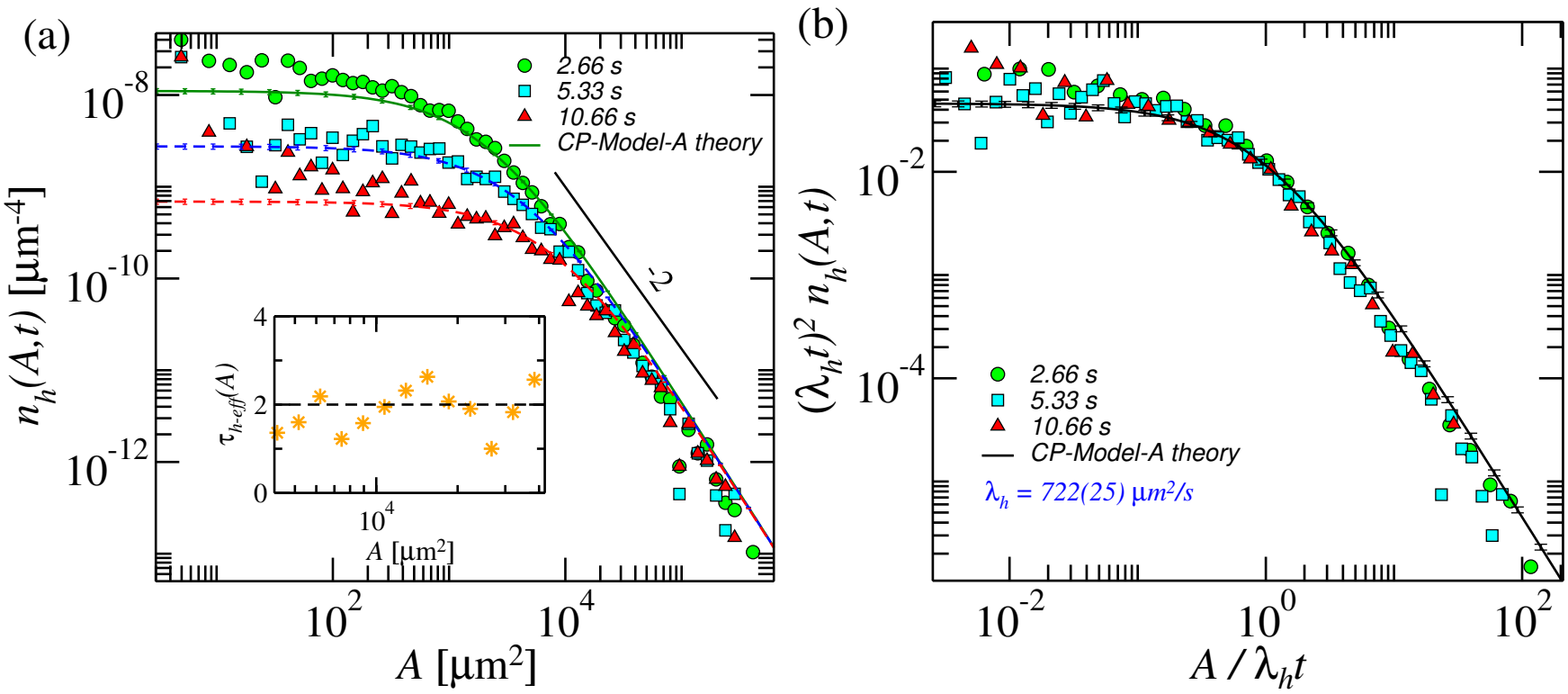
$$2c_h = (16\pi\sqrt{3})^{-1}$$

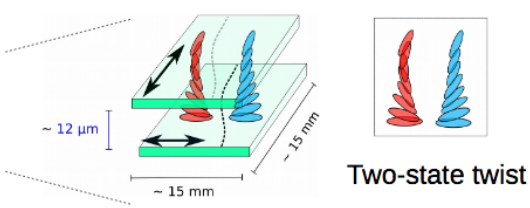
Arenzon, Bray, LFC & Sicilia 07

Experiments vs. theory

Number density of (finite) domain areas per unit area

$T_0 \to \infty$ **and** $T = 0$

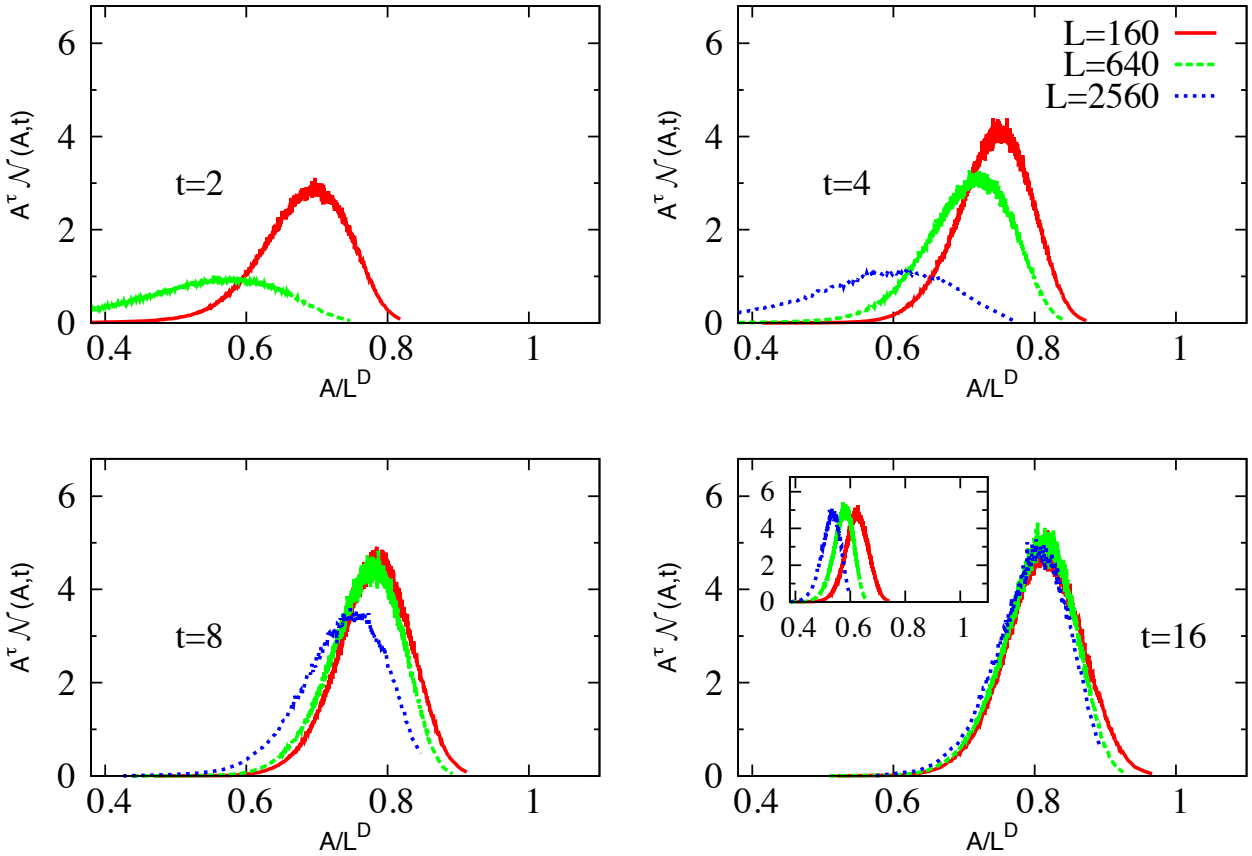




R. A. L. Almeida & K. Takeuchi 19

Is it critical percolation ?

Number density of the domain areas of percolated clusters

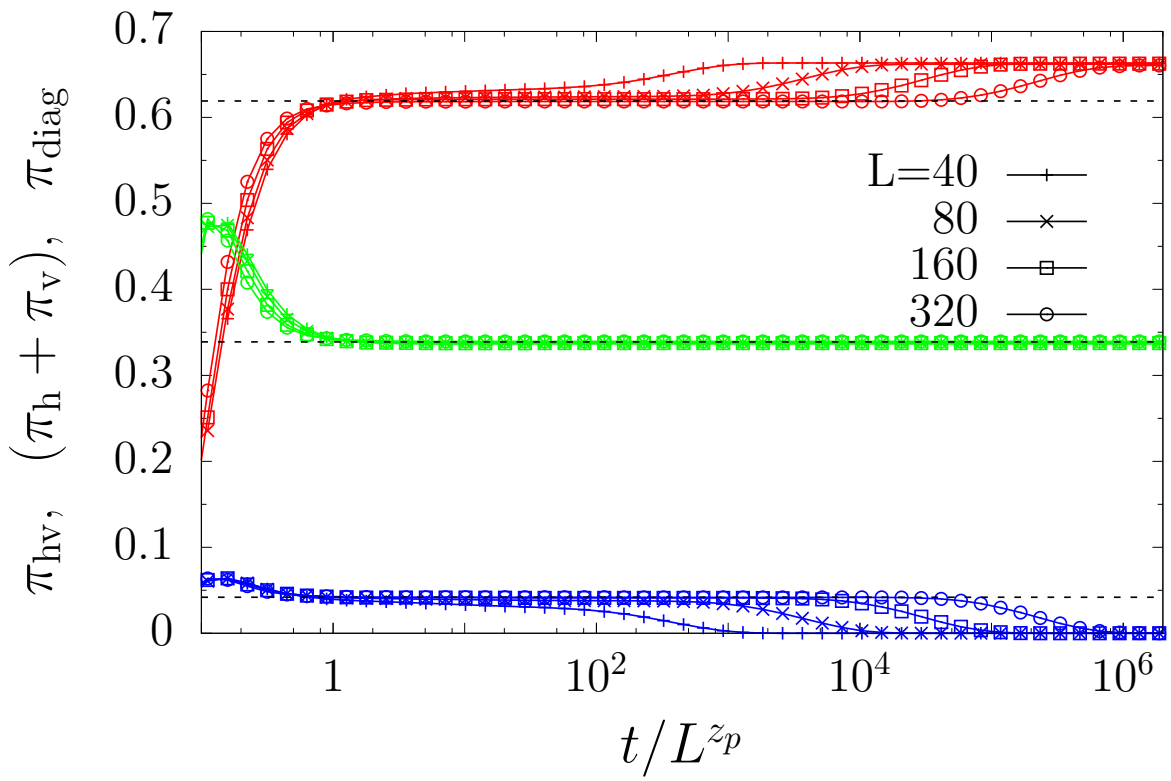


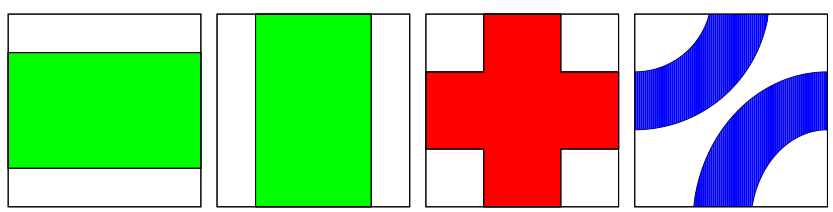
At $t = 16$ MCs the bump converged to a stationary form that satisfies the finite size scaling of critical site percolation with $\tau_A = 2.05$ and $D_A = 1.9$

Insert: failure of collapse if critical Ising exponents are used.

Is it critical percolation ?

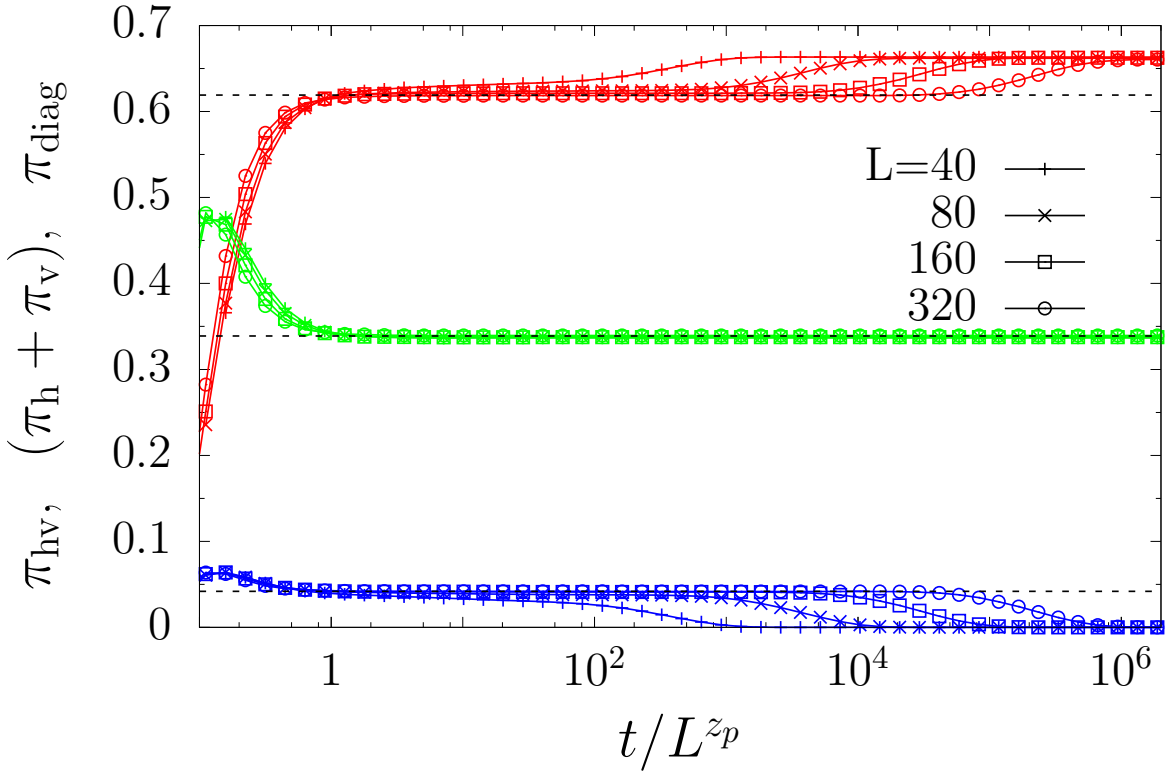
The probabilities of percolation in different directions





Is it critical percolation ?

Probabilities as in critical percolation



$\pi_{hv} \approx 0.62$

$\pi_h + \pi_v \approx 0.34$

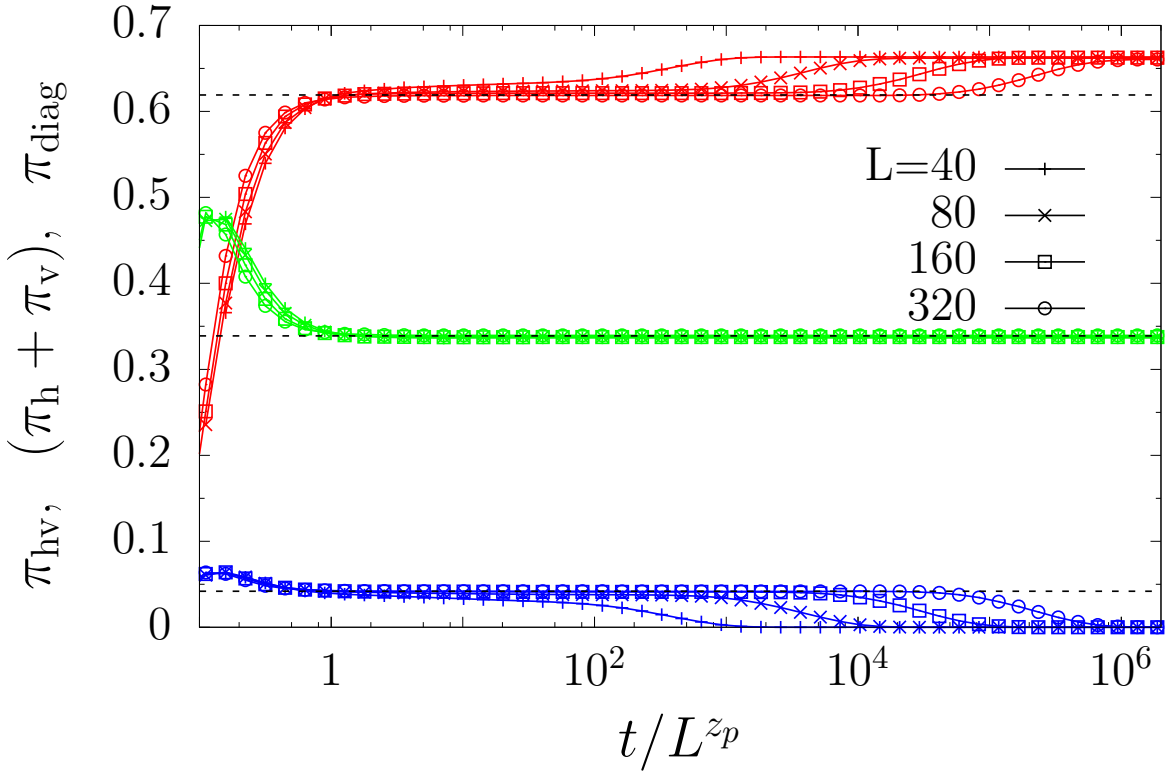
$\pi_d \approx 0.03$

Scaling with L^{z_p} and $z_p \approx 1/2$ in the approach to the constant values

Blanchard, Corberi, LFC, Picco & Tartaglia 14, 15

Is it critical percolation ?

The final configuration was decided at $t_p \simeq L^{z_p}$



$\pi_{hv} \approx 0.62$

$\pi_h + \pi_v \approx 0.34$

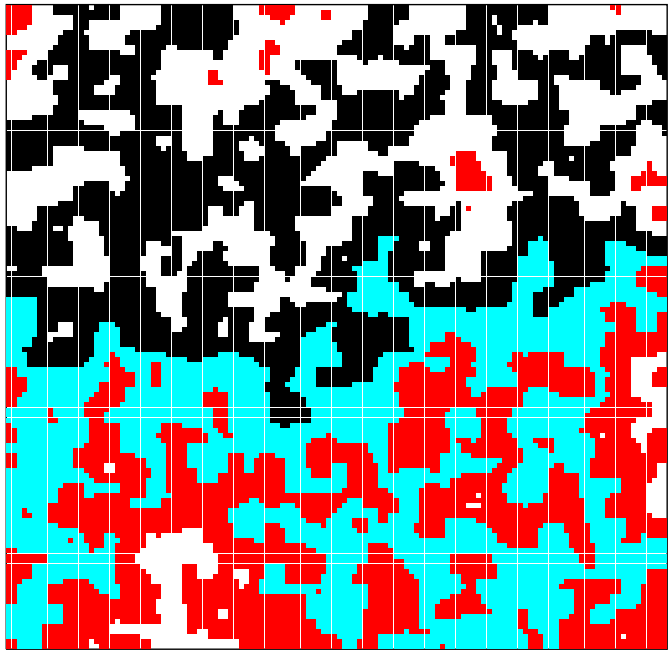
$\pi_d \approx 0.03$

Frozen stripe states with probability $\pi_h + \pi_v$

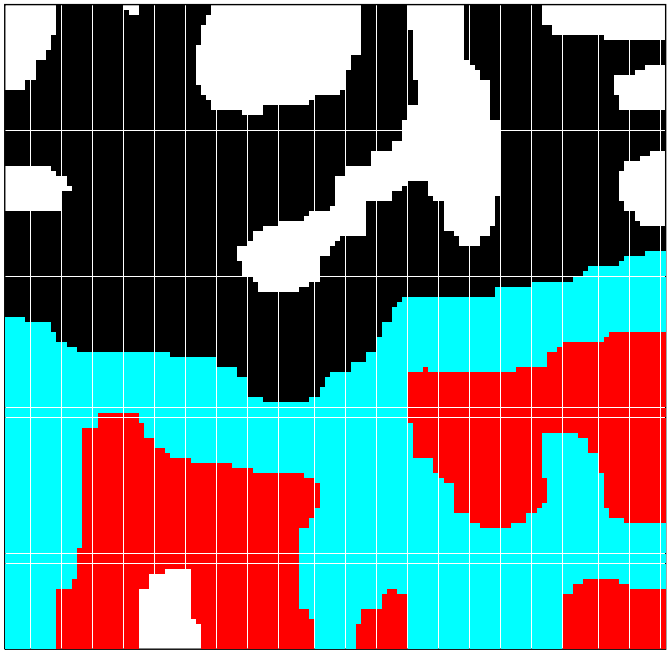
Barros, Krapivsky & Redner 09, Blanchard & Picco 13

Is it critical percolation ?

The final configuration was decided at t_p



t=7.46144



t=128.0

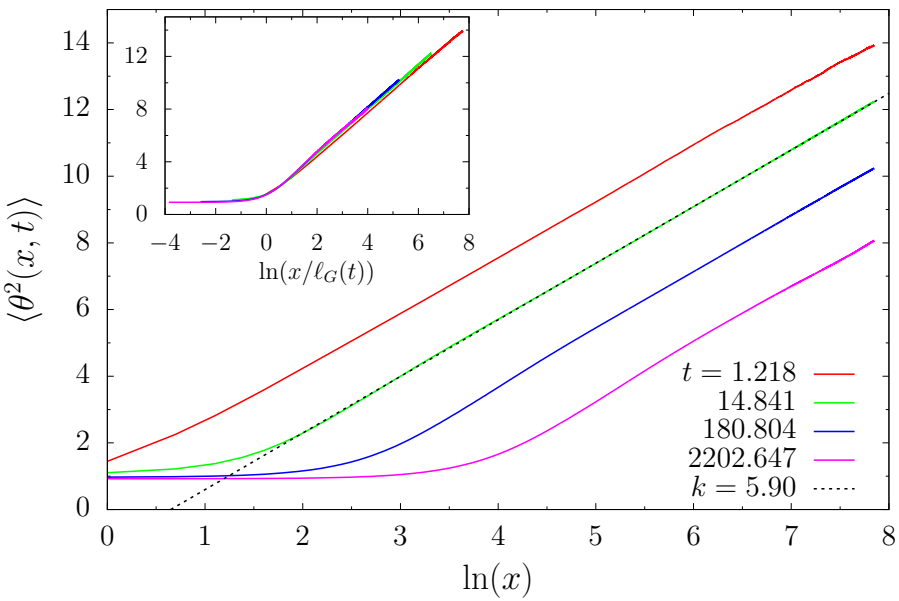
stripe states with the probabilities of critical percolation:

a spanning cluster along the two Cartesian directions $\equiv$ order $\pi_{hv} \approx 0.66$

spanning cluster along along only one of them $\equiv$ domain walls $1 - \pi_{hv} \approx 0.34$

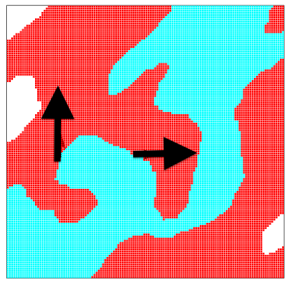
Is it critical percolation ?

The winding angle



$$\theta^2 = \mathrm{ct} + \frac{4\kappa}{8+\kappa}\ln x$$

x curvilinear coordinate

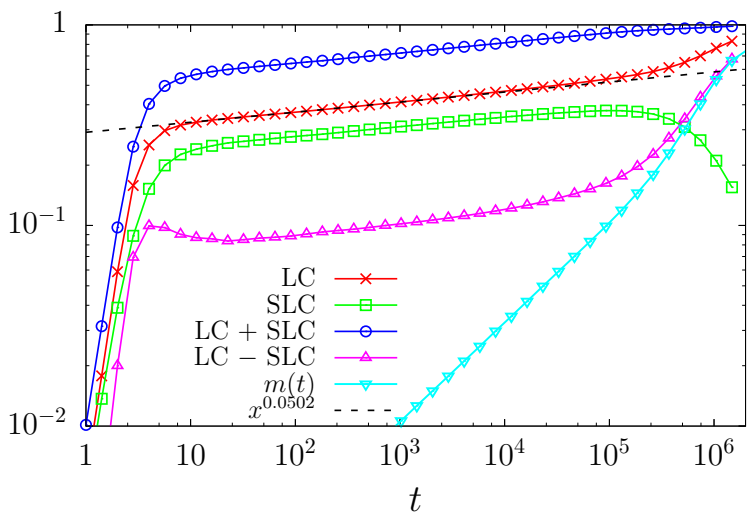


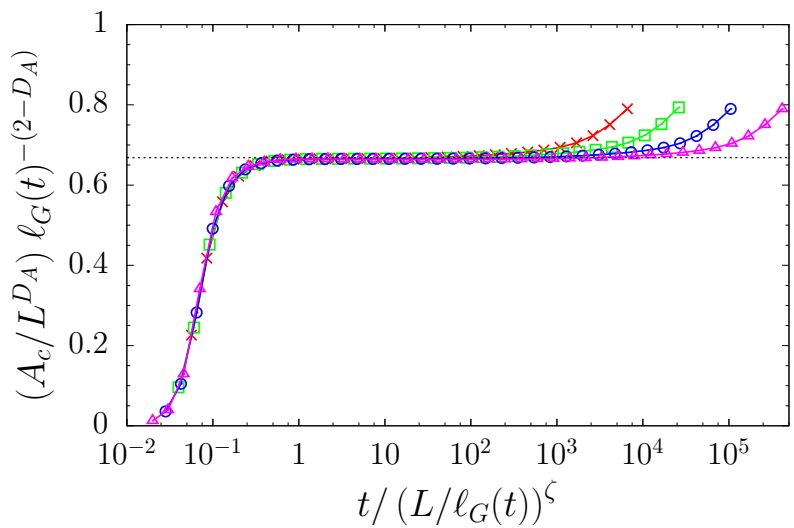
$\kappa \simeq 5.9$

In the inset: scaling with $\ell_G(t) \to l(t)$ estimated from the excess energy

The largest clusters

Algebraic growth after t_p in MC simulations





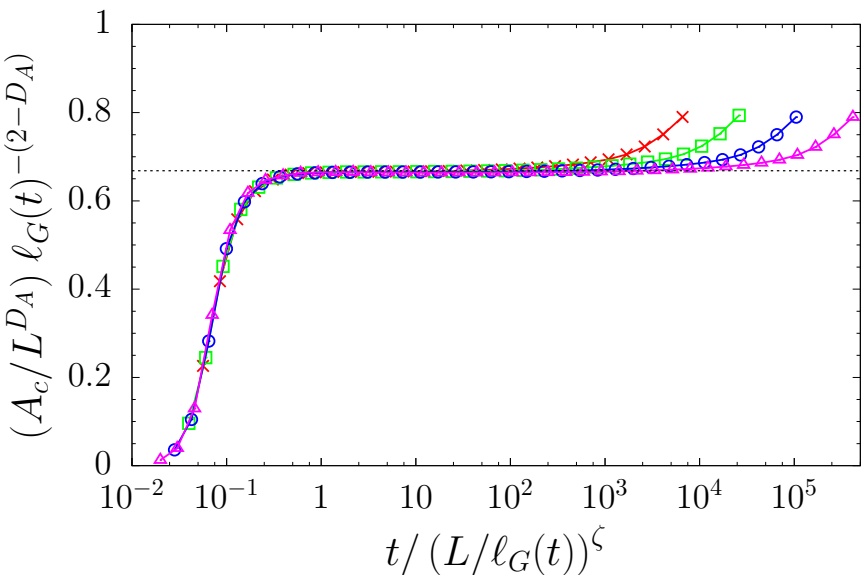
In critical percolation: $\dfrac{A_{\max}}{r_0^2} \simeq \left(\dfrac{L}{r_0}\right)^{D_A}$ with r_0 the lattice spacing.

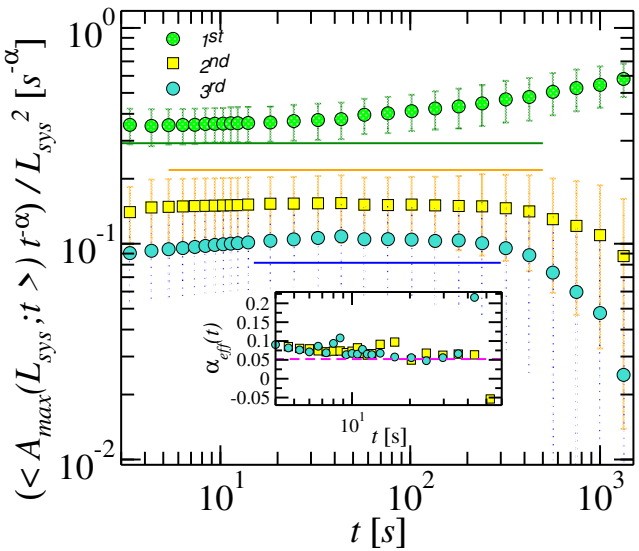
In coarsening : $r_0 \mapsto l(t)$ the growing length.

After substitution $\dfrac{A_{\max}}{L^2} \propto t^{\alpha}$ with $\alpha \equiv \dfrac{2-D_A}{z_d} = 0.0521$

The largest clusters

Algebraic growth after t_p in MC & experiments





In critical percolation: $\dfrac{A_{\max}}{r_0^2} \simeq \left(\dfrac{L}{r_0}\right)^{D_A}$ with r_0 the lattice spacing.

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After substitution $\dfrac{A_{\max}}{L^2} \propto t^{\alpha}$ with $\alpha \equiv \dfrac{2-D_A}{z_d} = 0.0521$

When ?

Let us call $t_p(L)$

the time needed to reach

the critical percolation state

Determination of $t_p(L)$

Cloning trick and measurement of the overlap

Quench a system from $T_0 \to \infty$ to $T = 0$ at $t = 0$.

Let it evolve at $T = 0$ until t_w.

Make a copy of the instantaneous configuration, $\sigma_i(t_w) = s_i(t_w)$.

Let the two clones evolve with different thermal noises.

Compute the time-dependent overlap

$$\boxed{q_{t_w}(t, L) = \frac{1}{L^d} \sum_{i=1}^{L^d} \langle s_i(t) \sigma_i(t) \rangle}$$

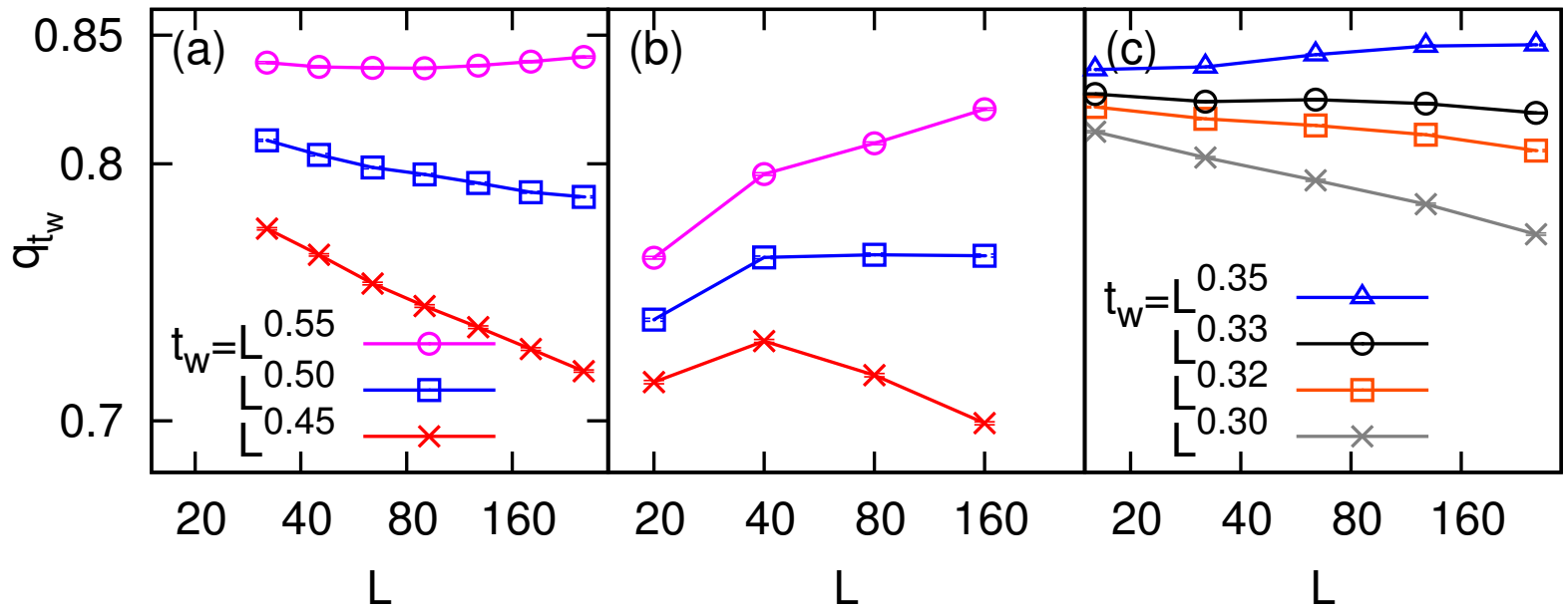
If $t_w < t_p(L)$ $\qquad \lim_{t \gg t_w} q_{t_w}(t, L) = 0$

If $t_w > t_p(L)$ $\qquad \lim_{t \gg t_w} q_{t_w}(t, L) > 0$

Determination of $t_p(L)$

The overlap

$\lim_{t\to\infty} q_{t_w(L)}(t, L)$ should reach a constant independent of L



Square FBC Kagome FBC Triangular PBC

$t_p(L) \simeq L^{0.5}$ $t_p(L) \simeq L^{0.33}$

Conclusions

Approach to critical percolation

The zero-temperature non-conserved (also the conserved) order parameter dynamics of the $2d$ Ising model, (both) starting from a totally uncorrelated $T_0 \to \infty$ paramagnetic initial state, approach uncorrelated critical percolation after a time

$t_p \simeq L^{z_p}$.

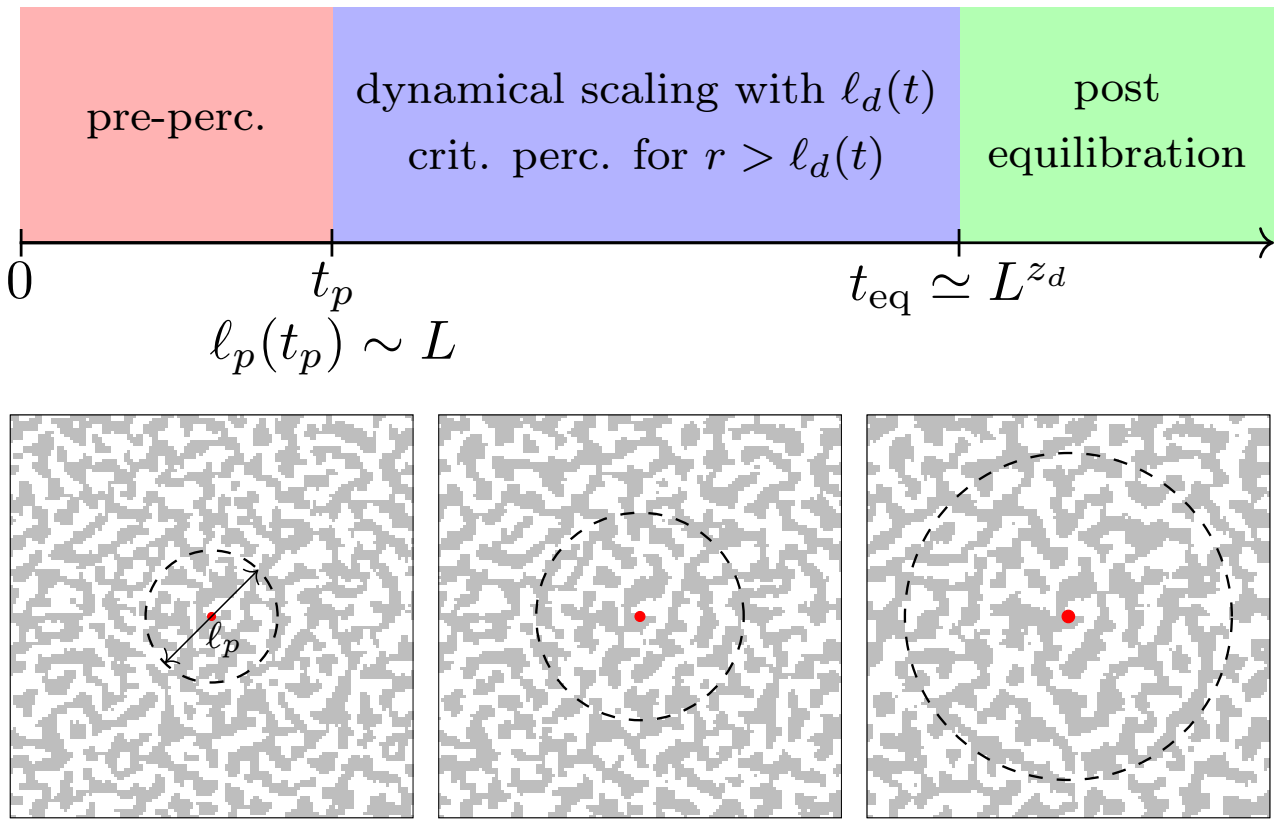
The exponent z_p depends upon the *effective connectivity* of the lattice and the *microscopic dynamics*.

For instance, $z_p \approx 0.5$ for the square lattice $2d$ Ising model with non-conserved order parameter dynamics.

Blanchard, Corberi, LFC & Picco 14 ; Tartaglia, LFC & Picco 15

Conclusions

Time scales & length scales



$$\ell_p(t) \quad \gg \quad l(t)$$

Conclusions

Approach to critical percolation : why is this feature interesting ?

A mechanism that went unnoticed in this context so-far.

Seems to be universal (NCOP, LCOP, NLCOP, Voter)

In RG language it suggests the first approach to a fixed point that is not fully attractive (critical percolation) and the subsequent departure from it.

Analytical challenge : how can one prove this claim ?

Manifold consequences :

metastability, blocked striped states at zero temperature ;

corrections to dynamic scaling.

Conclusions

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Analytical challenge: perhaps in the voter model.

Similar master equation to the one of the $1d$ Glauber chain

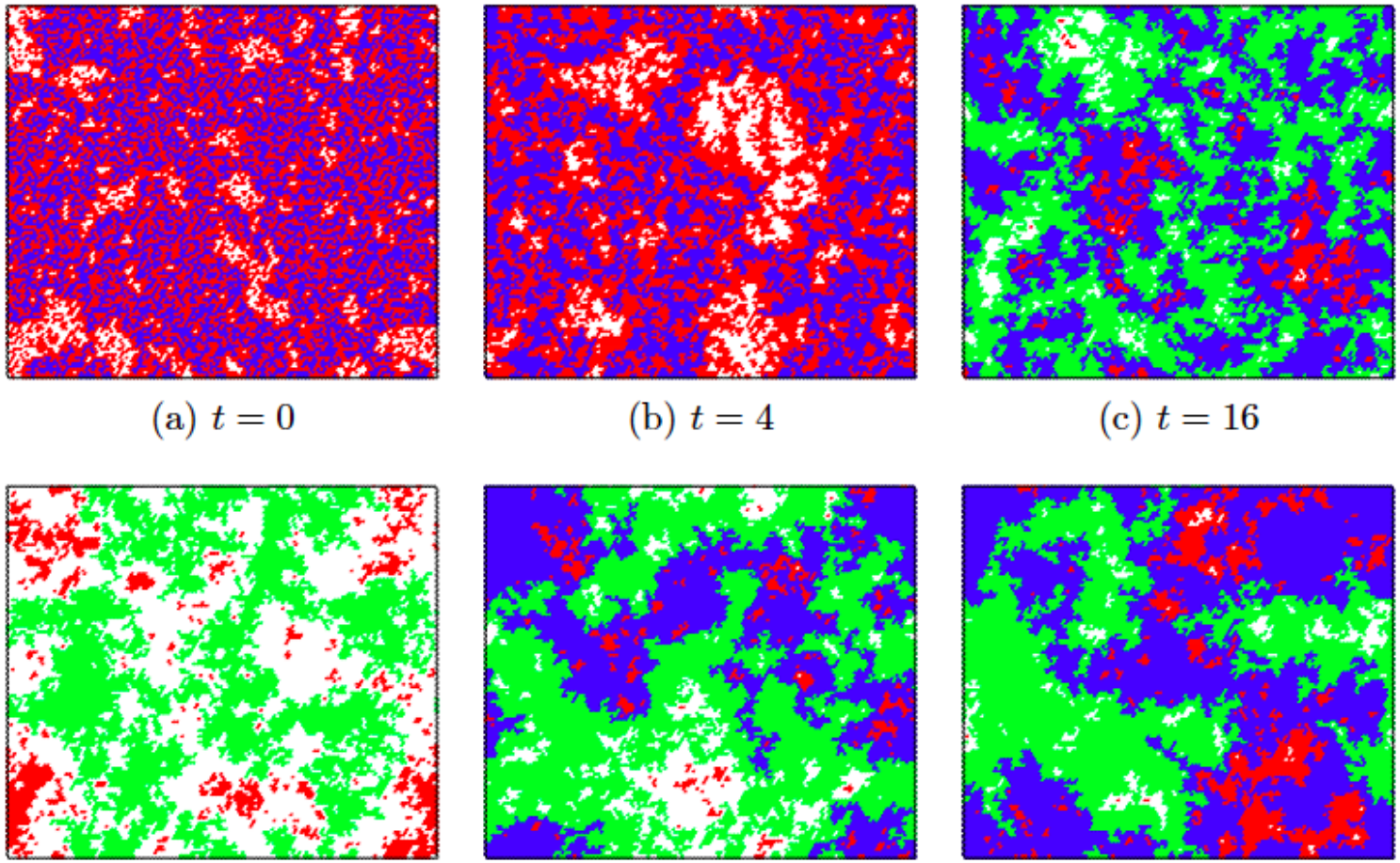
Krapivsky et al. 90s

Mapping to random walks **Cox & Griffeaths 80s**

But... finite L effects searched

Conclusions

Approach to critical percolation : why is this feature interesting ?

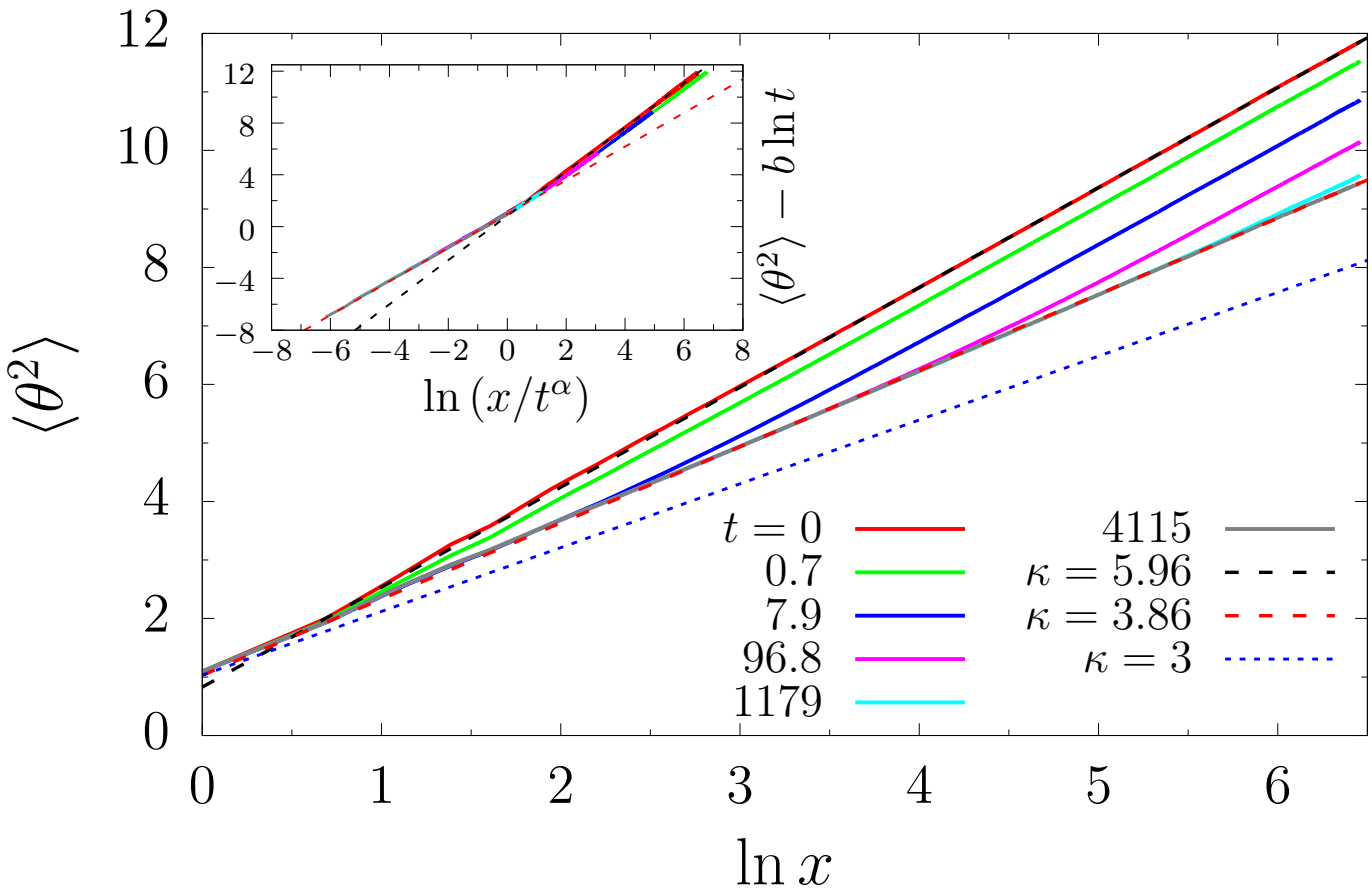


(a) $t = 0$ (b) $t = 4$ (c) $t = 16$

(d) $t = 64$ (e) $t = 256$ (f) $t = 1024$

Conclusions

Approach to critical percolation : why is this feature interesting ?



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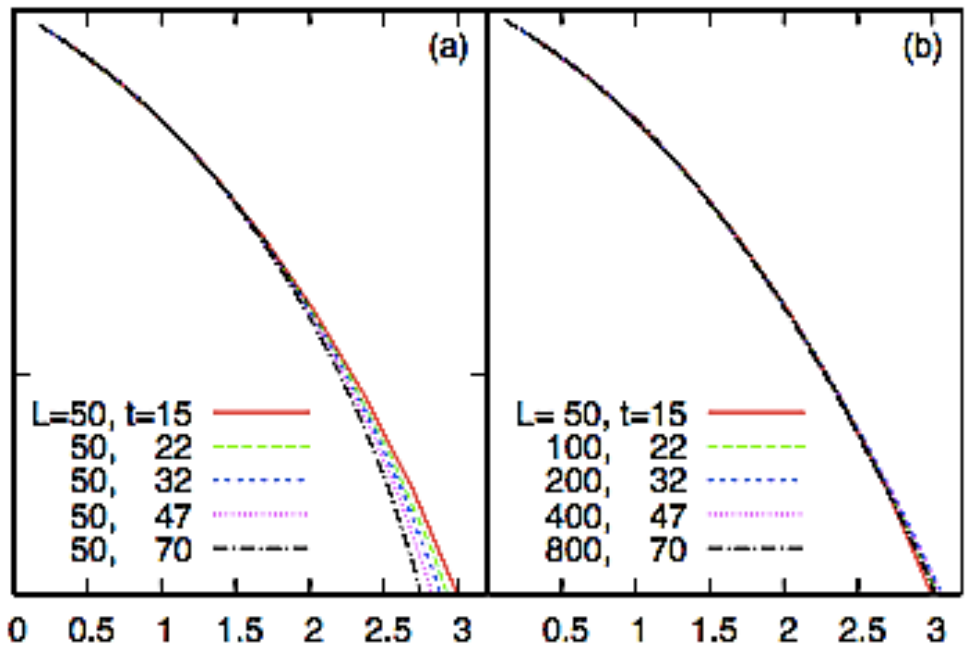
Correction to scaling

Linear-Log scale, zoom over $C \lesssim 0.1$

$C(r,t,L)$



$\frac{r}{l(t)}$

$$f\left(\frac{r}{l(t)}\right) \qquad \boxed{g\left(\frac{r}{l(t)}, \frac{\ell_p(t)}{L}\right)}$$

with $\ell_p(t) \equiv t^{1/z_p}$ and $l(t) \equiv t^{1/z_d}$

$z_p \simeq 1/2$ for the square & Kagome, and $z_p \simeq 1/3$ for the triangular lattice.

Conclusions

Early approach to percolation

$\ell_p(t) \simeq t^{1/z_p}$ is a new growing length-scale that brings about a new scaling variable to be taken into account in dynamic scaling.

Studies of the $2d$

Ising model with non-conserved order parameter dynamics

Voter model

Ising model with conserved (Kawasaki) order parameter dynamics

$\ell_p(t) \simeq t^{1/z_p}$ or $\ell_p(t) \simeq \xi_d^n(t)$

Blanchard, Corberi, LFC & Picco 14 ; Tartaglia, LFC & Picco 15

Summary

- Evidence for the approach to critical percolation at a time-scale that diverges with the system size as $t_p \simeq L^{z_p}$.
- The new growing length-scale, $\ell_p(t) \simeq t^{1/z_p}$ dominates at short times and is needed to improve the scaling of finite-size and finite-time data.

  This effect also exists at finite temperature. Metastability acquires a finite life-time.
- We derived the number density of hull & domain enclosed areas and interface length for $t > t_p$ and we showed that they satisfy dynamic scaling with respect to $l(t) \simeq t^{1/z_d}$ with $z_d = 2$ for times $t \gg t_p$.