ICFP Master

Quantum Field Theory I

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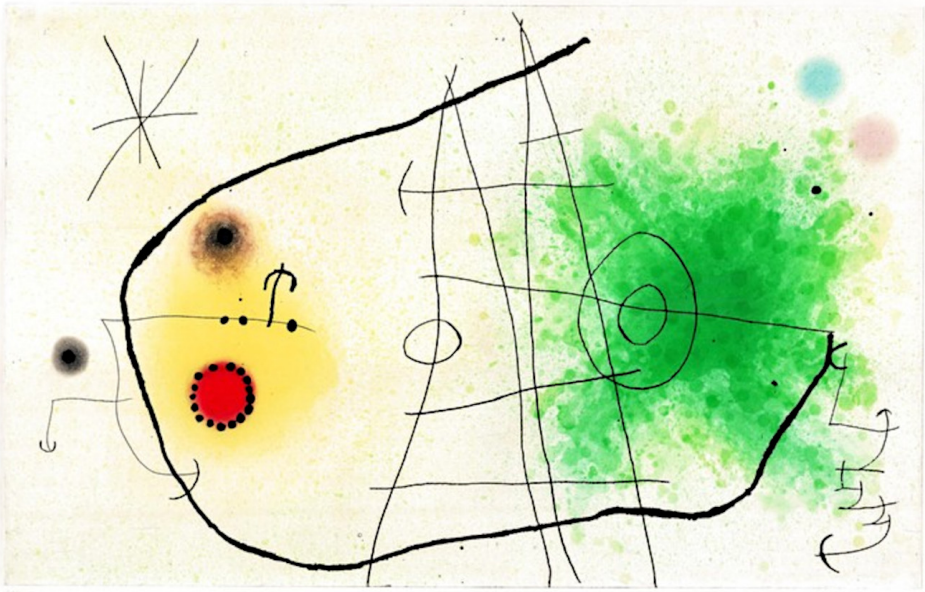

Joan Miró, *Partie de campagne*, 1967

Foreword

These lecture notes are based upon a series of courses given at the master program ICFP from 2024 by the author. Comments and suggestions are welcome.

Some standard references that can complement these notes, among many other published textbooks and online lecture notes, are:

- Peskin and Schroeder [1]
- Weinberg [2]
- Srednicki [3]
- Tom Banks [4]
- Itzykson and Zuber [5]
- David Tong's lecture notes: https://www.damtp.cam.ac.uk/user/tong/qft.html
- Timo Weigand's lecture notes: http://www.physics.umd.edu/grt/taj/624b/WeigandQFT.pdf

Conventions

- The space-time metric is chosen to be of signature $(+,-,\ldots,-)$.
- We work in natural units $\hbar = \mathsf{c} = \mathsf{G} = \mathsf{1}$

Acknowledgements

I would like to thank Daniel Petrov for a careful rereading of the notes. Needless to say that the remaining mistakes and typos are all my own responsability!

Latest update

March 23, 2026

References

[1] M. E. Peskin and D. V. Schroeder, *An Introduction to quantum field theory*. Addison-Wesley, Reading, USA, 1995.

[2] S. Weinberg, *The Quantum theory of fields. Vol. 1: Foundations*. Cambridge University Press, 6, 2005.

[3] M. Srednicki, *Quantum field theory*. Cambridge University Press, 1, 2007.

[4] T. Banks, *Modern Quantum Field Theory: A Concise Introduction*. Cambridge University Press, 2008.

[5] C. Itzykson and J. B. Zuber, *Quantum Field Theory*. International Series In Pure and Applied Physics. McGraw-Hill, New York, 1980.

Contents

Introduction

Quantum field theory is the common denominator of nearly all theoretical physics done today; one can even say that it justifies the very existence of theoretical physics departements. The common QFT langage has found applications in many seemingly different areas of physics, among which:

- Elementary particle physics: Standard Model and Beyond.
- Nuclear physics, in particular through effective field theory (EFT) methods.
- Primordial cosmology: inflation and cosmic microwave background.
- Dark matter models.
- String theory, both as 2d theories on the world-sheet of the strings, and as low-energy effective field theories.
- Quantum gravity, using the AdS/CFT correspondence.
- Black hole mergers in classical GR, using EFT methods.
- Critical phenomena in condensed matter.
- Exotic phases of matter (topological, Dirac,...).
- Mathematical physics (too many topics to be listed!).

This course will be focused on the framework in which QFT methods were first developed during the Twentieth Century, in order to obtain a consistent description of relativistic matter and interactions at the quantum level.

Special relativity and quantum mechanics. A consistent relativistic quantum theory is build on several assumptions, among which:

- The states of the system are described as positive-norm vectors in some Hilbert space.
- According to Wigner's theorem, Poincaré symmetries should be realized as unitary/anti-unitary operators acting on the Hilbert space, in order to preserve the probabilistic interpretation of quantum mechanics.

- In particular, time evolution is realized through the action of a unitary operator.
- Interactions between particles are realized *locally*: there is no notion of instantaneous action at distance.
- The number of particles in a relativistic theory cannot be conserved.
- The theory should be consistent with a relativistic notion of causality.

Somewhat suprisingly, these different requirements are deeply connected to each other. For instance, there seems to be a clash between the absence of finite-dimensional unitary representations of the Lorentz group and Wigner's theorem, which is solved by considering particle states, which transform into *infinite dimensional* unitary representations of the Poincaré group.

The absence of particle number conservation, which can be viewed as an immediate consequence of the relativistic dispersion relation $E = c\sqrt{\vec{p}^2 + m^2c^2}$, is deeply related to causality. Let us elaborate a bit more on this.

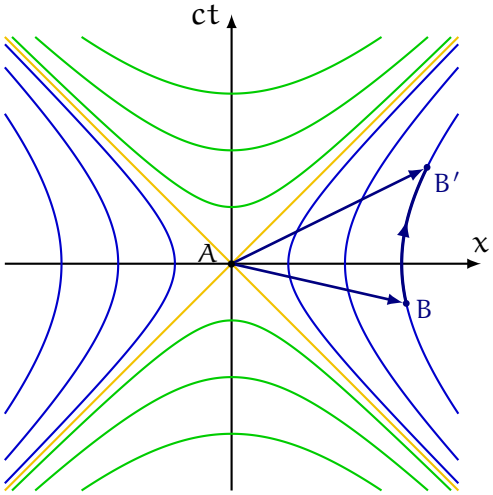

Figure 1: *Light-like separated events.*

In a relativistic theory, space-like separated events, *i.e.* such that $c^2(\Delta t)^2 - (\Delta\vec{x})^2 < 0$, cannot be in causal relation to each other, because time-ordering is only a Lorentz invariant concept for time-like or light-like separated events. On figure 1, event B occurs before event A; however after the action of a boost $B \mapsto B'$ it is just the opposite.

In order to diagnose whether this relativistic notion of causality can be encoded into a relativistic theory of quantum mechanics, let us consider the propagator between the point $(t = 0, \vec{0})$ and the point $(t, \vec{x})$, *i.e.* the probability amplitude that a particle prepared at $\vec{x} = \vec{0}$ and $t = 0$ is detected later at time $t > 0$ at position $\vec{x}$. This probabiblity amplitude, known as the propagator, is given by:

$$U(t, \vec{x}) = \left\langle \vec{x} \middle| \mathcal{U}(t) \middle| \vec{0} \right\rangle , \tag{1}$$

where $\mathcal{U}(t)$ is the time evolution operator, which is expected to be unitary. For a relativistic free particle of mass m, the evolution operator expressed in terms of the momentum operator $\vec{\mathcal{P}}$ should be given by:

$$\mathcal{U}(t) = \exp\left(-\frac{ict}{\hbar}\sqrt{\vec{\mathcal{P}}^2 + m^2c^2}\right) . \tag{2}$$

We can then express the propagator (1) as:

$$U(t,\vec{x}) = \int d^3p \int d^3p' \underbrace{\left\langle \vec{x}\middle|\vec{p}\right\rangle}_{(2\pi\hbar)^{-3/2}e^{i\vec{p}\cdot\vec{x}}} \left\langle \vec{p}\middle|\mathcal{U}(t)\middle|\vec{p}'\right\rangle \left\langle \vec{p}'\middle|\vec{0}\right\rangle = \int \frac{d^3p}{(2\pi\hbar)^3} e^{-ip_\mu x^\mu/\hbar}\Big|_{p^0=\sqrt{\vec{p}^2+m^2c^2}} \tag{3}$$

To simplify the expression, we choose spherical coordinates, such that $\vec{p}\cdot\vec{x} = \|\vec{p}\|\|\vec{x}\|\cos\theta$ and perform the angular integrals:

$$U(t,\vec{x}) = \frac{1}{(2\pi\hbar)^2}\frac{1}{i\|\vec{x}\|}\int_{-\infty}^{\infty} k dk\, e^{-\frac{i}{\hbar}\left(x^0p^0 - \|\vec{x}\|k\right)}\Big|_{p^0=\sqrt{k^2+m^2c^2}} . \tag{4}$$

We rewrite the integral in a suggestive form:

$$U(t,\vec{x}) = \frac{1}{(2\pi\hbar)^2}\frac{1}{i\|\vec{x}\|}\int_{-\infty}^{\infty} k dk\, e^{-i\frac{\|\vec{x}\|}{x^0}\left(\frac{(x^0)^2\sqrt{k^2+m^2c^2}}{\hbar\|\vec{x}\|} - \frac{kx^0}{\hbar}\right)} , \tag{5}$$

in order to give an estimate in the "deep" space-like limit $\|\vec{x}\|/x^0 \to \infty$, using the steepest descent method. Let define first

$$g(k) = \frac{(x^0)^2\sqrt{k^2+m^2c^2}}{\hbar\|\vec{x}\|} - \frac{kx^0}{\hbar} .$$

The stationnary point of g in the k-complex plane occurs for

$$g'(k_s) = 0 \implies \frac{x^0 k_s}{\|\vec{x}\|\sqrt{k_s^2+m^2c^2}} = 1 \implies k_s = \frac{imc\|\vec{x}\|}{\sqrt{\|\vec{x}\|^2 - (x^0)^2}} .$$

The leading behavior of the integral, which is what interests us, is then

$$U(t,\vec{x}) \sim e^{-i\|\vec{x}\|\, g(k_s)/x^0} \sim e^{-\frac{ik_s}{\hbar}\frac{(x^0)^2 - \|\vec{x}\|^2}{\|\vec{x}\|}} \sim e^{-\frac{\sqrt{\|\vec{x}\|^2 - (ct)^2}}{\lambda_c}} , \tag{6}$$

where $\lambda_c = \hbar/mc$ the reduced Compton wavelength of the particle, the scale below which special relativity effects become significant. Thus the wave-function of a particle prepared at $\vec{x} = \vec{0}$ spreads outside of the light-cone with characteristic scale λ_c.

This could be a real problem if we prepare the system in a highly localized state. Fortunately, the uncertainty principle comes to the rescue. If we want to confine the wave-function in a region of size $\Delta x = \lambda_c/2$ or smaller, we would have

$$\Delta p \geqslant \frac{\hbar}{2\Delta x} \geqslant mc , \tag{7}$$

leading to particle production as $\Delta E > mc^2$. The more we try to localize the wavefunction, the more particles get produced. As we will see later, this is precisely how quantum field theory ensures that causality holds, beyond this handwaving argument: production of particle/anti-particle pairs leads to destructive interference ensuring that space-like separated observables are not causally connected. This could not happen with a fixed number of particles.

Approaches to QFT. A quantum field theory is usually defined from a classical field theory with an action principle, *i.e.* a local functional in a set of local fields $\Phi^I(x^\mu)$, and their space-time derivatives. From here, two approaches are possible:

1. *Canonical quantization*: fields are promoted to operator-valued functions (more accurately distributions), and Poisson brackets are promoted to commutation or anti-commutation relations.

2. *Path integral*: amplitudes are defined formally as functional integrals over all field configurations in space-time, with a functional measure $\mathcal{D}\Phi \exp\big(-iS(\Phi)/\hbar\big)$.

In these notes, we will use mostly the first approach, which is simpler and usually more convenient to perform computations. The functional methods associated with the path integral formalism will be used mostly to discuss general features of QFTs, for instance the Ward–Takahashi identities.

It is a fortunate fact of life, but an unfortunate fact for theoretical physics students, that interesting quantum field theories are interacting.

1. In certain cases exact computations can be made in interacting QFTs, involving typically enlarged space-time symmetries (conformal symmetry, supersymmetry), lower dimensionality or a combination of both. This is a fascinating area of research on its own leading to deeper insights about what a quantum field theory is.

2. In all other cases, including the Standard Model of Particle physics, it is impossible to solve analytically the theory, thus approximation methods need to be used. As we shall see, this has far more reaching consequences than perturbation theory in quantum mechanics.

3. An alternative approach is to use numerical methods to perform the full non-perturbative computation of the path integral. Even space-time grids with a modest number of nodes require enormous computational power, and certain aspects are difficult to handle (like fermions), but this approach becomes more and more useful as computers become more and more powerful.

The goal of this course is to provide a flavor of what a quantum field theory is, particularly in the second, perturbative approach. The main working example that we will use is quantum electrodynamics (a.k.a. QED), the theory of electrons interacting with photons, which was historically the very reason to develop the QFT framework.

Part 1

Free spin zero and one-half fields

In this first part, quantization of free fields of spin zero and one-half fields is presented; spin one and gauge invariance will be covered in chapter 3. In most of these lectures, the space-time will be considered as four-dimensional, with signature $(+,-,-,-)$. In other words, The space-time metric and line element will be taken to be

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad , \quad ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = dt^2 - d\vec{x}^2 \, , \tag{1.1}$$

with $\vec{x} = (x, y, z)$. We will use the indices i, j, k or a, b, c for space directions. Note that with this metric, we have $x^i y_i = -\vec{x} \cdot \vec{y}$ or equivalently $x \cdot y = x^0 y^0 - \vec{x} \cdot \vec{y}$. This signature is known as the *West Coast signature*, and makes the transition from quantum mechanics smoother. The *East Coast signature* $(-,+,+,+)$ is preferred in the context of general relativity, as the space-like coordinates have the "correct" signature. As a consequence it makes easier the Wick rotation to Euclidian space.

To streamline calculations we use everywhere *Planck units*, chosen such that $c = 1$ and $\hbar = 1$ and, whenever needed, $k_B = 1$ and $G = 1$. As the end of the computations dimensionally correct expressions can be recovered by dimensional analysis, as you will see during the tutorials.

1.1 Particles and the Poincaré group

Elementary particles are expected to transform in *irreducible representations* of the symmetry group of space-time, the *Poincaré group*, *i.e.* endowed with a linear action of the Poincaré group leaving no vector sub-space invariant; otherwise, these particles could be considered as made of more fondamental constituents, each carrying its own representation. Composite particles, like a proton, a nucleus or an atom, can be also considered as transforming into an irreducible representation, provided that the experiments under consideration do not involve energy scales high enough to probe their inner structure.

1.1.1 The Poincaré group

The *Poincaré group* $\mathsf{P}(1,3)$ is the isometry group of Minkowski space-time. Each element $\mathcal{G}$ can be uniquely written as an affine transformation parameterized by a 4×4 real matrix Λ and a 4-vector a:

$$x^\mu \xmapsto{\mathcal{G}(\Lambda,a)} \tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \quad , \qquad \eta_{\mu\nu}\,\Lambda^\mu{}_\rho\,\Lambda^\nu{}_\sigma = \eta_{\rho\sigma}\,. \tag{1.2}$$

The set of linear transformations $\mathcal{G}(\Lambda,0)$ is naturally a subgroup of the Poincaré group, the Lorentz group, which consists in $\mathsf{GL}(4,\mathbb{R})$ elements leaving the Minkowski metric invariant:

$$\Lambda^{\mathsf{T}}\cdot\eta\cdot\Lambda = \eta\,. \tag{1.3}$$

The Lorentz group is often denoted by $\mathsf{O}(1,3)$, in analogy with the $\mathsf{O}(d)$ orthogonal groups of Euclidean spaces. The composition law of the Poincaré group is as follows:

$$\mathcal{G}\big(\Lambda,a\big)\cdot\mathcal{G}\big(\tilde{\Lambda},\tilde{a}\big) = \mathcal{G}\big(\Lambda\tilde{\Lambda},\Lambda\tilde{a}+a\big)\,. \tag{1.4}$$

Let $\mathcal{G}\big(e,a\big)$, where e is the identity, be a translation by a vector a. For any element $\mathcal{G}(\tilde{\Lambda},\tilde{a})$ of the Poincaré group, we have

$$\mathcal{G}(\tilde{\Lambda},\tilde{a})\mathcal{G}\big(e,a\big)\mathcal{G}(\tilde{\Lambda},\tilde{a})^{-1} = \mathcal{G}(\tilde{\Lambda},\tilde{a})\mathcal{G}\big(e,a\big)\mathcal{G}(\tilde{\Lambda}^{-1},-\tilde{\Lambda}^{-1}\tilde{a}) = \mathcal{G}(e,\tilde{\Lambda}a)\,, \tag{1.5}$$

which is itself a translation. Hence the translation group $\{\mathcal{G}\big(e,a\big),a\in\mathbb{R}^{1,3}\}\simeq\mathbb{R}^{1,3}$ is stable by conjugation by any element of the Poincaré group, which is the definition of a normal subgroup. Therefore $\mathsf{P}(1,3)$ has the structure of a semi-direct product:

$$\mathsf{P}(1,3)\simeq\mathsf{O}(1,3)\ltimes\mathbb{R}^{1,3}\,. \tag{1.6}$$

Lorentz group. Equation (1.3) implies that

$$\big(\det\Lambda\big)^2 = 1 \implies \det\Lambda = \pm 1\,. \tag{1.7}$$

Proper Lorentz transformations have $\det\Lambda = 1$ and form a subgroup of $\mathsf{O}(1,3)$, denoted by $\mathsf{SO}(1,3)$, consisting of transformations preserving orientation; those with $\det\Lambda=-1$ are called *improper*. Given that the $(0,0)$ component of eqn. (1.2) reads:

$$1 = \big(\Lambda^0{}_0\big)^2 - \sum_{i=1}^{3}\big(\Lambda^i{}_0\big)^2\,, \tag{1.8}$$

a given Lorentz transformation has either $\Lambda^0{}_0\geqslant 1$ or $\Lambda^0{}_0\leqslant -1$. In the former case the transformation is called *orthochronous*, as it preserves the direction of time. Orthochronous Lorentz transformations form a subgroup of $\mathsf{O}(1,3)$, denoted by $\mathsf{O}(1,3)^+$.[1]

[1] Consider two orthochronous Lorentz transformations Λ and $\tilde{\Lambda}$. One has $(\Lambda\tilde{\Lambda})^0{}_0 = \Lambda^0{}_0\tilde{\Lambda}^0{}_0 + \sum_{i=1}^3\Lambda^0{}_i\tilde{\Lambda}^i{}_0$. By eqn. (1.8) the 3-vector of components $\tilde{\Lambda}^i{}_0$ has norm $\sqrt{(\tilde{\Lambda}^0{}_0)^2-1}$, while, from the relation $\eta^{\rho\sigma}\,\Lambda^\mu{}_\rho\,\Lambda^\nu{}_\sigma = \eta^{\mu\nu}$, the 3-vector of components $\tilde{\Lambda}^0{}_i$ has norm $\sqrt{(\Lambda^0{}_0)^2-1}$. It follows that $\big|\sum_{i=1}^3\Lambda^0{}_i\tilde{\Lambda}^i{}_0\big|\leqslant\tilde{\Lambda}^0{}_0\Lambda^0{}_0$ hence $(\Lambda\tilde{\Lambda})^0{}_0\geqslant 0$ hence, given that $\Lambda\tilde{\Lambda}$ is itself a Lorentz transformation, $(\Lambda\tilde{\Lambda})^0{}_0\geqslant 1$.

From this analysis it follows that the Lorentz group $O(1,3)$ has four connected components; among those one is a subgroup, the *restricted Lorentz group* $SO(1,3)^+$, *i.e.* the group of proper and orthochronous Lorentz transformations.

All elements of the Lorentz group are obtained by combining $SO(1,3)^+$ transformations with the following involutions:

- Space parity $\vec{x} \mapsto -\vec{x}$, which is associated with the $O(1,3)$ matrix

$$\mathcal{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{1.9}$$

  Any improper Lorentz transformation can be written as $\mathcal{P}\Lambda$, where $\Lambda \in SO(1,3)$.

- Time reversal $t \mapsto -t$, which is associated with the $O(1,3)$ matrix

$$\mathcal{T} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{1.10}$$

  Any non-orthochronous transformation can be written as $\mathcal{T}\Lambda$, where $\Lambda \in O(1,3)^+$.

For these reasons, we will focus below on the properties of $SO(1,3)^+$.

Lie algebra. Let us now construct the Lie algebra of $SO(1,3)^+$, *i.e.* the algebra of its infinitesimal generators. For a transformation close to the identity (which belongs to the restricted Lorentz group), one can write

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \alpha^{\mu}{}_{\nu} , \tag{1.11}$$

with

$$\eta^{\rho\sigma}\,\Lambda^{\mu}{}_{\rho}\,\Lambda^{\nu}{}_{\sigma} = \eta^{\mu\nu} \quad \Longrightarrow \quad \alpha^{\mu\nu} + \alpha^{\nu\mu} = 0 + \mathcal{O}(\alpha^2) . \tag{1.12}$$

Therefore $\alpha^{\mu\nu}$ (with either two indices up or down) is a 4×4 antisymmetric real matrix. These matrices define a 6-dimensional real vector space.

A convenient basis is provided by the set of matrices $\Sigma^{\rho\sigma}$, which are labelled by an antisymmetric pair of indices, and whose components, dictated by antisymmetry both in (μ,ν) and in (ρ,σ), are given by:[2]

$$(\Sigma^{\rho\sigma})^{\mu\nu} = i\,(\eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\sigma\mu}\eta^{\rho\nu}) , \tag{1.13}$$

so that a generic infinitesimal transformation can be decomposed as:

$$\alpha_{\mu\nu} = -\tfrac{i}{2}\,\Omega_{\rho\sigma}\,(\Sigma^{\rho\sigma})_{\mu\nu} , \tag{1.14}$$

[2]The factor of i has been added to stick to physicists' conventions – Hermitian generators for compact Lie (sub)groups. Mathematicians rather use anti-Hermitian generators.

in terms of an antisymmetric matrix Ω containing the parameters of the transformation.[3] One can check easily that the Σ matrices obey the following commutation relations:

$$[\,\Sigma^{\mu\nu},\,\Sigma^{\rho\sigma}\,] = i\Big(\eta^{\nu\rho}\Sigma^{\mu\sigma} + \eta^{\mu\sigma}\Sigma^{\nu\rho} - \eta^{\nu\sigma}\Sigma^{\mu\rho} - \eta^{\mu\rho}\Sigma^{\nu\sigma}\Big)\,. \tag{1.15}$$

Among the six generators, the components Σ^{ij} with $i,j = 1,2,3$ are Hermitian matrices and correspond to spatial rotations. Explicitely,

$$J_i = \tfrac{1}{2}\epsilon_{ijk}\Sigma^{jk}\ , \quad [\,J_i,\,J_j\,] = i\epsilon_{ijk}J_k\,, \tag{1.16}$$

thereby giving the $\mathfrak{su}_2$ algebra. The three other generators,

$$\Sigma_{0i} = K_i\ , \quad i = 1,2,3 \tag{1.17}$$

are anti-Hermitian and correspond naturally to boosts. Using these one gets the two other commutators of the Lorentz Lie algebra:

$$[\,K_i,\,K_j\,] = -i\epsilon_{ijk}J_k\ , \quad [\,J_i,\,K_j\,] = i\epsilon_{ijk}K_k\,, \tag{1.18}$$

the last one expressing the fact that $\vec{K}$ transforms as a vector under spatial rotations. It is enlightening to realize that one can combine these six generators into the generators of a pair of commuting $\mathfrak{su}_2$ algebras.[4] Let us define

$$L_i = \tfrac{1}{2}\big(J_i + iK_i\big)\ , \quad R_i = \tfrac{1}{2}\big(J_i - iK_i\big)\,, \tag{1.19}$$

both Hermitian, satisfying

$$[\,L_i,\,L_j\,] = i\epsilon_{ijk}L_k\ , \quad [\,R_i,\,R_j\,] = i\epsilon_{ijk}R_k\ , \quad [\,L_i,\,R_j\,] = 0\,. \tag{1.20}$$

While these two sets of generators are commuting with each other, they are not independent. It is immediate, using eq. (1.15), that

$$(L_i)^* = -R_i\,. \tag{1.21}$$

Irreductible representations of $SO(1,3)^+$ are labelled by $(j_1,j_2) \in \mathbb{Z}_+/2 \times \mathbb{Z}_+/2$, the spins associated with these two $\mathfrak{su}_2$ algebras. From the relation (1.21) it follows that the complex conjugate of a representation (j_1,j_2) transforms as the representation (j_2,j_1). The representation of interest in this chapter are:

- The trivial representation $(0,0)$;
- The two-dimensional spinorial representations $(1/2,0)$ and $(0,1/2)$, of complex dimension two, which are mapped to each other by complex conjugation;
- The vector representation $(1/2,1/2)$ which is mapped to itself by complex conjugation (real representation), of real dimension four.

[3] The one-half factor is there to avoid overcounting as we contract two antisymmmetric matrices.

[4] To be more precise, what we are doing here is to show that the complexified Lie algebra of the Lorentz group $\mathfrak{so}_{1,3}|_{\mathbb{C}}$ is isomorphic to $\mathfrak{su}_2|_{\mathbb{C}} \oplus \mathfrak{su}_2|_{\mathbb{C}}$, *i.e.* the the direct sum of two complexified $\mathfrak{su}_2$ algebras, see the Lie groups course for more details.

Lorentz transformations and SL(2,$\mathbb{C}$). In order to get an explicit construction of the $(1/2, 0)$ and $(0, 1/2)$ representations, let us define the following set of 2×2 matrices, arranged as a 4-vector:

$$\sigma^\mu \stackrel{\text{def.}}{=} (\mathbb{I},\ \sigma^i) \tag{1.22}$$

written using the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.23}$$

One can associate to the 4-vector of space-time coordinates $(t, \vec{x})$ (or to any 4-vector) the 2×2 Hermitian matrix

$$\mathbb{X} = x_\mu \sigma^\mu = t - \vec{x} \cdot \vec{\sigma} = \begin{pmatrix} t - z & -x + iy \\ -x - iy & t + z \end{pmatrix}. \tag{1.24}$$

The length square of the 4-vector $(t, \vec{x})$ is then given by $\det \mathbb{X} = t^2 - \vec{x}^2$. Let S be a 2×2 matrix of determinant one, *i.e.* an element of the matrix group $SL(2, \mathbb{C})$. The determinant of $\mathbb{X}$ (hence the length of 4-vectors in Minkowski space) is invariant under the action:

$$\mathbb{X} \longmapsto S\mathbb{X}S^\dagger, \tag{1.25}$$

which preserves the hermiticity of $\mathbb{X}$.

Given that, for any $S \in SL(2, \mathbb{C})$, the matrix $-S \in SL(2, \mathbb{C})$ has the same action on $\mathbb{X}$, $SL(2, \mathbb{C})$ is actually a *double cover* of the restricted Lorentz group :

$$SL(2, \mathbb{C})/\mathbb{Z}_2 \simeq SO(1, 3)^+, \tag{1.26}$$

i.e. $SO(1, 3)^+$ is isomorphic to the quotient of $SL(2, \mathbb{C})$ by the order two group generated by the inversion.The relation between $SO(1, 3)^+$ and $SL(2, \mathbb{C})$ is similar to the relation between $SO(3)$ and $SU(2)$. While the former admits only representations with integer spins the latter admits also representations with half-integers spins; for this reason the double cover of $SO(1, 3)^+$ it is also denoted by $\mathfrak{Spin}(1, 3)$, the *spin group* of Minkowski space-time.

The Lie algebra of the Lorentz group constructed above is, by construction, isomorphic to the Lie algebra $\mathfrak{sl}_2$ of the group $SL(2, \mathbb{C})$. Let us consider an infinitesimal $SL(2, \mathbb{C})$ transformation, parameterized by a 2×2 matrix $\omega \in \mathcal{M}_{\mathbb{C}}(2, 2)$:

$$S = \mathbb{I} + \omega. \tag{1.27}$$

At first order in ω, the condition $\det M = 1$ translates into:

$$\det M = 1 \implies \operatorname{Tr} \omega = 0. \tag{1.28}$$

Thus the $\mathfrak{sl}_2$ algebra, which has real dimension six, is given by $\operatorname{Span}_{\mathbb{C}}(\sigma^i)$, *i.e.* the set of linear combinations of Pauli matrices with complex coefficients.

1.1.2 Representations of the Lorentz group

We will now discuss the properties of the representations of the Lorentz group of smallest dimension: spinor, conjugate spinor and vector.

Two-component spinors. As we have seen before, the smallest non-trivial representations of the restricted Lorentz group (more precisely, of the double cover thereof) are a pair of conjugate two-dimensional spinorial representations, $(1/2, 0)$ and $(0, 1/2)$.

Using the isomorphism between the spin group $\mathsf{Spin}(1,3)$ and $\mathsf{SL}(2,\mathbb{C})$ described above, one can identify one of them, say the representation $(1/2, 0)$, with the set of $\mathsf{SL}(2,\mathbb{C})$ matrices $\{S \in \mathcal{M}_{\mathbb{C}}(2,2), \det S = 1\}$ which act on a complex vector space of two-component spinors χ:

$$\chi \longmapsto S\chi\,, \tag{1.29}$$

or in components (take care of the positions of the indices!)

$$\chi_\alpha \longmapsto S_\alpha{}^\beta\, \chi_\beta \qquad \alpha, \beta = 1, 2 \tag{1.30}$$

For reasons that will become clear soon, those *Weyl spinors* are called left-handed.

Exactly as the matrix group $\mathsf{O}(1,3)$ preserves the Minkowski metric η, the matrix group $\mathsf{SL}(2,\mathbb{C})$ preserves the following "spinor metric":

$$\epsilon = -i\sigma^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{1.31}$$

in the sense that

$$\forall S \in \mathsf{SL}(2,\mathbb{C})\,, \quad S^{\mathsf{T}} \epsilon S = \epsilon\,. \tag{1.32}$$

We check immediately that:

$$\epsilon^{-1} = \epsilon^{\mathsf{T}} = -\epsilon = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,. \tag{1.33}$$

As for the Minkowski metric, the components of ϵ are denoted by $\epsilon_{\alpha\beta}$, while those of its inverse $\epsilon^{\alpha\beta}$. Using this invariant tensor and its inverse, one can lower and raise spinor indices:

$$\chi_\alpha = \epsilon_{\alpha\beta}\chi^\beta\,, \quad \chi^\alpha = \epsilon^{\alpha\beta}\chi_\beta \tag{1.34}$$

Using eq. (1.32) one sees that:

$$\chi^\alpha \longmapsto \epsilon^{\alpha\beta} S_\beta{}^\gamma \epsilon_{\gamma\delta}\chi^\delta = \left(\epsilon^{\mathsf{T}} S \epsilon\right)^\alpha{}_\delta \chi^\delta = \left((S^{-1})^{\mathsf{T}}\right)^\alpha{}_\delta\, \chi^\delta\,. \tag{1.35}$$

Since $\epsilon S \epsilon^{\mathsf{T}} = (S^{-1})^{\mathsf{T}}$, using eqn. (1.32), χ_α and χ^α transform in equivalent representations. On can then easily construct spinor bilinears invariant under the action of the restricted Lorentz group:[5]

$$\chi\psi \stackrel{\text{def.}}{=} -\epsilon^{\alpha\beta}\chi_\alpha\psi_\beta \quad \longmapsto \quad -\underbrace{\epsilon^{\alpha\beta}S_\alpha{}^\gamma S_\beta{}^\delta}_{(S^{\mathsf{T}}\epsilon S)^{\gamma\delta}}\chi_\gamma\psi_\delta = -\epsilon^{\gamma\delta}\chi_\gamma\psi_\delta = \chi\psi\,. \tag{1.36}$$

The minus sign in the definition has been chosen for consistency with Hermitian conjugation, see below. It can also be written as

$$\chi\psi = \epsilon^{\beta\alpha}\chi_\alpha\psi_\beta = \overset{\searrow}{\chi}{}^\beta\psi_\beta\,, \tag{1.37}$$

with the upper index first. Anticipating a bit, the spinor components should be considered as anti-commuting variables. For this reason,

$$\chi\chi = -\epsilon^{\alpha\beta}\chi_\alpha\chi_\beta = -\chi_1\chi_2 + \chi_2\chi_1 = 2\chi_2\chi_1\,. \tag{1.38}$$

In order to get an explicit expression of the Lorentz generators in this representation, let us first introduce the following 4-vector of matrices:

$$\bar\sigma^\mu \stackrel{\text{def.}}{=} (\mathbb{I},\ -\sigma^i) \tag{1.39}$$

i.e. with a minus sign for the space components compared to σ^μ, see eqn. (1.22). In terms of those let us define the following set of six linearly independent 2×2 matrices, labelled by an antisymmetrized pair of space-time indices:

$$\left(\sigma^{\mu\nu}\right)_\alpha{}^\beta \stackrel{\text{def.}}{=} \frac{i}{4}\left(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu\right)_\alpha{}^\beta \tag{1.40}$$

Explicitly,

$$\sigma^{ab} = \tfrac{1}{2}\epsilon^{ab}{}_c\sigma^c\,, \quad \sigma^{0a} = -\sigma^{a0} = \tfrac{i}{2}\sigma^a \qquad a,b,c = 1,2,3. \tag{1.41}$$

The former are Hermitian, and the latter anti-Hermitian. It is easy to check that they obey the same algebra as the generators $\Sigma^{\mu\nu}$ of the Lorentz transformations, see eqns. (1.13.1.15):

$$[\,\sigma^{\mu\nu},\,\sigma^{\rho\tau}\,] = i\Big(\eta^{\nu\rho}\sigma^{\mu\tau} + \eta^{\mu\tau}\sigma^{\nu\rho} - \eta^{\nu\tau}\sigma^{\mu\rho} - \eta^{\mu\rho}\sigma^{\nu\tau}\Big)\,. \tag{1.42}$$

In terms of those, the action of an arbitrary Lorentz transformation on a left-handed spinor χ reads:

$$\chi_\alpha \ \longmapsto\ S_\alpha{}^\beta\chi_\beta = \left(e^{-\frac{i}{2}\Omega_{\mu\nu}\sigma^{\mu\nu}}\right)_\alpha{}^\beta\chi_\beta\,, \tag{1.43}$$

[5] In terms of group theory, the trivial representation occurs in the decomposition into irreducible representations $(1/2,0)\otimes(1/2,0) = \underline{(0,0)}\oplus(1,0)$. The exotic $(1,0)$ representation (an anti-self-dual two form) will not be discussed further.

where the anti-symmetric matrix $\Omega_{\mu\nu}$ contains the parameters of the transformation; while Ω_{ab} correspond to rotation angles, Ω_{0a} correspond to boost parameters. Regarding rotations, one can define the unit 3-vector $\vec{n}$ giving the axis of rotation through:

$$\frac{1}{2}\Omega_{ab}\epsilon^{ab}{}_{c} = \theta\, n_c\ , \quad \vec{n}^2 = 1\,, \tag{1.44}$$

such that a rotation acts as

$$\chi \ \longmapsto\ e^{-\frac{i}{2}\theta\,\vec{n}\cdot\vec{\sigma}}\chi\,. \tag{1.45}$$

Boosts are parameterized by the 3-vector $\vec{w}$ such that $w_a = \Omega_{0a} = -\Omega_{a0}$:

$$\chi \ \longmapsto\ e^{\frac{1}{2}\vec{w}\cdot\vec{\sigma}}\chi\,. \tag{1.46}$$

Conjugate spinor representation. The other spinorial representation $(0, 1/2)$ should transform in the complex-conjugate of the $(1/2, 0)$ representation; it corresponds to *right-handed* Weyl spinors.

It is useful to consider first the matrix $\mathbb{X}^{\mathcal{P}}$ associated with the image of the 4-vector of coordinates $(t, \vec{x})$ under a parity transformation $\vec{x} \mapsto -\vec{x}$:

$$\mathbb{X}^{\mathcal{P}} = x_\mu \bar{\sigma}^\mu = t + \vec{x}\cdot\vec{\sigma} = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} = \epsilon^{\mathsf{T}}\,\mathbb{X}^{\mathsf{T}}\,\epsilon\,. \tag{1.47}$$

According to (1.25), $\mathbb{X}^{\mathcal{P}}$ transforms then under the action of $SL(2,\mathbb{C})$ as:

$$\mathbb{X}^{\mathcal{P}} \ \longmapsto \epsilon^{\mathsf{T}} S^* \mathbb{X}^{\mathsf{T}} S^{\mathsf{T}} \epsilon = \underbrace{\epsilon^{\mathsf{T}} S^* \epsilon}_{\left(S^{-1}\right)^\dagger}\, \mathbb{X}^{\mathcal{P}}\, \underbrace{\epsilon^{\mathsf{T}} S^{\mathsf{T}} \epsilon}_{S^{-1}} \tag{1.48}$$

i.e. parity maps each $S \in SL(2,\mathbb{C})$ to $\left(S^{-1}\right)^\dagger \in SL(2,\mathbb{C})$. In other words, the $(1/2, 0)$ and $(0, 1/2)$ representations are mapped to each other under parity.

It is customary to use *dotted indices* for the conjugate spinors, as well as for the matrix indices contracted with them. We then introduce a spinor of components $\zeta^{\dot\alpha}$, $\alpha = 1, 2$, transforming as follows:

$$\zeta^{\dot\alpha} \ \longmapsto \left(\left(S^{-1}\right)^\dagger\right)^{\dot\alpha}{}_{\dot\beta}\,\psi^{\dot\beta}\,, \tag{1.49}$$

from which we deduce that

$$\zeta_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\zeta^{\dot\beta} \ \longmapsto\ \left(\epsilon (S^{-1})^\dagger \epsilon^{\mathsf{T}}\right)_{\dot\alpha}{}^{\dot\delta}\,\zeta_{\dot\delta}\,. \tag{1.50}$$

Since the relation (1.32) is equivalent to $S^* = \epsilon (S^{-1})^\dagger \epsilon^{\mathsf{T}}$ we can rewrite this as

$$\boxed{\zeta_{\dot\alpha} = \longmapsto\ \left(S^*\right)_{\dot\alpha}{}^{\dot\beta}\zeta_{\dot\beta}\,,} \tag{1.51}$$

transforming indeed in the complex-conjugate representation.[6] In particular, the Hermitian conjugate of a left-handed spinor transforms as a right-handed spinor and vice-versa:

$$\zeta_{\dot\alpha} = \longmapsto \ \left(S^*\right)_{\dot\alpha}{}^{\dot\beta}\zeta_{\dot\beta}\ , \quad \zeta^\dagger_\alpha = \longmapsto \ S_\alpha{}^\beta\zeta^\dagger_\beta\,. \tag{1.52}$$

As with left-handed spinors, one can construct bilinear expressions transforming as scalars under the action of the restricted Lorentz group:

$$\zeta\eta \overset{\text{def.}}{=} \epsilon^{\dot\alpha\dot\beta}\zeta_{\dot\alpha}\eta_{\dot\beta} \ \longmapsto \ \underbrace{\epsilon^{\dot\alpha\dot\beta}S^*_{\dot\alpha}{}^{\dot\gamma}S^*_{\dot\beta}{}^{\dot\delta}}_{(S^*\ {}^{\mathsf{T}}\epsilon S^*)^{\dot\gamma\dot\delta}}\zeta_{\dot\gamma}\eta_{\dot\delta} = \epsilon^{\dot\gamma\dot\delta}\zeta_{\dot\gamma}\eta_{\dot\delta} = \zeta\eta\,. \tag{1.53}$$

It can also be written as

$$\zeta\eta = \epsilon^{\dot\alpha\dot\beta}\zeta_{\dot\alpha}\eta_{\dot\beta} = \overset{\nearrow}{\zeta_{\dot\alpha}}\eta^{\dot\alpha}\,, \tag{1.54}$$

with the lower index first. This is compatible with hermitian conjugation, as one can check explicitly, taking into account that spinor variables are anti-commuting:

$$(\chi\psi)^\dagger = (-\chi_1\psi_2+\chi_2\psi_1)^\dagger = -\psi_2^\dagger\chi_1^\dagger+\psi_1^\dagger\chi_2^\dagger = \chi_1^\dagger\psi_2^\dagger-\chi_2^\dagger\psi_1^\dagger = \epsilon^{\dot\alpha\dot\beta}\chi^\dagger_{\dot\alpha}\psi^\dagger_{\dot\beta} = \chi^\dagger\psi^\dagger\,. \tag{1.55}$$

Finally, let us give the expression of the generators of the restricted Lorentz group in the $(0,1/2)$ representation:

$$\left(\bar\sigma^{\mu\nu}\right)^{\dot\alpha}{}_{\dot\beta} \overset{\text{def.}}{=} \frac{i}{4}\left(\bar\sigma^\mu\sigma^\nu-\bar\sigma^\nu\sigma^\mu\right)^{\dot\alpha}{}_{\dot\beta}\,, \tag{1.56}$$

which are constructed such that, if a spinor in the $(1/2,0)$ representation transforms with $S(\Lambda)$, a conjugate spinor ($(0,1/2)$ representation) transforms with $S(\Lambda)^{\dagger\,-1}$. Explicitly,

$$\bar\sigma^{ab} = \tfrac{1}{2}\epsilon^{ab}{}_c\sigma^c\ , \quad \bar\sigma^{0a} = -\tfrac{i}{2}\sigma^a \qquad a,b,c = 1,2,3. \tag{1.57}$$

The index structure of (1.56) indicates that it acts on a right-handed spinor $\zeta^{\dot\alpha}$ (with index up); as one can check explicitly, this is compatible with the fact that $\zeta^{\dot\alpha}$ transforms with $(S^{-1})^\dagger$, see eqn. (1.49), *i.e.* in the same way under rotations but with opposite signs under boosts. The action of an arbitrary Lorentz transformation reads:

$$\zeta^{\dot\alpha} \ \longmapsto \ \left(e^{-\frac{i}{2}\Omega_{\mu\nu}\bar\sigma^{\mu\nu}}\right)^{\dot\alpha}{}_{\dot\beta}\,\zeta^{\dot\beta}\,. \tag{1.58}$$

Comparing the Lorentz generators (1.57) to those of the $(1/2,0)$ representation, see eqn. (1.41), one can see that while the left- and right- spinors transform in the same way under space rotations, following eqn. (1.45), they transform *with opposite signs under boosts* – compare with eqn. (1.46):

$$\zeta \ \longmapsto \ e^{-\frac{1}{2}\vec w\cdot\vec\sigma}\,\zeta\,. \tag{1.59}$$

[6] The $(1/2,0)$ and $(0,1/2)$ representations are not equivalent since one cannot find an invertible matrix M such that $S = M^{-1}S^*M$ for all $S\in SL(2,\mathbb{C})$.

To summarize, there are two kinds of spinors with respect to the Lorentz group, associated with the representations $(1/2,0)$ and $(0,1/2)$. They are exchanged mathematically by complex conjugation, and physically by parity. The associated property of a spinor (being of one kind or the other) is called the *chirality.*[7]

Vectors. So far we have constructed the two-dimensional spinorial representations $(1/2,0)$ and $(0,1/2)$, which are complex conjugate to each other. By tensoring those, we get the self-conjugate representation $(1/2,0)\otimes(0,1/2)=(1/2,1/2)$ which corresponds, as we shall see below, to the vector representation of the restricted Lorentz group.

To understand how to build such representation, let us remind that we started the construction of the $(1/2,0)$ representations by considering the following transformation law:

$$\mathbb{X}=x_\mu\sigma^\mu \;\longmapsto\; S\mathbb{X}S^\dagger = x_\mu S\sigma^\mu S^\dagger\,. \tag{1.60}$$

Since x^μ is a 4-vector, the action of the Lorentz group on the spinors should induce the map $x_\mu \mapsto \Lambda_\mu{}^\nu x_\nu$, with as usual $\Lambda_\mu{}^\nu = (\Lambda^{-1})^\nu{}_\mu$. This requirement is satisfied if:

$$S\sigma^\mu S^\dagger = \left(\Lambda^{-1}\right)^\mu{}_\nu\,\sigma^\nu \tag{1.61}$$

We have indeed:

$$S\mathbb{X}S^\dagger = x_\mu S\sigma^\mu S^\dagger = x_\mu\left(\Lambda^{-1}\right)^\mu{}_\nu\,\sigma^\nu = \Lambda_\nu{}^\mu\, x_\mu\,\sigma^\nu\,. \tag{1.62}$$

Thus the $SL(2,\mathbb{C})$ action on σ^μ induces the Lorentz transformation of a 4-vector (naturally the transformation (1.61) can be explicitly checked).

This property implies that the following fermion bilinear built out of a left-handed and a right-handed fermion transforms as a 4-vector:

$$\chi\sigma^\mu\zeta = \chi^\alpha\sigma^\mu_{\alpha\dot\beta}\zeta^{\dot\beta} \quad\longmapsto\quad \chi S^{-1}\sigma^\mu S^{-1\,\dagger}\zeta = \Lambda^\mu{}_\nu\,\chi\sigma^\nu\zeta\,. \tag{1.63}$$

The Hermitian conjugate of this expression is given by $\left(\chi\sigma^\mu\zeta\right)^\dagger = \zeta^{\dagger\,\alpha}\,\sigma^\mu_{\alpha\dot\alpha}\,\chi^{\dagger\,\dot\alpha}$.

A second fermion bilinear giving a 4-vector is obtained by noticing that $\bar\sigma^\mu$, defined in (1.39), can be obtained from σ^μ by raising indices with the inverse spinor "metric" ϵ^{-1}:

$$\sigma^\mu_{\alpha\dot\alpha}\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta} = \left(\bar\sigma^\mu\right)^{\dot\beta\beta}\,. \tag{1.64}$$

Exercise: prove this property.
Notice that, on the right-hand side, the un-dotted and dotted indices have been inverted, *i.e.* $\bar\sigma^\mu = \epsilon\sigma^{\mu\,\mathsf{T}}\epsilon^{\mathsf{T}}$. We can then construct the fermion bilinear:

$$\zeta\bar\sigma^\mu\chi = \zeta_{\dot\alpha}\left(\bar\sigma^\mu\right)^{\dot\alpha\beta}\chi_\beta\,. \tag{1.65}$$

[7]Since these spinors are only distinguished by their behavior under boosts, this distinction does not exist in non-relativistic physics.

which transforms as a vector since, from a similar reasoning as above, using the transformation of $\mathbb{X}^{\mathcal{P}}$ one gets:

$$S^\dagger \bar{\sigma}^\mu S = \Lambda^\mu{}_\nu \bar{\sigma}^\nu \tag{1.66}$$

We don't expect (1.63) and (1.65) to be independent vectors, since $(1/2, 1/2)$ defines a single four-dimensional representation of the Lorentz group. We have indeed

$$\zeta \bar{\sigma}^\mu \chi = \zeta_{\dot{\beta}} \, (\bar{\sigma}^\mu)^{\dot{\beta}\beta} \chi_\beta = \zeta_{\dot{\beta}} \chi_\beta \sigma^\mu_{\alpha\dot{\alpha}} \epsilon^{\alpha\beta} \epsilon^{\dot{\alpha}\dot{\beta}} = \zeta^{\dot{\alpha}} \chi^\alpha \sigma^\mu_{\alpha\dot{\alpha}} = -\chi \sigma^\mu \zeta \, . \tag{1.67}$$

Clifford algebra and Dirac spinors. The $(1/2, 0)$ and $(0, 1/2)$ spinorial representations are mapped to each other under parity. In order to build physical theories invariant under parity (like quantum electrodynamics, as we shall see), it is useful to combine these two-components spinors into a single four-dimensional spinor, corresponding to a *reducible* representation of the restricted Lorentz group.

Historically, this representation was built by Dirac by defining a set of complex matrices γ^μ obeying what is known as the *Clifford algebra*:

$$\{ \gamma^\mu, \gamma^\nu \} \stackrel{\text{def.}}{=} \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} \, \mathbb{I} \tag{1.68}$$

In particular, we have

$$\left(\gamma^0\right)^2 = \mathbb{I} \quad , \qquad \left(\gamma^i\right)^2 = -\mathbb{I} \, , \quad i = 1, 2, 3 \, . \tag{1.69}$$

Let us consider the commutator of two of these matrices, and construct

$$S^{\mu\nu} \stackrel{\text{def.}}{=} \frac{i}{4} [\gamma^\mu, \gamma^\nu] = \frac{i}{2} (\gamma^\mu \gamma^\nu - \eta^{\mu\nu}) \, , \tag{1.70}$$

which is antisymmetric in its space-times indices (μ, ν). On can check easily that:

$$[S^{\mu\nu}, \gamma^\rho] = \frac{i}{2} [\gamma^\mu \gamma^\nu, \gamma^\rho] = i \, (\eta^{\rho\nu} \gamma^\mu - \eta^{\rho\mu} \gamma^\nu) \tag{1.71}$$

From which we get, after a straightforward but slightly tedious calculation,

$$[S^{\mu\nu}, S^{\rho\sigma}] = i \Big(\eta^{\nu\rho} S^{\mu\sigma} + \eta^{\mu\sigma} S^{\nu\rho} - \eta^{\nu\sigma} S^{\mu\rho} - \eta^{\mu\rho} S^{\nu\sigma} \Big) \, . \tag{1.72}$$

In other words, every representation of the Clifford algebra furnishes a representation of the $\mathfrak{so}_{1,3}$ algebra, which is not necessarily irreducible.

The non-trivial representation of the Clifford algebra (1.68) of the smallest dimension has dimension four in space-time dimension $d = 4$. The Weyl spinor tools developed previously give a natural basis for this representation, known as the *Weyl basis* or *chiral representation*:

$$\gamma^{\mu}_{\rm w} = \begin{pmatrix} 0 & \sigma^{\mu} \\ \bar{\sigma}^{\mu} & 0 \end{pmatrix} \tag{1.73}$$

More explicitely,

$$\gamma^{0}_{\rm w} = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \ , \quad \gamma^{i}_{\rm w} = \begin{pmatrix} 0 & \sigma^{i} \\ -\sigma^{i} & 0 \end{pmatrix} . \tag{1.74}$$

The first one is Hermitian, while the other ones are anti-Hermitian. One can notice that:

$$\gamma^{0}\gamma^{\mu}\gamma^{0} = (\gamma^{\mu})^{\dagger} \ . \tag{1.75}$$

In terms of the gamma matrices, it is immediate that:

$$S^{\mu\nu}_{\rm w} = \frac{i}{4}\,[\,\gamma^{\mu}_{\rm w},\,\gamma^{\nu}_{\rm w}\,] = \frac{i}{4}\begin{pmatrix} \sigma^{\mu}\bar{\sigma}^{\nu} - \sigma^{\nu}\bar{\sigma}^{\mu} & 0 \\ 0 & \bar{\sigma}^{\mu}\sigma^{\nu} - \bar{\sigma}^{\nu}\sigma^{\mu} \end{pmatrix} = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar{\sigma}^{\mu\nu} \end{pmatrix} , \tag{1.76}$$

see eqns. (1.40) and (1.56). It indicates that the complex vector space on which these matrices act is a space of *bispinors* transforming in the reducible $(1/2, 0) \oplus (0, 1/2)$ representation:

$$\Psi_{\rm w} \stackrel{\rm def.}{=} \begin{pmatrix} \chi_{\alpha} \\ \zeta^{\dot{\alpha}} \end{pmatrix} \tag{1.77}$$

Take care of the index structure, which is chosen to be consistent with the transformations (1.43) and (1.58). One has then, for an arbitrary Lorentz transformation:

$$\Psi \ \longmapsto \ \exp\left(-\frac{i}{2}\Omega_{\mu\nu}S^{\mu\nu}\right)\Psi \tag{1.78}$$

We have dropped the subscript w here, since this is evidently true in any basis. The action of parity $\mathcal{P} : \vec{x} \mapsto -\vec{x}$ exchanges left- and right- handed Weyl spinors so its action on a Dirac spinor in the Weyl basis can be represented as the matrix multiplication:

$$\Psi_{\rm w} \ \stackrel{\mathcal{P}}{\longmapsto} \gamma^{0}\,\Psi_{\rm w}\,, \tag{1.79}$$

using the explicit form of γ^{0} in the Weyl basis, see eqn. (1.74).

It is useful to define projectors on the irreducible $(1/2, 0)$ and $(0, 1/2)$ representations; let us define first the matrix:

$$\gamma^{5}_{\rm w} = \begin{pmatrix} -\mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix} \tag{1.80}$$

The defining properties of the γ^5 matrix, independently of the basis of the Clifford algebra chosen, is that it anti-commutes with all the Dirac matrices and squares to the unit matrix:

$$\left\{\gamma^5, \gamma^\mu\right\} = 0\,, \mu = 0, \ldots, 3\,, \quad \left(\gamma^5\right)^2 = \mathbb{I}\,. \tag{1.81}$$

A basis-independent definition is provided by:

$$\boxed{\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3} \tag{1.82}$$

One can check indeed that by anticommutation,

$$\gamma^5\gamma^0 = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0 = -i(\gamma^0)^2\gamma^1\gamma^2\gamma^3 = -i\gamma^1\gamma^2\gamma^3 = -\gamma^0\gamma^5\,, \tag{1.83}$$

and similarly for the other ones. One has then by moving the second γ^0 to the left,

$$\left(\gamma^5\right)^2 = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 = (\gamma^0)^2\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 = \gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 = \cdots = \mathbb{I}\,. \tag{1.84}$$

In terms of γ^5, the projectors on left-handed (resp. right-handed) Weyl spinors are:

$$P_{\substack{L\\R}} = \frac{1 \mp \gamma^5}{2}\,. \tag{1.85}$$

In order to construct scalar and vectors from bilinears of Dirac spinors, let us write first the transformation of such spinor in terms of the action of $SL(2,\mathbb{C})$. According to eqns. (1.30,1.49) the associated 4×4 matrix $D(\Lambda)$ is given, in the Weyl basis, by:

$$\Psi_w = \begin{pmatrix}\chi\\ \zeta\end{pmatrix} \longmapsto D(\Lambda)_w\Psi_w = \begin{pmatrix}S(\Lambda) & 0\\ 0 & S^{-1\,\dagger}(\Lambda)\end{pmatrix}\begin{pmatrix}\chi\\ \zeta\end{pmatrix} \tag{1.86}$$

Therefore for the Hermitian conjugate one has

$$\Psi_w^\dagger = \begin{pmatrix}\chi^\dagger & \zeta^\dagger\end{pmatrix} \longmapsto \begin{pmatrix}\chi^\dagger & \zeta^\dagger\end{pmatrix}\begin{pmatrix}S^\dagger(\Lambda) & 0\\ 0 & S^{-1}(\Lambda)\end{pmatrix}\,, \tag{1.87}$$

and $\Psi_w^\dagger\Psi_w$ does not transform as a scalar quantity, which is not suprising since, for instance, $\chi^\dagger$ transforms as a right-handed spinor. Let us define the *Dirac conjugate* of a Dirac spinor:

$$\boxed{\overline{\Psi} \stackrel{\text{def.}}{=} \Psi^\dagger\gamma^0} \tag{1.88}$$

Using for example the Weyl basis, one has

$$\overline{\Psi}_w = \begin{pmatrix}\chi^\dagger & \zeta^\dagger\end{pmatrix}\begin{pmatrix}0 & \mathbb{I}\\ \mathbb{I} & 0\end{pmatrix} = \begin{pmatrix}\zeta^\dagger & \chi^\dagger\end{pmatrix}\,. \tag{1.89}$$

So a scalar quantity is obtained by left-multiplying a Dirac spinor by its Dirac conjugate, or the Dirac conjugate of another spinor:

$$\overline{\Psi}_1\Psi_2 \;\longmapsto \Psi_1^\dagger \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \begin{pmatrix} S^\dagger & 0 \\ 0 & S^{-1} \end{pmatrix} \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}}_{\begin{pmatrix} S^{-1} & 0 \\ 0 & S^\dagger \end{pmatrix}} \begin{pmatrix} S & 0 \\ 0 & S^{-1\,\dagger} \end{pmatrix} \Psi_2 = \overline{\Psi}_1\Psi_2\,. \tag{1.90}$$

Likewise under parity,

$$\overline{\Psi}_1\Psi_2 \;\stackrel{\mathcal{P}}{\longmapsto} \Psi_1^\dagger \underbrace{\gamma^0\gamma^0}_{\mathbb{I}}\gamma^0\Psi^2 = \overline{\Psi}_1\Psi_2\,. \tag{1.91}$$

One checks that $\overline{\Psi}\Psi$ is self-conjugate under Hermitian conjugation, since γ^0 is a Hermitian matrix: $(\overline{\Psi}\Psi)^\dagger = \Psi^\dagger\gamma^{0\,\dagger}\Psi = \overline{\Psi}\Psi$. We can also consider

$$\overline{\Psi}_1\gamma^5\Psi_2 \;\longmapsto \overline{\Psi}_1 \begin{pmatrix} S^{-1} & 0 \\ 0 & S^\dagger \end{pmatrix} \begin{pmatrix} -\mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix} \begin{pmatrix} S & 0 \\ 0 & S^{-1\,\dagger} \end{pmatrix} \Psi_2 = \overline{\Psi}_1\gamma^5\Psi_2\,. \tag{1.92}$$

However under parity one gets

$$\overline{\Psi}_1\gamma^5\Psi_2 \;\stackrel{\mathcal{P}}{\longmapsto} \overline{\Psi}_1\gamma^0\gamma^5\gamma^0\Psi^2 = -\overline{\Psi}_1\Psi_2\,, \tag{1.93}$$

using $\left\{\,\gamma^0,\,\gamma^5\,\right\} = 0$. Therefore $\overline{\Psi}_1\gamma^5\Psi_2$ is a *pseudo-scalar* rather than a scalar quantity w.r.t. $\mathrm{O}(3)$. Under Hermitian conjugation,

$$\left(\overline{\Psi}\gamma^5\Psi\right)^\dagger = \Psi^\dagger\gamma^{5\,\dagger}\gamma^{0\,\dagger}\Psi = \Psi^\dagger\gamma^5\gamma^0\Psi = -\Psi^\dagger\gamma^0\gamma^5\Psi = -\overline{\Psi}\gamma^5\Psi\,, \tag{1.94}$$

showing that this quantity is also anti-Hermitian rather than Hermitian.

Slashed notation. In the same way as space-time four-vectors have been constructed from bilinears of Weyl fermions, it is possible to built 4-vectors from bilinears of Dirac fermions. As the notation suggests, one has:

$$\begin{aligned} \overline{\Psi}_1\gamma^\mu\Psi_2 \;&= \Psi_1^\dagger \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \Psi_2 \\ &\longmapsto \quad \Psi_1^\dagger\gamma^0 \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \begin{pmatrix} S^\dagger & 0 \\ 0 & S^{-1} \end{pmatrix} \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} S & 0 \\ 0 & S^{-1\,\dagger} \end{pmatrix} \Psi_2 \\ &= \overline{\Psi}_1 \begin{pmatrix} 0 & S^{-1}\sigma^\mu S^{-1\,\dagger} \\ S^\dagger\bar{\sigma}^\mu S & 0 \end{pmatrix} \Psi_2 = \Lambda^\mu{}_\nu\,\overline{\Psi}_1\gamma^\nu\Psi_2\,, \end{aligned} \tag{1.95}$$

using the transformations (1.61,1.66). Under parity,

$$\overline{\Psi}_1\gamma^\mu\Psi_2 \;\stackrel{\mathcal{P}}{\longmapsto} \overline{\Psi}_1\gamma^0\gamma^\mu\gamma^0\Psi_2 = \begin{cases} +\overline{\Psi}_1\gamma^0\Psi_2 \\ -\overline{\Psi}_1\gamma^i\Psi_2 \end{cases} \tag{1.96}$$

as expected for a four-vector. Similarly, $\overline{\Psi}_1\gamma^5\gamma^\mu\Psi_2$ transforms as a pseudo-quadrivector:

$$\overline{\Psi}_1\gamma^5\gamma^\mu\Psi_2 \longmapsto \Lambda^\mu{}_\nu\overline{\Psi}_1\gamma^5\gamma^\nu\Psi_2 \quad , \qquad \overline{\Psi}_1\gamma^5\gamma^\mu\Psi_2 \overset{\mathcal{P}}{\longmapsto} \begin{cases} -\overline{\Psi}_1\gamma^0\Psi_2 \\ +\overline{\Psi}_1\gamma^i\Psi_2 \end{cases} \tag{1.97}$$

where we have used $\{\gamma^5, \gamma^\mu\} = 0$.

Let v^μ be some arbitrary four-vector. One can obtain a Lorentz scalar (resp. a pseudo-scalar) by contracting with $\overline{\Psi}\gamma^\mu\Psi$ (resp. with $\overline{\Psi}\gamma^5\gamma^\mu\Psi$).[8] It is convenient to use a short-hand notation known as the *Feynman slash*:

$$\not{v} \overset{\text{def.}}{=} v_\mu\gamma^\mu \quad , \qquad \overline{\Psi}\not{v}\Psi = v_\mu\overline{\Psi}\gamma^\mu\Psi\,. \tag{1.98}$$

1.2 Free classical field theories

After all this preparation we are ready to consider classical field theories associated with spin zero and spin one-half Lorentz representations. We will start with some reminders about classical field theories in general.

In the Lagrangian formalism, a field theory is defined by its action, a local and real functional of the fields $\{\Phi^I\}$ and their derivatives $\{\partial_\mu\Phi^I\}$, that can be expressed as a four-dimensional integral of a Lagrangian density:

$$S[\Phi^I] = \int d^4x\, \mathcal{L}\left(\Phi^I, \partial_\mu\Phi^I\right). \tag{1.99}$$

The Lagrangian $L = \int d^3x\, \mathcal{L}$ has the dimension of an energy, or equivalently of an inverse length, therefore (i) the Lagrangian density $\mathcal{L}$ has dimension of (energy)^4 and the action S is dimension-less in natural units.

Classical solutions of the field theory defined by the action functional (1.99) correspond to stationnary points of this action. Under an infinitesimal variation of the fields $\Phi^I \mapsto \Phi^I + \delta\Phi^I$, we have:

$$\begin{aligned}
\delta S[\Phi^I] &= \int d^4x\, \left(\mathcal{L}\left(\Phi^I(x) + \delta\Phi^I(x), \partial_\mu\Phi^I(x) + \partial_\mu\delta\Phi^I(x)\right) - \mathcal{L}\left(\Phi^I(x), \partial_\mu\Phi^I(x)\right)\right) \\
&= \int d^4x\, \left(\frac{\partial\mathcal{L}}{\partial\Phi^I(x)}\delta\Phi^I(x) + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\Phi^I(x)\right)}\partial_\mu\delta\Phi^I(x)\right) + \mathcal{O}\left(\delta\Phi^2\right) \\
&\overset{\text{IBP}}{=} \int d^4x\, \left(\frac{\partial\mathcal{L}}{\partial\Phi^I(x)}\delta\Phi^I(x) - \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\Phi^I(x)\right)}\right)\delta\Phi^I(x) + \int d^4x\, \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\Phi^I(x)\right)}\delta\Phi^I(x)\right) \\
&\qquad + \mathcal{O}\left(\delta\Phi^2\right)
\end{aligned} \tag{1.100}$$

[8] Of course if a^μ is a pseudo-vector the former contraction gives a pseudo-scalar and the latter a scalar.

Assuming that the fields and all their derivatives vanish fast enough at infinity, one obtains from the vanishing of the first-order variation of the action the Euler–Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial \Phi^I}(x) - \frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial\big(\partial_\mu \Phi^I\big)}(x) = 0\,. \tag{1.101}$$

Without entering into much detail, the Euler–Lagrange equations can be also phrased in terms of the *functional derivative*. For a local functional $F[\phi] = \int d^4x\, f[\Phi(x)]$, one defines

$$\frac{\delta F[\phi]}{\delta \phi(x)} \stackrel{\text{def.}}{=} \lim_{\epsilon\to 0}\frac{F[\phi + \epsilon\delta_x] - F[\phi]}{\epsilon}\,, \tag{1.102}$$

in terms of the Dirac distribution δ_x, such that the the stationnary point condition is simply

$$\frac{\delta S[\Phi^I]}{\delta \Phi^I(x)} = 0\,. \tag{1.103}$$

*
* *

A fundamental role is played by symmetries in quantum field theories, both space-time symmetries and *internal symmetries* acting on field space. Given the general principles of QFT, they essentially determine fully the content of a given particle physics model.

Noether theorem. A continuous transformation is a symmetry of a classical field theory defined by an action principle if, under an infinitesimal transformation of the fields $\delta_\alpha \Phi^I$, the Lagrangian density $\mathcal{L}$ shifts by a total derivative:

$$\mathcal{L} \;\stackrel{\alpha}{\longmapsto}\; \mathcal{L} + \partial_\mu \Lambda^\mu_\alpha\,, \tag{1.104}$$

in such a way that the equations of motion are unchanged.[9]

The variation of the action under $\Phi^I(x) \mapsto \Phi^I(x) + \delta_\alpha \Phi^I(x)$, where $\Phi^I(x)$ solves the equations of motion, is given by

$$\delta S = \int d^4x \Big(\underbrace{\frac{\partial \mathcal{L}}{\partial \Phi^I} - \frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial\big(\partial_\mu \Phi^I\big)}}_{=\,0}\Big)\delta_\alpha \Phi^I + \int d^4x\, \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial\big(\partial_\mu \Phi^I\big)}\delta_\alpha \Phi^I\right) \stackrel{!}{=} \int d^4x\, \partial_\mu \Lambda^\mu_\alpha \tag{1.105}$$

We have then the conserved current:

$$J^\mu_\alpha \stackrel{\text{def.}}{=} \frac{\partial \mathcal{L}}{\partial\big(\partial_\mu \Phi^I\big)}\delta_\alpha \Phi^I - \Lambda^\mu_\alpha\,, \quad \partial_\mu J^\mu_\alpha = 0\,, \tag{1.106}$$

[9] There can be actually more general transformations that do not leave the action invariant but preserve the equations of motion.

and the associated conserved Noether charge by integrating over space:

$$Q_\alpha \stackrel{\text{def.}}{=} \int d^3x\; J^0_\alpha \quad , \qquad \frac{d}{dt} Q_\alpha = 0\,. \tag{1.107}$$

Stress-energy tensor. A fundamental object in any field theory is the *stress-energy tensor*, that can be approached from several ways, one of them, which will not be developped here, being through the coupling to an arbitrary space-time metric. Let us consider first invariance under space-time translations:

$$\Phi^I(x) \;\stackrel{\alpha}{\longmapsto}\; \Phi^I(x-\alpha) = \Phi^I(x) - \alpha^\mu \partial_\mu \Phi^I(x) + \mathcal{O}(\alpha^2)\,, \tag{1.108}$$

under which the Lagrangian density, which is by itself a scalar, transforms in the same way:

$$\mathcal{L}\big(\Phi^I(x), \partial_\mu \Phi^I(x)\big) \;\stackrel{\alpha}{\longmapsto}\; \mathcal{L}\big(\Phi^I(x-\alpha), \partial_\mu \Phi^I(x-\alpha)\big) = \mathcal{L}\big(\Phi^I(x), \partial_\mu \Phi^I(x)\big) - \alpha^\mu \partial_\mu \mathcal{L} + \mathcal{O}(\alpha^2)\,. \tag{1.109}$$

We obtain then the conserved current:

$$J^\mu_\alpha = -\alpha_\nu T^{\mu\nu} \quad , \qquad T^{\mu\nu} \stackrel{\text{def.}}{=} \frac{\partial \mathcal{L}}{\partial\big(\partial_\mu \Phi^I\big)} \partial^\nu \Phi^I - \eta^{\mu\nu} \mathcal{L}\,. \tag{1.110}$$

Note that this definition of the stress-energy tensor is not unique; in particular, this one is not necessarily symmetric. The $(0,0)$ component of the stress-energy tensor gives naturally the energy density of the field:

$$\epsilon = T^{00} = \frac{\partial \mathcal{L}}{\partial \dot\Phi^I} \dot\Phi^I - \mathcal{L}\,, \tag{1.111}$$

while the $(0,i)$ components give the components of the space momentum density:

$$p^i = T^{0i} = \frac{\partial \mathcal{L}}{\partial \dot\Phi^I} \partial^i \Phi^I\,. \tag{1.112}$$

Hamiltonian formulation. The Hamiltonian formalism in field theory is formulated in a very similar way as in classical mechanics.

We first define the momentum field $\Pi_I(x)$ *conjugate* to the field Φ^I through the usual relation:

$$\Pi_I(x) \stackrel{\text{def.}}{=} \frac{\delta L}{\delta \dot\Phi^I(x)} = \frac{\partial \mathcal{L}}{\partial \dot\Phi^I}(x)\,. \tag{1.113}$$

and define the Hamiltonian density $\mathcal{H}$ (which is just the energy density expressed in canonical variables) through a Legendre transform, then the Hamiltonian H by integrating over the whole space:

$$\mathcal{H}(\Phi^I, \Pi_I) = \Pi_I \dot\Phi^I - \mathcal{L}\;, \quad H = \int d^3x\; \mathcal{H}\,. \tag{1.114}$$

The field theory in Hamiltonian form naturally satisfies the Hamilton equations of motion:

$$\dot{\Phi}^I(x) = \frac{\delta H}{\delta \Pi_I(x)} = \frac{\partial \mathcal{H}}{\partial \Pi_I}(x) \quad , \qquad \dot{\Pi}_I(x) = -\frac{\delta H}{\delta \Phi^I(x)} = -\frac{\partial \mathcal{H}}{\partial \Phi_I}(x)\,. \tag{1.115}$$

Poisson brackets. Let us consider two functionals in the conjugate variables (ϕ^I, Π_I), written as integrals of local densities over space:

$$A = \int d^3x\, a(\phi^I, \Pi_I) \quad , \qquad B = \int d^3x\, b(\phi^I, \Pi_I)\,. \tag{1.116}$$

We define the *equal time* Poisson bracket between these functionnals as:

$$\{\, A,\, B\,\} \stackrel{\text{def.}}{=} \int d^3x \left(\frac{\delta A}{\delta \Phi^I(t,\vec{x})} \frac{\delta B}{\delta \Pi_I(t,\vec{x})} - \frac{\delta A}{\delta \Pi_I(t,\vec{x})} \frac{\delta B}{\delta \Phi^I(t,\vec{x})} \right)\,. \tag{1.117}$$

Let us consider the canonical variables in field space expressed as functionals: $\Phi^I(t,\vec{x}) = \int d^3y\, \Phi^I(t,\vec{y})\delta^{(3)}(\vec{x}-\vec{y})$ and $\Pi_J(t,\vec{x}) = \int d^3y\, \Pi_J(t,\vec{y})\delta^{(3)}(\vec{x}-\vec{y})$. We find evidently the canonical Poisson brackets:

$$\begin{aligned} \left\{\, \Phi^I(t,\vec{x}),\, \Pi_J(t,\vec{y}) \,\right\} &= \int d^3z\, \left(\delta^{IK}\delta^{(3)}(\vec{x}-\vec{z})\delta_{JK}\delta^{(3)}(\vec{y}-\vec{z}) - 0 \right) \\ &= \delta^I{}_J\, \delta^{(3)}(\vec{x}-\vec{y})\,. \end{aligned} \tag{1.118}$$

1.2.1 Scalar fields

We consider a set of real scalar fields Φ^I, transforming in the trivial representation of the Lorentz group. A Lorentz-invariant functional can be built using any function of the fields themselves and of the quantities $\partial_\mu \Phi^I \partial^\mu \Phi^J$. At low energies/large scales we expect the terms with the lowest number of derivatives to dominate the dynamics.[10] A rather general Lagrangian density would therefore be of the form:

$$\mathcal{L} = G_{IJ}(\Phi^K)\partial_\mu \Phi^I \partial^\mu \Phi^J - V(\Phi^K)\,. \tag{1.119}$$

The set of functions $G_{IJ}(\Phi^K)$ can be thought as a (positive-definite) metric on field space, where Φ^K are understood as coordinates. If we assume that the potential V has a local minimum, that we set at $\Phi^I = 0$, one can expand the Lagrangian around it and keep only terms at most quadratic in the fields. Upon diagonalizing the (constant) metric G_{IJ} and rescaling the fields, one gets:

$$\mathcal{L} = \tfrac{1}{2}\delta_{IJ}\partial_\mu \Phi^I \partial^\mu \Phi^J - V_0 - a_I \Phi^I - \tfrac{1}{2} m^2_{IJ} \Phi^I \Phi^J\,, \tag{1.120}$$

[10] For instance, in 4 dimensions a term with 4 space-time derivatives would be written like $\Lambda^{-4}(\partial\Phi\partial\Phi)^2$ by dimensional analysis, where Λ is is an energy scale, *a priori* close to the Planck scale.

in terms of a symmetric, positive semi-definite square-mass matrix m^2_{IJ}. The constant term V_0 does not affect the equations of motion and merely shifts the energy density by a constant, therefore will be ignored, at least for now. Likewise, the linear terms will just move the position of the minimum, as can be seen by the field redefinition $\tilde{\Phi}^I = \Phi^I - b^J$ with $a_I = m^2_{IJ} b^J$ and will be also ignored.

In a space-time of dimension d, the Lagrangian density $\mathcal{L}$ should have dimension L^{-d}, therefore, looking at the kinetic term, a scalar field has dimension $L^{1-d/2}$. While it is dimension-less in dimension two, which has applications for 2d conformal field theories, it has dimension L^{-1} in our four-dimensional spacetime (or equivalently the dimension of a mass).

Klein–Gordon field. In the case of a single free scalar field, the Lagrangian reduces to the well-known massive Klein–Gordon model:

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \tfrac{1}{2}m^2\Phi^2 \tag{1.121}$$

This theory does not have any internal continuous symmetry whenever $m \neq 0$, but is invariant under $\Phi \mapsto -\Phi$.[11] The equation of motion is evidently given by the Klein–Gordon equation:

$$\big(\Box + m^2\big)\Phi(t,\vec{x}) = 0 \quad , \qquad \Box = \partial_t^2 - \vec{\nabla}^2 \, . \tag{1.122}$$

The general solution can be expressed as a superposition of plane waves, with the dispersion relation $p^2 - m^2 = 0$, where $p^\mu = (p^0, \vec{p})$. Let us define

$$E_p = \sqrt{\vec{p}^2 + m^2} \, . \tag{1.123}$$

We have then, given that the field Φ takes values in $\mathbb{R}$:[12]

$$\Phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_{\vec{p}} e^{-ip\cdot x} + a^\dagger_{\vec{p}} e^{ip\cdot x} \right) \quad , \quad p^\mu = (E_p, \vec{p}) \, . \tag{1.124}$$

In the Hamiltonian formalism, the corresponding field momentum can also be developed in terms of its Fourier modes:

$$\Pi = \dot{\Phi} = -i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{E_p}{2}} \left(a_{\vec{p}} e^{-ip\cdot x} - a^\dagger_{\vec{p}} e^{ip\cdot x} \right) \, . \tag{1.125}$$

Under a Lorentz transformation Λ, the field Φ, being a scalar, should transform according to:

$$\Phi(x) \overset{\Lambda}{\mapsto} \Phi(\Lambda^{-1}x) \, , \tag{1.126}$$

[11] In the massless case, the Lagrangian is invariant under constant shifts of the field, $\Phi \mapsto \Phi + \alpha$, corresponding to the Noether current $J^\mu = \partial^\mu\Phi$.

[12] We have used Hermitian rather than complex conjugation in this expression, in view of the study of the quantum theory later on.

i.e. the value of Φ at the space-time point of coordinates x^μ is given by its value at its pre-image under the transformation Λ. In the infinitesimal form,

$$\begin{aligned}\delta\Phi &= \Phi(\Lambda^{-1}x) - \Phi(x) = \Phi\left(x^\mu + \tfrac{i}{2}\Omega_{\rho\sigma}\,(\Sigma^{\rho\sigma})^\mu{}_\nu\, x^\nu + \mathcal{O}(\Omega^2)\right) - \Phi(x^\mu) \\ &= -\tfrac{1}{2}\Omega_{\rho\sigma}\,(\eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\sigma\mu}\eta^{\rho\nu})\,x_\nu\partial_\mu\Phi(x) + \mathcal{O}(\Omega^2) \\ &= -\Omega^\rho{}_\sigma x^\sigma\partial_\rho\Phi(x) + \mathcal{O}(\Omega^2)\,. \end{aligned} \tag{1.127}$$

The Lagrangian density (1.121), being a local scalar quantity as well, should transform in a similar way:

$$\delta\mathcal{L} = -\Omega^\rho{}_\sigma x^\sigma\partial_\rho\mathcal{L} = -\partial_\rho\left(\Omega^\rho{}_\sigma x^\sigma\mathcal{L}\right)\,, \tag{1.128}$$

using that $\operatorname{Tr}\Omega = 0$ in the last step. We obtain then from the Noether theorem:

$$\mathcal{J}^\mu = \partial^\mu\Phi\big(-\Omega^\rho{}_\sigma x^\sigma\partial_\rho\Phi\big) + \Omega^\mu{}_\sigma x^\sigma\mathcal{L} = -\Omega_{\rho\sigma}x^\sigma\left(\partial^\mu\Phi\partial^\sigma\Phi - \eta^{\rho\mu}\mathcal{L}\right) = -\Omega_{\rho\sigma}x^\sigma T^{\rho\mu}\,. \tag{1.129}$$

So one has overall six conserved Noether currents associated with the six generators of the Lorentz group (3 rotations, 3 boosts):

$$\boxed{(\mathcal{J}^{\rho\sigma})^\mu = x^\rho T^{\sigma\mu} - x^\sigma T^{\rho\mu}} \tag{1.130}$$

Free O(n) vector model and complex scalars. Another interesting case is a model with n real scalar fields of equal mass m. We consider the Lagrangian:

$$\mathcal{L} = \tfrac{1}{2}\sum_{I=1}^{n}\partial_\mu\Phi^I\partial^\mu\Phi^I - \tfrac{1}{2}m^2\sum_{I=1}^{n}\left(\Phi^I\right)^2\,. \tag{1.131}$$

The equations of motion are just given by n copies of the Klein–Gordon equation with equal masses: $(\Box + m^2)\Phi^I = 0$.

The fields $\Phi^{1,\cdots,n}$ can be seen as the components of an $\mathbb{R}^n$ vector Φ in field space, thereby transforming in the vector representation of $O(n)$:

$$\Phi \mapsto M\,\Phi\,, \quad M^T M = \mathbb{I}_n\,, \tag{1.132}$$

leaving the Lagrangian density (1.131) invariant. Infinitesimal transformations of the fields are expressed in terms of an anti-symmetric $n\times n$ real matrix ϖ:

$$\Phi^I \;\longmapsto\; \Phi^I + \varpi^I{}_J\Phi^J\,, \quad \varpi_{IJ} = \delta_{IK}\varpi^K{}_J = -\varpi_{JI}\,. \tag{1.133}$$

Like for the Lorentz algebra one can choose a basis of imaginary antisymmetric matrices as generators of the $\mathfrak{o}_n$ algebra,

$$(H^{IJ})^{KL} = i\left(\delta^{IK}\delta^{JL} - \delta^{JK}\delta^{IL}\right)\,, \tag{1.134}$$

and expand $\varpi^I{}_J = -\frac{i}{2}(\alpha_{KL}H^{KL})^I{}_J$. The associated Noether currents are given by

$$\mathcal{J}^{KL}_\mu = \left(H^{KL}\right)^I{}_J\Phi^J\partial_\mu\Phi_I\,, \quad \mathcal{J}^{KL} = -\mathcal{J}^{LK}\,. \tag{1.135}$$

In the case of a two-field model, it is useful to assemble the fields Φ^1 and Φ^2 into a complex-valued field:

$$\Phi \stackrel{\text{def.}}{=} \frac{\Phi^1 + i\Phi^2}{\sqrt{2}}\,, \tag{1.136}$$

in terms of which the Lagrangian (1.131) becomes:

$$\mathcal{L} = \partial_\mu \Phi^* \partial^\mu \Phi - m^2\, \Phi^* \Phi \tag{1.137}$$

It is convenient to view Φ and Φ^* as independent fields, for instance while deriving the equations of motion or computing Noether currents. The appropriate viewpoint is to first analytically continue Φ^1 and Φ^2 to independent complex-valued fields. Then the independent fields Φ and Φ^* are obtained by a unitary change of basis in field space:

$$\begin{pmatrix} \Phi \\ \Phi^* \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ 1/\sqrt{2} & -i/\sqrt{2} \end{pmatrix} \begin{pmatrix} \Phi^1 \\ \Phi^2 \end{pmatrix}\,. \tag{1.138}$$

We get for instance the equation of motion for Φ^*:

$$\frac{\partial \mathcal{L}}{\partial \Phi^*} = -m^2 \Phi = \partial_\mu \frac{\partial \mathcal{L}}{\partial\big(\partial_\mu \Phi^*\big)} = \Box \Phi\,. \tag{1.139}$$

The canonical momenta associated with Φ and Φ^* are:

$$\Pi_\Phi = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = \dot{\Phi}^* \quad , \qquad \Pi_{\Phi^*} = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}^*} = \dot{\Phi}\,. \tag{1.140}$$

The theory of a complex scalar field defined by the Lagrangian (1.137) is invariant under the $U(1)$ transformation $\Phi \mapsto \exp(i\theta)\Phi$, $\Phi^* \mapsto \exp(-i\theta)\Phi^*$. The associated Noether current is given by:

$$\mathcal{J}^\mu = i\big(\Phi \partial^\mu \Phi^* - \Phi^* \partial^\mu \Phi\big)\,, \tag{1.141}$$

which can naturally be deduced from (1.135). As mentioned earlier, this theory is invariant under $O(2)$, rather than under $SO(2) \sim U(1)$. The extra $\mathbb{Z}_2$ symmetry acts on the complex field Φ as:

$$\Phi \longmapsto \Phi^*\,. \tag{1.142}$$

This transformation, which maps a field of $U(1)$ charge $q = +1$ to a field of opposite charge, is actually a manifestation of a fundamental discrete symmetry of Nature, *charge conjugation*, that maps every particle to the corresponding anti-particle, as we will see later.

The mode expansion of the complex scalar field is given in terms of two *independent* sets of modes $\{a_{\vec{p}}\}$ and $\{b_{\vec{p}}\}$:

$$\Phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_{\vec{p}} e^{-ip\cdot x} + b^\dagger_{\vec{p}} e^{ip\cdot x} \right) , \tag{1.143a}$$

$$\Phi^\dagger(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(b_{\vec{p}} e^{-ip\cdot x} + a^\dagger_{\vec{p}} e^{ip\cdot x} \right) . \tag{1.143b}$$

1.2.2 Free Spinor fields

As for the scalar fields, Lagrangian densities for spinor fields are written in terms of Lorentz scalars built out of the fields and their derivatives. Let us first consider a single Weyl spinor, for instance a left-handed spinor χ, *i.e.* an anti-commuting field transforming in the $(1/2, 0)$ representation of $\mathsf{Spin}(1,3)$. A general bilinear Lagrangian density is provided by:

$$\mathcal{L} = i\chi^\dagger \bar{\sigma}^\mu \partial_\mu \chi - \frac{m}{2}(\chi\chi + \chi^\dagger\chi^\dagger) = i\chi^\dagger_{\dot{\alpha}} \bar{\sigma}^{\mu\,\dot{\alpha}\alpha} \partial_\mu \chi_\alpha - \frac{m}{2}\left(\epsilon^{\beta\alpha}\chi_\alpha\chi_\beta + \epsilon^{\dot{\alpha}\dot{\beta}}\chi^\dagger_{\dot{\alpha}}\chi^\dagger_{\dot{\beta}}\right) . \tag{1.144}$$

By construction this expression is Hermitian, up to a total derivative, provided that $m \in \mathbb{R}$.[13] We have indeed:

$$\left(i\chi^\dagger_{\dot{\alpha}} \bar{\sigma}^{\mu\,\dot{\alpha}\alpha} \partial_\mu \chi_\alpha \right)^\dagger = -i\partial_\mu \chi^\dagger_{\dot{\alpha}} \chi_\alpha \bar{\sigma}^{\mu\,\dot{\alpha}\alpha} = i\chi^\dagger_{\dot{\alpha}} \partial_\mu \chi_\alpha \bar{\sigma}^{\mu\,\dot{\alpha}\alpha} - i\partial_\mu \left(\chi^\dagger_{\dot{\alpha}} \chi_\alpha \bar{\sigma}^{\mu\,\dot{\alpha}\alpha} \right) . \tag{1.145}$$

The equations of motion following from this Lagrangian read:

$$0 = \frac{\partial \mathcal{L}}{\partial \chi^\dagger_{\dot{\alpha}}} = i\bar{\sigma}^{\mu\,\dot{\alpha}\alpha} \partial_\mu \chi_\alpha - m\chi^{\dagger\,\dot{\alpha}} , \tag{1.146a}$$

$$0 = -\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \chi_\alpha)} + \frac{\partial \mathcal{L}}{\partial \chi_\alpha} = i \underbrace{\bar{\sigma}^{\mu\,\dot{\alpha}\alpha}}_{\sigma^\mu_{\beta\dot{\beta}}\,\epsilon^{\beta\alpha}\epsilon^{\dot{\beta}\dot{\alpha}}} \partial_\mu \chi^\dagger_{\dot{\alpha}} - m\epsilon^{\beta\alpha}\chi_\beta$$

$$\implies i\sigma^\mu_{\beta\dot{\beta}} \partial_\mu \chi^{\dagger\,\dot{\beta}} - m\chi_\beta = 0 . \tag{1.146b}$$

As one can see, those equations mix the spinor χ with its Hermitian conjugate $\chi^\dagger$; in the quantized theory, it will mean mixing of the particle with its own anti-particle, in a sense that will be made more precise below.

Weyl spinor field. In the massless case, the coupling between the spinor field and its conjugate disappears and χ obeys what is known as the *Weyl equation*:

$$\bar{\sigma}^\mu \partial_\mu \chi = \bar{\sigma}^{\mu\,\dot{\alpha}\alpha} \partial_\mu \chi_\alpha = 0 \tag{1.147}$$

[13] Otherwise a phase can be absorbed in a redefinition of the spinor field χ.

Multiplying both terms on the left by $\sigma^\nu \partial_\nu$, one gets $\Box\chi = 0$, *i.e.* χ satisfies the massless Klein–Gordon equation.

The Lagrangian density from which the Weyl equation of motion follows, is given for a left-handed spinor by:

$$\mathcal{L}_{\rm L} = i\chi^\dagger \bar\sigma^\mu \partial_\mu \chi \,. \tag{1.148}$$

It is invariant under chiral $U(1)$ rotations of the spinor field:

$$\chi \longmapsto e^{i\theta}\chi \,, \quad \theta \in [0, 2\pi[\,, \tag{1.149}$$

with a corresponding Hermitian Noether current:

$$\mathcal{J}^\mu_{\rm w} = \chi^\dagger \bar\sigma^\mu \chi \,. \tag{1.150}$$

This symmetry is explicitly broken in the presence of a mass term as in eqn. (1.144). More generally, a set of n Weyl spinors of the same chirality is invariant under the action of $U(n)$ rotations.

Let us consider plane-wave solutions of the Weyl equation (1.147), with four-momentum $p^\mu = (p^0, \vec p)$:

$$\chi(x) = v(\vec p) e^{-ip\cdot x} \quad \Longrightarrow \quad \bar\sigma^\mu p_\mu v(\vec p) = \big(p^0 + \vec p \cdot \vec\sigma\big) v(\vec p) = 0 \,. \tag{1.151}$$

For $p^0 > 0$ (positive energy solutions), writting $\vec p = p^0\, \vec n$ with $\vec n^2 = 1$ using the dispersion relation $p^2 = 0$, one has

$$v(\vec p) = -\frac{\vec p \cdot \vec\sigma}{p_0}\, v(\vec p) = -\vec n \cdot \vec\sigma \; v(\vec p) \,. \tag{1.152}$$

Hence $v(\vec p)$ is an eigenvector of

$$h \stackrel{\rm def.}{=} \vec n \cdot \vec\sigma/2 \,, \tag{1.153}$$

the projection of the spin along the direction of the momentum $\vec p$, called the *helicity* of the particle, with eigenvalue $-1/2$.

Likewise, a right-handed spinor ζ in the $(0, 1/2)$ representation of $Spin(1,3)$ is described by the Lagrangian density

$$\mathcal{L}_{\rm R} = i\zeta^\dagger \sigma^\mu \partial_\mu \zeta \tag{1.154}$$

and satisfies the equation of motion:

$$\sigma^\mu\, \partial_\mu \zeta = \sigma^\mu_{\alpha\dot\alpha}\, \partial_\mu \zeta^{\dot\alpha} = 0 \,. \tag{1.155}$$

With a similar reasoning as before, for a plane-wave solution of the form $\zeta(x) = \varphi(\vec p) e^{-ip\cdot x}$, one gets

$$\vec n \cdot \vec\sigma \; \varphi(\vec p) = \varphi(\vec p) \,, \tag{1.156}$$

i.e. $\varphi(\vec p)$ is a helicity eigenvector with eigenvalue $+1/2$. We have thus realized that *massless* fermions of definite chirality have definite helicity:

- Massless Weyl fermions of helicity $h = -1/2$, called *left-handed*, transform in the $(1/2, 0)$ representation of $\mathrm{Spin}(1,3)$
- Massless Weyl fermions of helicity $h = +1/2$, called *right-handed*, transform in the $(0, 1/2)$ representation of $\mathrm{Spin}(1,3)$

For this reason, one can use interchangeably the concepts of helicity and chirality *for a massless Weyl fermion*. For a massive particle the concept of helicity is not Lorentz-invariant: it is always possible with an appropriate boost to reverse the direction of the momentum with respect to the observer.

An arbitrary field theory based of Weyl fermions is not invariant under space parity. While this may not be an issue – after all, the standard model of particle physics is not invariant under parity, more later – it is desirable to build a parity-invariant theory for spinor fields, in view of describing quantum electrodynamics, which is expected to be parity-invariant, since the corresponding classical theory is.

Dirac spinor field. As explained in section 1.1.2 a parity-invariant theory can be built in terms of bi-spinors $\Psi_{\mathrm{W}} = \left(\begin{smallmatrix}\chi\\ \zeta\end{smallmatrix}\right)$. First, the kinetic terms of left-handed and right-handed Weyl spinors, see eqns. (1.148,1.155), can be assembled into a Dirac kinetic term,

$$\mathcal{L}_{\mathrm{L}} + \mathcal{L}_{\mathrm{R}} = i\chi^\dagger \bar\sigma^\mu \partial_\mu \chi + i\zeta^\dagger \sigma^\mu \partial_\mu \zeta = i \begin{pmatrix}\chi^\dagger & \zeta^\dagger\end{pmatrix} \begin{pmatrix}0 & \mathbb{I}\\ \mathbb{I} & 0\end{pmatrix} \begin{pmatrix}0 & \sigma^\mu\\ \bar\sigma^\mu & 0\end{pmatrix} \partial_\mu \begin{pmatrix}\chi\\ \zeta\end{pmatrix} = i\overline{\Psi} \not\partial \Psi\,. \quad (1.157)$$

This could have been obtained directly from symmetry considerations. Since the Dirac bilinear $\overline{\Psi}\Psi$ is Lorentz- and parity-invariant, one can add a corresponding mass term, known as a *Dirac mass* term, and obtain the standard Dirac Lagrangian density:

$$\mathcal{L}_{\mathrm{D}} = \overline{\Psi}\big(i\not\partial - m\big)\Psi \quad (1.158)$$

Unlike the Klein–Gordon Lagrangian, see eqn. (1.121), it contains only a single derivative. In a space-time of dimension d a spinor field has dimension $L^{(1-d)/2}$.[14] The Dirac field naturally obeys the famous Dirac equation:

$$\frac{\partial \mathcal{L}_{\mathrm{D}}}{\partial \overline{\Psi}} = 0 \quad \Longrightarrow \quad \big(i\not\partial - m\big)\Psi = 0 \quad (1.159)$$

It implies that Ψ satisfies as well the massive Klein–Gordon equation:

$$0 = \big(-i\not\partial - m\big)\big(i\not\partial - m\big)\Psi = -\big(\underbrace{\gamma^\mu\gamma^\nu}_{\{\gamma^\mu,\gamma^\nu\}/2}\partial_\mu\partial_\nu + m^2\big)\Psi = -(\Box + m^2)\Psi\,. \quad (1.160)$$

Using eqn. (1.75), one finds that the conjugate of the Dirac equation is given by:

$$\overline{\Psi}\big(i\overleftarrow{\not\partial} + m\big) = 0\,, \quad (1.161)$$

[14] Spinors in arbitrary dimension will be considered in the tutorials.

where we have used the convenient shorthand notation:

$$\overline{\Psi}\overleftarrow{\not{\partial}} \stackrel{\text{def.}}{=} \partial_\mu \overline{\Psi}\gamma^\mu . \tag{1.162}$$

It is instructive to split the Dirac Lagrangian density in terms of the two-component Weyl spinors that constitute the bi-spinor Ψ:

$$\mathcal{L}_{\text{D}} = i\chi^\dagger \bar{\sigma}^\mu \partial_\mu \chi + i\zeta^\dagger \sigma^\mu \partial_\mu \zeta - m(\chi^\dagger \zeta + \zeta^\dagger \chi) . \tag{1.163}$$

One notices first that the Dirac mass term mixes the left- and right-handed spinors χ and ζ, in a way ensuring parity invariance. Second, the $U(1)_L \times U(1)_R$ chiral symmetry is broken to a diagonal subgroup by the mass term:

$$\Psi \mapsto e^{i\theta}\Psi \quad \Longleftrightarrow \quad (\chi,\, \zeta) \mapsto (e^{i\theta}\chi,\, e^{i\theta}\zeta) . \tag{1.164}$$

The corresponding Noether current is easily found to be:

$$\mathcal{J}^\mu_{\text{D}} = \overline{\Psi}\gamma^\mu\Psi . \tag{1.165}$$

Exercise: prove that this current is Hermitian. Show that the equation of motion (1.159) ensures that $\partial_\mu \mathcal{J}^\mu_{\text{D}} = 0$*, i.e. that the current is conserved as expected.*

In the massless theory, the full $U(1)_L \times U(1)_R$ symmetry is manifest, at least classically, which is obvious in the Weyl form of the Lagrangian density. The "diagonal" or *vector* combination of these two $U(1)$ actions corresponds to the current (1.167). The "anti-diagonal", or *axial* $U(1)$ acts as

$$\Psi \mapsto e^{i\theta\gamma^5}\Psi \quad \Longleftrightarrow \quad (\chi,\, \zeta) \mapsto (e^{-i\theta}\chi,\, e^{i\theta}\zeta) . \tag{1.166}$$

The corresponding Noether current, known as the *axial current*, is a pseudo-quadrivector given by:

$$\mathcal{J}^\mu_{\text{D}} = \overline{\Psi}\gamma^\mu\gamma^5\Psi . \tag{1.167}$$

Exercise: prove that this current is Hermitian. Show that the equation of motion (1.159) ensures that the current is conserved if $m = 0$*.*
The conservation of this current may not hold in the quantum theory.

Finally let us derive the Noether current associated with Lorentz transformations. The field Ψ should transform according to:

$$\Psi(x) \overset{\Lambda}{\mapsto} D(\Lambda)\Psi(\Lambda^{-1}x) , \tag{1.168}$$

where $D(\Lambda)$ is the 4×4 matrix associated with the Lorentz transformation Λ, see eqn. (1.86).

$$\begin{aligned}\delta\Psi &= D(\Lambda)\Phi(\Lambda^{-1}x) - \Phi(x) = \left(1 - \tfrac{i}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right)\Psi\left(x^\mu + \tfrac{i}{2}\Omega_{\rho\sigma}\,(\Sigma^{\rho\sigma})^\mu{}_\nu\, x^\nu + \mathcal{O}(\Omega^2)\right) - \Psi(x^\mu) \\ &= -\Omega_{\rho\sigma}\left(x^\sigma\partial^\rho + \tfrac{i}{2}S^{\rho\sigma}\right)\Psi(x) + \mathcal{O}(\Omega^2) .\end{aligned} \tag{1.169}$$

The Lagrangian density (1.121), being a local scalar quantity, should transform as:

$$\delta\mathcal{L} = -\Omega^{\rho}{}_{\sigma} x^{\sigma} \partial_{\rho}\mathcal{L} = -\partial_{\rho}\left(\Omega^{\rho}{}_{\sigma} x^{\sigma}\mathcal{L}\right) . \tag{1.170}$$

However for the Dirac field $\mathcal{L} = 0$ whenever Ψ satisfies its equation of motion. We obtain then from the Noether theorem the set of conserved currents:

$$(\mathcal{J}^{\rho\sigma})^{\mu} = \overline{\Psi}\gamma^{\mu}\left(x^{\sigma}\big(-i\partial^{\rho}\big) + \tfrac{1}{2}S^{\rho\sigma}\right)\Psi , \tag{1.171}$$

and the corresponding set of conserved charges:

$$\boxed{Q^{\rho\sigma} = \int d^{3}x\, \Psi^{\dagger}\left(x^{\sigma}\big(-i\partial^{\rho}\big) + \tfrac{1}{2}S^{\rho\sigma}\right)\Psi .} \tag{1.172}$$

Compared to the case of a scalar field, see eqn. (1.130), there is an extra term taking into account the mixing of the spinor components.

Charge conjugation and Majorana spinor field. So far we have constructed two free theories of massive spinors. First the Lagrangian density (1.144) describing a complex two-component spinor in the $(1/2, 0)$ representation with a mass term known as *Majorana mass*, that couples the spinor to its Hermitian conjugate and breaks the $U(1)_L$ symmetry of the Weyl kinetic term (1.148). Second, the Lagrangian density (1.158) for a massive Dirac spinor, *i.e.* a 4-component bi-spinor in the $(1/2, 0) \oplus (0, 1/2)$, with a *Dirac mass* that couples the left- and right-handed spinors and preserves a diagonal $U(1)$ symmetry.

To understand better the former theory, and how it can be related naturally to the latter, let us describe how *charge conjugation*, that maps every particle to the corresponding anti-particle as mentioned earlier, acts on the bi-spinor fields. Anticipating a bit, electrons will be described by quantized Dirac fields and the $U(1)$ symmetry (1.164) will be used to couple them to electromagnetism; so a positron should transform under this $U(1)$ symmetry with an opposite charge. Let us define the charge-conjugation matrix, in the Weyl basis:

$$C \overset{\text{def.}}{=} \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} = i\gamma^{2}\gamma^{0} , \quad C^{-1} = \begin{pmatrix} \epsilon^{\alpha\beta} & 0 \\ 0 & \epsilon_{\dot{\alpha}\dot{\beta}} \end{pmatrix} = -C . \tag{1.173}$$

We define then the charge-conjugate of a Dirac spinor Ψ, expressed as (1.77) in the Weyl basis, as follows:

$$\boxed{\Psi^{C} \overset{\text{def.}}{=} C\overline{\Psi}^{T} = \begin{pmatrix} \zeta^{\dagger}_{\alpha} \\ \chi^{\dagger\,\dot{\alpha}} \end{pmatrix} .} \tag{1.174}$$

It is easy to see that Ψ^{C} transforms in the same way as Ψ under Lorentz transformations, *i.e.* in the $(1/2, 0) \oplus (0, 1/2)$ representation, see the tutorials.

Under the $U(1)$ symmetry (1.164), as expected $\Psi^c \mapsto e^{-i\theta}\Psi^c$, *i.e.* with an opposite phase compared to Ψ. Since Ψ and Ψ^c transform in the same way under the action of the Lorentz-group, one can impose a Lorentz-invariant condition giving what is known as a *Majorana spinor* [1], a concept introduced by the Italian physicist Ettore Majorana:

$$\Psi_M = \Psi_M^c \iff \Psi_M = \begin{pmatrix} \chi_\alpha \\ \chi^{\dagger\dot{\alpha}} \end{pmatrix} . \tag{1.175}$$

Majorana spinors have half the number of degrees of freedom of Dirac spinors and cannot carry a charge; they transform in a real four-dimensional representation of $Spin(1,3)$, as can be seen by choosing an appropriate basis of gamma matrices, the Majorana basis.
Exercise: show that the Dirac-conjugate of a Majorana spinor is given by $\overline{\Psi}_M = -\Psi_M^T C^{-1}$.

The Lagrangian density for a free Majorana massive spinor can be obtained rather directly from the Dirac Lagrangian, by adding a factor $1/2$ to take into account that the bi-spinor is morally speaking a real rather than complex spinor field:

$$\mathcal{L}_M = -\frac{1}{2}\Psi_M^T C^{-1}\left(i\partial\!\!\!/ - m\right)\Psi_M , \tag{1.176}$$

where Ψ_M satisfies the constraint (1.175). Using the identity (1.64), one gets the same as eqn. (1.144), up to a total derivative:

$$\mathcal{L}_M = i\chi^\dagger\bar{\sigma}^\mu\partial_\mu\chi - \frac{m}{2}\left(\chi\chi + \chi^\dagger\chi^\dagger\right) . \tag{1.177}$$

Plane wave solutions. Let us construct classical solutions of the Dirac equation (1.159) in the massive case ($m \neq 0$). *Positive energy* solutions with momentum $\vec{p}$ are of the form:

$$\Psi(x) = u(\vec{p})e^{-ip\cdot x} \quad , \qquad p = (E_p, \vec{p}) , \tag{1.178}$$

with $E_p = \sqrt{\vec{p}^2 + m^2}$ as before. The spinor $u(\vec{p})$ satisfies:

$$\boxed{\left(p\!\!\!/ - m\right)u(\vec{p}) = 0 ,} \tag{1.179}$$

Its Dirac conjugate is obtained using eqn. (1.75)

$$0 = u^\dagger(\vec{p})\left(p_\mu\gamma^{\mu\,\dagger} - m\right) = u^\dagger(\vec{p})\left(p_\mu\gamma^0\gamma^\mu\gamma^0 - m\right) = \bar{u}(\vec{p})(p\!\!\!/ - m)\gamma^0 , \tag{1.180}$$

hence

$$\boxed{\bar{u}(\vec{p})(p\!\!\!/ - m) = 0} \tag{1.181}$$

In the Weyl representation, see eqn. (1.74), the identity (1.179) can be written in matrix form as:

$$\begin{pmatrix} -m & \sigma^\mu p_\mu \\ \bar{\sigma}^\mu p_\mu & -m \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{cases} \sigma^\mu p_\mu u_2 = m u_1 \\ \bar{\sigma}^\mu p_\mu u_1 = m u_2 \end{cases} , \tag{1.182}$$

where $u_{1,2}$ are two-component constant spinors. These two equations are compatible with each other thanks to the obvious identity:

$$p_\mu \sigma^\mu p_\nu \bar{\sigma}^\nu = (p_0)^2 - (\vec{p}\cdot\vec{\sigma})^2 = p^2 = m^2 \,. \tag{1.183}$$

This identity also suggests that a solution of the system (1.182) is given by:

$$u(\vec{p}) = \begin{pmatrix} \frac{1}{m} p_\mu \sigma^\mu \zeta \\ \zeta \end{pmatrix} , \tag{1.184}$$

where ζ is arbitrary.

To make this solution more symmetric, let us choose a Weyl spinor η such that

$$\sqrt{p_\mu \bar{\sigma}^\mu}\,\eta = \zeta \,, \tag{1.185}$$

that we can also choose to be normalized as $\eta^\dagger \eta = 1$. We have then:

$$\tfrac{1}{m} p_\mu \sigma^\mu \zeta = \tfrac{1}{m} \sqrt{p_\mu \sigma^\mu p_\nu \bar{\sigma}^\nu}\, \sqrt{p_\mu \sigma^\mu}\,\eta = \sqrt{p_\mu \sigma^\mu}\,\eta \,, \tag{1.186}$$

therefore the general solution, expressed in terms of an arbitrary spinor η, reads:

$$\boxed{u(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta \\ \sqrt{p\cdot\bar{\sigma}}\,\eta \end{pmatrix}} \tag{1.187}$$

Associating such field configuration to a particle of mass m – as will be made explicit later – one can go to its rest frame by a Lorentz transform, in which case eqn. (1.187) becomes:

$$u(\vec{p}) = \sqrt{m} \begin{pmatrix} \eta \\ \eta \end{pmatrix} . \tag{1.188}$$

A basis of the two-dimensional vector space of solutions to eqn. (1.179) is provided by:

$$\eta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \qquad \eta_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{1.189}$$

According to eqn. (1.76), the two-component spinor η transforms under rotations around the particle position as $\eta \mapsto \exp(-\frac{i}{2}\theta\,\vec{n}\cdot\vec{\sigma})\eta$, *i.e.* η is an $SU(2)$ spinor as in non-relativistic quantum mechanics and $\eta_{1,2}$ correspond to the eigenstates $|\pm\rangle$ of S_z.

From this solution in the rest frame one can naturally reach a solution with arbitrary 4-momentum $p^\mu = (E, \vec{p})$ with an appropriate Lorentz transform. Let us consider for instance a 3-momentum along z, *i.e.* $p^\mu = (E, 0, 0, p^3)$ with $E = m\cosh w$ and $p^3 = -m\sinh w$. One checks explicitly that the bi-spinor associated with the boosted 4-momentum reads:

$$u(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta \\ \sqrt{p\cdot\bar{\sigma}}\,\eta \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \sqrt{E-p^3} & 0 \\ 0 & \sqrt{E+p^3} \end{pmatrix}\eta \\ \begin{pmatrix} \sqrt{E+p^3} & 0 \\ 0 & \sqrt{E-p^3} \end{pmatrix}\eta \end{pmatrix} = \sqrt{m} \begin{pmatrix} \begin{pmatrix} e^{w/2} & 0 \\ 0 & e^{-w/2} \end{pmatrix}\eta \\ \begin{pmatrix} e^{-w/2} & 0 \\ 0 & e^{w/2} \end{pmatrix}\eta \end{pmatrix} . \tag{1.190}$$

From the spinor transformation rules under boosts, see eqns. (1.46,1.59), one gets the same:

$$u(\vec{p}) = \sqrt{m}\begin{pmatrix} e^{+w\sigma^3/2}\eta \\ e^{-w\sigma^3/2}\eta \end{pmatrix} = \sqrt{m}\begin{pmatrix} \begin{pmatrix} e^{w/2} & 0 \\ 0 & e^{-w/2} \end{pmatrix}\eta \\ \begin{pmatrix} e^{-w/2} & 0 \\ 0 & e^{w/2} \end{pmatrix}\eta \end{pmatrix} . \tag{1.191}$$

Naturally the same holds for an arbitrary 3-momentum $\vec{p}$, reached by a suitable Lorentz transformation. We obtain then a basis in an arbitrary frame:

$$u_s(\vec{p}) \overset{\text{def.}}{=} \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta_s \\ \sqrt{p\cdot\bar{\sigma}}\,\eta_s \end{pmatrix} \quad , \qquad s = 1,2 \tag{1.192}$$

The Dirac equation (1.159) also admits *negative energy* plane-wave solutions which are of the form:[15]

$$\Psi(x) = v(\vec{p})e^{ip\cdot x} \quad , \qquad p = (E_p, \vec{p}\,) , \tag{1.193}$$

The spinor $v(\vec{p})$ satisfies in this case:

$$\boxed{\quad (\not{p} + m)v(\vec{p}) = 0 . \quad} \tag{1.194}$$

With a similar reasoning as before, the general solution is of the form:

$$\boxed{\quad v(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta \\ -\sqrt{p\cdot\bar{\sigma}}\,\eta \end{pmatrix} . \quad} \tag{1.195}$$

The general solution of the Dirac equation can be expanded as a sum over of positive- and negative-energy modes in terms of two sets of Fourier coefficients $\{b^s_{\vec{p}},\, c^s_{\vec{p}} \in \mathbb{C}\}$ as follows:

$$\Psi(x) = \sum_{s=1}^{2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(b^s_{\vec{p}}\,u_s(\vec{p})e^{-ip\cdot x} + c^{s\,\dagger}_{\vec{p}}\,v_s(\vec{p})e^{ip\cdot x}\right) , \tag{1.196}$$

where, as for scalar fields, it is understood that $p^\mu = (E_p, \vec{p}\,)$.

In the massless case there is naturally no notion of rest frame. For positive energy plane-waves one can choose a frame in which $p^\mu = (E, 0, 0, E)$, with, $E > 0$, in which case the solution of the Dirac equation $\not{\partial}\Psi = 0$ can be obtained from (1.187) as

$$u(\vec{p}) = \sqrt{2E}\begin{pmatrix} \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)\eta \\ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\eta \end{pmatrix} , \tag{1.197}$$

so a basis is provided by

$$u_1(\vec{p}) = \sqrt{2E}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad , \qquad u_2(\vec{p}) = \sqrt{2E}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} . \tag{1.198}$$

As noticed before, below eqn. (1.151), solutions with definite helicity have definite chirality.

[15] Indeed their time dependence is of the form $\psi \sim e^{-i(-E_p)t}$ with $E_p > 0$ according to our conventions.

Bilinear spinorial identities. Before ending the presentation of the classical theory we will derive some useful identities involving bilinear expressions in the Dirac spinors $u(\vec{p})$ and $v(\vec{p})$.

We consider first the Lorentz-invariant inner product $\bar{u}_s(\vec{p})u_{s'}(\vec{p})$ between to basis spinors. Using (1.183) one finds that:

$$\bar{u}_s(\vec{p})u_{s'}(\vec{p}) = \begin{pmatrix}\eta_s^\dagger\sqrt{p\cdot\sigma} & \eta_s^\dagger\sqrt{p\cdot\bar{\sigma}}\end{pmatrix}\gamma^0\begin{pmatrix}\sqrt{p\cdot\sigma}\,\eta_{s'}\\ \sqrt{p\cdot\bar{\sigma}}\,\eta_{s'}\end{pmatrix} = \eta_s^\dagger\big(\sqrt{p\cdot\sigma\,p\cdot\bar{\sigma}}+\sqrt{p\cdot\bar{\sigma}\,p\cdot\sigma}\big)\eta_{s'}$$
$$\Longrightarrow\quad \bar{u}_s(\vec{p})u_{s'}(\vec{p}) = 2m\delta_{ss'} \tag{1.199}$$

Similarly one has

$$\bar{v}_s(\vec{p})v_{s'}(\vec{p}) = -2m\delta_{ss'}\,. \tag{1.200}$$

while

$$\bar{v}_s(\vec{p})u_{s'}(\vec{p}) = \bar{u}_s(\vec{p})v_{s'}(\vec{p}) = 0\,. \tag{1.201}$$

Next we consider the non-Lorentz-invariant inner product $u_s^\dagger(\vec{p})u_{s'}(\vec{p})$ that will appear in the expression of the Hamiltonian. One gets:

$$u_s^\dagger(\vec{p})u_{s'}(\vec{p}) = \begin{pmatrix}\sqrt{p\cdot\sigma}\,\eta_s^\dagger & \sqrt{p\cdot\bar{\sigma}}\,\eta_s^\dagger\end{pmatrix}\begin{pmatrix}\sqrt{p\cdot\sigma}\,\eta_{s'}\\ \sqrt{p\cdot\bar{\sigma}}\,\eta_{s'}\end{pmatrix} = \big(\underbrace{p\cdot\sigma+p\cdot\bar{\sigma}}_{2p_0}\big)\eta_s^\dagger\eta_{s'}$$
$$\Longrightarrow\quad u_s^\dagger(\vec{p})u_{s'}(\vec{p}) = 2p_0\delta_{ss'} \tag{1.202}$$

Similarly one has

$$v_s^\dagger(\vec{p})v_{s'}(\vec{p}) = 2p_0\delta_{ss'}\,. \tag{1.203}$$

We consider now the inner products between u and v type spinors. Let us define $\tilde{p} = (p^0,-\vec{p})$; the useful identities are then given by:

$$u_s^\dagger(\vec{p})v_{s'}(-\vec{p}) = \eta_s^\dagger\Big(\underbrace{\sqrt{p\cdot\sigma\,\tilde{p}\cdot\sigma}}_{\sqrt{(p^0)^2-\vec{p}^2}=m} - \underbrace{\sqrt{p\cdot\bar{\sigma}\,\tilde{p}\cdot\bar{\sigma}}}_{m}\Big)\eta_{s'} = 0\,,\quad v_s^\dagger(\vec{p})u_{s'}(-\vec{p}) = 0 \tag{1.204}$$

Finally, two matrix identities involving spin sums will be very useful for later use. We notice first that:

$$\begin{aligned}\sum_{s=1}^{2}u_s(\vec{p})\bar{u}_s(\vec{p}) &= \sum_{s=1}^{2}\begin{pmatrix}\sqrt{p\cdot\sigma}\,\eta_s\\ \sqrt{p\cdot\bar{\sigma}}\,\eta_s\end{pmatrix}\begin{pmatrix}\eta_s^\dagger\sqrt{p\cdot\sigma} & \eta_s^\dagger\sqrt{p\cdot\bar{\sigma}}\end{pmatrix}\gamma^0\\ &= \begin{pmatrix}\sqrt{p\cdot\sigma}\,\big(\sum\eta_s\eta_s^\dagger\big)\sqrt{p\cdot\bar{\sigma}} & \sqrt{p\cdot\sigma}\,\big(\sum\eta_s\eta_s^\dagger\big)\sqrt{p\cdot\sigma}\\ \sqrt{p\cdot\bar{\sigma}}\,\big(\sum\eta_s\eta_s^\dagger\big)\sqrt{p\cdot\bar{\sigma}} & \sqrt{p\cdot\bar{\sigma}}\,\big(\sum\eta_s\eta_s^\dagger\big)\sqrt{p\cdot\sigma}\end{pmatrix}\\ &= \begin{pmatrix}\sqrt{p\cdot\sigma\,p\cdot\bar{\sigma}} & p\cdot\sigma\\ p\cdot\bar{\sigma} & \sqrt{p\cdot\bar{\sigma}\,p\cdot\sigma}\end{pmatrix},\end{aligned} \tag{1.205}$$

therefore, as $p \cdot \sigma\, p \cdot \bar{\sigma} = m^2$, one finds finally:

$$\sum_{s=1}^{2} u_s(\vec{p})\bar{u}_s(\vec{p}) = \not{p} + m \tag{1.206}$$

In the same way, starting from (1.195) one finds that:

$$\sum_{s=1}^{2} v_s(\vec{p})\bar{v}_s(\vec{p}) = \not{p} - m \tag{1.207}$$

1.3 Canonical quantization

In the Hamiltonian formalism, a classical field theory, like Klein–Gordon or Dirac, is promoted to a quantum theory using *canonical quantization*. An alternative approach based on the Lagrangian formalism, the *path integral* method, will be discussed later in these lectures. One of its advantages is to be manifestly Lorentz-invariant (rather than indirectly).

1.3.1 Quantization in the Schrödinger picture

As in ordinary non-relativistic quantum mechanics, fields and their conjugates are promoted to time-independent and local operators acting on some state space, and equal time Poisson brackets (1.117) between phase-space functionals are replaced by commutators between the corresponding operators.

Real scalar field. Consider a free real-valued scalar field. The field operator $\Phi(\vec{x})$ and the conjugate momentum operator $\Pi(\vec{x})$ are *time-independent* self-ajoint operators which satisfy the commutation relations:

$$[\,\Phi(\vec{x}),\, \Pi(\vec{y})\,] = i\delta^{(3)}(\vec{x}-\vec{y})\ , \quad [\,\Phi(\vec{x}),\, \Phi(\vec{y})\,] = [\,\Phi(\vec{x}),\, \Phi(\vec{y})\,] = 0\,. \tag{1.208}$$

As in the classical theory, the field operator Φ and the conjugate operator Π can be decomposed in terms of plane-waves, the Fourier coefficients being promoted to (non-local) operators $\{a_{\vec{p}},\, a^\dagger_{\vec{p}}\}$:[16]

$$\Phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}} \right)\ , \tag{1.209}$$

[16] Be careful of the signs in the exponentials, due to the replacement $p \cdot x \to -\vec{p}\cdot\vec{x}$ as we drop time-dependence here.

and

$$\Pi(\vec{x}) = -i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{E_p}{2}} \left(a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} - a^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}} \right) . \tag{1.210}$$

Using

$$\int d^3x \; e^{i(\vec{p}-\vec{q})\cdot\vec{x}} = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q}) \, , \tag{1.211}$$

these can be inverted into:

$$a_{\vec{q}} = \frac{1}{\sqrt{2}} \int d^3x \, e^{-i\vec{q}\cdot\vec{x}} \left(\sqrt{E_p}\Phi(\vec{x}) + \frac{i}{\sqrt{E_p}}\Pi(\vec{x}) \right) , \tag{1.212a}$$

$$a^\dagger_{\vec{q}} = \frac{1}{\sqrt{2}} \int d^3x \, e^{i\vec{q}\cdot\vec{x}} \left(\sqrt{E_p}\Phi(\vec{x}) - \frac{i}{\sqrt{E_p}}\Pi(\vec{x}) \right) . \tag{1.212b}$$

Since

$$\left[a_{\vec{p}}, \, a^\dagger_{\vec{q}} \right] = -i \int d^3x \int d^3y e^{-i(\vec{p}\cdot\vec{x} - \vec{q}\cdot\vec{y})} \left[\Phi(\vec{x}), \, \Pi(\vec{y}) \right] , \tag{1.213}$$

we get immediately the commutator:

$$\boxed{\left[a_{\vec{p}}, \, a^\dagger_{\vec{q}} \right] = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q})} \tag{1.214}$$

while

$$\left[a_{\vec{p}}, \, a_{\vec{q}} \right] = 0 \quad , \qquad \left[a^\dagger_{\vec{p}}, \, a^\dagger_{\vec{q}} \right] = 0 \, . \tag{1.215}$$

The next step is to construct the state space of theory by expanding the Hamiltonian operator in terms of the modes. Denoting by V_0 the value of the potential $V(\Phi)$ around which we expand the theory, one starts with:

$$\begin{aligned} H =& \frac{1}{2} \int d^3x \left(\Pi^2 + (\vec{\nabla}\Phi)^2 + m^2\Phi^2 + V_0 \right) \\ =& \int d^3x \left(\int \frac{d^3p d^3q}{(2\pi)^6} \frac{1}{4\sqrt{E_p E_q}} \Big\{ - \left(\sqrt{E_p E_q} + \vec{p}\cdot\vec{q} \right) \left(a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} - a^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}} \right) \left(a_{\vec{q}} e^{i\vec{q}\cdot\vec{x}} - a^\dagger_{\vec{q}} e^{-i\vec{q}\cdot\vec{x}} \right) \right. \\ & \left. \left. + m^2 \left(a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}} \right) \left(a_{\vec{q}} e^{i\vec{q}\cdot\vec{x}} + a^\dagger_{\vec{q}} e^{-i\vec{q}\cdot\vec{x}} \right) \Big\} + V_0 \right) \end{aligned} \tag{1.216}$$

This can be simplified using

$$\begin{aligned} & \int \frac{d^3p}{(2\pi)^3} \frac{1}{4E_p} \Big(\underbrace{(-E_p^2 + \vec{p}^2 + m^2)}_{-p^2+m^2=0} (a_{\vec{p}} a_{-\vec{p}} + a^\dagger_{\vec{p}} a^\dagger_{-\vec{p}}) + \underbrace{(E_p^2 + \vec{p}^2 + m^2)}_{2E_p^2} (a_{\vec{p}} a^\dagger_{\vec{p}} + a^\dagger_{\vec{p}} a_{\vec{p}}) \Big) \\ & = \int \frac{d^3p}{(2\pi)^3} \frac{E_p}{2} \left(a_{\vec{p}} a^\dagger_{\vec{p}} + a^\dagger_{\vec{p}} a_{\vec{p}} \right) . \end{aligned} \tag{1.217}$$

Since the commutation relations (1.214) are similar to those of a set of decoupled harmonic oscillators, one is tempted to normal-order the operators, *i.e.* to move the annihilation operators to the right, in order to reach:

$$H \stackrel{?}{=} \int \frac{d^3p}{(2\pi)^3} E_p a^\dagger_{\vec{p}} a_{\vec{p}} + (2\pi)^3 \delta^{(3)}(\vec{0}) \int \frac{d^3p}{(2\pi)^3} \frac{E_p}{2} + \int d^3x \, V_0 \,. \tag{1.218}$$

The vacuum. In analogy with the harmonic oscillator, one can define the *vacuum state* $|0\rangle$ of the quantum theory as the normalized state annihilated by all annihilation operators:

$$a_{\vec{p}} \, |0\rangle = 0 \,, \quad \forall \vec{p} \in \mathbb{R}^3 \,. \tag{1.219}$$

Since, for any (positive-norm) state $|\psi\rangle$, the first term of eqn. (1.218) satisfies:

$$\langle \psi | \int \frac{d^3p}{(2\pi)^3} E_p a^\dagger_{\vec{p}} a_{\vec{p}} \, |\psi\rangle = \int \frac{d^3p}{(2\pi)^3} E_p \, \| a_{\vec{p}} \, |\psi\rangle \|^2 \geqslant 0 \,, \tag{1.220}$$

$|0\rangle$ is by definition is the state of lowest energy, in other words the ground state of the system.

The last two terms in (1.218) could be interpreted as the energy of this ground state; however this expression is highly divergent, if not meaningless. There are actually two rather different sources of divergence. The first one is rather mundane: for any physical system with a non-vanishing constant energy density, the total energy in an infinite volume diverges. Putting the system in a box of size L, the identity (1.211) becomes, for each dimension:

$$\int_{-L/2}^{L/2} dx \, e^{i(p-q)x} = \tfrac{2}{p-q} \sin \tfrac{(p-q)L}{2} \xrightarrow{L\to\infty} 2\pi \, \delta(p-q) \,. \tag{1.221}$$

This type of divergence is called an *infrared divergence*, as it affects only the long-distance, or low-energy, behavior of the system; it does not usually signal a problem with the theory but rather an innacurate consideration of the physical system. A more physical quantity would be given by the energy density of the vacuum per unit volume ρ_E:

$$\rho_{\text{vac}} = V_0 + \int \frac{d^3p}{(2\pi)^3} \frac{\sqrt{p^2+m^2}}{2} \,. \tag{1.222}$$

We are not done yet as the second integral over momenta diverges quartically (*i.e.* as $\int p^3 dp$) for large values of the momentum. This type of divergence is called a *UV divergence*.

A first attitude towards this divergence would be to consider that the reference energy is arbitrary and can be modified *ad libitum*; it might however be doubtful to do so by an infinite amount. Besides this, the vacuum energy density, acts as a source term for gravity; to be more precise, its contribution to the stress-energy tensor is similar to that of a cosmological constant term[17] thus eventually determines the fate of our Universe! It is known for a few

[17] This might be a bit too naive however, as we consider *classical* gravity coupled to the *quantum* fields of the Standard model.

decades that the dominant contributing factor to the expansion of the universe behaves exactly like a vacuum energy density of about $(10^{-3}\mathrm{eV})^4$ (the so-called *dark energy*). This prompts us to look at this problem more closely.

After a second thought, the divergence of the momentum integral is not very surprising. It occurs for very large p, *i.e.* very large energies. We had secretely assumed that the quantum field theory we aim to define is valid up to arbitrarily high energies, up to the Planck scale (10^{28} eV) and beyond, even though we expect that our current understanding of particle physics, in terms of a quantum field theory for Standard Model fields, falls apart there; quantum gravity effect should play a role, and the elementary degrees of freedom may be different, for instance microscopic strings in the context of string theory.

Nonetheless, to make the theory handleable in the intermediate steps of the calculation, let us choose an energy scale λ, higher than the energies considered in the physical process under consideration. Since we expect the integral in eqn. (1.222) to be an extrapolation to arbitrarily high energies of a physical description of the theory valid at energies below the scale λ, one can regularize this expression by modifying its high-energy behavior, without (hopefully) affecting the physics at lower energies. The more drastic way is to truncate the integral at λ, which is called in this context a *UV cutoff*:

$$\int \frac{d^3p}{(2\pi)^3}\frac{\sqrt{p^2+m^2}}{2} \longrightarrow \int_{\|\vec{p}\|<\lambda} \frac{d^3p}{(2\pi)^3}\frac{\sqrt{p^2+m^2}}{2} \sim \lambda^4 . \tag{1.223}$$

The key observation is that the parameter V_0 in eqn. (1.222) is not by itself physical: it represents the minimal value of the potential in the *classical* theory so does not correspond to a measurable quantity in the quantum theory. If we can measure the true energy density of the vacuum ε in some experiment, for instance through its interaction with a gravitationnal field, in the limit where the latter can be considered as classical, one can tune the arbitrary parameter V_0 of the Lagrangian such that, for each value of the regulator λ:

$$\mathcal{E} = V_0(\lambda) + \int_{\|\vec{p}\|<\lambda} \frac{d^3p}{(2\pi)^3}\frac{\sqrt{p^2+m^2}}{2} \tag{1.224}$$

From the *effective theory* point of view, $V_0(\lambda)$ can be thought as incorporating the effects of physics at energy scales above λ, which have been "integrated out".

At the end of the computation, one can remove the cutoff by taking the limit $\lambda \to \infty$, while keeping the "true" energy density of the vacuum $\mathcal{E}$ fixed, by adjusting $V_0(\lambda)$ accordingly, such that the physics is eventually independent of the cutoff scale λ. The price to pay is that we lose some predictive power of the theory: the vacuum energy density $\mathcal{E}$, rather than being a prediction, becomes an input parameter from some experiment.[18]

[18] As we will see later, this is a general property of the *renormalization* process in quantum field theories. In favorable situtations like quantum electrodynamics, the number of input parameters that should be taken from experiments is finite and the theory maintains a predictive power, while in less favorable situations, like a putative quantum field theory for gravity this is not the case, calling for an alternative framework.

From now on, since we will not be interested in gravitational dynamics, we will set $\mathcal{E} = 0$, thus consider that the Hamiltonian of the system is given by:

$$H = \int \frac{d^3p}{(2\pi)^3} E_p a^\dagger_{\vec{p}} a_{\vec{p}} \,. \tag{1.225}$$

Likewise, for the space momentum, from eqn. (1.112) one starts with the classical expression:

$$P^i = \int d^3x \, \dot{\Phi} \partial^i \Phi \tag{1.226}$$

to get the momentum operator:

$$P^i = \int \frac{d^3p}{(2\pi)^3} p^i a^\dagger_{\vec{p}} a_{\vec{p}} \,. \tag{1.227}$$

In this case, the formally divergent ordering term is proportionnal to $\int \vec{p} d^3p$ hence vanishes thanks to isotropy of space. Naturally, the vacuum state does not carry any momentum: $P^i |0\rangle = 0$.

Altogether, one can define a 4-momentum operator of components $P^\mu = (H, P^i)$, generating space-time translations:

$$\boxed{P^\mu = \int \frac{d^3p}{(2\pi)^3} p^\mu a^\dagger_{\vec{p}} a_{\vec{p}} \,.} \tag{1.228}$$

This operator should be a vector operator under the action of the Lorentz group. Specifically, according to Wigner's theorem, to each Lorentz transformation $\Lambda \in SO(1,3)^+$ is associated a unitary operator $U(\Lambda)$ such that

$$U(\Lambda)^\dagger P^\mu U(\Lambda) \overset{!}{=} \Lambda^\mu{}_\nu P^\nu \,. \tag{1.229}$$

This means that one should satisfy the operator relation

$$\begin{aligned} \Lambda^\mu{}_\nu P^\nu &= \int \frac{d^3p}{(2\pi)^3} p^\mu \Big(U(\Lambda)^\dagger a^\dagger_{\vec{p}} U(\Lambda) \Big) \Big(U(\Lambda)^\dagger a_{\vec{p}} U(\Lambda) \Big) \\ &= \int \frac{d^4p}{(2\pi)^3} \delta(p^2 - m^2) \Theta(p^0) p^\mu \Big(U(\Lambda)^\dagger \sqrt{2E_p} a^\dagger_{\vec{p}} U(\Lambda) \Big) \Big(U(\Lambda)^\dagger \sqrt{2E_p} a_{\vec{p}} U(\Lambda) \Big) \,, \end{aligned} \tag{1.230}$$

where we have introduced in the second line the following measure invariant under the action of the restricted Lorentz group:

$$\int \frac{d^4p}{(2\pi)^4} \, 2\pi \, \delta\left(p^2 - m^2\right) \Theta(p^0) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p_0} \Big|_{p_0 = E_p} \,, \tag{1.231}$$

where Θ is the step function. Then, using the change of variables $p = \Lambda q$ in the integral, the equation (1.230) is solved by:

$$\sqrt{E_{\Lambda q}} U(\Lambda)^\dagger a_{\overrightarrow{\Lambda q}} U(\Lambda) = \sqrt{E_p} \, a_{\vec{q}} \,, \tag{1.232}$$

where we use the shorthand notation $\Lambda q = (E_{\Lambda q}, \overrightarrow{\Lambda q})$.

We also expect the vacuum state to be invariant under Lorentz transformation, since it should not admit a preferred frame:

$$U(\Lambda)\,|0\rangle = |0\rangle \ . \tag{1.233}$$

This, together with translation invariance, motivates the name of *Poincaré-invariant vacuum.*

Single particle states. The first excited states are obtained by acting with a single creation operator $a^\dagger_{\vec{p}}$ on the vacuum. In general one has a linear superposition of the form:

$$|\alpha\rangle = \int \frac{d^3p}{(2\pi)^3}\alpha(\vec{p})a^\dagger_{\vec{p}}\,|0\rangle \ , \tag{1.234}$$

with norm squared:

$$\langle\alpha|\alpha\rangle = \int \frac{d^3p}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3}\,\alpha^*(\vec{q})\alpha(\vec{p})\,\langle 0|\,\underbrace{a_{\vec{q}}a^\dagger_{\vec{p}}}_{\cancel{a^\dagger_{\vec{p}}a_{\vec{q}}}+(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})}\,|0\rangle = \int \frac{d^3p}{(2\pi)^3}\,|\alpha(\vec{p})|^2 \ , \tag{1.235}$$

so this state is normalizable provided the complex-valued function α belongs to $\mathcal{L}^2(\mathbb{R}^3)$. It is not generically an eigenstate of the 4-momentum operator $P^\mu = (H, P^i)$,

$$P^\mu\,|\alpha\rangle = \int \frac{d^3p}{(2\pi)^3}\,p^\mu a^\dagger_{\vec{p}}a_{\vec{p}}\,|\alpha\rangle = \int \frac{d^3p}{(2\pi)^3}\,\alpha(\vec{p})p^\mu a^\dagger_{\vec{p}}\,|0\rangle \ , \tag{1.236}$$

but of the quadratic operator $P_\mu P^\mu$:

$$P_\mu P^\mu\,|\alpha\rangle = \int \frac{d^3q}{(2\pi)^3}\,q_\mu a^\dagger_{\vec{q}}a_{\vec{q}}P^\mu\,|\alpha\rangle = \int \frac{d^3p}{(2\pi)^3}\,\alpha(\vec{p})p_\mu p^\mu a^\dagger_{\vec{p}}\,|0\rangle = m^2\,|\alpha\rangle \ . \tag{1.237}$$

The unbounded operator P^μ is obviously Hermitian, and self-adjoint on an appropriate domain of definition, therefore it has a real spectrum. As the momentum operator in quantum mechanics, it has no eigenvectors. Nonetheless, as nearly every physicist we consider through misuse of language the "momentum eigenvectors":

$$|\vec{p}\rangle \overset{\text{def.}}{=} \sqrt{2E_p}a^\dagger_{\vec{p}}\,|0\rangle \quad , \quad P^\mu\,|\vec{p}\rangle = \sqrt{2E_p}\int \frac{d^3q}{(2\pi)^3}q^\mu a^\dagger_{\vec{q}}a_{\vec{q}}a^\dagger_{\vec{p}}\,|0\rangle = p^\mu\,|\vec{p}\rangle \ , \tag{1.238}$$

which have thus an energy $E_p = \sqrt{p^2+m^2}$. Importantly, the "vectors" $|\vec{p}\rangle$ are not part of the Hilbert space of the quantum theory as they are not normalizable:

$$\langle\vec{q}|\vec{p}\rangle = 2E_p(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q}) \tag{1.239}$$

The states $|\vec{p}\rangle$ have been normalized with respect to the Lorentz-invariant measure (1.231), in the sense that $\int \frac{d^3p}{(2\pi)^2 2E_p} \langle \vec{q}|\vec{p}\rangle = 1$.

The actual physical states are given by the wave-packets (1.234) with $\alpha \in \mathcal{L}^2(\mathbb{R}^3)$. The resolution of the identity in the one-particle sector can be written similarly in a manifestly Lorentz-invariant way:

$$\mathbb{I}_1 = \int \frac{d^3p}{(2\pi)^3} a^\dagger_{\vec{p}} |0\rangle \left(a^\dagger_{\vec{p}} |0\rangle\right)^\dagger = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} |\vec{p}\rangle \langle \vec{p}| \ . \tag{1.240}$$

These excitations of the quantum field are interpreted as scalar particles of mass m and 3-momentum $\vec{p}$. Since, as in ordinary quantum mechanics, the (time-independent) Hamiltonian (1.225) is the time translation generator, they evolve according to Schrödinger equation:

$$|\vec{p}, t\rangle = e^{-iHt} |\vec{p}\rangle = e^{-iE_p t} |\vec{p}\rangle \ . \tag{1.241}$$

The action of Lorentz transformations on these states follows directly from eqn. (1.232) and from invariance of the vacuum, eqn. (1.233). One has

$$U(\Lambda) |\vec{p}\rangle = \left(U(\Lambda)\sqrt{2E_p}\, a^\dagger_{\vec{p}} U(\Lambda)^\dagger\right) U(\Lambda) |0\rangle = \sqrt{2E_{\Lambda p}} a^\dagger_{\Lambda p} |0\rangle = \left|\vec{\Lambda p}\right\rangle \ , \tag{1.242}$$

thus these particles transform indeed scalars under the Lorentz group.

Let us consider the action of the field operator $\Phi(\vec{x})$ on the vacuum $|0\rangle$. Using the expansion (1.209) one has:

$$|\vec{x}\rangle \stackrel{\text{def.}}{=} \Phi(\vec{x}) |0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} e^{-i\vec{p}\cdot\vec{x}} a^\dagger_{\vec{p}} |0\rangle \tag{1.243}$$

One has then:

$$\langle \vec{p}|\vec{x}\rangle = \int \frac{d^3q}{(2\pi)^3} \sqrt{\frac{E_{\vec{p}}}{E_{\vec{q}}}} e^{-i\vec{q}\cdot\vec{x}} \langle 0| a_{\vec{p}} a^\dagger_{\vec{q}} |0\rangle = e^{-i\vec{p}\cdot\vec{x}}. \tag{1.244}$$

Thus, as the notation $|\vec{x}\rangle$ suggests, one can say that $\Phi(\vec{x})$ creates a particle at $\vec{x}$ out of the vacuum (as we get the Fourier transform of the Dirac distribution centered at $\vec{x}$). Using translation invariance of the vacuum, this result could be deduced directly, up to a (possibly momentum-dependent) normalization:

$$\langle \vec{p}|\vec{x}\rangle = \langle \vec{p}| \Phi(\vec{x}) |0\rangle = \underbrace{\langle \vec{p}| e^{-i\vec{P}\cdot\vec{X}}}_{e^{-i\vec{p}\cdot\vec{x}}\langle\vec{p}|} \Phi(0) \underbrace{e^{i\vec{P}\cdot\vec{X}} |0\rangle}_{|0\rangle} = e^{-i\vec{p}\cdot\vec{x}} \langle \vec{p}| \Phi(0) |0\rangle \ . \tag{1.245}$$

In the case of the free theory, one has $\langle \vec{p}| \Phi(0) |0\rangle = 1$.

Let us stress that $|\vec{x}\rangle$, like $|\vec{p}\rangle$, does not belong to the Hilbert space of the theory. One has

$$\langle \vec{x}|\vec{y}\rangle = \int \frac{d^3q}{(2\pi)^3} \frac{d^3p}{(2\pi)^3} \langle 0| a_{\vec{q}} a^\dagger_{\vec{p}} |0\rangle e^{i(\vec{q}\cdot\vec{y} - \vec{p}\cdot\vec{x})} = \int \frac{d^3p}{(2\pi)^3} e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} = (2\pi)^3 \delta^{(3)}(\vec{x}-\vec{y}) \, . \tag{1.246}$$

More correctly, one can choose a complex *wave-function* $\psi \in \mathcal{L}^2(\mathbb{R}^3)$ and define the corresponding normalizable state:

$$|\psi\rangle \overset{\text{def.}}{=} \int d^3x\, \psi(\vec{x})\, |\vec{x}\rangle \ , \quad \big\| |\psi\rangle \big\|^2 = \int d^3x \int d^3y\, \psi^*(\vec{y})\psi(\vec{x})\, \langle \vec{y}|\vec{x}\rangle = \int d^3x\, |\psi(\vec{x})|^2 < \infty \, . \tag{1.247}$$

One should consider Φ as an *operator-valued distribution* [2]; loosely speaking it means that, acting on a test function ψ, it gives a well-defined operator acting on the Hilbert space:

$$\Phi : \psi \ \mapsto \ \int d^3x\, \psi(\vec{x})\Phi(\vec{x}) \, . \tag{1.248}$$

This statement is one of the Wightman axioms, due to Arthur Wightman, one of the founders of axiomatic quantum field theory.

Multi-particle states. Other states can be obtained from the action of an arbitrary number of creation operators on the ground state. Let us define:

$$|\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_n\rangle \overset{\text{def.}}{=} \sqrt{2E_1 2E_2 \cdots 2E_n}\, a^\dagger_{\vec{p}_1} a^\dagger_{\vec{p}_2} \cdots a^\dagger_{\vec{p}_n} |0\rangle \tag{1.249}$$

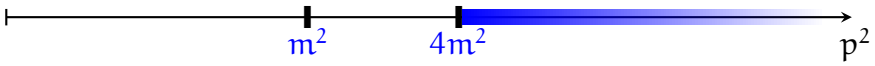

Figure 1.1: *Single- vs. two-particle states.*

Importantly, the creation operators $a^\dagger_{\vec{p}}$ commute among themselves, see eqn. (1.215), so for any permutation $\sigma \in S_n$,

$$\big|\vec{p}_{\sigma(1)}, \vec{p}_{\sigma(2)}, \ldots, \vec{p}_{\sigma(n)}\big\rangle = |\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_n\rangle \ . \tag{1.250}$$

In other words, the particles corresponding to excitations of a free scalar field follow *Bose-Einstein statistics*. Let us consider for example a two-particle state is given by:

$$|\vec{p}_1, \vec{p}_2\rangle = \sqrt{2E_{p_1} 2E_{p_2}}\, a^\dagger_{\vec{p}_1} a^\dagger_{\vec{p}_2} |0\rangle \ . \tag{1.251}$$

It is naturally an eigenstate of P^μ with the total 4-momentum as an eigenvalue:

$$P^\mu |\vec{p}_1, \vec{p}_2\rangle = \sqrt{2E_{p_1} 2E_{p_2}} \int \frac{d^3q}{(2\pi)^3} p^\mu\, a^\dagger_{\vec{q}} \underbrace{a_{\vec{q}} a^\dagger_{\vec{p}_1}}_{a^\dagger_{\vec{p}_1} a_{\vec{q}} + (2\pi)^3 \delta^{(3)}(\vec{q}-\vec{p}_1)} a^\dagger_{\vec{p}_2} |0\rangle = (p_1 + p_2)^\mu |\vec{p}_1, \vec{p}_2\rangle \tag{1.252}$$

thus satisfies

$$P^\mu P_\mu \,|\vec{p}_1, \vec{p}_2\rangle = (p_1 + p_2)^2 \,|\vec{p}_1, \vec{p}_2\rangle = s \,|\vec{p}_1, \vec{p}_2\rangle \ , \tag{1.253}$$

where we have introduced s, one of the Mandelstam kinematic invariants. Unlike single-particle states, that have a definite value of p^2, multi-particle states like this two-particle states belong to a continuum with $s \geqslant 4m^2$, as can be obtained by going to the center of momentum frame, see fig. 2.1.

The number of particles of a given multi-particle state is an eigenstate of the particule number operator defined as:

$$\mathcal{N} = \int \frac{d^3p}{(2\pi)^3} a^\dagger_{\vec{p}} a_{\vec{p}} \,, \tag{1.254}$$

$$\begin{aligned}
\mathcal{N}\,|\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_n\rangle &= \sqrt{2E_1 \cdots 2E_n}\, a^\dagger_{\vec{p}_1} \int \frac{d^3p}{(2\pi)^3}\, a^\dagger_{\vec{p}}\, \underbrace{a_{\vec{p}} a^\dagger_{\vec{p}_1}}_{(2\pi)^3\delta^{(3)}(\vec{p}_1-\vec{p}) + a^\dagger_{\vec{p}_1} a_{\vec{p}}}\, a^\dagger_{\vec{p}_2} \cdots a^\dagger_{\vec{p}_n}\,|0\rangle \\
&= |\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_n\rangle + \sqrt{2E_1 \cdots 2E_n}\, a^\dagger_{\vec{p}_1} \int \frac{d^3p}{(2\pi)^3} a^\dagger_{\vec{p}} a^\dagger_{\vec{p}_1}\, \underbrace{a_{\vec{p}} a^\dagger_{\vec{p}_2}}_{(2\pi)^3\delta^{(3)}(\vec{p}_2-\vec{p}) + a^\dagger_{\vec{p}_2} a_{\vec{p}}} \cdots a^\dagger_{\vec{p}_n}\,|0\rangle \\
&= n\,|\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_n\rangle \ .
\end{aligned} \tag{1.255}$$

Since $[\mathcal{N}, H] = 0$, the number of particles is a constant of motion in the free theory; generically, one expects that this is no longer the case in the presence of interactions, as particles can be created or annihilated. In the n-particle sector, the resolution of the identity, which generalises (1.240), is given by:

$$\mathbb{I}_n = \frac{1}{n!} \int \prod_{k=1}^{n} \frac{d^3p_k}{(2\pi)^3} \frac{1}{2E_1 \cdots 2E_n} \,|\vec{p}_1, \ldots, \vec{p}_1\rangle \,\langle \vec{p}_1, \ldots, \vec{p}_n| \ . \tag{1.256}$$

the $n!$ factor avoids overcounting as states related by permutations of the momenta are identical.

Let $\mathcal{H}$ be the one-particle Hilbert space of the theory and, for a given permutation $\sigma \in S_n$, $\mathcal{Q}(\sigma)$ the corresponding unitary operator acting on $\mathcal{H}^{\otimes n}$. We define the symmetrized product:

$$\mathcal{S}_n\big(\mathcal{H}\big) = \{|\alpha\rangle \in \mathcal{H} \otimes \cdots \otimes \mathcal{H} \,,\ \forall \sigma \in S_n \,,\quad \mathcal{Q}(\sigma)\,|\alpha\rangle = \alpha\} \ . \tag{1.257}$$

Creation operators are maps from $\mathcal{S}_n\big(\mathcal{H}\big)$ to $\mathcal{S}_{n+1}\big(\mathcal{H}\big)$, while annihilation operators from $\mathcal{S}_n\big(\mathcal{H}\big)$ to $\mathcal{S}_{n-1}\big(\mathcal{H}\big)$. The space of states of the theory has the structure of a *Fock space*, which is defined, for bosonic particles, as the formal sum:

$$\mathcal{F}_B\big(\mathcal{H}\big) \overset{\text{def.}}{=} \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \mathcal{S}_n\big(\mathcal{H}^{\otimes n}\big) \,, \tag{1.258}$$

the first factor being associated to the vacuum state $|0\rangle$. The resolution of the identity on $\mathcal{F}_{\text{B}}(\mathcal{H})$ reads:

$$\mathbb{I} = |0\rangle\langle 0| + \sum_{n=1}^{\infty} \frac{1}{n!} \int \prod_{k=1}^{n} \frac{d^3p_k}{(2\pi)^3} \frac{1}{2E_1 \cdots 2E_n} |\vec{p}_1, \ldots, \vec{p}_1\rangle \langle \vec{p}_1, \ldots, \vec{p}_n| \,. \tag{1.259}$$

Each term of the sum can be understood as the orthogonal projector on the corresponding sub-space with n particles.

Complex scalar field. For a complex scalar field, the story is mostly similar technically, but with new interesting physical aspects. From the expression of the conjugate momenta, eqn. (1.140), one gets the canonical commutation relations:

$$[\,\Phi(\vec{x}),\, \Pi_{\Phi}(\vec{y})\,] = \left[\,\Phi^\dagger(\vec{x}),\, \Pi_{\Phi^\dagger}(\vec{y})\,\right] = i\delta^{(3)}(\vec{x}-\vec{y}) \tag{1.260a}$$

$$[\,\Phi(\vec{x}),\, \Pi_{\Phi^\dagger}(\vec{y})\,] = \left[\,\Phi^\dagger(\vec{x}),\, \Pi_{\Phi}(\vec{y})\,\right] = 0\,. \tag{1.260b}$$

Starting from (1.143) we have an expansion into creation and annihilation operators:

$$\Phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + b^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}}\right) , \tag{1.261}$$

and

$$\Pi_{\Phi}(\vec{y}) = i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{E_p}{2}} \left(a^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}} - b_{\vec{p}} e^{i\vec{p}\cdot\vec{x}}\right) , \tag{1.262}$$

as well as their Hermitian conjugates. It implies the following commutation relations between the modes operators:

$$\left[\, a_{\vec{p}},\, a^\dagger_{\vec{q}} \,\right] = \left[\, b_{\vec{p}},\, b^\dagger_{\vec{q}} \,\right] = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q})\,, \tag{1.263a}$$

$$\left[\, a_{\vec{p}},\, b^\dagger_{\vec{q}} \,\right] = \left[\, b_{\vec{p}},\, a^\dagger_{\vec{q}} \,\right] = 0\,, \tag{1.263b}$$

showing that the two type of mode operators, $a_{\vec{p}}$ and $b_{\vec{p}}$, are independent from each other. *Exercise: show it!*

One can therefore construct one-particle states from the vacuum of two types, either as $\sqrt{2E_p}\, a^\dagger_{\vec{p}}\,|0\rangle$ as $\sqrt{2E_p}\, b^\dagger_{\vec{p}}\,|0\rangle$. To understand better what it means, let us recall that the field theory of the complex scalar admits a $U(1)$ symmetry, and the conserved charge from the Noether current (1.141) can be promoted to a self-adjoint operator generating the transformation.

$$Q = i \int d^3x \left(\Phi \Pi_{\Phi} - \Phi^\dagger \Pi_{\Phi^\dagger}\right). \tag{1.264}$$

By plugging in the mode expansions (1.261) and integrating over space one gets:

$$\begin{aligned} Q &= -\frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\big(a_{\vec{p}}a^\dagger_{\vec{p}} - b^\dagger_{\vec{p}}b_{\vec{p}} - a_{\vec{p}}b_{-\vec{p}} + b^\dagger_{\vec{p}}a^\dagger_{-\vec{p}} + a^\dagger_{\vec{p}}a_{\vec{p}} - b_{\vec{p}}b^\dagger_{\vec{p}} - a^\dagger_{\vec{p}}b^\dagger_{-\vec{p}} + b_{\vec{p}}a_{-\vec{p}}\big) \\ &= \int \frac{d^3p}{(2\pi)^3}\big(b^\dagger_{\vec{p}}b_{\vec{p}} - a^\dagger_{\vec{p}}a_{\vec{p}}\big)\,. \end{aligned} \tag{1.265}$$

One finds that the one-particle states satisfy:

$$Q\sqrt{2E_p}a^\dagger_{\vec{p}}\,|0\rangle = -\sqrt{2E_p}\int \frac{d^3p}{(2\pi)^3}a^\dagger_{\vec{p}}a_{\vec{p}}a^\dagger_{\vec{p}}\,|0\rangle = -\sqrt{2E_p}a^\dagger_{\vec{p}}\,|0\rangle\ , \tag{1.266a}$$

$$Q\sqrt{2E_p}b^\dagger_{\vec{p}}\,|0\rangle = \sqrt{2E_p}\int \frac{d^3p}{(2\pi)^3}b^\dagger_{\vec{p}}b_{\vec{p}}b^\dagger_{\vec{p}}\,|0\rangle = +\sqrt{2E_p}b^\dagger_{\vec{p}}\,|0\rangle\ . \tag{1.266b}$$

These one-particle states are both interpreted as scalar particles of the same mass m, but with opposite $U(1)$ Noether charge. If we couple this complex scalar field to the electromagnetic field, as we will do later, these particles will have opposite electric charge. One can then think of the "a-type" particle as the *anti-particle* of the "b-type" particle. While the field operator $\Phi(\vec{x})$ creates a particle at $\vec{x}$, $\Phi^\dagger(\vec{x})$ creates an anti-particle.

CPT invariance. The field theory of a free complex scalar field possesses two types of continuous symmetries: invariance under the action of the restricted Lorentz group $SO(1,3)^+$, through the unitary operators $U(\Lambda)$, and invariance under the global $U(1)$ symmetry generated by the Hermitian operator (1.265). On top of this, the theory admits three discrete $\mathbb{Z}_2$ symmetries:

- Space parity $\vec{x} \mapsto -\vec{x}$, associated with a unitary operator $\mathcal{P}$ such that

$$\mathcal{P}e^{i\vec{P}\cdot\vec{x}}\mathcal{P} = e^{-i\vec{P}\cdot\vec{x}}\ , \quad \mathcal{P}^2 = 1\,. \tag{1.267}$$

- Charge conjugation that maps particles to anti-particles. It is associated with a unitary operator $\mathcal{C}$ that acts on the field operator $\Phi(\vec{x})$ as[19]

$$\mathcal{C}\Phi(\vec{x})\mathcal{C} = \Phi^\dagger(\vec{x})\ , \quad \mathcal{C}\Phi^\dagger(\vec{x})\mathcal{C} = \Phi(\vec{x})\ , \quad \mathcal{C}^2 = 1\,. \tag{1.268}$$

- Time reversal symmetry $t \mapsto -t$, that is associated to an *anti-unitary* operator $\mathcal{T}$ such that:

$$\mathcal{T}e^{-iHt}\mathcal{T} = e^{iHt}\ , \quad \mathcal{T}^2 = 1\,, \tag{1.269}$$

see the tutorials for details.[20]

[19] In full generality, one can add a phase factor without spoiling invariance: $\mathcal{C}\Phi(\vec{x})\mathcal{C} = \eta\Phi^\dagger(\vec{x})$, $|\eta| = 1$.

[20] Basically the argument goes as follows. Equation (1.269) implies that $iH\mathcal{T} = \mathcal{T}(-iH)$. If $\mathcal{T}$ was unitary it would mean $\{\mathcal{T}, H\} = 0$, implying that to every energy eigenstate $|E\rangle$ one would associate $\mathcal{T}\,|E\rangle$ which is an energy eigenstate of energy $-E$, so the spectrum would be unbounded from below.

The Standard Model of particle physics is not invariant under space parity, as only left-handed neutrinos participate into weak interactions. For less simple reasons, it turns out that the combination $\mathcal{CP}$, which combines space-parity with charge conjugation, is also violated by the Standard Model.

Theorem 1 (CPT theorem). *Any Lorentz-invariant local quantum field theory with a spectrum unbounded from below is invariant under the action of the discrete symmetry $\mathcal{CPT}$.*

This had beend argued for in various ways, starting from Schwinger in the fifties, including the axiomatic approach to QFT where it was shown to be a consequence of the Wightman axiom that will be discussed in the next part [4].

Dirac spinor field. We now shift to Dirac fields and first notice that the conjugate momentum to the spinor field Ψ that follows from (1.158) is:

$$\Pi_\Psi = \frac{\partial \mathcal{L}}{\partial \dot{\Psi}} = i\Psi^\dagger \,. \tag{1.270}$$

Notice that the conjugate momentum does not involve any time derivative; this is consistent with having a time evolution governed by a first-order equation. The Hamiltonian of the system reads then:[21]

$$H = \int d^3x \left(\Pi_\Psi \dot{\Psi} - \mathcal{L}\right) = \int d^3x \, \overline{\Psi}\big(- i\gamma^j \partial_j + m\big)\Psi \,. \tag{1.271}$$

One considers then the mode expansions:

$$\Psi(x) = \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(b^s_{\vec{p}} u_s(\vec{p}) e^{i\vec{p}\cdot\vec{x}} + c^{s\,\dagger}_{\vec{p}} v_s(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} \right) , \tag{1.272a}$$

$$\Pi_\Psi(x) = i\Psi^\dagger = i \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(b^{s\,\dagger}_{\vec{p}} u^\dagger_s(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} + c^s_{\vec{p}} v^\dagger_s(\vec{p}) e^{i\vec{p}\cdot\vec{x}} \right) . \tag{1.272b}$$

In terms of those modes, one can express the Hamiltonian of the system as:

$$\begin{aligned} H = \sum_{r,s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \int d^3x \, \left(b^{r\,\dagger}_{\vec{p}} \bar{u}_r(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} + c^r_{\vec{p}} \bar{v}_r(\vec{p}) e^{i\vec{p}\cdot\vec{x}} \right) \\ \times \, \left(\big(-\gamma^j q_j + m\big) b^s_{\vec{q}} u_s(\vec{q}) e^{i\vec{q}\cdot\vec{x}} + \big(\gamma^j q_j + m\big) c^{s\,\dagger}_{\vec{q}} v_s(\vec{q}) e^{-i\vec{q}\cdot\vec{x}} \right) \end{aligned} \tag{1.273}$$

[21] While the result is correct, the procedure is a bit suspicious as $\Pi_{\overline{\Psi}} = 0$ and the Legendre transform is singular; this is because we are dealing with a constrained system (the phase space variables are not independent) and for canonical quantization one should start with *Dirac brackets* rather than Poisson brackets [3].

After performing the integral over $\vec{x}$ the various terms can be simplified using the bilinear spinorial identities derived previously. One has first:

$$\sum_{r,s=1}^{2} b_{\vec{p}}^{r\,\dagger} b_{\vec{p}}^{s} \bar{u}_r(\vec{p}) \underbrace{\big(-\gamma^j p_j + m\big) u_s(\vec{p})}_{\gamma^0 p_0 u_s(\vec{p})} = E_p \sum_{r,s=1}^{2} b_{\vec{p}}^{r\,\dagger} b_{\vec{p}}^{s} u_r^\dagger(\vec{p}) u_s(\vec{p}) = 2E_p^2 \sum_{s=1}^{2} b_{\vec{p}}^{s\,\dagger} b_{\vec{p}}^{s}\,, \quad (1.274)$$

where we have used eqn. (1.179) then eqn. (1.202). We get in a similar way using eqn. (1.194) then eqn. (1.203):

$$\sum_{r,s=1}^{2} c_{\vec{p}}^{r} c_{\vec{p}}^{s\,\dagger} \bar{v}_r(\vec{p}) \underbrace{\big(\gamma^j p_j + m\big) v_s(\vec{p})}_{-\gamma^0 p_0 v_s(\vec{p})} = -E_p \sum_{r,s=1}^{2} c_{\vec{p}}^{r} c_{\vec{p}}^{s\,\dagger} v_r^\dagger(\vec{p}) v_s(\vec{p}) = -2E_p^2 \sum_{s=1}^{2} c_{\vec{p}}^{s} c_{\vec{p}}^{s\,\dagger}\,, \quad (1.275)$$

Finally, the cross-terms are proportional to $u_r^\dagger(\vec{p}) v_s(-\vec{p})$ or $v_r^\dagger(\vec{p}) u_s(-\vec{p})$ hence vanish thanks to eqn. (1.204). We obtain then when the dust settles:

$$H = \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} E_p \left(b_{\vec{p}}^{s\,\dagger} b_{\vec{p}}^{s} - c_{\vec{p}}^{s} c_{\vec{p}}^{s\,\dagger} \right). \quad (1.276)$$

To quantize the Dirac field, we could attempt to replace the Poisson bracket by a commutator between the field operator and its conjugate momentum:

$$\big[\, \Psi(\vec{x})^a,\, \Pi_\Psi(\vec{y})^b \,\big] = i \big[\, \Psi(\vec{x})^a,\, \Psi^\dagger(\vec{y})^b \,\big] \stackrel{?}{=} i\delta^{ab}\delta^{(3)}(\vec{x}-\vec{y})\,. \quad (1.277)$$

As we will see, this leads to severe inconsistencies of the resulting theory. First, let us check that this commutation relation means that the non-vanishing commutators between the modes are:

$$\Big[\, b_{\vec{p}}^{r},\, b_{\vec{q}}^{s\,\dagger} \,\Big] = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p}-\vec{q})\,, \quad \Big[\, c_{\vec{p}}^{r},\, c_{\vec{q}}^{s\,\dagger} \,\Big] = -(2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p}-\vec{q})\,, \quad (1.278)$$

with a minus sign in the second commutator. We have indeed

$$\begin{aligned}
\big[\, \Psi(\vec{x})^a,\, \Psi^\dagger(\vec{y})^b \,\big] &= \sum_{r,s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \\
&\Big(e^{i(\vec{p}\cdot x - \vec{q}\cdot\vec{y})} u_r(\vec{p})^a u_s^\dagger(\vec{q})^b \underbrace{\Big[\, b_{\vec{p}}^{r},\, b_{\vec{q}}^{s\,\dagger} \,\Big]}_{(2\pi)^3\delta^{rs}\delta^{(3)}(\vec{p}-\vec{q})} + e^{-i(\vec{p}\cdot\vec{x} - \vec{q}\cdot\vec{y})} v_r(\vec{p})^a v_s^\dagger(\vec{q})^b \underbrace{\Big[\, c_{\vec{p}}^{r\,\dagger},\, c_{\vec{q}}^{s} \,\Big]}_{(2\pi)^3\delta^{rs}\delta^{(3)}(\vec{p}-\vec{q})} \Big) \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \Big(e^{ip\cdot(\vec{x}-\vec{y})} \Big(\sum_{s=1}^{2} u_s(\vec{p})^a u_s^\dagger(\vec{p})^b \Big) + e^{-ip\cdot(\vec{x}-\vec{y})} \Big(\sum_{s=1}^{2} v_s(\vec{p})^a v_s^\dagger(\vec{p})^b \Big) \Big) \quad (1.279)
\end{aligned}$$

We can now use the spin sums (1.206,1.207):

$$\Big(\sum_{s=1}^{2} u_s(\vec{p}) u_s^\dagger(\vec{p}) \Big)^{ab} = \Big(\sum_{s=1}^{2} u_s(\vec{p}) \bar{u}_s(\vec{p}) \gamma^0 \Big)^{ab} = \Big((\not{p} + m)\gamma^0 \Big)^{ab}, \quad (1.280a)$$

$$\Big(\sum_{s=1}^{2} v_s(\vec{p}) v_s^\dagger(\vec{p}) \Big)^{ab} = \Big(\sum_{s=1}^{2} v_s(\vec{p}) \bar{v}_s(\vec{p}) \gamma^0 \Big)^{ab} = \Big((\not{p} - m)\gamma^0 \Big)^{ab}. \quad (1.280b)$$

After the change of variables $\vec{p} \mapsto -\vec{p}$ in the second term of the integral (1.279) we reach:

$$\left[\Psi(\vec{x})^a, \Pi_\Psi(\vec{y})^b\right] = \int \frac{d^3p}{(2\pi)^3}\frac{e^{i\vec{p}\cdot(\vec{x}-\vec{y})}}{2E_p}\left(p^0\delta^{ab} + \cancel{\left((p_i\gamma^i + m)\gamma^0\right)^{ab}} + p^0\delta^{ab} + \cancel{\left((-p_i\gamma^i - m)\gamma^0\right)^{ab}}\right) = \delta^{ab}\delta^{(3)}(\vec{x}-\vec{y})\,. \quad (1.281)$$

The minus sign of the second commutator in (1.278) was crucial to get this result. We seem to face an insoluble dilemna:

- As for the quantization of a complex scalar field, we may define the vacuum of the theory by the conditions $b^r_{\vec{p}}\,|0\rangle = 0$ and $c^r_{\vec{p}}\,|0\rangle = 0$. We notice then that there exist states of negative norm squared:[22]

$$\langle 0|\ \underbrace{c^r_{\vec{q}}c^{r\,\dagger}_{\vec{p}}}_{c^{r\,\dagger}_{\vec{p}}c^r_{\vec{q}} - (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})}\ |0\rangle = -(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\,,$$

  thus spoiling the unitarity of the theory.

- We could decide instead to define the vacuum state by the conditions $b^r_{\vec{p}}\,|0\rangle = 0$ and $c^{r\,\dagger}_{\vec{p}}\,|0\rangle = 0$. However, the spectrum of the Hamiltonian (1.276) is then unbounded from below! Setting aside the ordering constant, for instance for any $n \in \mathbb{Z}_{>0}$ the state $(c^r_{\vec{p}})^n\,|0\rangle$ would have an energy $E = -nE_p$.

To cut the Gordian knot one needs to change drastically the rules of canonical quantization: trade the commutators (1.278) for *anti-commutators* (with positive signs on the right-hand side for both):

$$\boxed{\left\{\, b^r_{\vec{p}},\, b^{s\,\dagger}_{\vec{q}}\,\right\} = (2\pi)^3\delta^{rs}\delta^{(3)}(\vec{p}-\vec{q})\ , \quad \left\{\, c^r_{\vec{p}},\, c^{s\,\dagger}_{\vec{q}}\,\right\} = (2\pi)^3\delta^{rs}\delta^{(3)}(\vec{p}-\vec{q})\,,} \quad (1.282)$$

where we have defined

$$\{\,A,\,B\,\} \stackrel{\text{def.}}{=} AB + BA\,. \quad (1.283)$$

Likewise eqn. (1.277) is replaced by:

$$\boxed{\left\{\,\Psi(\vec{x})^a,\,\Psi^\dagger(\vec{y})^b\,\right\} = \delta^{ab}\delta^{(3)}(\vec{x}-\vec{y})\,.} \quad (1.284)$$

Using these anti-commutation relations the Hamiltonian can be normal-ordered to:

$$H = \sum_{s=1}^{2}\int \frac{d^3p}{(2\pi)^3}E_p\left(b^{s\,\dagger}_{\vec{p}}b^s_{\vec{p}} + c^{s\,\dagger}_{\vec{p}}c^s_{\vec{p}}\right) - 2(2\pi)^3\delta^{(3)}(\vec{0})\int\frac{d^3p}{(2\pi)^3}E_p\,. \quad (1.285)$$

[22] Of course a more correct statement would be to construct a normalizable wave-packet from those one-particle states, leading to the same conclusion.

The zero-point energy is infinite for the same reasons as before. Notice though that it is negative in the present case.[23] We will consider from now on the Hamiltonian:

$$H = \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} E_p \left(b_{\vec{p}}^{s\,\dagger} b_{\vec{p}}^{s} + c_{\vec{p}}^{s\,\dagger} c_{\vec{p}}^{s} \right) \tag{1.286}$$

Likewise we have a momentum operator defined by

$$\vec{P} = -i \int d^3x\, \Psi^\dagger(\vec{x}) \vec{\nabla} \Psi(\vec{x}) = \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \vec{p} \left(b_{\vec{p}}^{s\,\dagger} b_{\vec{p}}^{s} + c_{\vec{p}}^{s\,\dagger} c_{\vec{p}}^{s} \right) . \tag{1.287}$$

Exercise: show it.

The vacuum state of the theory is defined in the usual way:

$$b_{\vec{p}}^{s} |0\rangle = 0 \quad , \qquad c_{\vec{p}}^{s} |0\rangle = 0 . \tag{1.288}$$

We define then the one-particle states:

$$|\vec{p}, s\rangle \stackrel{\text{def.}}{=} \sqrt{2E_s}\, b_{\vec{p}}^{s\,\dagger} |0\rangle \quad , \quad H |\vec{p}, s\rangle = E_p |\vec{p}, s\rangle \quad , \quad \vec{P} |\vec{p}, s\rangle = \vec{p} |\vec{p}, s\rangle . \tag{1.289}$$

They are interpreted as spin one-half particles of mass m and momentum $\vec{p}$, in a spin state s. The same can be said for the states $c_{\vec{p}}^{s\,\dagger} |0\rangle$. Using (1.282) one checks that both states have positive norm squared.

To confirm that these particles have spin one-half, let us move to the rest frame of the particle, such that $p^\mu = (m, \vec{0})$, and consider a rotation around the z axis. The operators generating the action of Lorentz transformations on the Hilbert space are obtained from the corresponding Noether charge, see eqn. (1.172), up to possible ordering ambiguities:

$$Q^{\rho\sigma} = \int d^3x\, \Psi^\dagger \left(x^\sigma \big(-i\partial^\rho \big) + \tfrac{1}{2} S^{\rho\sigma} \right) \Psi . \tag{1.290}$$

A rotation of an angle θ around an axis specified by the unit vector $\vec{n}$ is parametrized by $\Omega_{ij} = \theta \epsilon_{ijk} n^k$ hence, for a rotation around the z axis, we consider the transformation:

$$b_{0}^{s\,\dagger} |0\rangle \;\mapsto\; \exp\left(-i\theta \int d^3x\, \Psi^\dagger S^{12} \Psi \right) b_{\vec{0}}^{s\,\dagger} |0\rangle . \tag{1.291}$$

[23] One may wonder whether cancellation between the divergences due to bosons and fermions could occur. This is indeed the case in supersymmetric theories, a pattern that persists at the level of loop corrections to some extent.

We have, ignoring a term that annihilates the vacuum:

$$\begin{aligned}
\left[\int d^3x\,\Psi^\dagger S^{12}\Psi,\, b^{s\,\dagger}_{\vec 0}\right] &= \sum_{r,t=1}^{2}\frac{1}{2}\int d^3x\int\frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\int\frac{d^3q}{(2\pi)^3\sqrt{2E_q}} \\
&\qquad\left[\left(b^{r\,\dagger}_{\vec p}u^\dagger_r(\vec p)e^{-i\vec p\cdot\vec x}+c^r_{\vec p}v^\dagger_r(\vec p)e^{i\vec p\cdot\vec x}\right)\begin{pmatrix}\sigma^3 & 0\\ 0 & \sigma^3\end{pmatrix}\left(b^t_{\vec q}u_t(\vec q)e^{i\vec q\cdot\vec x}+c^{t\,\dagger}_{\vec q}v_t(\vec q)e^{-i\vec q\cdot\vec x}\right),\, b^{s\,\dagger}_{\vec 0}\right] \\
&= \frac{1}{2}\sum_{r,t}\int d^3x\int\frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\int\frac{d^3q}{(2\pi)^3\sqrt{2E_q}}e^{-i\vec p\cdot\vec x}u^\dagger_r(\vec p)\begin{pmatrix}\sigma^3 & 0\\ 0 & \sigma^3\end{pmatrix}u_t(\vec q)e^{i\vec q\cdot\vec x}\left[b^{r\,\dagger}_{\vec p}b^t_{\vec q},\, b^{s\,\dagger}_{\vec 0}\right]+\cdots \\
&= \frac{1}{4m}\sum_r u^\dagger_r(\vec 0)\begin{pmatrix}\sigma^3 & 0\\ 0 & \sigma^3\end{pmatrix}u_s(\vec 0)b^{r\,\dagger}_{\vec 0} = \frac{1}{4}\sum_r\begin{pmatrix}\eta^\dagger_r & \eta^\dagger_r\end{pmatrix}\begin{pmatrix}\sigma^3 & 0\\ 0 & \sigma^3\end{pmatrix}\begin{pmatrix}\eta_s\\ \eta_s\end{pmatrix}b^{r\,\dagger}_{\vec 0} = \frac{1}{2}\sum_r\eta^\dagger_r\sigma_3\eta_s b^{r\,\dagger}_{\vec 0}
\end{aligned} \tag{1.292}$$

where we have used

$$\left[b^{r\,\dagger}_{\vec p}b^t_{\vec q},\, b^{s\,\dagger}_{\vec 0}\right] = b^{r\,\dagger}_{\vec p}b^t_{\vec q}b^{s\,\dagger}_{\vec 0} - \underbrace{b^{s\,\dagger}_{\vec 0}b^{r\,\dagger}_{\vec p}}_{-b^{r\,\dagger}_{\vec p}b^{s\,\dagger}_{\vec 0}}b^t_{\vec q} = b^{r\,\dagger}_{\vec p}\left\{b^t_{\vec q},\, b^{s\,\dagger}_{\vec 0}\right\} = (2\pi)^3\delta^{ts}\delta^{(3)}(\vec q)b^{r\,\dagger}_{\vec p}\,. \tag{1.293}$$

As before one can choose in the rest frame an orthonormal basis of spinors that diagonalize rotation around the z axis: $\sigma_3\eta_1=\eta_1$ and $\sigma_3\eta_2=-\eta_2$, in which case, taking advantage of the Lorentz invariance of the vacuum, one finds the transformations:

$$b^{1\,\dagger}_0|0\rangle\mapsto e^{-i\theta/2}b^{1\,\dagger}_0|0\rangle\ ,\quad b^{2\,\dagger}_0|0\rangle\mapsto e^{+i\theta/2}b^{2\,\dagger}_0|0\rangle\ , \tag{1.294}$$

hence transforming indeed as a spinor with J_z eigenvalues $+1/2$ and $-1/2$ respectively. For a one-particle state $c^{s\,\dagger}_0|0\rangle$ the story is similar except that the commutator (1.293) is replaced by:

$$\left[c^r_{\vec p}c^{t\,\dagger}_{\vec q},\, c^{s\,\dagger}_{\vec 0}\right] = -\left[c^{t\,\dagger}_{\vec q}c^r_{\vec p},\, c^{s\,\dagger}_{\vec 0}\right] = -(2\pi)^3\delta^{rs}\delta^{(3)}(\vec q)c^{r\,\dagger}_{\vec p}\,. \tag{1.295}$$

One notices that the J_z eigenvalues of the one-particle states $c^{s\,\dagger}_0|0\rangle$ are opposite to those of the one-particle states $b^{s\,\dagger}_0|0\rangle$.[24]

As for the complex scalar field, $b^{s\,\dagger}_{\vec p}|0\rangle$ and $c^{s\,\dagger}_{\vec p}|0\rangle$ are distinguished by their charge under the action of the $U(1)$ symmetry $\Psi\mapsto e^{i\theta}\Psi$. The associated Noether charge is obtained from eqn. (1.167), by similar steps as in the previous computations:

$$Q_D = \int d^3x\,\mathcal{J}^0_D = \int d^3x\,\Psi^\dagger\Psi = \sum_{s=1}^{2}\int\frac{d^3p}{(2\pi)^3}\left(b^{s\,\dagger}_{\vec p}b^s_{\vec p}-c^{s\,\dagger}_{\vec p}c^s_{\vec p}\right)\ \cancel{+\text{const.}}\,, \tag{1.296}$$

[24]For massless fermions, as there is no notion of rest frame, the one-particle states are not characterized by their spin (in the sense of their $SU(2)$ irreducible representation) but rather by their *helicity*, as discussed below (1.151), associated with rotations in the plane orthogonal to their momentum $\vec p$.

where we choose to drop the normal-ordering constant. We have:

$$Q_D b_{\vec{p}}^{s\,\dagger} |0\rangle = +b_{\vec{p}}^{s\,\dagger} |0\rangle \quad , \qquad Q_D c_{\vec{p}}^{s\,\dagger} |0\rangle = -c_{\vec{p}}^{s\,\dagger} |0\rangle \ . \tag{1.297}$$

Therefore one can think of $b_{\vec{p}}^{s\,\dagger} |0\rangle$ as a particle and of $c_{\vec{p}}^{s\,\dagger} |0\rangle$ as the corresponding anti-particle whenever the Q_D eigen-value is interpreted as the electric charge.

A state made of two particles can be obtained as:

$$|\vec{p}_1, s_1, \vec{p}_2, s_2\rangle \overset{\text{def.}}{=} \sqrt{2E_{p_1} 2E_{p_2}}\, b_{\vec{p}_1}^{s_1\,\dagger} b_{\vec{p}_2}^{s_2\,\dagger} |0\rangle \ . \tag{1.298}$$

Crucially, the anticommutation relations (1.282) imply that this state is antisymmetric under the exchange of the momenta and spin state of the two particles:

$$|\vec{p}_2, s_2, \vec{p}_1, s_1\rangle = -|\vec{p}_1, s_1, \vec{p}_2, s_2\rangle \ . \tag{1.299}$$

More generally, one can define n-particle states:

$$|\vec{p}_1, s_1, \ldots, \vec{p}_n, s_n\rangle \overset{\text{def.}}{=} \sqrt{2E_{p_n} \cdots 2E_{p_n}}\, b_{\vec{p}_1}^{s_1\,\dagger} \cdots b_{\vec{p}_n}^{s_n\,\dagger} |0\rangle \ , \tag{1.300}$$

which are totally antisymmetric:

$$\forall \sigma \in S_n \ , \quad \left|\vec{p}_{\sigma(1)}, s_{\sigma(1)}, \ldots, \vec{p}_{\sigma(n)}, s_{\sigma(n)}\right\rangle = \epsilon_\sigma |\vec{p}_1, s_1, \ldots, \vec{p}_n, s_n\rangle \ , \tag{1.301}$$

where ϵ_σ is the signature of the permutation σ; therefore these particle follow *Fermi–Dirac statistics*.

Let us summarize what we have found here. Due to the structure of the Lagrangian for a Dirac spinor, which is essentially fixed by Lorentz invariance, using Fermi–Dirac statistics (rather than Bose–Einstein) was forced upon us by general principles of quantum mechanics: unitarity and energy spectrum bounded from below. This observation illustrates a very general and deep statement relating the spin of a particle (being elementary or composite) to its bosonic or fermionic nature.

Theorem 2 (Spin-statistics). *Fields corresponding to particles of integer spin are quantized according to Bose–Einstein statistics, while fields corresponding to particles of half-integer spin are quantized according to Fermi–Dirac statistics.*

A general proof can be found for instance in Weinberg [5]. Among the known elementary particles, electrons and quark are fermions, while photons, gluons, $W^\pm$ and Z and the Higgs are bosons. A bound state of particles, at energies low enough, behaves like a particle whose spin, thereby whose statistics, is given by the angular momentum of its ground state. For instance a meson, which is composed of an equal number of quarks and anti-quarks, is a boson.

1.3.2 Quantization in the Heisenberg picture

As in ordinary non-relativistic quantum-mechanics, it is possible to use, rather than the Schrödinger picture, the Heisenberg picture for which time dependence lies in the operators rather than the states. In the context of a relativistic theory, this framework is essential to examine the question of causality, among other things. In a theory characterized by a Hamiltonian H, assumed to be time-independent, one associates to any operator $\mathcal{O}_S$ in the Schrödinger picture, the corresponding operator in the Heisenberg picture following:

$$\mathcal{O}_H \stackrel{\text{def.}}{=} e^{iHt}\,\mathcal{O}\,e^{-iHt}\,. \tag{1.302}$$

The two representations agree at $t = 0$ (which is of course an arbitrary reference); the subscript H will be remove whenever there will be no ambiguity.

Let us illustrate first the formalism with the theory of a real scalar field. Under the map (1.302), the commutation relations (1.208) become equal-time commutators:

$$[\,\Phi(t,\vec{x}),\,\Pi(t,\vec{y})\,] \;=\; i\delta^{(3)}(\vec{x}-\vec{y}) \tag{1.303}$$

$$[\,\Phi(t,\vec{x}),\,\Phi(t,\vec{y})\,] \;=\; \big[\,\Phi(t,\vec{x}),\,\Phi(t,\vec{y})\,\big] \;=\; 0 \tag{1.304}$$

Indeed we have for instance:

$$\begin{aligned} e^{iHt}\,[\,\Phi(\vec{x}),\,\Pi(\vec{y})\,]\,e^{-iHt} &= e^{iHt}\Phi(\vec{x})e^{-iHt}e^{iHt}\Pi(\vec{y}) - e^{iHt}\Phi(\vec{x})e^{-iHt}e^{iHt}\Pi(\vec{y})e^{-iHt} \\ &= [\,\Phi(t,\vec{x}),\,\Pi(t,\vec{y})\,]\,, \end{aligned} \tag{1.305}$$

and the right-hand-sides of (1.208) are naturally invariant. One easily checks that the operators $\Phi(t,\vec{x})$ and $\Pi(t,\vec{x})$ satisfy the Hamilton equation of motion of a Klein–Gordon field. One has first:

$$\begin{aligned} \dot{\Phi}(t,\vec{x}) = i\,[\,H,\,\Phi(t,\vec{x})\,] &= i\left[\frac{1}{2}\int d^3y\,\big(\Pi^2(t,\vec{y}) + (\vec{\nabla}\Phi(t,\vec{y}))^2 + m^2\Phi^2(t,\vec{y})\big),\,\Phi(t,\vec{x})\right] \\ &= \frac{i}{2}\int d^3y\,\big[\,\Pi^2(t,\vec{y}),\,\Phi(t,\vec{x})\,\big] = \Pi(t,\vec{x}) \end{aligned} \tag{1.306}$$

Then:

$$\begin{aligned} \dot{\Pi}(t,\vec{x}) = i\,[\,H,\,\Pi(t,\vec{x})\,] &= \frac{i}{2}\int d^3y\,\left(\Big[\,(\vec{\nabla}\Phi(t,\vec{y}))^2,\,\Pi(t,\vec{x})\,\Big] + m^2\,\big[\,\Phi^2(t,\vec{y}),\,\Pi(t,\vec{x})\,\big]\right) \\ &= -\int d^3y\,\left(\vec{\nabla}\Phi(t,\vec{y})\cdot\vec{\nabla}\delta^{(3)}(\vec{x}-\vec{y}) + m^2\Phi(t,\vec{y})\delta^{(3)}(\vec{x}-\vec{y})\right) \\ &= \Delta\Phi(t,\vec{x}) - m^2\Phi(t,\vec{x})\,. \end{aligned} \tag{1.307}$$

Let us consider next how the mode expansion (1.209) of the field operator Φ looks like in the

Heisenberg picture. We need first to compute, in the Schrödinger picture:

$$\left[H,\, a_{\vec{p}}\right] = \int \frac{d^3q}{(2\pi)^3} E_q \left[a^\dagger_{\vec{q}} a_{\vec{q}},\, a_{\vec{p}}\right] = -E_p a_{\vec{p}} \tag{1.308a}$$

$$\left[H,\, a^\dagger_{\vec{p}}\right] = \int \frac{d^3q}{(2\pi)^3} E_q \left[a^\dagger_{\vec{q}} a_{\vec{q}},\, a^\dagger_{\vec{p}}\right] = +E_p a^\dagger_{\vec{p}} \tag{1.308b}$$

From which we deduce that (given that the Heisenberg and Schrödinger pictures coincide by definition at $t = 0$):

$$\dot{a}_{\vec{p},\,H} = i\left[H,\, a_{\vec{p},\,H}\right] = -iE_p\, a_{\vec{p},\,H} \implies a_{\vec{p},\,H} = e^{-iE_p t} a_{\vec{p}} \tag{1.309a}$$

$$\dot{a}^\dagger_{\vec{p},\,H} = i\left[H,\, a^\dagger_{\vec{p},\,H}\right] = +iE_p\, a^\dagger_{\vec{p},\,H} \implies a^\dagger_{\vec{p},\,H} = e^{iE_p t} a_{\vec{p}} \tag{1.309b}$$

We obtain then the mode expansion of the field operator $\Phi(x) = \Phi(t, \vec{x})$ in the Heisenberg picture. Dropping the H index on Φ, one has:

$$\Phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_{\vec{p}} e^{-ip\cdot x} + a^\dagger_{\vec{p}} e^{ip\cdot x}\right) \tag{1.310}$$

Exercise: show that the Lorentz transformation $U(\Lambda)^\dagger \Phi(x) U(\Lambda) = \Phi(\Lambda^{-1}x)$ *follows from eqn. (1.232).*

In the Heisenberg picture, the inversion formulæ (1.212) giving the modes in terms of the field operator and its conjugate can be expressed in a convenient way. For a pair of solutions $(f, g) : \mathbb{R}^{1,3} \to \mathbb{C}^2$ of the Klein–Gordon equation, we introduce an indefinite hermitian inner product:

$$\langle f | g \rangle \overset{\text{def.}}{=} i \int d^3x\, f^\dagger \overleftrightarrow{\partial_0}\, g = i \int d^3x \left(f^\dagger \partial_0 g - \partial_0 f^\dagger g\right)\Big|_{\text{fixed } t} . \tag{1.311}$$

It is independent of the time t chosen for the computation, provided that f and g satisfy the Klein–Gordon equation (with identical masses):

$$\frac{d\langle f | g \rangle}{dt} = i \int d^3x \left(f^\dagger(\Delta - m^2) g - (\Delta - m^2) f^\dagger g\right) = i \int d^3x\, \vec{\nabla} \cdot \left(f^\dagger \vec{\nabla} g - \vec{\nabla} f^\dagger g\right) = 0\,, \tag{1.312}$$

assuming that f and g decay fast enough at infinity. More generally, it is independent of time slicing. Next, we observe that relativistically normalized plane-waves are orthonormal w.r.t. this inner product:

$$\left\langle \tfrac{e^{-ip\cdot x}}{\sqrt{2E_p}} \middle| \tfrac{e^{-iq\cdot x}}{\sqrt{2E_q}} \right\rangle = \frac{1}{\sqrt{2E_p 2E_q}} \int d^3x \left(E_q e^{i(p-q)\cdot x} + E_p e^{-i(p-q)\cdot x}\right) = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q})\,, \tag{1.313}$$

for on-shell positive frequency 4-momenta, *i.e.* $p^0 = E_p = \sqrt{\vec{p}^2 + m^2}$. With one mode of positive frequency and one of negative frequency one gets:

$$\left\langle \tfrac{e^{-ip\cdot x}}{\sqrt{2E_p}} \middle| \tfrac{e^{+iq\cdot x}}{\sqrt{2E_q}} \right\rangle = 0\,. \tag{1.314}$$

This allows to extract easily $a_{\vec{p}}$ and $a^\dagger_{\vec{p}}$ from $\Phi(x)$:

$$a_{\vec{p}} = \left\langle \tfrac{e^{-ip\cdot x}}{\sqrt{2E_p}} \middle| \Phi(x) \right\rangle \tag{1.315a}$$

$$a^\dagger_{\vec{p}} = -\left\langle \tfrac{e^{ip\cdot x}}{\sqrt{2E_p}} \middle| \Phi(x) \right\rangle \tag{1.315b}$$

The story is essentially the same for the other field theories considered in these notes. For the Dirac field operator, we have the non-trivial equal-time anti-commutation relation:

$$\left\{ \Psi(t,\vec{x})^a,\, \Psi^\dagger(t,\vec{y})^b \right\} = \delta^{ab}\, \delta^{(3)}(\vec{x}-\vec{y})\,. \tag{1.316}$$

We have then the following expression for the Heisenberg picture Dirac field operator $\Psi(x) = e^{iHt}\Psi(\vec{x})e^{-iHt}$:

$$\Psi(x) = \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(b^s_{\vec{p}} u_s(\vec{p}) e^{-ip\cdot x} + c^{s\,\dagger}_{\vec{p}} v_s(\vec{p}) e^{ip\cdot x} \right) \tag{1.317}$$

The spinorial counterparts of the scalar inversion formulae (1.315) are a bit more difficult to get since we have to deal with the polarization spinors. Using the bilinear identities (1.199, 1.200) one finds that:

$$\frac{\bar{u}_s(\vec{p})}{2m} \left\langle \tfrac{e^{-ip\cdot x}}{\sqrt{2E_p}} \middle| \Psi(x) \right\rangle = \frac{1}{2m} \sum_{r=1}^{2} \bar{u}_s(\vec{p}) u_r(\vec{p}) b^r_{\vec{p}} = b^s_{\vec{p}}\,, \tag{1.318a}$$

$$\frac{\bar{v}_s(\vec{p})}{2m} \left\langle \tfrac{e^{ip\cdot x}}{\sqrt{2E_p}} \middle| \Psi(x) \right\rangle = -\frac{1}{2m} \sum_{r=1}^{2} \bar{v}_s(\vec{p}) v_r(\vec{p}) c^{r\,\dagger}_{\vec{p}} = c^{s\,\dagger}_{\vec{p}}\,. \tag{1.318b}$$

These identities can be written in a more useful way taking advantage of the first-order nature of the Dirac equation. One has for instance

$$\begin{aligned}
\bar{u}_s(\vec{p}) \left\langle e^{-ip\cdot x} \middle| \Psi(x) \right\rangle &= u^\dagger_s(\vec{p}) \int d^3x\, e^{ip\cdot x} \big(i\gamma^0 \partial_0 + \gamma^0 E_p \big) \Psi(x) \\
&= u^\dagger_s(\vec{p}) \frac{1}{\sqrt{2E_p}} \int d^3x\, e^{ip\cdot x} \big(-i\gamma^i \partial_i + m + \gamma^0 E_p \big) \Psi(x) \\
&\overset{\text{IBP}}{=} u^\dagger_s(\vec{p}) \int d^3x\, e^{ip\cdot x} \underbrace{\big(-\gamma^i p_i + m + \gamma^0 E_p \big)}_{\gamma^0(\gamma^\mu p_\mu + m)\gamma^0} \Psi(x) \\
&= \int d^3x\, e^{ip\cdot x} \bar{u}_s(\vec{p}) (\not{p} + m) \gamma^0 \Psi(x) = 2m \int d^3x\, e^{ip\cdot x} \bar{u}_s(\vec{p}) \gamma^0 \Psi(x)\,,
\end{aligned} \tag{1.319}$$

using the identity (1.181) in the last step.

Overall, and going through the same steps for the other creation and annihilation modes, one obtains the useful relations:

$$b^s_{\vec{p}} = \int d^3x \, \frac{e^{ip\cdot x}}{\sqrt{2E_p}} \bar{u}^s(\vec{p}) \gamma^0 \Psi(t, \vec{x}) \,, \tag{1.320a}$$

$$b^{s\,\dagger}_{\vec{p}} = \int d^3x \, \frac{e^{-ip\cdot x}}{\sqrt{2E_p}} \overline{\Psi}(t, \vec{x}) \gamma^0 u^s(\vec{p}) \,, \tag{1.320b}$$

$$c^s_{\vec{p}} = \int d^3x \, \frac{e^{ip\cdot x}}{\sqrt{2E_p}} \overline{\Psi}(t, \vec{x}) \gamma^0 v^s(\vec{p}) \,, \tag{1.320c}$$

$$c^{s\,\dagger}_{\vec{p}} = \int d^3x \, \frac{e^{ip\cdot x}}{\sqrt{2E_p}} \bar{v}^s(\vec{p}) \gamma^0 \Psi(t, \vec{x}) \,. \tag{1.320d}$$

Causality. Any sensible relativistic physical theory should respect causality; since we did not impose this as a constraint at this stage, this property should be somehow "built-in" in the construction of quantum field theories.

Let us recall some basic facts. Since time-ordering is not a Lorentz-invariant concept for space-like separated space-time points (with $(\Delta t)^2 - (\Delta \vec{x})^2 < 0$), there exists a notion of causality only for time-like separated events (with $(\Delta t)^2 - (\Delta \vec{x})^2 > 0$). Therefore there could not be any causality relation between space-like separated events. In the context of a quantum theory, it means that *observables associated with space-like separated measurements should commute*, otherwise one measurement could influence the other one.[25]

In the context of the Klein–Gordon theory let us consider the commutator of field operators $\Phi(x)$ at two separated space-time points, not necessarily at egal time. An explicit computation gives:

$$\begin{aligned}
[\,\Phi(x),\,\Phi(y)\,] &= \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_p 2E_q}} \left[\, a_{\vec{p}} e^{-ip\cdot x} + a^\dagger_{\vec{p}} e^{ip\cdot x},\, a_{\vec{q}} e^{-iq\cdot y} + a^\dagger_{\vec{q}} e^{iq\cdot y} \,\right] \\
&= \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_p 2E_q}} \left(e^{i(q\cdot y - p\cdot x)} \left[\, a_{\vec{p}},\, a^\dagger_{\vec{q}} \,\right] + e^{-i(q\cdot y - p\cdot x)} \left[\, a^\dagger_{\vec{p}},\, a_{\vec{q}} \,\right] \right) \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left(e^{-ip\cdot(x-y)} - e^{-ip\cdot(y-x)} \right) .
\end{aligned} \tag{1.321}$$

Let us assume that x and y are space-like separated, *i.e.* that $(x-y)^2 < 0$. By a Lorentz transformation, it is possible to reach a frame with $x^0 - y^0 = 0$. The integration measure

[25] This should not be confused with quantum entanglement; quantum states are not local objects (in a QFT, a state is defined on a whole space-like slice of the ambient space-time). In the classic gedankenexperiment with two entangled spin states, the space-separated spin operators associated with each of the two spins commute with each other so the order of the measurements do not matter. This experiment preserves causality, as the result of the measurement realized by the one observer cannot be transmitted to the other observer faster than light.

in the last step of eqn. (1.321) being Lorentz invariant, one obtains, using the change of variables $\vec{p} \to -\vec{p}$ in the second term :

$$[\,\Phi(x),\,\Phi(y)\,]\big|_{(x-y)^2<0} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\left(e^{i\vec{p}\cdot(\vec{y}-\vec{x})} - e^{-i\vec{p}\cdot(\vec{y}-\vec{x})}\right) = 0\,. \tag{1.322}$$

As a consequence, the same holds for the commutator of any local expressions in the field and its derivatives and the theory preserves causality. Beyond the case of a free theory, where this property can be obtained from a direct computation, in a general interacting QFT this property, known as *micro-causality*, becomes one of the postulates of axiomatic QFT [6].

It is instructive to consider the how the same property is realized for a complex scalar field. While $[\,\Phi,\,\Phi\,] = \left[\,\Phi^\dagger,\,\Phi^\dagger\,\right] = 0$ regardless of the space-like or time-like nature of the separation (technically, because there are two sets of modes, one of particles and the other for anti-particles), one has:

$$\begin{aligned}\left[\,\Phi(x),\,\Phi(y)^\dagger\,\right] &= \int \frac{d^3p}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_p 2E_q}}\left[\,a_{\vec{p}}e^{-ip\cdot x} + b^\dagger_{\vec{p}}e^{ip\cdot x},\,a^\dagger_{\vec{q}}e^{-iq\cdot y} + b_{\vec{q}}e^{iq\cdot y}\,\right] \\ &= \int \frac{d^3p}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_p 2E_q}}\left(e^{i(q\cdot y-p\cdot x)}\left[\,a_{\vec{p}},\,a^\dagger_{\vec{q}}\,\right] + e^{-i(q\cdot y-p\cdot x)}\left[\,b^\dagger_{\vec{p}},\,b_{\vec{q}}\,\right]\right)\end{aligned} \tag{1.323}$$

Hence as before $\left[\,\Phi(x),\,\Phi(y)^\dagger\,\right] = 0$ for $(x-y)^2 < 0$. Remarkably, one sees that causality holds thanks to the cancelation between the particle and anti-particle contributions. If one considers the expectation value $\langle 0|\left[\,\Phi(x),\,\Phi(y)^\dagger\,\right]|0\rangle$, as we will do below, the first term of the commutator corresponds to the amplitude of the process where a particle is created at y and destroyed at x, while the second term corresponds to an anti-particle created at x and destroyed at y; both amplitudes should be equal whenever x and y are space-like separated.[26] It demonstrates that a QFT approach, as opposed to relativistic quantum mechanics, is essential to ensure causality.

Let us consider now the case of a Dirac field. The relevant quantity to define micro-causality is an anti-commutator between the field operator and its Dirac conjugate, rather

[26]The same can be said of course for a real scalar field, except that in this case the particle is its own anti-particle.

than a commutator:

$$\begin{aligned}
&\big\{\,\Psi(x),\,\overline{\Psi}(y)\,\big\} = \\
&\sum_{r,s}\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_p 2E_q}}\Big\{\,b^r_{\vec p}u_r(\vec p)e^{-ip\cdot x}+c^{r\,\dagger}_{\vec p}v_r(\vec p)e^{ip\cdot x},\;b^{s\,\dagger}_{\vec q}\bar u_s(\vec q)e^{iq\cdot y}+c^s_{\vec q}\bar v_s(\vec q)e^{-iq\cdot y}\,\Big\}\\
&\qquad=\sum_{r,s}\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_p 2E_q}}\Big(\Big\{\,b^r_{\vec p},\,b^{s\,\dagger}_{\vec q}\,\Big\}u_r(\vec p)\bar u_s(\vec q)e^{-ip\cdot x+iq\cdot y}\\
&\qquad\qquad+\Big\{\,c^{r\,\dagger}_{\vec p},\,c^s_{\vec p}\,\Big\}v_r(\vec p)\bar v_s(\vec p)e^{ip\cdot x-iq\cdot y}\Big)\\
&\qquad=\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\Big(\underbrace{\sum_s u_s(\vec p)\bar u_s(\vec p)}_{\not p+m}\,e^{-ip\cdot(x-y)}+\underbrace{\sum_s v_s(\vec p)\bar v_s(\vec p)}_{\not p-m}\,e^{ip\cdot(x-y)}\Big)\\
&\qquad=(i\not\partial+m)\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\big(e^{-ip\cdot(x-y)}-e^{ip\cdot(x-y)}\big)
\end{aligned}\tag{1.324}$$

Hence, as for scalar fields,

$$\big\{\,\Psi(x),\,\overline{\Psi}(y)\,\big\}\big|_{(x-y)^2<0}=0\,,\tag{1.325}$$

thanks to the cancelation between the particle and the anti-particle contributions. Causality follows from this result as local observables in the QFT are local expressions of integer spin, for instance made from the spinor bilinear $\bar\Psi\Psi$ and its derivatives.

Propagators. Another way to look at causality in a quantum field theory is to consider *propagators*. Let us consider, for a real scalar field, the amplitude:

$$\Delta(x-y)=\langle 0|\,\Phi(x)\Phi(y)\,|0\rangle=\langle x|y\rangle\ ,\tag{1.326}$$

which is the probability amplitude to detect at a point of space-time coordinates x a particle created at y.[27] As the notation on the left-hand side suggests, it only depends on the difference $(x-y)$, thanks to translation invariance of the vacuum:

$$\begin{aligned}
\langle 0|\,\Phi(x)\Phi(y)\,|0\rangle&=\langle 0|\,e^{iP\cdot x}\Phi(0)e^{-iP\cdot x}e^{iP\cdot y}\Phi(0)e^{-iP\cdot y}\,|0\rangle=\langle 0|\,e^{iP\cdot(x-y)}\Phi(0)e^{-iP\cdot(x-y)}\Phi(0)\,|0\rangle\\
&=\langle 0|\,\Phi(x-y)\Phi(0)\,|0\rangle\ .
\end{aligned}\tag{1.327}$$

Second, its dependence is also constrained by Lorentz invariance. One has

$$\Delta\big(\Lambda^{-1}(x-y)\big)=\langle 0|\,\Phi\big(\Lambda^{-1}(x-y)\big)\Phi(0)\,|0\rangle=\langle 0|\,U(\Lambda)^\dagger\Phi(x-y)U(\Lambda)\Phi(0)\,|0\rangle\tag{1.328}$$

$$=\langle 0|\,\Phi(x-y)\underbrace{U(\Lambda)\Phi(0)U(\Lambda)^\dagger}_{\Phi(0)}|0\rangle=\Delta(x-y)\,.\tag{1.329}$$

[27] This quantity should be thought as a distribution rather than a function, and should be, as such, convoluted with appropriate test-functions which correspond physically to normalizable wave-functions.

These properties are completely general. For a free scalar field obeying the Klein–Gordon equation, an explicit computation, very similar to the previous ones, gives:

$$\Delta(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_p 2E_q}} \langle 0| \underbrace{a_{\vec{p}} a^\dagger_{\vec{q}}}_{\cancel{a^\dagger_{\vec{q}} a_{\vec{p}}} + (2\pi)^2\delta^{(3)}(\vec{p}-\vec{q})} |0\rangle\, e^{-i(p\cdot x - q\cdot y)} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p} e^{-ip\cdot(x-y)} \tag{1.330}$$

with the 4-momentum on-shell, *i.e.* $p = (E_p, \vec{p})$. It can be naturally written in a Lorentz-invariant way:

$$\Delta(x-y) = \int \frac{d^4p}{(2\pi)^3}\delta(p^2-m^2)\Theta(p^0)e^{-ip\cdot(x-y)}\,. \tag{1.331}$$

Written in this form, it is obvious that D satisfies the Klein–Gordon equation:

$$\big(\Box + m^2\big)\Delta(x) = -\int \frac{d^4p}{(2\pi)^3}\delta(p^2-m^2)\Theta(p^0)\big(p^2-m^2\big)e^{-ip\cdot x} = 0\,. \tag{1.332}$$

One notices also that the commutator (1.321) can be written in terms of D:

$$[\,\Phi(x),\,\Phi(y)\,] = \Delta(x-y) - \Delta(y-x)\,. \tag{1.333}$$

The expression (1.330) of the propagator allows to probe its behaviour for spacelike-separated points, exactly as was done in the introduction. Let us repeat the argument here. We work in spherical coordinates taking $\vec{x}$ as a z-axis, *i.e.* such that $\vec{p}\cdot\vec{x} = p\|\vec{x}\|\cos\theta$. We have then:

$$\begin{aligned}
\Delta(x) &= \frac{1}{8\pi^2}\int_0^\infty \frac{p^2 dp}{\sqrt{p^2+m^2}} e^{-i\sqrt{p^2+m^2}x^0}\int_{-1}^{1} d(\cos\theta) e^{ip\|\vec{x}\|\cos\theta} \\
&= -\frac{i}{8\pi^2\|\vec{x}\|}\int_0^\infty \frac{p\,dp}{\sqrt{p^2+m^2}} e^{-i\sqrt{p^2+m^2}x^0}\left(e^{ip\|\vec{x}\|} - e^{-ip\|\vec{x}\|}\right) \qquad (1.334)\\
&= -\frac{im}{8\pi^2\|\vec{x}\|}\int_{-\infty}^{\infty} \frac{u\,du}{\sqrt{u^2+1}} e^{-im\|\vec{x}\|\left(\sqrt{u^2+1}\,\frac{x^0}{\|\vec{x}\|} - u\right)}\,. \qquad (1.335)
\end{aligned}$$

The phase of the integrand has a stationnary point at

$$\frac{u}{\sqrt{u^2+1}}\frac{x^0}{\|\vec{x}\|} = 1 \quad\Longrightarrow\quad u^2 = -\frac{\|x\|^2}{\|x\|^2 - (x^0)^2} \tag{1.336}$$

For space-like separations, one can get an estimate of the integral for large values of $m\|\vec{x}\|$ using the steepest descent method. Importantly, the leading behavior of the two-point function is given by:

$$\Delta(x) \overset{\|\vec{x}\|\gg\lambda\!\!\!^{-}_c}{\sim} e^{-\|\vec{x}\|/\lambda\!\!\!^{-}_c}\,, \tag{1.337}$$

where $\lambda\!\!\!^{-}_c = \hbar/mc$, which is equal to $1/m$ in natural units, is the reduced Compton wavelength of the particle. It means that, somehow, if we could locate particles within regions of size $\Delta x \ll \lambda\!\!\!^{-}_c$, we would be able to detect particle positions as space-like separated but correlated

events. Fortunately, the uncertainty principle saves the day. If $\Delta x \ll \lambda\!\!\!^{-}_c$, $\Delta p \gg mc$ hence $\Delta E_p > mc^2$, leading to particle production. This crude reasoning is made more precise by the discussion below eqn. (1.323): particle/anti-particle pair production is essential for causality to hold, thanks to the totally destructive interference bewteen their respective contributions to the commutator.

A physically more important quantity, of which we will have ample use later, is the *Feynman propagator*, named after the American physicist Richard Feynman. Let us define the time-ordered product of two bosonic local operators:

$$T\,\mathcal{O}_B(x)\mathcal{P}_B(y) = \begin{cases} \mathcal{O}_B(x)\mathcal{P}_B(y) & \text{if} \quad x^0 \geqslant y^0 \\ \mathcal{P}_B(y)\mathcal{O}_B(x) & \text{if} \quad x^0 < y^0 \end{cases} \tag{1.338}$$

We define then

$$D(x-y) \overset{\text{def.}}{=} \langle 0|\, T\Phi(x)\Phi(y)\,|0\rangle = \Theta(x_0-y_0)\Delta(x-y) + \Theta(y_0-x_0)\Delta(y-x) \tag{1.339}$$

We expect that $D(x-y)$ is Lorentz-invariant for time-like separations, since time-ordering is a Lorentz-invariant concept only there. For space-like separations, $\Phi(x)$ and $\Phi(y)$ commute so time ordering has no effect and Lorentz invariance is maintained. Using eqn. (1.330),

$$D(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\left(\Theta(x^0)e^{-ip\cdot x} + \Theta(-x^0)e^{ip\cdot x}\right) . \tag{1.340}$$

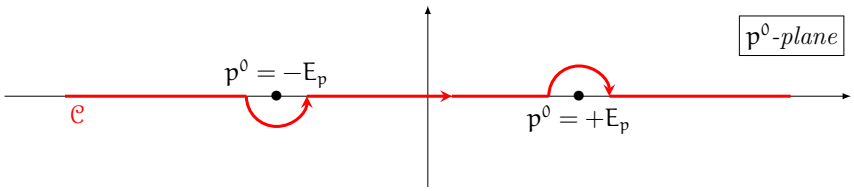

Figure 1.2: *Feynman contour.*

Property 1 *The scalar Feynman propagator (1.339) can be represented by the integral:*

$$D(x) = \int_{\mathcal{C}} \frac{dp^0}{2\pi} \int \frac{d^3p}{(2\pi)^3} \frac{ie^{-ip\cdot x}}{p^2 - m^2}\,, \tag{1.341}$$

where the contour of integration $\mathcal{C}$ in the complex p^0-plane is represented on figure. 1.2.

We notice that the integrand in (1.341) has two simple poles on the real axis:

$$\frac{1}{p^2-m^2} = \frac{1}{(p^0)^2 - \vec{p}^2 - m^2} = \frac{1}{2E_p}\left(\frac{1}{p^0 - E_p} - \frac{1}{p^0 + E_p}\right) . \tag{1.342}$$

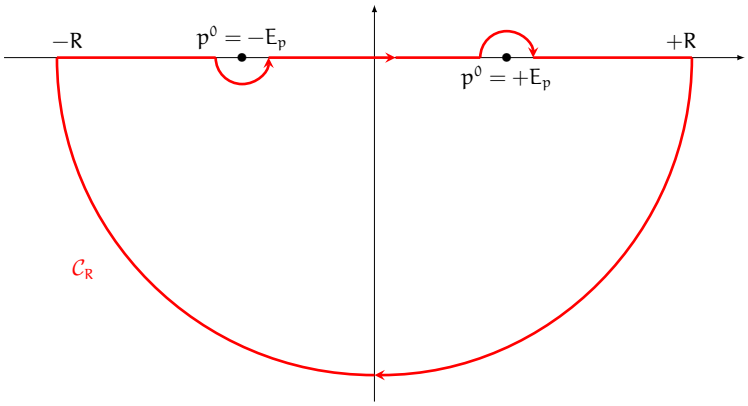

Figure 1.3: *Contour for* $x^0 > 0$.

The contour $\mathcal{C}$ turns below the pole at $p^0 = -E_p$ and above the pole at $p^0 = E_p$.

Let us consider that $x^0 > 0$. Then $|e^{-ip\cdot x}| = \exp\left(x^0 \operatorname{Im}(p^0)\right)$ falls exponentially for large negative values of p^0 and one can evaluate the integral (1.341) using Cauchy's residue theorem by considering the contour $\mathcal{C}_R$ closed below the real axis and running clockwise, see fig. 1.3. In the limit $R \to \infty$, the contribution of the big half-circle vanishes and one obtains:

$$\int_{\mathcal{C}} \frac{dp^0}{2\pi} \frac{ie^{-ip^0x^0}}{(p^0)^2 - \vec{p}^2 - m^2} = \operatorname*{Res}_{p^0=E_p} \left(\frac{e^{-ip^0x^0}}{(p^0)^2 - \vec{p}^2 - m^2} \right) = \frac{e^{-iE_px^0}}{2E_p}\,. \tag{1.343}$$

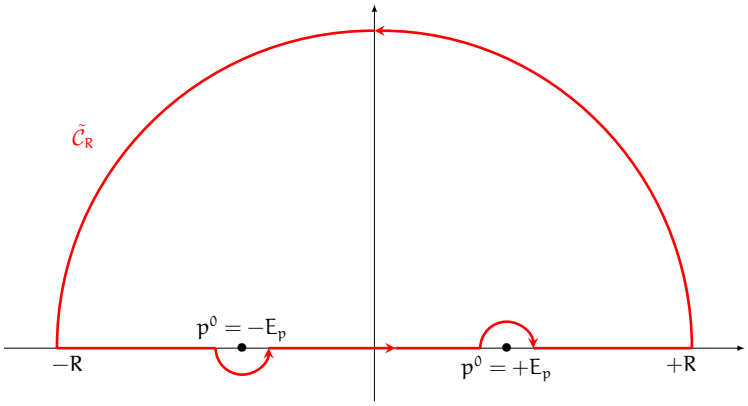

Figure 1.4: *Contour for* $x^0 < 0$.

Second, let us consider that $x^0 < 0$. In this case, $|e^{-ip\cdot x}| = \exp\left(x^0 \operatorname{Im}(p^0)\right)$ falls off exponentially for large positive values of p^0. Therefore, one can evaluate the integral (1.341) using Cauchy's residue theorem by considering the contour $\tilde{\mathcal{C}}_R$ closed above the real axis and running counter-clockwise, see fig. 1.4. In the limit $R \to \infty$, one obtains:

$$\int_{\mathcal{C}} \frac{dp^0}{2\pi} \frac{ie^{-ip^0x^0}}{(p^0)^2 - \vec{p}^2 - m^2} = -\operatorname*{Res}_{p^0=-E_p} \left(\frac{e^{-ip^0x^0}}{(p^0)^2 - \vec{p}^2 - m^2} \right) = \frac{e^{iE_px^0}}{2E_p} \tag{1.344}$$

Adding these two pieces together, we find indeed eqn. (1.340) after doing the change of variables $\vec{p} \to -\vec{p}$ in the second term there. □

Another equivalent way to write this integral representation of the Feynman propagator is to keep the p^0 integral running along the real axis, but displacing the poles infinitesimally above and below the real axis in order to pick the same residues as before for $x^0 > 0$ and $x^0 < 0$. This is known as the $i\epsilon$ prescription, see fig. 1.5.

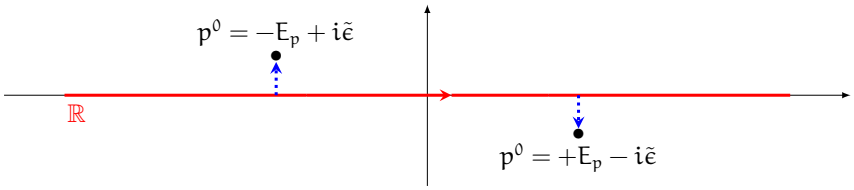

Figure 1.5: $i\epsilon$ *prescription.*

This prescription is implemented using the following representation of the Feynman propagator:

$$D(x) = \int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ip\cdot x}}{p^2 - m^2 + i\epsilon} \tag{1.345}$$

where the integral is done *before* taking the limit $\epsilon \to 0^+$. One benefit of this prescription is to make Lorentz invariance manifest.[28] Its Fourier transform is simply:

$$\widehat{D}(p) = \int d^4x\, e^{ip\cdot x} \langle 0|\, T\Phi(x)\Phi(y)\, |0\rangle = \frac{i}{p^2 - m^2 + i\epsilon} \tag{1.346}$$

Finally, notice that the Feynman propagator is a Green function for the Klein–Gordon equation. One has indeed, regardless of the specific choice of contour in the p^0-plane:

$$(\Box + m^2)D(x) = (\Box + m^2) \int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ip\cdot x}}{p^2 - m^2} = -i \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} = -i\delta^{(4)}(x)\,. \tag{1.347}$$

Other choices of contour (more specifically, of ways of bypassing the singularities) give the retarded or advanced Green function, as you will see during the tutorials.

The same computation of the Feynman propagator can be repeated for a complex scalar field. We first notice that, with or without time-ordering, we have:

$$\langle 0|\, \Phi(x)\Phi(y)\, |0\rangle = \langle 0|\, \Phi^\dagger(x)^\dagger \Phi(y)\, |0\rangle = 0\,, \tag{1.348}$$

[28] The parameter $\tilde{\epsilon}$ on the figure is identified with $\epsilon/2E_p$, since

$$p^2 - m^2 + i\epsilon = (p_0 - E_p + \tfrac{i\epsilon}{2E_p})(p_0 + E_p - \tfrac{i\epsilon}{2E_p}) + \mathcal{O}(\epsilon^2/E_p^2)\,.$$

because the mode expansion of Φ, see eqn. (1.261), contains particle creation operators and anti-particle annihilation operators. A more fundamental reason, which extends to an interacting theory, is the existence of the $\mathsf{U}(1)$ symmetry $\Phi \mapsto e^{i\theta}\Phi$. Let Q be the operator associated to its Noether charge, see eqn. (1.265). The vacuum is expected to be invariant under the $\mathsf{U}(1)$ action (this is trivially true in the free theory):

$$e^{iQ\theta}\,|0\rangle = |0\rangle \tag{1.349}$$

For an arbitrary n-point correlation function, we have

$$\begin{aligned}\langle 0|\,\Phi(x_1)\Phi(x_2)\cdots\Phi(x_n)\,|0\rangle &= \langle 0|\,e^{-iQ\theta}\Phi(x_1)\Phi(x_2)\cdots\Phi(x_n)e^{iQ\theta}\,|0\rangle \\ &= \langle 0|\,\underbrace{e^{-iQ\theta}\Phi(x_1)e^{iQ\theta}}_{e^{i\theta}\Phi(x_1)}\,e^{-iQ\theta}\Phi(x_2)\cdots e^{-iQ\theta}\Phi(x_n)e^{iQ\theta}\,|0\rangle \\ &= e^{in\theta}\,\langle 0|\,\Phi(x_1)\Phi(x_2)\cdots\Phi(x_n)\,|0\rangle \end{aligned} \tag{1.350}$$

hence necessarily $\langle 0|\,\Phi(x_1)\Phi(x_2)\cdots\Phi(x_n)\,|0\rangle = 0$. This extends trivially to theories with multiple field carrying different charges: *all non-neutral correlation functions vanish.*[29] Coming back to the two-point function or propagator, we are left with:

$$\langle 0|\,T\Phi(x)\Phi(y)^\dagger\,|0\rangle = D(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{ie^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon}\,, \tag{1.351}$$

i.e. the same result as for a real scalar field, as well as its Hermitian conjugate.

To close this first part of these lecture notes, let us consider the Feynman propagator for a Dirac field. First, because fermionic fields satisfy anti-commutation rather than commutation relations, the definition (1.338) of the time-ordering needs to be modified to:

$$T\,\mathcal{O}_F(x)\mathcal{P}_F(y) = \begin{cases} \mathcal{O}_F(x)\mathcal{P}_F(y) & \text{if} \quad x^0 \geqslant y^0 \\ -\mathcal{P}_F(y)\mathcal{O}_F(x) & \text{if} \quad x^0 < y^0 \end{cases} \tag{1.352}$$

As in the previous case, the $\mathsf{U}(1)$ symmetry $\Psi \mapsto e^{i\theta}\Psi$ forbids non-zero values for the correlators $\langle\Psi\Psi\rangle$ and $\langle\bar\Psi\bar\Psi\rangle$. We consider then the propagator:

$$\boxed{\quad D_F^{ab}(x-y) \overset{\text{def.}}{=} \langle 0|\,T\,\Psi^a(x)\bar\Psi^b(y)\,|0\rangle\ . \quad} \tag{1.353}$$

The computation is very similar to previous ones, with an extra minus sign from time ordering. One starts with:

$$\begin{aligned}\langle 0|\,\Psi^a(x)\bar\Psi^b(y)\,|0\rangle &= \sum_{r,s}\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_p 2E_q}}\,\langle 0|\,b_{\vec p}^{\,r}b_{\vec q}^{\,s\,\dagger}\,|0\rangle\,u_r^a(\vec p)\bar u_s^b(\vec q)e^{iq\cdot y-ip\cdot x} \\ &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip\cdot(x-y)}\sum_s u_s^a(\vec p)\bar u_s^b(\vec p) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip\cdot(x-y)}\big(\not p^{ab}+m\delta^{ab}\big)\end{aligned} \tag{1.354}$$

[29] For non-abelian symmetry groups, the correlator should transform in the trivial representation.

as well as

$$\langle 0|\,\bar{\Psi}^b(y)\Psi^a(x)\,|0\rangle = \sum_{r,s}\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_p 2E_q}}\,\langle 0|\,c^r_{\vec{p}}c^{s\,\dagger}_{\vec{q}}\,|0\rangle\,\bar{v}^b_r(\vec{p})v^a_s(\vec{q})e^{iq\cdot y-ip\cdot x}$$
$$= \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{ip\cdot(x-y)}\sum_s \bar{v}^b_s(\vec{p})v^a_s(\vec{p}) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{ip\cdot(x-y)}\big(\not{p}^{ab}-m\delta^{ab}\big) \quad (1.355)$$

Such that (notice the relative minus sign):

$$D_F^{ab}(x) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\left(\Theta(x)e^{-ip\cdot x}\big(\not{p}^{ab}+m\delta^{ab}\big)-\Theta(-x)e^{ip\cdot x}\big(\not{p}^{ab}-m\delta^{ab}\big)\right) \quad (1.356)$$

Looking at the expression (1.340) of the scalar Feynman propagator $D(x)$, we realize that:

$$D_F^{ab}(x) = (i\not{\partial}^{ab}+m\delta^{ab})\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\left(\Theta(x^0)e^{-ip\cdot x}+\Theta(-x^0)e^{ip\cdot x}\right) = (i\not{\partial}^{ab}+m\delta^{ab})D(x)\,. \quad (1.357)$$

Therefore, using the representation (1.345) of the latter, one can represent the Feynman propagator for the Dirac field as a 4-dimensional integral with a $i\epsilon$ prescription:

$$D_F(x) = \int\frac{d^4p}{(2\pi)^4}\frac{i(\not{p}+m)e^{-ip\cdot x}}{p^2-m^2+i\epsilon} \quad (1.358)$$

In passing, eqn. (1.357) indicates that D_F is also a Green function for the Dirac equation:

$$(i\not{\partial}-m)D_F(x) = (i\not{\partial}-m)(i\not{\partial}+m)D(x) = -(\Box+m^2)D(x) = i\delta^{(4)}(x)\,. \quad (1.359)$$

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Part 2

Interacting quantum field theories

Elementary particles interact with each other, otherwise our world would be quite boring and lifeless. Handling interactions in the quantum field theory framework is difficult in many ways.

The first question is identifying the right degrees of freedom, in other words the particles associated with excitations of a quantum field. The answer may depend of the energies probed by the experiments under consideration: for instance, in the context of strong interactions, the relevant degrees of freedom at very high energies (far above 200 MeV) are quarks and gluons, while at low energies the relevant degrees of freedom are hadrons, *i.e.* bound states like protons or neutrons.

The second question is whether the QFT describing the interacting particles can be formulated, to start with, in terms of a least action principle based on some classical Lagrangian. Physicists have found in the last decades compelling evidence that interacting supersymmetryc QFTs that do not admit any Lagrangian description make perfect sense. Even if a Lagrangian description exists, it should include in principle all possible interaction terms that are not forbidden by the symmetries of the theory, space-time and internal, which are in infinite number. Here the ideas of renormalization come to rescue: in most cases the strength of their couplings depend on the energy scale and if one is, say, interested in the low-energy behavior of the theory, only a handful of interaction terms – if not a single one – matter.

Third, assuming that the QFT under consideration admits a Lagrangian with few interaction terms as a starting point, we do not know how to solve the theory, except in particular cases where lower dimensionality and/or extra symmetries like supersymmetry help. The key is to identify a region – if any – where these interactions become weak: for instance, low-energies for quantum electrodynamics (QED) and high energies for quantum chromodynamics (QCD). Whenever this is the case one can treat the interactions as perturbations of a free-theory, using methods close to those that you are familiar with in the context of quantum mechanics. This program is not without difficulties, as the presence of ultraviolet and infrared divergences, the convergence of the perturbation series, the extremely hard technical issues involved in the actual computation of perturbative corrections, etc. but is incredibly successful, providing high-precision predictions for experiments in particle accelerators.

2.1 Some general aspects of quantum field theories

It is a longstanding dream of theoretical physicists to exactly solve quantum theories using as an input some set of sensible assumptions, as well as the symmetries of the theory (Poincaré symmetry, its possible conformal and supersymmetric extensions, and internal symmetries) together with the representations carried by the fields. At a rather modest level, we will derive here some general properties of the propagator.

Let us start with some simple general properties that any sensible quantum field theory is expected to satisfy (these are the Wightman axioms in a simplified presentation):

1. The state space of the theory is a (separable) Hilbert space $\mathcal{H}$ admitting the action of a unitary representation of the (double-cover of) the Poincaré group: $\mathcal{G}(\Lambda, a) \to U(\Lambda, a)$. We denote by P^μ and $J^{\mu\nu}$ the corresponding Hermitian generators.

2. There exists a unique vacuum state $|\Omega\rangle$. As this vacuum should be identical in all inertial frames, it should be invariant under the action of the Poincaré group:

$$P^\mu |\Omega\rangle = 0 \quad , \qquad J^{\mu\nu} |\Omega\rangle = 0 \, . \tag{2.1}$$

   This vacuum is denoted by $|\Omega\rangle$, whereas $|0\rangle$ is used for free theories.

3. The spectrum of the 4-momentum operator P^μ should belong to the future light-cone:

$$p^0 \geqslant 0 \, , \quad p_\mu p^\mu \geqslant 0 \, . \tag{2.2}$$

   In any sensible quantum system, the spectrum of the Hamiltonian should be bounded from below; one can always shift the ground-state energy to zero. The operator $P_\mu P^\mu$ can be thought as the mass squared operator, whose eigenvalues are the mass squared of the excitations of the fields; particles with negative mass squared are unphysical.

4. There exists a set of local field operators $\Phi_D(x)$, which are more accurately operator-valued distributions, carrying a finite-dimensional representation R_D of the double-cover of the Lorentz group:[1]

$$U(\Lambda, a)^\dagger \Phi_D(x) U(\Lambda, a) = R_D(\Lambda, a) \Phi_D(\Lambda^{-1} x - \Lambda^{-1} a) \, . \tag{2.3}$$

5. The set of states:

$$\int d^4x_1 \cdots d^4x_n \, f_1(x_1) \cdots f_n(x_n) \, \Phi_{D_1}(x_1) \cdots \Phi_{D_n}(x_n) \, |\Omega\rangle \, , \tag{2.4}$$

   constructed with an appropriate choice of test functions $\{f_k\}$, is dense in $\mathcal{H}$.

[1] All non-trivial finite-dimensional representations of the double-cover of the Lorentz group, as the spinor representations $(1/2, 0)$ and $(0, 1/2)$, are not unitary. This is not a problem as those representations are carried by field operators, while particle states carry representations of the Poincaré group, that are infinite-dimensional and unitary.

6. Finally, in order to impose causality wr impose the micro-causality condition on all pair of local operators. For respectively pairs of bosonic and fermionic field operators,

$$\left[\,\Phi_{D}(x),\,\tilde{\Phi}_{\tilde{D}}(\tilde{x})\,\right]\Big|_{(x-\tilde{x})^2<0}=0\,,\quad,\qquad\left\{\,\Psi_{D}(x),\,\tilde{\Psi}_{\tilde{D}}(\tilde{x})\,\right\}\Big|_{(x-\tilde{x})^2<0}=0\,,\tag{2.5}$$

Note that the spin-statistics theorem mentioned earlier can be deduced from these axioms.

2.1.1 Källén–Lehmann spectral representation

Let us consider a generic field theory for some field Φ_{D}, *i.e.* transforming in some irreducible representation D_{D} of the Lorentz group. We want to characterize the states $\Phi_{D}(x)\,|\Omega\rangle$ and $\Phi_{D}^{\dagger}(x)\,|\Omega\rangle$, obtained by the action of the field operator and of its Hermitian conjugate on the vacuum of the interacting theory.

Let us start with their overlap with the vacuum, or vacuum expectation value (VEV thereafter). Using translation invariance of the vacuum, the VEV of $\Phi(x)$ satisfies:

$$\langle\Omega|\,\Phi_{D}(x)\,|\Omega\rangle=\langle\Omega|\,e^{iP\cdot x}\Phi_{D}(0)e^{-iP\cdot x}\,|\Omega\rangle=\langle\Omega|\,\Phi_{D}(0)\,|\Omega\rangle=v\,,\tag{2.6}$$

which is a constant. Using eqns. (2.1,2.3), one finds that, for any $\Lambda\in SO(1,3)$:

$$\begin{aligned}v&=\langle\Omega|\,\Phi_{D}(0)\,|\Omega\rangle=\langle 0|\,U(\Lambda)U(\Lambda)^{\dagger}\Phi_{D}(0)U(\Lambda)U^{\dagger}(\Lambda)\,|\Omega\rangle=\langle\Omega|\,U^{\dagger}(\Lambda)\Phi_{D}(0)U(\Lambda)\,|\Omega\rangle\\&=R_{D}(\Lambda)v\,.\end{aligned}\tag{2.7}$$

Whenever Φ_{D} transforms in a non-trivial irreducible representation, either $v=0$ (since by definition the representation space does not admit any invariant subspace), or the assumption about Poincaré invariance of the vacuum is violated. When Φ is a scalar field (*i.e.* transforming in the trivial representation), v is just a constant, which can be removed by redefining the field: $\Phi\to\Phi-v$, which is such that its vacuum expectation value vanishes.

From now, for simplicity of the presentation, we consider a quantum field theory of a single complex scalar field Φ. Recall that, in the free theory, $\Phi^{\dagger}(x)\,|0\rangle$, the action of $\Phi^{\dagger}$ on the vacuum, creates a single-particle state at x, which can be expanded in terms of one-particle states of definite (on-shell) 4-momentum p:

$$\Phi^{\dagger}(x)\,|0\rangle=\int\frac{d^3x}{(2\pi)^3}\frac{e^{ip\cdot x}}{\sqrt{2E_{p}}}\,|p\rangle\ \ ,\quad|p\rangle=\sqrt{2E_{p}}a_{\vec{p}}^{\dagger}\,|0\rangle\ .\tag{2.8}$$

In the interacting theory, things get of course more complicated. Since the 4-momentum operator P^{μ} is self-adjoint, one can consider a set of "eigenvectors" (in the generalized sense):

$$P^{\mu}\,|p;u\rangle=p^{\mu}\,|p;u\rangle\ ,\tag{2.9}$$

where the extra label u takes care of a possible degeneracy of the spectrum of P^{μ}. Depending on the situation, it could be discrete or continuous. By eqn. (2.2), *i.e.* positivity of the spectrum, $p_{\mu}p^{\mu}\geqslant 0$, nonetheless the states $|p;u\rangle$ are not necessarily one-particle states. Typically the spectrum of p^2 looks like this:

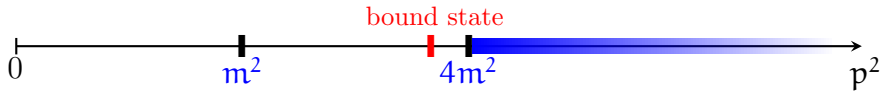


Figure 2.1: *Spectrum of* p^2

The eigenvalue at $p^2 = m^2$ is interpreted as a one-particle state; note that m may no longuer be the same than the mass parameter that appears in the Lagrangian of the theory. Starting at $p^2 = 4m^2$, there is a continuum of two-particle states. Unlike in the case of free theories, there may be *bound states* made of two particles, just below the two-particle states threshold. In the general case, there may be several one-particle states with different masses as well as several bound states below the continuum threshold at $p^2 = 4m^2$.

We consider then the pair of wave-functions:[2]

$$\langle p;u|\,\Phi^\dagger(x)\;|\Omega\rangle = \langle p;u|\,e^{iP\cdot x}\Phi^\dagger(0)\overbrace{e^{-iP\cdot x}\,|\Omega\rangle}^{|\Omega\rangle} = e^{ip\cdot x}\,\langle p;u|\,\Phi^\dagger(0)\,|\Omega\rangle = \sqrt{Z_u(p)}e^{ip\cdot x} \quad (2.10a)$$

$$\langle p;u|\,\Phi\;(x)\,|\Omega\rangle = \langle p;u|\,e^{iP\cdot x}\Phi\;(0)e^{-iP\cdot x}\,|\Omega\rangle = e^{ip\cdot x}\,\langle p;u|\,\Phi\;(0)\,|\Omega\rangle = \sqrt{\tilde{Z}_u(p)}e^{ip\cdot x} \quad (2.10b)$$

In the free theory, following eqn. (2.8) one has $Z(p) = \tilde{Z}(p) = 1$.

Whenever the $U(1)$ symmetry $\Phi \mapsto e^{i\theta}\Phi$ is preserved by the interactions, there exists a corresponding Hermitian generator Q such that

$$e^{-i\theta Q}\Phi(x)e^{i\theta Q} = e^{i\theta}\Phi(x)\;, \quad e^{-i\theta Q}\Phi^\dagger(x)e^{i\theta Q} = e^{-i\theta}\Phi^\dagger(x)\,, \qquad (2.11)$$

or equivalently

$$[\,Q,\,\Phi(x)\,] = -\Phi(x) \quad , \qquad \big[\,Q,\,\Phi^\dagger(x)\,\big] = \Phi(x)\,. \qquad (2.12)$$

The operator Q commutes with the 4-momentum generator P^μ, so one can label their common eigenvectors as $|p;q;\nu\rangle$, q being the $U(1)$ charge and ν other possible quantum numbers. One has:

$$\langle p;q;\nu|\,e^{-i\theta Q}\Phi^\dagger(x)e^{i\theta Q}\,|\Omega\rangle = e^{i\theta}\,\langle p;q;\nu|\,\Phi^\dagger(x)\,|\Omega\rangle = e^{-i\theta q}\,\langle p;q;\nu|\,\Phi^\dagger(x)\,|\Omega\rangle\;, \qquad (2.13)$$

either using (2.11) or the fact that $|p;q;\nu\rangle$ is an eigenstate of Q, together with invariance of the vacuum. Therefore one has $\tilde{Z}_{q;\nu} = 0$ unless $q = -1$. Likewise, $Z_{q;\nu} = 0$ unless $q = 1$. In other words, the states created by $\Phi^\dagger$ or Φ on the vacuum, associated respectively to particles and anti-particles, has only non-trivial overlaps with momentum states carying the same charges.

Next, following Källén and Lehmann [1,2], we consider the two-point correlation function. As explained earlier, due to the global $U(1)$ symmetry of the complex scalar QFT the only

[2]The strange notation used for the momentum-dependent constants on the RHS will become clearer later on.

non-zero correlators are $\langle \Phi\Phi^\dagger\rangle$ and $\langle \Phi^\dagger\Phi\rangle$. Let us consider the resolution of the identity using the basis of 4-momentum eigenstates:

$$\begin{aligned}\mathbb{I} &= \sum_u \int_0^\infty dm^2 \sum_u \int \frac{d^4p}{(2\pi)^4}\Theta(p^0)2\pi\delta(p^2-m^2)\,|p;u\rangle\,\langle p;u| \\ &= \sum_u \int_0^\infty dm^2 \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\,|p;u\rangle\,\langle p;u|\,\Big|_{p^0=E_p=\sqrt{\vec{p}^2+m^2}}\,. \end{aligned} \tag{2.14}$$

where we have used the spectrum positivity hypothetis (2.2) to restrict the integral over m^2 to positive values only. We stress again that the m^2 here is just the value of p^2, and the states $|p;u\rangle$ are not necessarily single-particle states. We have then:

$$\begin{aligned}\langle\Omega|\,\Phi(x)\Phi^\dagger(y)\,|\Omega\rangle &\overset{\text{transl. inv.}}{=} \langle\Omega|\,\Phi(x-y)\Phi^\dagger(0)\,|\Omega\rangle \\ &= \sum_u \int_0^\infty dm^2 \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\,\langle\Omega|\,\Phi(x-y)\,|p;u\rangle\,\langle p;u|\,\Phi^\dagger(0)\,|\Omega\rangle \qquad (2.15)\\ &= \int_0^\infty dm^2 \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\sum_u |Z_u(p)|e^{-ip\cdot(x-y)}\Big|_{p^0=E_p} \qquad (2.16)\end{aligned}$$

This expression involves the non-negative function of m^2 known as the *spectral density*:

$$\rho(m^2) \overset{\text{def.}}{=} \sum_u |Z_u(p)|\,\Big|_{p^0=E_p} \geqslant 0\,. \tag{2.17}$$

Let us show, as the definition suggests, that it depends only on p^2, or equivalently on m^2. Since Φ is a scalar field, we have, for any Lorentz transformation Λ:

$$\begin{aligned}\langle\Omega|\,\Phi(x)\Phi^\dagger(y)\,|\Omega\rangle &= \langle\Omega|\,U(\Lambda)\Phi(x)U(\Lambda)^\dagger U(\Lambda)\Phi^\dagger(y)U(\Lambda)^\dagger\,|\Omega\rangle \\ &= \langle\Omega|\,\Phi(\Lambda x)\Phi^\dagger(\Lambda y)\,|\Omega\rangle = \int_0^\infty dm^2 \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\sum_u |Z_u(p)|e^{-ip\cdot\Lambda(x-y)} \\ &= \int_0^\infty dm^2 \int \frac{d^3q}{(2\pi)^3}\frac{1}{2E_q}\sum_u |Z_u(\Lambda q)|e^{-iq\cdot(x-y)} \qquad (2.18)\end{aligned}$$

where we have used the Lorentz-invariance of the measure. We conclude that the spectral density, being invariant under Lorentz transformations, must depend on $p^2 = m^2$ only. We obtain then the *Källén–Lehmann spectral representation* of the propagator in the interacting scalar theory:

$$\langle\Omega|\,\Phi(x)\Phi^\dagger(y)\,|\Omega\rangle = \int_0^\infty dm^2\,\rho(m^2)\,\Delta_{m^2}(x-y)\,, \tag{2.19}$$

where

$$\Delta_{m^2}(x-y) \overset{\text{def.}}{=} \int \frac{d^3p}{(2\pi)^3}\frac{1}{2p^0}e^{-ip\cdot(x-y)}\Big|_{p_0=\sqrt{p^2+m^2}} \tag{2.20}$$

is the free Klein–Gordon propagator for a scalar field of mass m^2. With a similar reasoning, one gets:

$$\langle\Omega|\,\Phi^\dagger(x)\Phi(y)\,|\Omega\rangle = \int_0^\infty dm^2\,\tilde\rho(m^2)\,\Delta_{m^2}(x-y)\,, \tag{2.21}$$

written in terms of

$$\tilde\rho(m^2) \stackrel{\text{def.}}{=} \sum_u |\tilde Z_u(p)|\Big|_{p^0=E_p}\,. \tag{2.22}$$

Importantly, in the presence of a global $U(1)$ symmetry the spectral densities $\rho(m^2)$ and $\tilde\rho(m^2)$ are related to particles and anti-particles states respectively, see the discussion above. Let us show that they must be related through the micro-causality condition (2.5). Evaluating the commutator at space-like separation in the vacuum, one has:

$$0 = \langle\Omega|\,\big[\,\Phi(x),\,\Phi^\dagger(y)\,\big]\,|\Omega\rangle\,\Big|_{(x-y)^2<0} = \int_0^\infty dm^2\big(\rho(m^2)-\tilde\rho(m^2)\big)\Delta_{m^2}(y-x)\,. \tag{2.23}$$

Since this quantity is Lorentz-invariant and $x-y$ is space-like, one can go to a frame where $x^0=y^0$. One finds then, that for any value of $\vec x-\vec y$,

$$0 = \int dm^2 \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{i\vec p\cdot(\vec x-\vec y)}\,\big(\rho(m^2)-\tilde\rho(m^2)\big)\;; \tag{2.24}$$

This condition is solved for $\rho(m^2)=\tilde\rho(m^2)$; with the help of the CPT theorem one can show that it is the only solution. *A similar Källén–Lehmann spectral representation can be derived for an interacting Dirac spinor field, as you will see during the tutorials.*

Feynman propagator. The same arguments lead to a spectral representation for the Feynman propagators. Using the $i\epsilon$ prescription, one has

$$D(x-y) = \langle\Omega|\,T\Phi(x)\Phi^\dagger(y)\,|\Omega\rangle = \int_0^\infty dm^2\,\rho(m^2)\int\frac{d^4p}{(2\pi)^4}\frac{ie^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon} \tag{2.25}$$

or in momentum space:

$$\widehat D(p^2) = \int_0^\infty dm^2\,\frac{i\rho(m^2)}{p^2-m^2+i\epsilon}\,. \tag{2.26}$$

The momentum space Feynman propagator $\widehat D(p)$ has an interesting analytic structure in the p^2 complex plane. Since we do not integrate over p^2 here, one can ignore the $i\epsilon$ prescription, *i.e.* send ϵ to zero. As mentioned earlier, on general grounds the spectral density $\rho(m^2)$ is expected to have isolated contributions, at $m^2=m_i^2$, corresponding to one-particle states, as well as continuum of multiple-particle states starting at the threshold $m_T^2=4m_1^2$, where we have chosen m_1 to be the mass of the lightest state. Explicitely,

$$\rho(m^2) = \sum_i \delta(m^2-m_i^2)Z_i + \Theta(m^2-m_T^2)\rho_c(m^2)\,, \tag{2.27}$$

where both the coefficients Z_i and the density of the continuum ρ_c are positive.[3] We have then:

$$\widehat{D}(p^2) = \sum_i \frac{iZ_i}{p^2 - m^2} + \int_{m_T^2}^{\infty} dm^2 \, \frac{i\rho_c(m^2)}{p^2 - m^2} \tag{2.28}$$

Thus, in the p^2-plane, the function $\widehat{D}(p^2)$ has two types of singularities:

1. Simple poles at $p^2 = m^2$, with residues iZ_i where $Z_i > 0$.
2. A branch cut on the positive real axis starting at $p^2 = m_T^2$ with discontinuity:[4]

$$\mathrm{disc}_{\mathrm{Im}\, p^2=0} \widehat{D}(p^2) = \lim_{\mu^2 \to 0} \widehat{D}(p^2 + i\mu^2) - \widehat{D}(p^2 - i\mu^2) = 2\pi\Theta(p^2 - m_T^2)\rho_c(p^2) \tag{2.29}$$

The propagator $\widehat{D}(p^2)$ is essentially fixed by its analytic properties. Let us assume that $\widehat{D}(p^2)$ decays at infinity. For p^2 in the upper-half plane, one can start with the integral representation:

$$\widehat{D}(p^2) = \frac{1}{2i\pi} \oint_{\mathcal{C}_{p^2}} \frac{\widehat{D}(z)dz}{z - p^2}\,, \tag{2.30}$$

where the contour $\mathcal{C}_{p^2}$ circles around $z = p^2$ counterclockwise, see the left panel on fig. 2.2. Then, by the contour manipulation displayed on the figure, one has

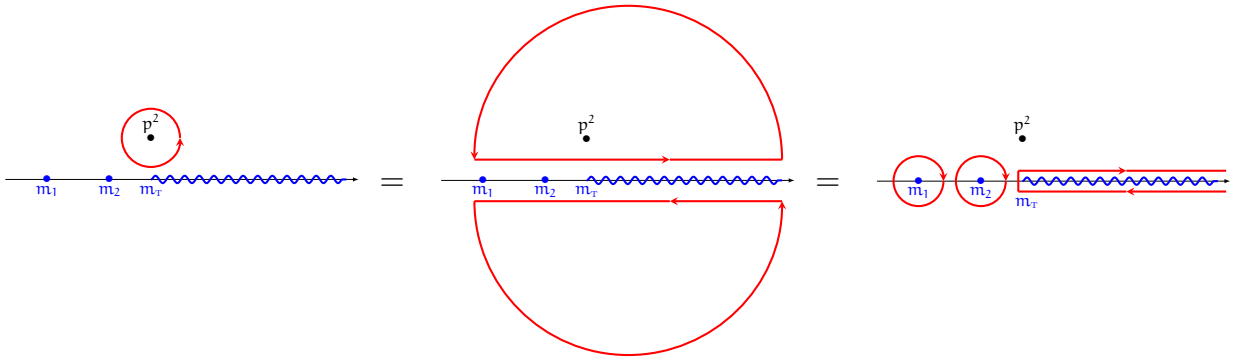


Figure 2.2: *Analytic structure of the propagator.*

$$\widehat{D}(p^2) = \sum_i \frac{\widehat{D}(m_i^2)}{p^2 - m_i^2} - \frac{1}{2i\pi} \int_{m_T^2}^{\infty} \frac{\mathrm{disc}_{\mathrm{Im}\, p^2=0} \widehat{D}(p^2)}{p^2 - m^2} \tag{2.31}$$

[3] From eqn. (2.17), $Z = |Z_u(p)|\big|_{p^2=m^2}$.

[4] We have used here a standard result of complex analysis. For any function f analytic in the upper half-plane and decaying fast enough at infinity, one has (P being the Riemann principal value):

$$\lim_{\alpha \to 0^+} \int_{-\infty}^{+\infty} \frac{f(x)dx}{x - x_0 + i\alpha} = P \int_{-\infty}^{+\infty} \frac{f(x)dx}{x - x_0} - i\pi f(x_0)\,.$$

Here $f(x) = -i\rho_c(m^2)$, $\alpha = -\mu^2$ and $x_0 = p^2$.

If $i\widehat{D}(p^2)$ is real for $p^2 \in \mathbb{R}$, one identifies $\widehat{D}(m_i^2) = iZ_i$ and $\operatorname{disc}\widehat{D}(m^2) = 2\pi\rho_c(m^2) > 0$, recovering eqn. (2.26). These are known as *Källén–Lehmann dispersion relations* in analogy with the Kramers-Kronig dispersion relations.

2.1.2 The S-matrix

The principal object of interest in quantum field theories is the *S-matrix*, which constitutes the main bridge between theory and experiments. We will discuss here some of its general properties before developping tools to compute it explicitely in some approximation scheme. It is so pivotal that there is a longstanding research program from the seventies whose main ideas is to solve quantum field theories directly from the properties of the S-matrix (as unitarity, analyticity, crossing symmetries, etc.), without using directly any underlying Lagrangian description, see the lectures notes of Alexander Zhiboedov (https://courses.ipht.fr/?q=en/node/248) for a recent overview.

A typical scattering process is displayed on fig. 3.2. Imagine that two well-separated electrons, represented by wave-packets centered at momenta $\vec{p}_1$ et $\vec{p}_2$ (on the left) are prepared in the infinite past, sent towards each other and interact in the dashed region at the center around $t = 0$. We would like to compute the probablity amplitude, in the appropriate sense, of measuring, far away in the future (on the right) two outgoing wave-packets associated with electrons of momenta $\vec{k}_1$ and $\vec{k}_2$. As we shall see, these amplitudes are directly related to measurable quantities like cross-sections.

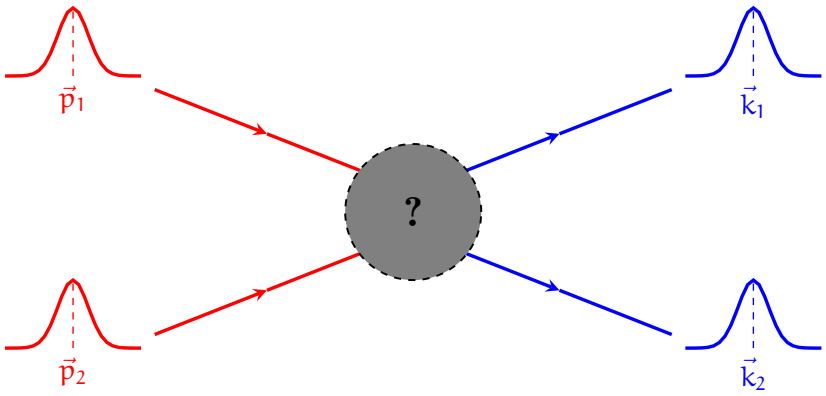


Figure 2.3: *Scattering process.*

This calls for a few comments:

- In the infinite past/future, while the two wave-packets do not interact with each other, it does not mean that the theory is free: the electrons still interact with the quantum field. There is no way to "turn off" the interactions in the theory.
- In theories like QCD that are strongly coupled at low energies, excitations of the elementary fields (quark and gluons) do not appear as asymptotic states in scattering processes but rather bound states (for instance protons in the LHC).

- Even in weakly coupled theories like QED, bound states can appear in the outgoing states, for instance a hydrogen atom by scattering of an electron and a proton.

For simplicity of the presentation, in the following we will idealize the incoming and outgoing wave-packets as plane-waves of definite momentum.

Definition 1 (in an out states). *Let $\mathcal{H}$ be the Hilbert space of an **interacting** quantum field theory, and let $|\alpha\rangle_{\text{IN}} \in \mathcal{H}$ (resp. $|\alpha\rangle_{\text{OUT}} \in \mathcal{H}$) be a set of states, in the Heisenberg picture. The states $|\alpha\rangle_{\text{IN}}$ (resp. $|\alpha\rangle_{\text{OUT}}$) are said to be incoming or **in** states (resp. outgoing or **out** states) if, for any operator $\mathcal{O}(t)$, the matrix elements ${}_{\text{IN}}\langle\alpha|\,\mathcal{O}(t)\,|\beta\rangle_{\text{IN}}$ (resp. ${}_{\text{OUT}}\langle\alpha|\,\mathcal{O}(t)\,|\beta\rangle_{\text{OUT}}$) coincide for $t \to -\infty$ (resp. $t \to +\infty$) with those of the free theory.*

In particular conserved quantitites, which are, by definition, associated with time-independent operators in the Heisenberg picture, can be used to label in or out states like in the free theory. For instance, for an in state made of n particles with momenta $p_1, \dots, p_n$ one has

$$P^\mu\,|p_1, \dots, p_n\rangle_{\text{IN/OUT}} = (p_1 + \cdots + p_n)^\mu\,|p_1, \dots, p_n\rangle_{\text{IN/OUT}}\,. \tag{2.32}$$

Likewise, if the theory contains also a $U(1)$ symmetry, one can label in or out states by the charges of the particles $e_1, \dots, e_n$:

$$Q\,|p_1, \dots, p_n;\, e_1, \dots, e_n\rangle_{\text{IN/OUT}} = (e_1 + \cdots + e_n)\,|q_1, \dots, p_n;\, e_1, \dots, e_n\rangle_{\text{IN/OUT}}\,, \tag{2.33}$$

and, in the case of Dirac or Weyl fields, by their spin states.

In the following we will consider the *asymptotic completeness hypothesis*. Let $\mathcal{H}_{\text{IN}}$ (resp. $\mathcal{H}_{\text{OUT}}$) the span of the *in* vectors $|\alpha\rangle_{\text{IN}}$ (resp. of the *out* vectors $|\alpha\rangle_{\text{OUT}}$). We will assume that $\mathcal{H} \simeq \mathcal{H}_{\text{IN}} \simeq \mathcal{H}_{\text{OUT}}$, *i.e.* we can label states in the Hilbert space of the interacting theory using those of the free theory that it is related to in the infinite past or future. Said differently, either an orthonormal basis of $\mathcal{H}_{\text{IN}}$ or of $\mathcal{H}_{\text{OUT}}$ gives a resolution of the identity:[5]

$$\mathbb{I} = \sum_\alpha |\alpha\rangle_{\text{OUT}}\,{}_{\text{OUT}}\langle\alpha| = \sum_\alpha |\alpha\rangle_{\text{IN}}\,{}_{\text{IN}}\langle\alpha| \quad , \quad {}_{\text{IN}}\langle\beta|\alpha\rangle_{\text{IN}} = {}_{\text{OUT}}\langle\beta|\alpha\rangle_{\text{OUT}} = \delta_{\alpha\beta} \tag{2.34}$$

Definition 2 (S-matrix). *Let assume that we have prepared, in the far past, a set of of incoming particles in the in state $|\alpha\rangle_{\text{IN}}$. The probability amplitude to detect outgoing particles in the far future in the out state $|\beta\rangle_{\text{OUT}}$ is given by the overlap ${}_{\text{OUT}}\langle\beta|\alpha\rangle_{\text{IN}}$. By the completeness hypothesis, one can expand the in state on the basis of out states:*

$$|\alpha\rangle_{\text{IN}} = \sum_\gamma |\gamma\rangle_{\text{OUT}} S_{\gamma\alpha}\,. \tag{2.35}$$

The unitary matrix realizing the change of basis between an orthonormal basis of incoming states and an orthonormal basis of outgoing states is called the S-matrix.

[5]The sum over α is a symbolic shorthand for the sum of the projectors onto the identity, one-particle subspace, two-particles subspaces, etc. each expanded into a basis of particle momenta.

$$ {}_{\text{OUT}}\langle\beta|\alpha\rangle_{\text{IN}} = S_{\beta\alpha}\,. \tag{2.36} $$

Unitarity of the S-matrix ensures that probability is conserved by time evolution, as it should be in a quantum mechanical system.

A formal definition of the corresponding S-matrix operator can be achieved as follows. One can define formally maps between vectors of the free Hilbert space $|\alpha\rangle_{\text{FREE}} \in \mathcal{H}_0$ and vectors of the in/out Hilbert spaces (which are both isomorphic to $\mathcal{H}$:

$$ \Xi_{\mp} : \left\{ \begin{array}{lcl} \mathcal{H}_0 & \to & \mathcal{H}_{\text{IN/OUT}} \\ |\alpha\rangle_{\text{FREE}} & \mapsto & \Xi_{\mp}|\alpha\rangle_{\text{FREE}} = |\alpha\rangle_{\text{IN/OUT}} \end{array} \right. \tag{2.37} $$

These are called *Møller operators*. By definition 1, the in and out states have identical matrix elements as the free states asymptotically in the far past/future:

$$ e^{-itH}|\alpha\rangle_{\text{IN/OUT}} \xrightarrow{t\to\mp\infty} e^{-itH_{\text{FREE}}}|\alpha\rangle_{\text{FREE}}\,, \tag{2.38} $$

so[6]

$$ \Xi_{\mp} = \lim_{t\to\mp\infty} e^{itH}e^{-itH_{\text{FREE}}}\,. \tag{2.39} $$

We can then define the S-matrix operator as:

$$ S = \Xi_+^\dagger\Xi_- \quad\Longleftrightarrow\quad S_{\beta\alpha} \;=\; {}_{\text{OUT}}\langle\beta|\alpha\rangle_{\text{IN}} \;=\; {}_{\text{FREE}}\langle\beta|\,\Xi_+^\dagger\Xi_-\,|\alpha\rangle_{\text{FREE}} \;=\; {}_{\text{FREE}}\langle\beta|\,S\,|\alpha\rangle_{\text{FREE}} \tag{2.40} $$

Scattering amplitudes. It is useful to split the S-matrix operator as:

$$ S = \mathbb{I} + i\mathcal{T}\,, \tag{2.41} $$

to isolate the operator $i\mathcal{T}$ that corresponds to situations where the particles actually scatter, rather than just flying by. Then, the matrix elements $\mathcal{M}_{\beta\,\alpha}$, called *scattering amplitudes*, are defined by factorizing out the momentum conservation factor:

$$ {}_{\text{FREE}}\langle\beta|\,\mathcal{T}\,|\alpha\rangle_{\text{FREE}} \;=\; (2\pi)^4\delta^{(4)}\big(\textstyle\sum_\beta p_\beta - \sum_\alpha q_\alpha\big)\mathcal{M}_{\beta\,\alpha} \tag{2.42} $$

We may also denote them by $\mathcal{M}_{\alpha\to\beta}$.

Optical theorem. As noticed earlier, one of the most important properties of the S-matrix operator is *unitarity*. Using the decomposition (2.41), one finds that:

$$ \mathbb{I} = S^\dagger S \;\;\Longrightarrow\;\; -i\big(\mathcal{T} - \mathcal{T}^\dagger\big) = \mathcal{T}^\dagger\mathcal{T}\,. \tag{2.43} $$

[6]This is the late /early time limit of the interaction picture evolution operator defined in subsection 2.2.1.

An immediate corollary of this statement is known as the optical theorem. Let us consider a 2→2 scattering. From the left-hand side of eqn. (2.43) one gets:

$$ {}_{\text{F}}\langle \vec{p}_1,\, \vec{p}_2| - i\big(\mathcal{T} - \mathcal{T}^\dagger\big)\,|\vec{q}_1,\, \vec{q}_2\rangle_{\text{F}} = -i(2\pi)^4\delta^{(4)}\big(p_1 + p_2 - q_1 - q_2\big)\ \ \big(\mathcal{M}_{q_1 q_2 \to p_1 p_2} - \overline{\mathcal{M}}_{p_1 p_2 \to q_1 q_2}\big) \tag{2.44} $$

To put the right-hand side in a useful form, one inserts a multi-particle resolution of the identity in the free theory Hilbert space:

$$ \mathbb{I} = |0\rangle\,\langle 0| + \int \frac{d^3r}{(2\pi)^3 2E_r}\,|\vec{r}\rangle_{\text{F}\,\text{F}}\langle\vec{r}| + \int \frac{d^3r_1}{(2\pi)^3 2E_{r_1}}\frac{d^3r_2}{(2\pi)^3 2E_{r_2}}\,|\vec{r}_1\,\vec{r}_2\rangle_{\text{F}\,\text{F}}\langle\vec{r}_1\,\vec{r}_2| + \ (n \geqslant 3 \text{ particle states}) \tag{2.45} $$

One gets then:

$$ \begin{aligned} {}_{\text{F}}\langle \vec{p}_1,\, \vec{p}_2|\,\mathcal{T}^\dagger\mathcal{T}\,|\vec{q}_1,\, \vec{q}_2\rangle_{\text{F}} &= \sum_n \prod_{k=1}^{n} \int \frac{d^3r_k}{2E_{r_k}}\,{}_{\text{F}}\langle \vec{p}_1,\, \vec{p}_2|\,\mathcal{T}^\dagger\,|\vec{r}_1,\, \ldots,\, \vec{r}_n\rangle_{\text{F}\,\text{F}}\langle\vec{r}_1,\, \ldots,\, \vec{r}_n|\,\mathcal{T}\,|\vec{q}_1,\, \vec{q}_2\rangle_{\text{F}} \\ &= \sum_n \prod_{k=1}^{n} \int \frac{d^3r_k}{2E_{r_k}} (2\pi)^4\delta^{(4)}\big(p_1 + p_2 - \textstyle\sum r_k\big)(2\pi)^4\delta^{(4)}\big(q_1 + q_2 - \textstyle\sum r_k\big)\overline{\mathcal{M}_{p_1 p_2 \to r_1 \cdots r_n}}\mathcal{M}_{q_1 q_2 \to r_1 \cdots r_n} \end{aligned} \tag{2.46} $$

One can then simplify one momentum conservation constraint between the LHS and the RHS. Setting the final momenta to be equal to the initial ones, one gets:

$$ 2\,\text{Im}\,\mathcal{M}_{p_1 p_2 \,\to\, p_1 p_2} = \sum_n \int \prod_{k=1}^{n} \frac{d^3r_k}{2E_{r_k}} (2\pi)^4\delta^{(4)}\big(p_1 + p_2 - \textstyle\sum r_k\big)\,\overline{\mathcal{M}_{p_1 p_2 \,\to\, r_1 \cdots r_n}}\;\mathcal{M}_{p_1\, p_2 \,\to\, r_1 \cdots r_n} \tag{2.47} $$

This theorem teaches us something important about the analytic structure of the scattering amplitudes. The imaginary part of a given amplitude is given by a product of amplitudes involving all possible numbers of on-shell intermediate states, see fig. (2.4).

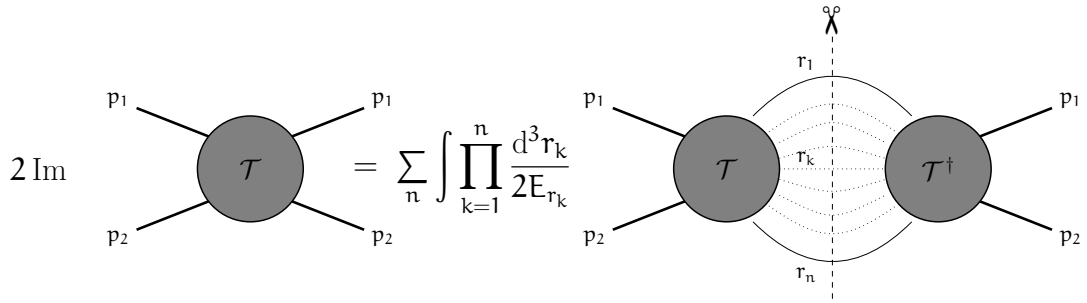


Figure 2.4: *Optical theorem.*

At the perturbative level, it can be connected to the *Cutkosky's cutting rules* which allow to compute the imaginary part, or discontinuity, of a given Feynman diagram by a set of

graphical rules. A more physical and common translation of the optical theorem in terms of scattering amplitudes will be given later.

In/out creation and annihilation operators. We expect the vacuum of the $\mathcal{H}_{\text{IN}}$ and $\mathcal{H}_{\text{OUT}}$ to be the same Poincaré-invariant vacuum $|\Omega\rangle$ that we have defined previously. It has necessarily to be mapped to itself by the evolution of the system, hence $S|\Omega\rangle = |\Omega\rangle$. As in the free scalar theory, in or out one-particle states can be created out of the vacuum by suitable creation operators:

$$|\vec{p}\rangle_{\text{IN}} = \sqrt{2E_p}\, a^\dagger_{\vec{p},\,\text{IN}}\,|\Omega\rangle \ , \quad |\vec{p}\rangle_{\text{OUT}} = \sqrt{2E_p}\, a^\dagger_{\vec{p},\,\text{OUT}}\,|\Omega\rangle \ . \tag{2.48}$$

Likewise, anti-particles states are created by acting with $b^\dagger_{\vec{p},\,\text{IN/OUT}}$, Dirac one-particle states of spin polarization s by acting with $b^{s\,\dagger}_{\vec{p},\,\text{IN/OUT}}$, etc. Using these modes we construct the in/out field operators as usual. For a complex interacting scalar field for instance, one gets:

$$\Phi_{\text{IN/OUT}}(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_{\vec{p},\,\text{IN/OUT}}e^{-ip\cdot x} + b^\dagger_{\vec{p},\,\text{IN/OUT}}e^{-ip\cdot x}\right) . \tag{2.49}$$

We now claim that *the matrix elements* of the field operator $\Phi(x)$ of the interacting theory are related to those of $\Phi_{\text{IN/OUT}}(x)$ asymptotically as:

$$\Phi(x) \overset{t\to\mp\infty}{\longrightarrow} \sqrt{Z}\Phi_{\text{IN/OUT}}(x)\,, \tag{2.50}$$

where, as in the beginning of this chapter, $\langle\Omega|\,\Phi(0)\,|\vec{p}\rangle = \sqrt{Z}$.[7] We stress that this is only true in the weak sense, *i.e.* for matrix elements, but not as an operator equation; for instance Φ^2 is not related to $\Phi^2_{\text{IN/OUT}}$ in such simple way. Combining eqns. (2.49,2.50), we obtain

$$a_{\vec{p},\,\text{IN/OUT}} = \frac{1}{\sqrt{Z}}\lim_{t\to\mp\infty}\left\langle \frac{e^{-ip\cdot x}}{\sqrt{2E_p}}\middle|\Phi(x)\right\rangle = \frac{1}{\sqrt{Z}}\lim_{t\to\mp\infty}\int d^3x\,\frac{e^{ip\cdot x}}{\sqrt{2E_p}}\,i\overleftrightarrow{\partial_0}\,\Phi(x)\,, \tag{2.51a}$$

$$b_{\vec{p},\,\text{IN/OUT}} = \frac{1}{\sqrt{Z}}\lim_{t\to\mp\infty}\left\langle \frac{e^{-ip\cdot x}}{\sqrt{2E_p}}\middle|\Phi^\dagger(x)\right\rangle = \frac{1}{\sqrt{Z}}\lim_{t\to\mp\infty}\int d^3x\,\frac{e^{ip\cdot x}}{\sqrt{2E_p}}\,i\overleftrightarrow{\partial_0}\,\Phi^\dagger(x)\,. \tag{2.51b}$$

For a real scalar field, we just get eqn. (2.51a).

Let us make a few checks to convince ourselves that these relations make sense, as before at the level of matrix elements.

- The average of $a_{\vec{p},\,\text{IN/OUT}}$ in the vacuum is given by

$$\langle\Omega|\,a_{\vec{p},\,\text{IN/OUT}}\,|\Omega\rangle = \sqrt{\frac{E_p}{2Z}}\int d^3x\, e^{ip\cdot x}\,\langle\Omega|\,\Phi(0)\,|\Omega\rangle \ . \tag{2.52}$$

[7] Here, the 4-momentum being on-shell, Z is is a constant. For any $\Lambda \in SO(1,3)^+$, $\langle\Omega|\,\Phi(0)\,|p\rangle = \langle\Omega|\,U^\dagger(\Lambda)\Phi(0)U(\Lambda)\,|p\rangle = \langle\Omega|\,\Phi(0)\,|\Lambda p\rangle$ so one can go to the rest frame $\Lambda p = (m,\vec{0})$.

By shifting the field Φ by a constant value, one can ensure that $\langle\Omega|\,\Phi(0)\,|\Omega\rangle{=}0$, thereby that $\langle\Omega|\,a_{\vec{p},\,\text{IN/OUT}}\,|\Omega\rangle = 0$ as it should. The same holds for $b_{\vec{p},\,\text{IN/OUT}}$.

- The action on in/out one-particle states satisfies

$$\begin{aligned}\langle\Omega|\,a_{\vec{p},\,\text{IN/OUT}}\,|\vec{q}\rangle_{\text{IN/OUT}} &= \frac{1}{\sqrt{Z}}\lim_{t\to\mp\infty}\int d^3x\,\frac{e^{ip\cdot x}}{\sqrt{2E_p}}\,i\overleftrightarrow{\partial_0}\,\langle\Omega|\,\Phi(x)\,|\vec{q}\rangle_{\text{IN/OUT}}\\ &= \frac{1}{\cancel{\sqrt{Z}}}\lim_{t\to\mp\infty}\int d^3x\,\frac{e^{ip\cdot x}}{\sqrt{2E_p}}\,i\overleftrightarrow{\partial_0}\,e^{-iq\cdot x}\cancel{\sqrt{Z}}\\ &= (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\sqrt{2E_p}\,,\end{aligned}\tag{2.53}$$

  which is the expected result. We have used the wave-function (2.10) using the following facts:

  1. If we consider stable particles, we expect that, as far as single particle states are concerned, $|\vec{p}\rangle_{\text{IN/OUT}} = |\vec{p}\rangle$.
  2. Compared to eqn. (2.10), which was written for arbitrary 4-momentum p, the states $|\vec{p}\rangle$ are on-shell, *i.e.* satisfy $p^2 = m^2$, thus Z is a constant.

  These points understood, we check that the outcome of (2.53) is the expected result; here also the limit $t \to \pm\infty$ does not play any role.

- The action on in/out two-particle states satisfies:

$$\begin{aligned}\langle\Omega|\,a_{\vec{p},\,\text{IN/OUT}}\,|\vec{q}_1,\vec{q}_2\rangle_{\text{IN/OUT}} &= \frac{1}{\sqrt{Z}}\lim_{t\to\mp\infty}\int d^3x\,\frac{e^{ip\cdot x}}{\sqrt{2E_p}}\,i\overleftrightarrow{\partial_0}\,\overbrace{\langle\Omega|\,\Phi(x)\,|\vec{q}_1,\vec{q}_2\rangle_{\text{IN/OUT}}}^{\langle\Omega|\Phi(0)|\vec{q}_1,\vec{q}_2\rangle_{\text{IN/OUT}}e^{-i(q_1+q_2)\cdot x}}\\ &= \frac{\langle\Omega|\,\Phi(0)\,|\vec{q}_1,\vec{q}_2\rangle}{\sqrt{2ZE_p}}(2\pi)^3\delta^{(3)}(\vec{q}_1+\vec{q}_2-\vec{p})\big(E_p+(q_1+q_2)^0\big)\lim_{t\to\mp\infty}e^{-i(q_1^0+q_2^0-p^0)t}\,,\end{aligned}\tag{2.54}$$

  where the matrix element $\langle\Omega|\,\Phi(0)\,|\vec{q}_1,\vec{q}_2\rangle$ has no *a priori* reason to vanish.

  Crucially a two-particle state of 4-momentum $q = q_1 + q_2$ with 3-momentum $\vec{p}$ (as imposed by the delta-function constraint) has energy $q^0 = \sqrt{\vec{p}^2+q^2}$ with $q^2 \geqslant 4m^2$, while $p^0 = \sqrt{\vec{p}^2+m^2}$, hence $q^0 > p^0$ (strict inequality). Then, following the Riemann-Lebesgue lemma,[8] the matrix elements of a wave-packet constructed from those two-particle states tend to zero while $t \to \mp\infty$, because the oscillations become wilder in the far past/future. The same holds for other multi-particle states.

Another way to phrase the result is that, in the fully interacting quantum field theory, the candidate creation operators $a^\dagger_{\vec{p}} = \langle e^{ip\dot{x}}/\sqrt{2E_p}|\Phi(x)\rangle$, which are time-dependent (because the field Φ no longer satisfies the Klein–Gordon equation), do not create only one-particle states

[8] For an integrable function $f:\mathbb{R}^n\to\mathbb{C}$, $\lim_{t\to\infty}|\int d^nx\,f(x)e^{ixt}| = 0$. Here we mean that the two-particle momentum states are normalizable wave-packets built on $|\vec{q}_1,\vec{q}_2\rangle_{\text{IN/OUT}}$.

out of the vacuum $|\Omega\rangle$, but also two-particle states, etc. However all these extra contributions go away in the far past/future limits $t \to \mp\infty$, as these operators tend to the corresponding in/out creation operators up to the wave-function normalization factor $\sqrt{Z}$ (in the weak sense of matrix elements):

$$a^\dagger_{\vec{p}}(t) = \left\langle \frac{e^{ipx}}{\sqrt{2E_p}} \middle| \Phi(x) \right\rangle \xrightarrow{t\to\mp\infty} \sqrt{Z}\, a^\dagger_{\vec{p},\,\text{IN/OUT}}\,, \tag{2.55}$$

which is how eqn. (2.50) for the field operator Φ should be understood. The same type of relation holds for the other creation and annihilation operators of the interacting complex scalar field QFT.

If we consider a Dirac field in an interacting quantum field theory, one has similarly, starting from eqns. (1.320) the weak relations:

$$b^s_{\vec{p},\,\text{IN/OUT}} = \frac{1}{\sqrt{Z}} \lim_{t\to\mp\infty} \int d^3x \frac{e^{ip\cdot x}}{\sqrt{2E_p}} \bar{u}^s(\vec{p})\gamma^0\Psi(x)\,, \tag{2.56a}$$

$$c^s_{\vec{p},\,\text{IN/OUT}} = \frac{1}{\sqrt{Z}} \lim_{t\to\mp\infty} \int d^3x \frac{e^{ip\cdot x}}{\sqrt{2E_p}} \bar{\Psi}(x)\gamma^0 v^s(\vec{p})\,,\,, \tag{2.56b}$$

as well as their Hermitian conjugates.

Using these insights we will now develop a pratical way to handle S-matrix elements in an interacting quantum field theory.

LSZ reduction. The LSZ reduction, named after Harry Lehmann, Kurt Symanzik and Wolfhart Zimmermann [3], is a very general relation between scattering amplitudes and correlation functions in quantum field theories.

In general, S-matrix elements can be decomposed into its connected and non-connected parts. The case of a $3 \to 3$ scattering is depicted on fig. 2.5.

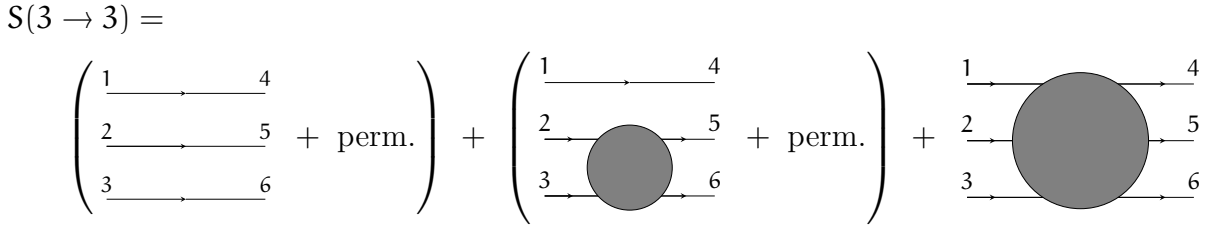


Figure 2.5: *Connected S-matrix.*

The first type of terms are are of no interest as the three particles miss each other and do not interact at all. The second type of terms are partially connected; they correspond to situations where some particles interact, but not all of them.

The last term is the *connected S-matrix* in which all particles participate into the interactions. We will focus on the connected S-matrix, which is more fundamental as a partial scattering event, where some particles do not participate, can be reduced to a scattering of less particles. In practice, the connected S-matrix will contain a single overall momentum conservation constraint $\delta^{(4)}(\sum p_i - \sum q_j)$.

For simplicity of the presentation, we will consider S-matrix elements of a real scalar interacting QFT, such that incoming and outgoing states are solely labeled by their momenta; generalizations to complex scalars and spinors will follow thereafter. The starting point is the S-matrix element:

$$S_{pq} = {}_{\text{OUT}}\langle \vec{p}_1, \dots, \vec{p}_\ell | \vec{q}_1, \dots, \vec{q}_n \rangle_{\text{IN}} \,. \tag{2.57}$$

The idea is to use the relation (2.51a) and its Hermitian conjugate to "pull out" particles from the incoming and outgoing states, step by step. As a first step, let us write:

$$\begin{aligned} {}_{\text{OUT}}\langle \vec{p}_1, \dots, \vec{p}_\ell | \vec{q}_1, \dots, \vec{q}_n \rangle_{\text{IN}} &= \sqrt{2E_{p_1}}\, {}_{\text{OUT}}\langle \vec{p}_2, \dots, \vec{p}_\ell |\, a_{\vec{p}_1,\text{OUT}}\, | \vec{q}_1, \dots, \vec{q}_n \rangle_{\text{IN}} \\ &= \sqrt{2E_{p_1}} \Big({}_{\text{OUT}}\langle \vec{p}_2, \dots, \vec{p}_\ell | \left(a_{\vec{p}_1,\text{OUT}} - a_{\vec{p}_1,\text{IN}} \right) | \vec{q}_1, \dots, \vec{q}_n \rangle_{\text{IN}} \\ &\qquad + {}_{\text{OUT}}\langle \vec{p}_2, \dots, \vec{p}_\ell |\, a_{\vec{p}_1,\text{IN}}\, | \vec{q}_1, \dots, \vec{q}_n \rangle_{\text{IN}} \Big) \,. \end{aligned} \tag{2.58}$$

In the second term, the action of $a_{\vec{p}_1,\text{IN}}$ on $|\vec{q}_1, \dots, \vec{q}_n\rangle_{\text{IN}}$ will annihilate one of the incoming particles, thus giving contributions in $\delta^{(3)}(\vec{p}_1 - \vec{q}_j)$. These terms will not contribute to the connected S-matrix, since the j-th incoming particle does not participate in the scattering. Focusing on the connected part one has then, using eqn. (2.55):

$$\begin{aligned} S_{pq} &= \sqrt{\frac{2E_{p_1}}{Z}} \lim_{\tau\to\infty} {}_{\text{OUT}}\langle \vec{p}_2, \dots, \vec{p}_\ell | \int_{-\tau}^{\tau} dt\, \partial_t a_{\vec{p}_1}(t)\, | \vec{q}_1, \dots, \vec{q}_n \rangle_{\text{IN}} + \text{disc.} \\ &= \frac{i}{\sqrt{Z}} \int d^4x\, \partial_0 \Big(e^{ip_1\cdot x} \overleftrightarrow{\partial_0}\, {}_{\text{OUT}}\langle \vec{p}_2, \dots, \vec{p}_\ell |\, \Phi(x)\, | \vec{q}_1, \dots, \vec{q}_n \rangle_{\text{IN}} \Big) + \text{disc.} \\ &= \frac{i}{\sqrt{Z}} \int d^4x\; e^{ip_1\cdot x}\, {}_{\text{OUT}}\langle \vec{p}_2, \dots, \vec{p}_\ell | \left(\partial_0^2 + (p_1^0)^2 \right) \Phi(x)\, | \vec{q}_1, \dots, \vec{q}_n \rangle_{\text{IN}} \Big) + \text{disc.} \end{aligned} \tag{2.59}$$

Next we can integrate by parts twice, using the fact that the momentum p_1 is on-shell:

$$\begin{aligned} \int d^4x\, e^{ip_1\cdot x} \big(\partial_0^2 + (p_1^0)^2 \big) \langle \cdots \Phi(x) \cdots \rangle &= \int d^4x\, e^{ip_1\cdot x} \big(\partial_0^2 + \vec{p}_1^{\,2} + m^2 \big) \langle \cdots \Phi(x) \cdots \rangle \\ &\overset{\text{IBP}}{=} \int d^4x\, e^{ip_1\cdot x} \big(\partial_0^2 - \vec{\nabla}^2 + m^2 \big) \langle \cdots \Phi(x) \cdots \rangle \end{aligned} \tag{2.60}$$

In conclusion we have obtained:

$$S_{pq} = \frac{i}{\sqrt{Z}} \int d^4x_1\, e^{ip_1\cdot x_1} \left(\Box_{x_1} + m^2 \right) {}_{\text{OUT}}\langle \vec{p}_2, \dots, \vec{p}_\ell |\, \Phi(x_1)\, | \vec{q}_1, \dots, \vec{q}_n \rangle_{\text{IN}} + \text{disc.} \tag{2.61}$$

The idea is to keep on reducing S in order to obtain an $(n+m)$-point correlation function in the vacuum, up to the disconnected components. In order to see how it works, we perform one more step:

$$\begin{aligned}{}_{\text{OUT}}\langle \vec{p}_2,\dots,\vec{p}_\ell|\,\Phi(x_1)\,|\vec{q}_1,\dots,\vec{q}_n\rangle_{\text{IN}} &= \sqrt{2E_{q_1}}\,{}_{\text{OUT}}\langle \vec{p}_2,\dots,\vec{p}_\ell|\,\Phi(x_1)a^\dagger_{\vec{q}_1,\,\text{IN}}\,|\vec{q}_2,\dots,\vec{q}_n\rangle_{\text{IN}}\\ &= \sqrt{2E_{q_1}}\,{}_{\text{OUT}}\langle \vec{p}_2,\dots,\vec{p}_\ell|\left(\Phi(x_1)a^\dagger_{\vec{q}_1,\,\text{IN}} - a^\dagger_{\vec{q}_1,\,\text{OUT}}\Phi(x_1)\right)|\vec{q}_2,\dots,\vec{q}_n\rangle_{\text{IN}}\\ &\quad + \sqrt{2E_{q_1}}\,{}_{\text{OUT}}\langle \vec{p}_2,\dots,\vec{p}_\ell|\,a^\dagger_{\vec{q}_1,\,\text{OUT}}\Phi(x_1)\,|\vec{q}_2,\dots,\vec{q}_n\rangle_{\text{IN}} \qquad (2.62)\end{aligned}$$

As before, the third term with ${}_{\text{OUT}}\langle p_2,\dots,p_\ell|\,a^\dagger_{\vec{q}_1,\,\text{OUT}}$ will contribute to the disconnected part of the S-matrix. The first term can be rewritten, using the Hermitian conjugate of eqn (2.51a), as:

$$\begin{aligned}&\sqrt{2E_{q_1}}\,{}_{\text{OUT}}\langle \vec{p}_2,\dots,\vec{p}_\ell|\,\Phi(x_1)a^\dagger_{\vec{q}_1,\,\text{IN}}\,|\vec{q}_2,\dots,\vec{q}_n\rangle_{\text{IN}}\\ &\qquad = \frac{-i}{\sqrt{Z}}\lim_{y^0\to-\infty}\int d^3y\, e^{-iq_1\cdot y}\overleftrightarrow{\partial_{y^0}}\,{}_{\text{OUT}}\langle \vec{p}_2,\dots,\vec{p}_\ell|\,\Phi(x_1)\Phi(y)\,|\vec{q}_2,\dots,\vec{q}_n\rangle_{\text{IN}}\,, \qquad (2.63)\end{aligned}$$

remembering that $\Phi(x)$ is Hermitian. In the same way, regarding the second term one can write:

$$\begin{aligned}&\sqrt{2E_{q_1}}\,{}_{\text{OUT}}\langle \vec{p}_2,\dots,\vec{p}_\ell|\,a^\dagger_{\vec{q}_1,\,\text{OUT}}\Phi(x_1)\,|\vec{q}_2,\dots,\vec{q}_n\rangle_{\text{IN}}\\ &\qquad = \frac{-i}{\sqrt{Z}}\lim_{y^0\to+\infty}\int d^3y\, e^{-iq_1\cdot y}\overleftrightarrow{\partial_{y^0}}\,{}_{\text{OUT}}\langle \vec{p}_2,\dots,\vec{p}_\ell|\,\Phi(y)\Phi(x_1)\,|\vec{q}_2,\dots,\vec{q}_n\rangle_{\text{IN}} \qquad (2.64)\end{aligned}$$

We realize that, thanks to the limit $y^0\to\mp\infty$, both terms are time-ordered correctly and can be combined together as:

$$\begin{aligned}&\frac{i}{\sqrt{Z}}\int d^4y\,\partial_{y_0}\Big(e^{-iq_1\cdot y}\overleftrightarrow{\partial_{y^0}}\,{}_{\text{OUT}}\langle \vec{p}_2,\dots,\vec{p}_\ell|\,T\Phi(x_1)\Phi(y)\,|\vec{q}_2,\dots,\vec{q}_n\rangle_{\text{IN}}\Big)\\ &\qquad = \frac{i}{\sqrt{Z}}\int d^4y_1\, e^{-iq_1\cdot y_1}\left(\Box_{y_1}+m^2\right){}_{\text{OUT}}\langle \vec{p}_2,\dots,\vec{p}_\ell|\,T\Phi(x_1)\Phi(y_1)\,|\vec{q}_2,\dots,\vec{q}_n\rangle_{\text{IN}}\,, \qquad (2.65)\end{aligned}$$

using the same integration by parts as in the first step of the LSZ reduction.

We can of course iterate the process in order to reach both the in and out vacuum. One obtains then the LSZ reduction formula for a real scalar field:

$$\begin{aligned}{}_{\text{OUT}}\langle \vec{p}_1,\dots,\vec{p}_\ell|\vec{q}_1,\dots,\vec{q}_n\rangle_{\text{IN}} = \text{ disc. } + \left(\frac{i}{\sqrt{Z}}\right)^{\ell+n}\int\prod_{i=1}^{\ell}d^4x_i\,e^{ip_i\cdot x_i}\prod_{j=1}^{n}dy_j\,e^{-iq_j\cdot y_j}\\ \times\prod_{i=1}^{\ell}\left(\Box_{x_i}+m^2\right)\prod_{j=1}^{n}\left(\Box_{y_j}+m^2\right)\langle\Omega|\,T\,\Phi(x_1)\cdots\Phi(x_\ell)\Phi(y_1)\cdots\Phi(y_n)\,|\Omega\rangle\,\Big|_{\text{con.}}\end{aligned} \qquad (2.66)$$

Importantly, all the momenta p_i and q_j are on-shell in this expression, *i.e.* satisfy $p^2 = q^2 = m^2$,where m is the physical mass of the particle.

It is easy to adapt this construction to other quantum field theories, by following similar steps.

Let us first consider the case of a complex scalar field, using both equations (2.51a) and (2.51b) as well as their hermitian conjugates. One gets an expression similar to eqn. (2.66) where we have to use the following the replacement rules that distinguishes particles in/out states from anti-particles in/out states:

LSZ reduction for a complex scalar field

- Incoming particle of momentum p: $\frac{i}{\sqrt{Z}} e^{-ip\cdot x}\big(\Box_x + m^2\big)\langle T\cdots\Phi^\dagger(x)\cdots\rangle$
- Incoming anti-particle of momentum p: $\frac{i}{\sqrt{Z}} e^{-ip\cdot x}\big(\Box_x + m^2\big)\langle T\cdots\Phi\ (x)\cdots\rangle$
- Outgoing particle of momentum p: $\frac{i}{\sqrt{Z}} e^{+ip\cdot x}\big(\Box_x + m^2\big)\langle T\cdots\Phi\ (x)\cdots\rangle$
- Outgoing anti-particle of momentum p: $\frac{i}{\sqrt{Z}} e^{+ip\cdot x}\big(\Box_x + m^2\big)\langle T\cdots\Phi^\dagger(x)\cdots\rangle$

The LSZ reduction, as promized, has reduced the problem of computing scattering amplitudes, to the problem of computing N-point time-ordered correlation functions.

LSZ reduction for Dirac spinors. In the case of a Dirac field, both incoming and outgoing states are also characterized by their spin polarization.

To understand how it works, let us first consider pulling out from the out state the operator $b^s_{\vec{p},\,\text{OUT}}$ associated with an outgoing Dirac particle of 3-momentum $\vec{p}$ and spin polarization s. Using eqn. (2.56a), one has:

$$\begin{aligned}
{}_{\text{OUT}}\langle p, s;\cdots|\cdots\rangle_{\text{IN}} &= \text{disc.} + \sqrt{2E_p}\,{}_{\text{OUT}}\langle\cdots|\,\big(b^s_{\vec{p},\,\text{OUT}} - b^s_{\vec{p},\,\text{IN}}\big)\,|\cdots\rangle_{\text{IN}} \\
&= \text{disc.} + \frac{1}{\sqrt{Z}}\int d^4x\,\partial_0\Big(e^{ip\cdot x}\bar{u}^s(\vec{p})\,{}_{\text{OUT}}\langle\cdots|\,\gamma^0\Psi(x)\,|\cdots\rangle_{\text{IN}}\Big) \\
&= \text{disc.} + \frac{1}{\sqrt{Z}}\int d^4x\, e^{ip\cdot x}\bar{u}^s(\vec{p})\big(ip^0\gamma^0 + \gamma^0\partial_0\big)\,{}_{\text{OUT}}\langle\cdots|\,\Psi(x)\,|\cdots\rangle_{\text{IN}} \qquad (2.67)
\end{aligned}$$

This can be simplified using eqn. (1.181):

$$\begin{aligned}
\int d^3x\, e^{ip\cdot x}\bar{u}^s(\vec{p})p^0\gamma^0\cdots = \int d^3x\, e^{ip\cdot x}\bar{u}^s(\vec{p})\big(m+\vec{\gamma}\cdot\vec{p}\big)\cdots = \int d^3x\,\bar{u}^s(\vec{p})\big(me^{ip\cdot x} - i\vec{\gamma}\cdot\vec{\nabla}e^{ip\cdot x}\big)\cdots \\
\overset{\text{IBP}}{=} \int d^3x\, e^{ip\cdot x}\,\bar{u}^s(\vec{p})\big(m + i\vec{\gamma}\cdot\vec{\nabla}\big)\cdots \qquad (2.68)
\end{aligned}$$

Such that

$$ {}_{\text{OUT}}\langle p, s; \cdots | \cdots \rangle_{\text{IN}} = \text{disc.} + \frac{-i}{\sqrt{Z}} \int d^3x \, e^{ip\cdot x} \bar{u}^s(\vec{p}) \big(i\not{\partial} - m \big) \, {}_{\text{OUT}}\langle \cdots | \, \Psi(x) \, | \cdots \rangle_{\text{IN}} \tag{2.69} $$

The same can be repeated for an outgoing anti-particle, using (2.56b):

$$ {}_{\text{OUT}}\langle p, s; \cdots | \cdots \rangle_{\text{IN}} = \text{disc.} + \frac{1}{\sqrt{Z}} \int d^4x \, e^{ip\cdot x} \, {}_{\text{OUT}}\langle \cdots | \, \bar{\Psi}(x) \, | \cdots \rangle_{\text{IN}} \big(ip^0\gamma^0 + \gamma^0 \overleftarrow{\partial}_0 \big) v^s(\vec{p}) \tag{2.70} $$

Then using (1.194):

$$ \int d^3x \cdots e^{ip\cdot x}(\vec{p}) p^0 \gamma^0 v^s(\vec{p}) \overset{\text{IBP}}{=} \int d^3x \cdots \big(- i\gamma^i \overleftarrow{\partial}_i - m \big) v^s(\vec{p}) e^{ip\cdot x} \tag{2.71} $$

One gets

$$ {}_{\text{OUT}}\langle p, s; \cdots | \cdots \rangle_{\text{IN}} = \text{disc.} + \frac{-i}{\sqrt{Z}} \int d^3x \, {}_{\text{OUT}}\langle \cdots | \, \bar{\Psi}(x) \, | \cdots \rangle_{\text{IN}} \big(i \overleftarrow{\not{\partial}} + m \big) v^s(\vec{p}) e^{ip\cdot x} , \tag{2.72} $$

where for convenience we have made the space-time derivative acting on the left rather than on the right.

Naturally, the same can be done for incoming particles and anti-particles. We summarize the result below.

LSZ reduction for a Dirac field

- Incoming particle $|p, s\rangle$: $\frac{-i}{\sqrt{Z}} \langle T \cdots \bar{\Psi}(x) \cdots \rangle \big(i \overleftarrow{\not{\partial}} + m \big) u_s(\vec{p}) e^{-ip\cdot x}$
- Incoming anti-particle $|p, s\rangle$: $\frac{-i}{\sqrt{Z}} e^{-ip\cdot x} \, \bar{v}_s(\vec{p}) \big(i\not{\partial} - m \big) \, \langle T \cdots \Psi(x) \cdots \rangle$
- Outgoing particle $|p, s\rangle$: $\frac{-i}{\sqrt{Z}} e^{+ip\cdot x} \, \bar{u}_s(\vec{p}) \big(i\not{\partial} - m \big) \, \langle T \cdots \Psi(x) \cdots \rangle$
- Outgoing anti-particle $|p, s\rangle$: $\frac{-i}{\sqrt{Z}} \langle T \cdots \bar{\Psi}(x) \cdots \rangle \big(i \overleftarrow{\not{\partial}} + m \big) v^s(\vec{p}) e^{ip\cdot x}$

As for a scalar field, all the momenta are on-shell, and the resulting N-point correlation function is time-ordered.

LSZ in momentum space. The LSZ formulæ are expressed as Fourier transforms of differential operators acting on time-ordered correlation functions, so it is worthwile expressing them in momentum space.

Let us consider first the case of a real scalar field. The time-ordered correlators of interest are of the form:

$$\mathcal{G}_{\rm C}(x_1,\ldots,x_n) \stackrel{\rm def.}{=} \langle\Omega|\,\mathsf{T}\,\Phi(x_1)\cdots\Phi(x_n)\,|\Omega\rangle\Big|_{\rm connected}. \tag{2.73}$$

Due to translation invariance of the vacuum, this depends only on $n-1$ variables. Using the same trick as before,

$$\langle\Omega|\,\mathsf{T}\,e^{-iP\cdot x_n}\Phi(x_1)e^{iP\cdot x_n}e^{-iP\cdot x_n}\cdots e^{+iP\cdot x_n}\Phi(0)e^{-iP\cdot x_n}\,|\Omega\rangle \\ = \mathcal{G}(x_1-x_n,\ldots,x_{n-1}-x_n,0)\,. \tag{2.74}$$

The Fourier transform of $\mathcal{G}_{\rm C}$ is then:

$$\mathcal{G}_{\rm C}(p_1,\ldots,p_n) = \int d^4x_1\cdots d^4x_n\, e^{ip_1\cdot x_1}\cdots e^{ip_n\cdot x_n}\mathcal{G}_{\rm C}(x_1-x_n,\ldots,x_{n-1}-x_n,0)\,. \tag{2.75}$$

Using the change of variables $\tilde{x}_i = x_i - x_n$ for $i<n$, one gets:

$$\mathcal{G}_{\rm C}(p_1,\ldots,p_n) = (2\pi)^4\delta^{(4)}\big(\sum p_i\big)\int d^4\tilde{x}_1\cdots d^4\tilde{x}_{n-1}e^{i(p_1\cdot\tilde{x}_1+\cdots+p_{n-1}\cdot\tilde{x}_{n-1})}\mathcal{G}_{\rm C}\big(\tilde{x}_1,\ldots,\tilde{x}_{n-1},0\big)\,. \tag{2.76}$$

The first term on the RHS expresses momentum conservation. We define then the quantity $\widehat{\mathcal{G}}$ through:

$$\mathcal{G}_{\rm C}(p_1,\ldots,p_n) = (2\pi)^4\delta^{(4)}\big(\sum p_i\big)\widehat{\mathcal{G}}(p_1,\ldots,p_n)\,. \tag{2.77}$$

We can then express the LSZ formula (2.66) in terms of $\widehat{\mathcal{G}}$, taking care of the opposite signs for the momenta of incoming vs. outgoing states:

$$\begin{aligned}{}_{\rm OUT}\langle p_1,\ldots,p_\ell|q_1,\ldots,q_n\rangle_{\rm IN} = {\rm disc.} + \\ \prod_{k=1}^{\ell}\frac{p_k^2-m^2}{i\sqrt{Z}}\prod_{j=1}^{n}\frac{q_j^2-m^2}{i\sqrt{Z}}(2\pi)^4\delta^{(4)}\big(\sum p_i-\sum q_j\big) \\ \times\,\widehat{\mathcal{G}}(p_1,\ldots,p_\ell,-q_1,\ldots,-q_n)\Big|_{\rm on\text{-}shell}\end{aligned} \tag{2.78}$$

The way to understand this expression is as follows. One first multiplies the momentum space N-point function by the product of inverse propagators for all ingoing and outgoing states, before evaluating the resulting expression on shell, *i.e.* for external momenta satisying $p_1^2 = \cdots = q_n^2 = m^2$.

In order to get a non-zero answer $\widehat{\mathcal{G}}$ should have corresponding poles in its analytic structure. This will be clarified shortly when we will develop tools to actually compute it (perturbation theory).

Before that, let us give the recipe to get the other LSZ formulæ in momentum space. First, for a *complex scalar field*, one needs first to distinguish Φ and $\Phi^\dagger$ in the position space N-point correlation functions, and perform the reduction using the formulae (2.51). Let us define then:

$$\mathcal{G}_{\mathrm{C}}(x_1,\dots,x_n|y_1,\dots,y_\ell) \stackrel{\text{def.}}{=} \langle\Omega|\,\mathsf{T}\,\Phi(x_1)\cdots\Phi(x_n)\,|\,\Phi^\dagger(y_1)\cdots\Phi^\dagger(y_\ell)\,|\Omega\rangle\,\Big|_{\text{con.}} . \tag{2.79}$$

In terms of their Fourier transform, one gets the following rules:

LSZ reduction for a complex scalar field in momentum space

- Incoming particle of momentum p: $\frac{p^2-m^2}{i\sqrt{Z}}\widehat{\mathcal{G}}(\cdots|\cdots,-p,\cdots)$
- Incoming anti-particle of momentum p: $\frac{p^2-m^2}{i\sqrt{Z}}\widehat{\mathcal{G}}(\cdots,-p,\cdots|\cdots)$
- Outgoing particle of momentum p: $\frac{p^2-m^2}{i\sqrt{Z}}\widehat{\mathcal{G}}(\cdots,+p,\cdots|\cdots)$
- Outgoing anti-particle of momentum p: $\frac{p^2-m^2}{i\sqrt{Z}}\widehat{\mathcal{G}}(\cdots|\cdots,+p,\cdots)$

For a *Dirac field*, one starts with correlators in position space with both the field Ψ and its Dirac conjugate $\bar\Psi$.

$$\mathcal{G}_{\mathrm{D}}(x_1,\dots,x_\ell|\,y_1,\dots,y_n) \stackrel{\text{def.}}{=} \langle\Omega|\,\mathsf{T}\,\Psi(x_1)\cdots\Psi(x_\ell)\bar\Psi(y_1)\cdots\bar\Psi(y_n)\,|\Omega\rangle\ . \tag{2.80}$$

Considering first an outgoing particle, the relevant piece of computation, for the Fourier transform appearing in the LSZ reduction process, is:

$$-\frac{i}{\sqrt{Z}}\int d^4x e^{ip\cdot x}\bar u_s(\vec p)(i\not\partial - m)\langle\cdots\Psi(x)\cdots\rangle \stackrel{\text{IBP}}{=} -\frac{i}{\sqrt{Z}}\bar u_s(\vec p)(\not p - m)\int d^4x\, e^{ip\cdot x}\langle\cdots\Psi(x)\cdots\rangle\ . \tag{2.81}$$

For an ingoing anti-particle, the computation is very similar:

$$-\frac{i}{\sqrt{Z}}\int d^4x e^{-ip\cdot x}\bar v_s(\vec p)(i\not\partial - m)\langle\cdots\Psi(x)\cdots\rangle \stackrel{\text{IBP}}{=} -\frac{i}{\sqrt{Z}}\bar v_s(\vec p)(-\not p - m)\int d^4x\, e^{-ip\cdot x}\langle\cdots\Psi(x)\cdots\rangle\ . \tag{2.82}$$

Next for an ingoing particle, one considers:

$$-\frac{i}{\sqrt{Z}}\int d^4x\,\langle\cdots\bar\Psi(x)\cdots\rangle\,(i\overleftarrow{\not\partial} + m)u_s(\vec p)e^{-ip\cdot x} \stackrel{\text{IBP}}{=} -\frac{i}{\sqrt{Z}}\left(\int d^4x\, e^{-ip\cdot x}\langle\cdots\bar\Psi(x)\cdots\rangle\right)(-\not p + m)u_s(\vec p) \tag{2.83}$$

and finally for an outgoing anti-particle:

$$-\frac{i}{\sqrt{Z}}\int d^4x\,\langle\cdots\bar{\Psi}(x)\cdots\rangle\,(i\overleftarrow{\not{\partial}}+m)v_s(\vec{p})e^{ip\cdot x}$$
$$\overset{\text{IBP}}{=}-\frac{i}{\sqrt{Z}}\left(\int d^4x\,e^{ip\cdot x}\langle\cdots\bar{\Psi}(x)\cdots\rangle\right)(-\not{p}+m)v_s(\vec{p}) \quad (2.84)$$

To summarize, the rules of LSZ reduction for Dirac fields in momentum space are:

LSZ reduction for a Dirac field in momentum space

- Incoming particle of momentum p: $\widehat{\mathcal{G}}_D(\cdots|\cdots,-p,\cdots)\frac{-\not{p}+m}{i\sqrt{Z}}u_s(\vec{p})$
- Incoming anti-particle of momentum p: $\bar{v}_s(\vec{p})\,\frac{-\not{p}-m}{i\sqrt{Z}}\,\widehat{\mathcal{G}}_D(\cdots,-p,\cdots|\cdots)$
- Outgoing particle of momentum p: $\bar{u}_s(\vec{p})\,\frac{\not{p}-m}{i\sqrt{Z}}\,\widehat{\mathcal{G}}_D(\cdots,+p,\cdots|\cdots)$
- Outgoing anti-particle of momentum p: $\widehat{\mathcal{G}}_D(\cdots|\cdots,+p,\cdots)\frac{\not{p}+m}{i\sqrt{Z}}v_s(\vec{p})$

2.2 Perturbation theory

Thanks to the LSZ reduction, we have converted the problem of computing scattering amplitudes in QFT to the problem of computing time-ordered correlation functions of field operators in the (interacting) vacuum, which are given for instance in the case of the real scalar QFT by:

$$\mathcal{G}(x_1,\ldots,x_n)=\langle\Omega|\,T\Phi(x_1)\cdots\Phi(x_n)\,|\Omega\rangle\ . \quad (2.85)$$

Being able to compute such correlation functions exactly means solving the quantum field theory at hand.

As mentioned in the beginning of this section, no interacting QFT can be solved analytically without having extra space-time symmetries like supersymmetry or conformal symmetry. In most cases, in particular in quantum electrodynamics (QED) or more generally in Standard Model physics, we have essentially two approaches at our disposal:

1. putting the quantum field theory on the lattice and solving it numerically. This demands considerable computational power, even with moderate size grids, and is not without difficulties, for instance when fermions are involved.

2. using a perturbative approach whenever the coupling constants are small, which may be true only low or high energies. This approach is powerful but difficult to handle beyond the first few orders, and does not capture non-perturbative phenomena like instanton contributions, confinement, etc. unlike the former.

In these lectures we will focus on the latter.

The starting point of the perturbative approach is a field theory described by a Lagrangian density $\mathcal{L}$ that can be separated into two parts:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{INT}}\,, \tag{2.86}$$

where $\mathcal{L}_0$ is a Lagrangian density for a solvable theory (a free gaussian field theory) and a term $\mathcal{L}_{\text{INT}}$ containing interactions, beyond quadratic order.

A typical example is the famous ϕ^4, theory, a massive real scalar field with quadratic self-interactions:

$$\mathcal{L} \;=\; \underbrace{\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}m_0^2\Phi^2}_{\mathcal{L}_0}\; \underbrace{-\frac{\lambda_0}{4!}\,\Phi^4}_{\mathcal{L}_{\text{INT}}}\,. \tag{2.87}$$

In this theory, the parameter m_0 has the dimension of a mass, but, unlike in the free theory, it is not necessarily the same as the physical mass of the 1-particle states, as we will discuss later. The parameter λ, that controls the strength of the interactions, has dimension L^{d-4} in d-dimensions. Therefore, in a 4-dimensional space-time it is dimension-less and it makes sense to consider that λ is small, at least naively.

From this Lagrangian density, one can obtain the Hamiltonian of the system, that can be also split into free part and interactions:

$$H = \int d^3x\, \left(\Pi_I\dot{\Phi}^I - \mathcal{L}\right) = H_0 + H_{\text{INT}}. \tag{2.88}$$

Whenever the interactions do not involve derivatives of the fields, – *and only in this case!* – the interaction Hamiltonian is just given by minus the interaction Lagrangian:

$$H \;=\; H_0 \;\underbrace{-\int d^3x\,\mathcal{L}_{\text{INT}}}_{H_{\text{INT}}}\,. \tag{2.89}$$

As in quantum mechanics, the idea behind pertubation theory is to expand in power series the quantities of interest, here scattering amplitudes, in terms the parameter that controls the strength of the interactions. In quantum mechanics it is well-known to raise a number of issues:

1. Given a self-adjoint Hamiltonian H_0 and a Hermitian operator H_{INT}, is the operator $H = H_0 + H_{\text{INT}}$ self-adjoint on a certain domain?
2. Is the spectrum of $H = H_0 + H_{\text{INT}}$ bounded from below?[9]
3. What is the radius of convergence of the perturbative power series?
4. Whenever the parameter that controls the perturbative expansion is not dimensionless, how to make sense of perturbation theory?

New issues that are not present in quantum mechanics arise, like the presence of divergences in the perturbative corrections, and will be an important focus of this course.

[9] A simple counter-example is an x^3 perturbation of the harmonic oscillator.

2.2.1 Interaction picture

To prepare the ground for perturbation theory, we need to introduce the interaction picture which is a kind of "middle-ground" between the Schrödinger and Heisenberg pictures, also used in ordinary quantum mechanics. The idea is roughly to "factor out" the trivial time-dependence due to the free evolution of the system, while keeping the time-dependence triggered by the interactions.

Definition 3 (interaction picture). *Let* H *be the Hamiltonian of a quantum system. Assume that* H *can be split as* $\mathsf{H}_0 + \mathsf{H}_{\text{INT}}$*, where* H_0 *is solvable in the sense that its spectrum is computable. Let* $|\alpha(t)\rangle_{\text{S}} \in \mathcal{H}$ *be some state of the system defined in the Schrödinger picture and* $\mathcal{O}_{\text{S}}$ *some operator acting on the Hilbert space* $\mathcal{H}$*, also in the Schrödinger representation. The corresponding interaction picture state and operator are defined by the unitary transformation:*

$$|\alpha(t)\rangle_{\text{I}} \overset{\text{def.}}{=} e^{iH_0 t}\,|\alpha(t)\rangle_{\text{S}}\,, \tag{2.90a}$$

$$\mathcal{O}_{\text{I}} \overset{\text{def.}}{=} e^{iH_0 t}\mathcal{O}_{\text{S}}e^{-iH_0 t}\,. \tag{2.90b}$$

These two transformations are compatible in the sense that they do not change the value of matrix elements of observables. The interaction picture can also be conveniently related to the Heisenberg picture:

$$\mathcal{O}_{\text{H}} = e^{iHt}\mathcal{O}_{\text{S}}e^{-iHt} = e^{iHt}e^{-iH_0 t}\,\mathcal{O}_{\text{I}}\,e^{iH_0 t}e^{-iHt}\,. \tag{2.91}$$

Notice that $e^{iHt}e^{-iH_0 t} \neq e^{iH_{\text{INT}}}$ since H_{INT} and H_0 do not necessarily commute.

Property 1 (time evolution in interaction picture). *Let* $|\alpha(t)\rangle_{\text{I}} \in \mathcal{H}$ *describing the state of the system at instant* t *in the interaction picture. The time evolution of this state is governed by the interaction Hamiltonian expressed in the interaction picture:*

$$i\frac{d}{dt}\,|\alpha(t)\rangle_{\text{I}} \,=\, \mathsf{H}_{\text{I}}(t)\,|\alpha(t)\rangle_{\text{I}} \quad , \qquad \mathsf{H}_{\text{I}}(t) = \mathsf{H}_{\text{INT}}\Big|_{\text{I}} = e^{iH_0 t}\mathsf{H}_{\text{INT}}e^{-iH_0 t}\,. \tag{2.92}$$

Indeed,

$$\begin{aligned} i\frac{d}{dt}\,|\alpha(t)\rangle_{\text{I}}\,(t) &= -\mathsf{H}_0 e^{iH_0 t}\,|\alpha(t)\rangle_{\text{S}}\,(t) + e^{iH_0 t}\big(\mathsf{H}+\mathsf{H}_0\big)\,|\alpha(t)\rangle_{\text{S}}\,(t) \\ &= -\cancel{\mathsf{H}_0}\,|\alpha(t)\rangle_{\text{I}}\,(t) + e^{iH_0 t}\big(\mathsf{H}+\cancel{\mathsf{H}_0}\big)e^{-iH_0 t}\,|\alpha(t)\rangle_{\text{I}}\,(t) = \mathsf{H}_{\text{I}}(t)\,|\alpha(t)\rangle_{\text{I}}\,. \end{aligned} \tag{2.93}$$

Property 2 (evolution operator in interaction picture). *Let prepare the state of the system at* $t = 0$*. The solution of the time evolution equation (2.92) is given in terms of a unitary operator* $\mathsf{U}(t)$ *such that*

$$|\alpha(t)\rangle = \mathsf{U}(t)\,|\alpha(0)\rangle \quad , \qquad i\frac{d\mathsf{U}(t)}{dt} = \mathsf{H}_{\text{I}}(t)\mathsf{U}(t)\,. \tag{2.94}$$

The proof is completely trivial. Coming back to the original definition (2.90a), $U(t)$ can be written as:

$$U(t) = e^{iH_0 t} e^{-iHt}\,, \tag{2.95}$$

which already appeared in eqn. (2.91).

The unitary evolution operator $U(t)$ naturally satisfies a the composition law of the group $(\mathbb{R}, +)$. Given that, for $t_1 < t_2$:

$$|\alpha(t_2)\rangle_{_I} = U(t_2)\,|\alpha(0)\rangle = U(t_2)U(t_1)^{-1}\,|\alpha(t_1)\rangle_{_I}\,, \tag{2.96}$$

The evolution from t_1 to t_2 is implemented by the operator:[10]

$$U(t_2, t_1) \overset{\text{def.}}{=} U(t_2)U(t_1)^{-1} = e^{it_2 H_0} e^{-i(t_2-t_1)H} e^{-it_1 H_0}\,, \tag{2.97}$$

such that, for $t_1 < t_2 < t_3$,

$$U(t_3, t_2)U(t_2, t_1) = U(t_3, t_1)\,. \tag{2.98}$$

Solving (2.94) is elementary whenever the interaction Hamiltonian evaluated at $t = \tau$ commutes with the interaction Hamiltonian evaluated at $t = \tau' \neq \tau$. We have in this case:

$$U(t) = \exp\left(-i\int_0^t d\tau\, H_{_I}(\tau)\right) \quad \text{if} \qquad \forall \tau, \tau' \in \mathbb{R}_{\geqslant 0},\, [\,H_{_I}(\tau),\, H_{_I}(\tau')\,] = 0\,. \tag{2.99}$$

Whenever this is not the case, the solution is given in terms of the *Dyson series* involving the time-ordering operator already introduced. One has:

$$\boxed{U(t) \;=\; \mathsf{T}\,\exp\Big(-i\int_0^t d\tau\, H_{_I}(\tau)\Big)\,,} \tag{2.100}$$

where the time-ordered exponential can be usefully formulated in terms of its series expansion:

$$\begin{aligned}\mathsf{T}\,\exp\Big(-i\int_0^t d\tau\, H_{_I}(\tau)\Big) &\overset{\text{def.}}{=} 1 - i\int_0^t d\tau\, H_{_I}(\tau) \\ &+ \frac{(-i)^2}{2!}\int_0^t d\tau \int_0^t d\tau'\,(\Theta(\tau-\tau')H_{_I}(\tau)H_{_I}(\tau') + \Theta(\tau'-\tau)H_{_I}(\tau')H_{_I}(\tau)) + \cdots\end{aligned} \tag{2.101}$$

Naturally, both terms at quadratic order are identical when swapping integration variables in one of them; the same pattern persists at higher orders. We can rewrite the whole series as:

$$\mathsf{T}\,\exp\Big(-i\int_0^t d\tau\, H_{_I}(\tau)\Big) \overset{\text{def.}}{=} 1 - i\int_0^t d\tau\, H_{_I}(\tau) + (-i)^2\int_0^t d\tau \int_0^\tau d\tau'\, H_{_I}(\tau)H_{_I}(\tau') + \cdots \tag{2.102}$$

[10] Take care of the convention: the "arrival time" is on the left and the "departure time" on the right.

Differentiating this expression w.r.t. time, we get indeed what we expect:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{T}\,\exp\Big(-\mathrm{i}\int_0^t \mathrm{d}\tau\, H_{\mathrm{I}}(\tau)\Big) = \\ -\mathrm{i}H(t) + (-\mathrm{i})^2 H_{\mathrm{I}}(t)\int_0^t \mathrm{d}\tau' H_{\mathrm{I}}(\tau') + \cdots = -\mathrm{i}H_{\mathrm{I}}(t)\mathsf{T}\,\exp\Big(-\mathrm{i}\int_0^t \mathrm{d}\tau\, H_{\mathrm{I}}(\tau)\Big)\,, \quad (2.103)$$

i.e. with the un-integrated $H_{\mathrm{I}}(t)$ factor appearing *on the left* of each term of the series, thereby allowing to factorize the result without worrying about ordering issues. For the evolution between t_1 and t_2, from (2.97) we get as expected

$$U(t_2,t_1) = \mathsf{T}\,\exp\Big(-\mathrm{i}\int_{t_1}^{t_2} \mathrm{d}t\, H_{\mathrm{I}}(t)\Big)\,. \quad (2.104)$$

Interaction picture in QFT. Let us see how these general concepts of quantum mechanics apply to quantum field theories, taking as an example a real scalar field for the time being.

Consider $\Phi(0,\vec{x})$, the field operator Heisenberg picture evaluated at $t=0$ chosen as the arbitrary reference time, when it is identical to the Schrödinger picture operator. We define then the field operator in the interaction picture as:

$$\Phi_{\mathrm{I}}(t,\vec{x}) = e^{\mathrm{i}H_0 t}\Phi(0,\vec{x})e^{-\mathrm{i}H_0 t}\,. \quad (2.105)$$

and similarly for $\Pi_{\mathrm{I}}(t,\vec{x})$. One can relate at any time the Heisenberg picture and interaction picture operators through:

$$\Phi_{\mathrm{I}}(t,\vec{x}) = e^{\mathrm{i}H_0 t}e^{-\mathrm{i}Ht}\Phi(t,\vec{x})e^{\mathrm{i}Ht}e^{-\mathrm{i}H_0 t} = U(t)\Phi(t,\vec{x})U(t)^{-1}\,. \quad (2.106)$$

From the definition (2.105), one has:

$$\partial_t\Phi_{\mathrm{I}}(t,\vec{x}) = \mathrm{i}\,[\,H_0,\,\Phi_{\mathrm{I}}(t,\vec{x})\,] = e^{\mathrm{i}H_0 t}\mathrm{i}\,[\,H_0,\,\Phi(0,\vec{x})\,]\,e^{-\mathrm{i}H_0 t}\,. \quad (2.107)$$

Assuming that the interaction Lagrangian $\mathcal{L}_{\mathrm{INT}}$ does not contain derivatives for simplicity of the demonstration, one obtains from the canonical quantization rules that $\partial_t\Phi_{\mathrm{I}}(t,\vec{x}) = \Pi_{\mathrm{I}}(t,\vec{x})$. Differentiating once again w.r.t. to time, one gets:

$$\partial_t^2\Phi_{\mathrm{I}}(t,\vec{x}) = \partial_t e^{\mathrm{i}H_0 t}\Pi(0,\vec{x})e^{-\mathrm{i}H_0 t} = e^{\mathrm{i}H_0 t}\mathrm{i}\,[\,H_0,\,\Pi(0,\vec{x})\,]\,e^{-\mathrm{i}H_0 t}\,. \quad (2.108)$$

Since

$$H_0 = \frac{1}{2}\int \mathrm{d}^3x\,\Big(\Pi(0,\vec{x})^2 + \big(\vec{\nabla}\Phi(0,\vec{x})\big)^2 + m_0^2\Phi(0,\vec{x})^2\Big)\,, \quad (2.109)$$

One gets from the canonical commutation relations:

$$\partial_t^2\Phi_{\mathrm{I}}(t,\vec{x}) = \big(\Delta - m_0^2\big)\Phi_{\mathrm{I}}(t,\vec{x})\,. \quad (2.110)$$

In other words, the interaction picture field operator $\Phi_I(t,\vec{x})$ obeys the same Klein–Gordon equation of motion as the free field of mass m_0. As such it admits a free-field expansion in terms of creation an annihilation operators:

$$\Phi_I(t,\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_{\vec{p},I}e^{-ip\cdot x} + a^\dagger_{\vec{p},I}e^{ip\cdot x}\right), \tag{2.111}$$

where $E_p = \sqrt{\vec{p}^2 + m_0^2}$. Naturally the modes satisfy the standard commutation relations:

$$\left[\, a_{\vec{p},I},\, a^\dagger_{\vec{q},I}\,\right] = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\,. \tag{2.112}$$

Importantly the state satisfying $a_{\vec{p},I}\,|0\rangle = 0$ for all $\vec{p}$ is identified with the vacuum state of the free quantum field theory governed by the free Hamiltonian H_0 as Φ_I satisfies the free Klein–Gordon equation (note that $H_0 = H_{0,I}$ by construction).

We have learned from this analysis, which can be generalized to other QFTs, that the field operators in the interaction picture behave like free fields. To come back to our original motivation, a practical method to compute correlators like (2.85) will be obtained by mapping these quantities to correlators in the interaction picture.

Let us consider the example of the time-ordered two-point function, which is nothing but the Feynman propagator already considered in subsection. 2.1.1 from another viewpoint. For definiteness we consider the case $t_2 > t_1$:

$$\begin{aligned}\mathcal{G}(x_1,x_2) &= \langle\Omega|\,T\Phi(t_1,\vec{x}_1)\Phi(t_2,\vec{x}_2)\,|\Omega\rangle = \langle\Omega|\,\Phi(t_2,\vec{x}_2)\Phi(t_1,\vec{x}_1)\,|\Omega\rangle \\ &= \langle\Omega|\,U(t_2)^{-1}\Phi_I(t_2,\vec{x}_2)U(t_2)U(t_1)^{-1}\Phi_I(t_1,\vec{x}_1)U(t_1)\,|\Omega\rangle \\ &= \langle\Omega|\,U(t_2)^{-1}\Phi_I(t_2,\vec{x}_2)U(t_2,t_1)\Phi_I(t_1,\vec{x}_1)U(t_1)\,|\Omega\rangle \end{aligned} \tag{2.113}$$

Then we introduce an arbitrary time scale τ such that $-\tau < t_1 < t_2 < \tau$. We have:

$$\begin{aligned}\mathcal{G}(x_1,x_2) &= \langle\Omega|\,U(\tau)^{-1}U(\tau)U(t_2)^{-1}\Phi_I(t_2,\vec{x}_2)U(t_2,t_1)\Phi_I(t_1,\vec{x}_1)U(t_1)U(-\tau)^{-1}U(-\tau)\,|\Omega\rangle \\ &= \Big(\langle\Omega|\,U(\tau)^{-1}\Big)U(\tau,t_2)\Phi_I(t_2,\vec{x}_2)U(t_2,t_1)\Phi_I(t_1,\vec{x}_1)U(t_1,-\tau)\Big(U(-\tau)\,|\Omega\rangle\Big)\end{aligned} \tag{2.114}$$

Next, we would like to find a way to map the interacting vacuum $|\Omega\rangle$ to the free vacuum $|0\rangle$. For this purpose, let us consider the resolution of the identity associated with the *free* hamiltonian H_0. One has symbolically:

$$\mathbb{I} = |0\rangle\langle 0| + \sum_{n\neq 0}|n\rangle\langle n|\ , \quad H\,|n\rangle = E_n\,|n\rangle\ , \tag{2.115}$$

where $E_n > E_0$ for $n \neq 0$ by unicity of the vacuum. In the free theory, one can choose to adjust the parameter V_0 (the classical value of the potential at the minimum) such that

$E_0 = 0$; we will make this choice from now for convenience. We have then:

$$\begin{aligned} U(-\tau)\,|\Omega\rangle &= e^{-iH_0\tau}e^{iHt}\,|\Omega\rangle = e^{-iH_0\tau}\,|\Omega\rangle\, e^{iE_\Omega\tau} \qquad (2.116)\\ &= e^{-iH_0\tau}\,|0\rangle\,\langle 0|\Omega\rangle\, e^{iE_\Omega\tau} + \sum_{n\neq 0} e^{-iH_0\tau}\,|n\rangle\,\langle n|\Omega\rangle\, e^{iE_\Omega\tau} \\ &= e^{iE_\Omega\tau}\left(|0\rangle\,\langle 0|\Omega\rangle + \sum_{n\neq 0} e^{-iE_n\tau}\,|n\rangle\,\langle n|\Omega\rangle\right) \qquad (2.117)\end{aligned}$$

We realize, starting from these manipulations, that one can project the interacting vacuum $|\Omega\rangle$ onto the *free vacuum* $|0\rangle$ by taking the parameter τ to infinity, slightly below the real axis:

$$U(-\tau)\,|\Omega\rangle \overset{\tau\to+\infty(1-i\epsilon)}{\sim} |0\rangle\,\langle 0|\Omega\rangle\, e^{iE_\Omega\tau} \qquad (2.118)$$

In this limit, the contributions from the states different from the vacuum are all exponentially suppressed. As we will see below, this prescription boils down eventually to a choice of contour in the complex plane.

One assumes that the overlap $\langle 0|\Omega\rangle$ between the free and interacting vacuum does not vanish, which is a sensible assumption at least when the theory is not strongly coupled (in which case much is not under control). In exactly the same way, one finds

$$\langle\Omega|\,U(\tau)^{-1} \overset{\tau\to+\infty(1-i\epsilon)}{\sim} \langle\Omega|0\rangle\, e^{+iE_\Omega\tau}\,\langle 0| \qquad (2.119)$$

Coming back to eqn. (2.114)), the full propagator for $t_2 > t_1$ is given by:

$$\begin{aligned} &\mathcal{G}(x_1,x_2) \overset{\tau\to+\infty(1-i\epsilon)}{\sim} \\ &\qquad \langle 0|\,U(\tau,t_2)\Phi_I(t_2,\vec{x}_2)U(t_2,t_1)\Phi_I(t_1,\vec{x}_1)U(t_1,-\tau)\,|0\rangle\,\left|\langle 0|\Omega\rangle\right|^2 e^{2iE_\Omega\tau} \qquad (2.120)\end{aligned}$$

To have an explicit expression of the normalization factor appearing on the right, we use the fact that the interacting vacuum is normalized:

$$1 = \langle\Omega|\Omega\rangle = \langle\Omega|\,U(\tau)^{-1}U(\tau)U(-\tau)^{-1}U(-\tau)\,|\Omega\rangle \overset{\tau\to+\infty(1-i\epsilon)}{\sim} \left|\langle 0|\Omega\rangle\right|^2 e^{2iE_\Omega\tau}\,\langle 0|\,U(\tau;-\tau)\,|0\rangle \qquad (2.121)$$

Therefore we have obtained that (with the $-i\epsilon$ shift understood):

$$\langle\Omega|\,\Phi(t_2,\vec{x}_2)\Phi(t_1,\vec{x}_1)\,|\Omega\rangle = \frac{\langle 0|\,U(\infty,t_2)\Phi_I(t_2,\vec{x}_2)U(t_2,t_1)\Phi_I(t_1,\vec{x}_1)U(t_1,-\infty)\,|0\rangle}{\langle 0|\,U(\infty,-\infty)\,|0\rangle}\,; \qquad (2.122)$$

the term with $t_1 > t_2$ is treated exactly in the same way, with the role of t_1 and t_2 interchanged. Let us use the expression of the interaction picture evolution operator U in terms of Dyson series, see eqn. (2.100) . One has:

$$\begin{aligned} &U(\infty,t_2)\Phi_I(t_2,\vec{x}_2)U(t_2,t_1)\Phi_I(t_1,\vec{x}_1)U(t_1,-\infty) = \\ &\qquad \left(T\,e^{-i\int_{t_2}^{\infty}H_I(t)dt}\right)\Phi_I(t_2,\vec{x}_2)\left(T\,e^{-i\int_{t_2}^{t_1}H_I(t)dt}\right)\Phi_I(t_1,\vec{x}_1)\left(T\,e^{-i\int_{-\infty}^{t_1}H_I(t)dt}\right) \qquad (2.123)\end{aligned}$$

The $-i\epsilon$ prescription introduced above for the limit of large τ can be thought as a choice of contour in the complex t-plane, which is fine as long as no poles or cuts are crossed during the process, a property that we will be able to check in the explicit computation.

Crucially, as $-\infty < t_1 < t_2 < \infty$, everything is time-ordered correctly inside the expectation value (2.123). Therefore one can consider that the time-ordering symbol acts on the whole expression and reorder everything at will inside. Using this logic, one finds that

$$\langle\Omega|\,\mathsf{T}\,\Phi(x_1)\Phi(x_2)\,|\Omega\rangle = \frac{\langle 0|\,\mathsf{T}\,\Phi_I(x_1)\Phi_I(x_2)e^{-i\int_{-\infty}^{\infty}H_I(t)dt}\,|0\rangle}{\langle 0|\,\mathsf{T}\,e^{-i\int_{-\infty}^{\infty}dtH_I(t)dt}\,|0\rangle} \tag{2.124}$$

This reasoning can be easily extended to an arbitrary number of fields. Finally, to write this expression in a more covariant way, one can use the fact that, by locality, the interaction Hamiltonian is necessarily a space integral of a Hamiltonian density:

$$H_I(t) = \int d^3x\,\mathcal{H}_I(t,\vec{x}). \tag{2.125}$$

We reach then the very important result:

$$\langle\Omega|\,\mathsf{T}\,\Phi(x_1)\cdots\Phi(x_n)\,|\Omega\rangle = \frac{\langle 0|\,\mathsf{T}\,\Phi_I(x_1)\cdots\Phi_I(x_n)e^{-i\int d^4x\,\mathcal{H}_I(x)}\,|0\rangle}{\langle 0|\,\mathsf{T}\,e^{-i\int d^4x\,\mathcal{H}_I(x)}\,|0\rangle} \tag{2.126}$$

This is the starting point of perturbation theory. Everything on the right-hand side is expressed in terms of the interacting field Φ_I, which behaves as a free field, and of the free vacuum $|0\rangle$. A similar formula is obtained in other QFTs, since we did not use any specificities of the Klein–Gordon theory.

Whenever $\mathcal{H}_I$ does not contain derivative interactions, and only in this case, it is just the opposite of the interaction Lagrangian density expressed in terms of the interaction picture fields, *i.e.* $\mathcal{H}_I = -\mathcal{L}(\Phi_I)$.

Perturbative expansion. At this stage we did not make any assumption about the interactions, except that the overlap $\langle 0|\Omega\rangle$ between the free and interaction vacua does not vanish. Even if the correlator (2.126) involves only free fields, it is impossible to evaluate exactly in general due to the presence of the exponentiated interaction Hamiltonian. To make progress, we will assume that the interaction are weak in a meaningful way – as we will see, it is not a simple question! – and develop in terms of an expansion parameter.

Let us consider as an example the Φ^4 theory, whose Lagrangian is given by eqn. (2.87). In this case,

$$\int dt\,H_I(t) = -\int d^4x\,\mathcal{L}(\Phi_I,\partial_\mu\Phi_I) = \int d^4x\,\frac{\lambda_0}{4!}\,\Phi_I^4(x)\,, \tag{2.127}$$

such that we can expand inside the time-ordered correlator (2.126):

$$e^{-i\int d^4x\,\mathcal{H}_I(x)} = e^{-i\int d^4x\,\frac{\lambda_0}{4!}\,\Phi_I^4(x)} = 1 - i\frac{\lambda_0}{4!}\int d^4x\,\Phi_I^4(x) + \mathcal{O}(\lambda_0^2)\,. \tag{2.128}$$

Thus computing the order λ_0^k contribution to the n-point function (2.126) boils down to a computing a free-field $n+4k$-point function and integrating the result over $4k$ space-time variables.

2.2.2 Wick theorem and Feynman diagrams

To evaluate efficiently the correlators between the interaction picture fields, let us first split their expansion (2.111) into the sum of a positive-frequency and negative-frequency contributions:

$$\Phi_I(t,\vec{x}) = \underbrace{\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}} a_{\vec{p},I} e^{-ip\cdot x}}_{\Phi_I^+(t,\vec{x})} + \underbrace{\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}} a^\dagger_{\vec{p},I} e^{ip\cdot x}}_{\Phi_I^-(t,\vec{x})} \tag{2.129}$$

Importantly, as Φ_I^+ contains only free annihilation operators, it annihilates the free vacuum. Similarly Φ_I^- annihlates $\langle 0|$:

$$\Phi_I^+(x)\,|0\rangle = 0 \quad , \qquad \langle 0|\,\Phi_I^-(0) = 0\,. \tag{2.130}$$

As you have probably done in your QM course in order to study pertubations of the harmonic oscillators, it is convenient for the computations to move all the annihilation operators to the right.

Definition 4 (normal order). *Let us consider an operator $\mathcal{O}$ expressed as a string of free annihilation and creation operators. The corresponding normal-ordered operator, denoted by $:\!\mathcal{O}\!:$, has all the annihilation operators moved to the right and all the creation operators moved to the left.*

From now on we drop the index I of the interacting fields, since they behave like free fields. Let us consider a product of two field operators evaluated at different points and decompose them according to eqn. (2.129). One has

$$\begin{aligned}
\Phi(x)\Phi(y) &= \big(\Phi^+(x)+\Phi^-(x)\big)\big(\Phi^+(y)+\Phi^-(y)\big) \\
&= \Phi^+(x)\Phi^-(y) + \Phi^+(x)\Phi^+(y) + \Phi^-(x)\Phi^+(y) + \Phi^-(x)\Phi^-(y) \\
&= [\,\Phi^+(x),\,\Phi^-(y)\,] + \Phi^+(x)\Phi^+(y) + \Phi^-(y)\Phi^+(x) + \Phi^+(x)\Phi^+(y) + \Phi^-(x)\Phi^+(y) \\
&= [\,\Phi^+(x),\,\Phi^-(y)\,] + :\!\Phi(x)\Phi(y)\!:\,,
\end{aligned} \tag{2.131}$$

where we have normal-ordered the first term in the expansion using the commutator. Going through the similar steps for $\Phi(y)\Phi(x)$ one can decompose the time-ordered product as:

$$T\,\Phi(x)\Phi(y) = :\!\Phi(x)\Phi(y)\!: + \Theta(x_0-y_0)\,[\,\Phi^+(x),\,\Phi^-(y)\,] + \Theta(y_0-x_0)\,[\,\Phi^+(y),\,\Phi^-(x)\,] \tag{2.132}$$

The last two terms are just proportionnal to the identity, as the commutators $\left[\,a_{\vec{p}},\,a^\dagger_{\vec{q}}\,\right] = (2\pi)^3\delta^{(3)}\big(\vec{p}-\vec{q}\big)$, and their sum gives the Feynman propagator $D(x-y)$.

Let us introduce the convenient notation known as *Wick contraction*:

$$\mathsf{T}\,\Phi(x)\Phi(y) = :\Phi(x)\Phi(y): + \overbrace{\Phi(x)\Phi(y)} \qquad , \qquad \overbrace{\Phi(x)\Phi(y)} \stackrel{\text{def.}}{=} D(x-y)\,\mathbb{I} \tag{2.133}$$

Evaluating this expression on the vacuum,

$$\begin{aligned}\langle 0|\,\mathsf{T}\,\Phi(x)\Phi(y)\,|0\rangle = \overbrace{\langle 0|\,:\Phi(x)\Phi(y):\,|0\rangle}^{=0} \\ + \langle 0|\,\Big(\Theta(x_0-y_0)\,[\,\Phi^+(x),\,\Phi^-(y)\,] + \Theta(y_0-x_0)\,[\,\Phi^+(y),\,\Phi^-(x)\,]\,\Big)\,|0\rangle \\ = D(x-y)\end{aligned} \tag{2.134}$$

i.e. the Feynman propagator (1.339).

One can do the same for longer chains of operators. For instance, for a chain of 3 scalars operators, using the shorthand notation $\Phi_i = \Phi(x_i)$, one finds by direct computation the following identity:[11]

$$\mathsf{T}\,\Phi_1\Phi_2\Phi_3 = :\Phi_1\Phi_2\Phi_3: + \overbrace{\Phi_1\Phi_2}\Phi_3 + \overbrace{\Phi_1\Phi_3}\Phi_2 + \overbrace{\Phi_2\Phi_3}\Phi_1\,. \tag{2.135}$$

Evaluating this expression on the vacuum gives:

$$\langle 0|\,\mathsf{T}\,\Phi_1\Phi_2\Phi_3\,|0\rangle = \overbrace{\Phi_1\Phi_3}\,\langle 0|\,\Phi_2\,|0\rangle + \text{perm.} = 0\,, \tag{2.136}$$

shifting the field Φ by a constant, if needed, in order to have a non-zero VEV. We will assume in the following that it is always the case; this choice makes all the computations simpler.

To study the general general case of a chain of n operators, we will use the following useful theorem.

Theorem 1 (Wick theorem for real scalars). *Let Φ_i be the free real scalar field $\Phi(x)$ evaluated at the space-time coordinates $x = x_i$. The time-ordered product of a chain of field operators statisfies:*

$$\mathsf{T}\,\Phi_1\cdots\Phi_n \;=\; :\Phi_1\cdots\Phi_n: \;+\; \text{all possible Wick contractions}\; :\,. \tag{2.137}$$

[11] if say $t_1 > t_2 > t_3$, for reaching normal order of $\Phi_1\Phi_2\Phi_3$ we will have to consider the commutator $\big[\,\Phi_1^+,\,\Phi_2\Phi_3\,\big] = \Phi_3\,\big[\,\Phi_1^+,\,\Phi_2^-\,\big] + \Phi_2\,\big[\,\Phi_1^+,\,\Phi_3^-\,\big]$.

For instance, the theorem states that

$$\begin{aligned}\mathsf{T}\,\Phi_1\Phi_2\Phi_3\Phi_4 &= :\Phi_1\Phi_2\Phi_3\Phi_4: + \overbrace{\Phi_1\Phi_2}\,:\Phi_3\Phi_4: + \overbrace{\Phi_1\Phi_3}\,:\Phi_2\Phi_4: + \overbrace{\Phi_1\Phi_4}\,:\Phi_2\Phi_3: \\ &\quad + \overbrace{\Phi_2\Phi_3}\,:\Phi_1\Phi_4: + \overbrace{\Phi_2\Phi_4}\,:\Phi_1\Phi_3: + \overbrace{\Phi_3\Phi_4}\,:\Phi_1\Phi_2: \\ &\quad + \overbrace{\Phi_1\Phi_2}\,\overbrace{\Phi_3\Phi_4} + \overbrace{\Phi_1\Phi_3}\,\overbrace{\Phi_2\Phi_4} + \overbrace{\Phi_1\Phi_4}\,\overbrace{\Phi_2\Phi_3} \\ &= :\Phi_1\Phi_2\Phi_3\Phi_4: + \overbrace{\Phi_1\Phi_2}:\Phi_3\Phi_4: + \overbrace{\Phi_1\Phi_3}:\Phi_2\Phi_4: + \overbrace{\Phi_1\Phi_4}:\Phi_2\Phi_3: \\ &\quad + \overbrace{\Phi_2\Phi_3}:\Phi_1\Phi_4: + \overbrace{\Phi_2\Phi_4}:\Phi_1\Phi_3: + \overbrace{\Phi_3\Phi_4}:\Phi_1\Phi_2: \\ &\quad + \overbrace{\Phi_1\Phi_2}\,\overbrace{\Phi_3\Phi_4} + \overbrace{\Phi_1\Phi_3}\,\overbrace{\Phi_2\Phi_4} + \overbrace{\Phi_1\Phi_4}\,\overbrace{\Phi_2\Phi_3}\end{aligned} \tag{2.138}$$

Thus according to this theorem we obtain in this case:

- the fully normal-ordered term $:\Phi_1\Phi_2\Phi_3\Phi_4:$, with zero expectation value in the vacuum.
- terms of the form $\overbrace{\Phi_i\Phi_j}:\Phi_k\Phi_\ell:$, with zero expectation value in the vacuum.
- remaining terms in $\overbrace{\Phi_i\Phi_j}\overbrace{\Phi_k\Phi_\ell}$ proportionnal to the identity, thus contributing to the vacuum expectation value.

Let us give a sketch of a proof by induction. Assume that the theorem is true of to n fields.

Let us consider then the $(n+1)$ fields $\Phi_1, \ldots, \Phi_{n+1}$ and consider without loss of generality that $t_1 \geqslant t_2 \geqslant \cdots \geqslant t_{n+1}$. We have then:

$$\mathsf{T}\,\Phi_1 \cdots \Phi_{n+1} = \Phi_1 \cdots \Phi_{n+1} = \Phi_1 :\Phi_2 \cdots \Phi_{n+1}: \; + \text{ their Wick contractions } :\;:$$

If we split $\Phi_1 = \Phi_1^+ + \Phi_1^-$, the second term is already ordered correctly:

$$\Phi_1^- :\Phi_2 \cdots \Phi_{n+1}: \; + \text{ Wick contractions } :\;: = :\Phi_1^-\big(\Phi_2 \cdots \Phi_{n+1} + \text{ Wick contractions}\big): \tag{2.139}$$

For the first term we want to reach a normal-ordered expression by commuting Φ_1^+ with the rest. The first of these contributions is

$$\Phi_1^+ :\Phi_2 \cdots \Phi_n: = :\Phi_2 \cdots \Phi_n:\Phi_1^+ + [\,\Phi_1^+,\, :\Phi_2 \cdots \Phi_n:\,] \tag{2.140}$$

The first term of (2.140), combined with the second term of (2.139), gives $:\Phi_1\Phi_2 \cdots \Phi_{n+1}:$, while the commutator in the second term gives the sum of commutators with the individual fields:

$$[\,\Phi_1^+,\, :\Phi_2 \cdots \Phi_n:\,] = :[\,\Phi_1^+,\,\Phi_2\,]\,\Phi_3 \cdots \Phi_n: + :\Phi_2\,[\,\Phi_1^+,\,\Phi_3\,]\,\Phi_4 \cdots \Phi_n + \cdots: \tag{2.141}$$

The other contributions are treated in the same way.

The fully explicit proof of the Wick theorem is rather tedious and not really interesting, but hopefully you got the idea of how it works.

Corollary 1 (Correlation functions of free real scalars). *Let* Φ_i *be the free real scalar field* $\Phi(x)$ *evaluated at the space-time coordinates* $x = x_i$*.*

- *For* $N = 2n + 1$ *odd,*

$$\langle 0|\, T\, \Phi_1 \cdots \Phi_n\, |0\rangle = 0 \tag{2.142}$$

- *For* $N = 2n$ *even,*

$$\langle 0|\, T\, \Phi_1 \cdots \Phi_{2n}\, |0\rangle = \overbrace{\Phi_1 \Phi_2}\cdots \overbrace{\Phi_{2n-1}\Phi_{2n}} \;+\; \text{inequivalent pairings.} \tag{2.143}$$

Equivalent pairings are identified with the equivalence relations:

(i) by permutting the two fields inside a pair, as $\overbrace{\Phi_1\Phi_2} \sim \overbrace{\Phi_2\Phi_1}$

(i) by permutting pairs, as $\overbrace{\Phi_1\Phi_2}\overbrace{\Phi_3\Phi_4} \sim \overbrace{\Phi_3\Phi_4}\overbrace{\Phi_2\Phi_1}$

Equivalently, the result can be written as a sum over permutations, divided by the appropriate combinatorial factor in order to avoid overcounting:

$$\langle 0|\, T\, \Phi_1 \cdots \Phi_{2n}\, |0\rangle = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \overbrace{\Phi_{\sigma(1)} \Phi_{\sigma(2)}} \cdots \overbrace{\Phi_{\sigma(2n-1)} \Phi_{\sigma(2n)}}\,. \tag{2.144}$$

The generalization to the free complex scalar field QFT is rather straightforward. First, as explained in the comments around (1.348), only "neutral" correlators, *i.e.* with as much Φ's as $\Phi^\dagger$'s, of the type given in eqn. (2.73) can give a non-zero answer. This is because the only non-vanishing Wick contraction is between Φ and $\Phi^\dagger$. It leads to the following version Wick theorem.

Corollary 2 (Correlation functions of free complex scalars). *Let* Φ_i *the free complex scalar field operator* $\Phi(x)$ *evaluated at the space-time coordinates* $x = x_i$ *and* $\Phi_j^\dagger$ *its Hermitian conjugate evaluated at the space-time coordinates* $y = y_j$*. The only non-zero time-ordered correlators in the vacuum are the* $2n$*-point functions given by:*

$$\langle 0|\, T\, \Phi_1 \cdots \Phi_n \Phi_1^\dagger \cdots \Phi_n^\dagger\, |0\rangle = \overbrace{\Phi_1 \Phi_1^\dagger} \cdots \overbrace{\Phi_n \Phi_n^\dagger} \;+\; \text{inequivalent } \overbrace{\Phi \Phi^\dagger} \text{ pairings.} \tag{2.145}$$

In this case, it can be more easily written in terms of permutations as

$$\langle 0|\, T\, \Phi_1 \cdots \Phi_n \Phi_1^\dagger \cdots \Phi_n^\dagger\, |0\rangle = \sum_{\sigma \in S_n} \overbrace{\Phi_1 \Phi^\dagger_{\sigma(1)}} \cdots \overbrace{\Phi_n \Phi^\dagger_{\sigma(n)}}\,, \tag{2.146}$$

with no overcounting to be compensated for.

Feynman diagrams through examples. A Feynman diagram is a very convenient graphical representation of a given pairing of field operators contributing to a given correlation function.

It also relates the field point of view with the more intuitive particle intuition. As Julian Schwinger said, the Feynman diagrams *"like the silicon chip, would bring computation to the masses"*.

It is easier to present the method by looking at a few examples. Let us start with the quantum field theory of a free real scalar. The free 4-point correlation function is given by the sum of three terms:

$$\langle 0|\,\Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4)\,|0\rangle = \overbrace{\Phi_1\Phi_2}\overbrace{\Phi_3\Phi_4} + \overbrace{\Phi_1\Phi_3}\overbrace{\Phi_3\Phi_4} + \overbrace{\Phi_1\Phi_4}\overbrace{\Phi_2\Phi_3} \tag{2.147}$$

Figure 2.6: *Feynman diagram expansion of* $\langle\Phi_1\Phi_2\Phi_3\Phi_4\rangle$.

This sum is represented by the diagrams on fig 2.6. Obviously, each line correspond to a Feynman propagator between the corresponding space-time points represented by dots.

For computing scattering amplitudes, we will be more interested in the expression of the correlators momentum space. The starting point is

$$\mathcal{G}(p_1,\ldots,p_4) = \int dx_1^4\cdots dx_4^4\, e^{i(p_1\cdot x_1+\cdots+p_4\cdot x_4)}\,\langle 0|\,\Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4)\,|0\rangle$$

We recall that the Feynman propagator reads:

$$D(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{ie^{-ip\cdot(x-y)}}{p^2-m_0^2+i\epsilon}\,. \tag{2.148}$$

Therefore, the first contribution to (2.147) is Fourier-transformed to:

$$\begin{aligned}\overbrace{\Phi_1\Phi_2}\overbrace{\Phi_3\Phi_4} \;\longrightarrow\; &\int dx_1^4\cdots dx_4^4\, e^{i(p_1\cdot x_1+\cdots+p_4\cdot x_4)}\int\frac{d^4p}{(2\pi)^4}\frac{ie^{-ip\cdot(x_1-x_2)}}{p^2-m_0^2+i\epsilon}\int\frac{d^4q}{(2\pi)^4}\frac{ie^{-iq\cdot(x_3-x_4)}}{q^2-m_0^2+i\epsilon}\\ &= (2\pi)^4\delta^{(4)}(p_1+p_2)\frac{i}{p_1^2-m_0^2+i\epsilon}\;\times\;(2\pi)^4\delta^{(4)}(p_3+p_4)\frac{i}{p_3^2-m_0^2+i\epsilon}\end{aligned} \tag{2.149}$$

We get a product of two terms, that are easily intepreted by looking at the left panel of fig. 2.7. The first one corresponds to $\overbrace{\Phi_1\Phi_2}$, *i.e.* interpreted as the propagation of a particle of momentum p_1. The delta-function $\delta^{(4)}(p_1+p_2)$ just expresses momentum conservation: as this particle does not interact with other ones, its momentum does not change, so the

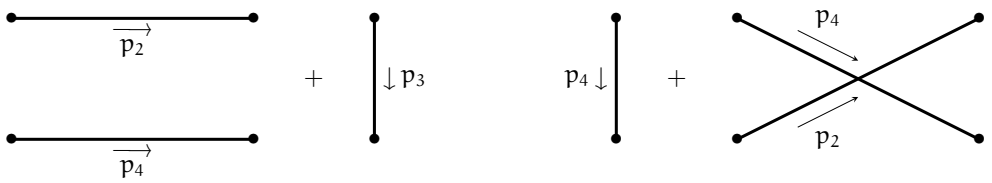


Figure 2.7: *Feynman diagram expansion of* $\langle \Phi_1 \Phi_2 \Phi_3 \Phi_4 \rangle$ *in momentum space.*

momentum $-p_1$ that goes in is the same as the momentum p_2 that goes out. The other sub-diagram is associated with $\Phi_3 \overbrace{\Phi_4}$ an is interpeted in the same way. As far as the Feynman diagrams are concerned, we will label the lines not by the position of the endpoints but by the momentum flowing from one end to the other one. Each line of the diagram is weighted by a free scalar Feynman propagator in momentum space,

$$\widehat{D}(p) = \frac{i}{p^2 - m_0^2 + i\epsilon}\,. \tag{2.150}$$

Next, let us compute a more interesting object, that will be – at last! – our first computation in an interacting QFT: the first correction to the propagator due to a ϕ^4 interaction. According to the general formula (2.126) we have to start from the expression:

$$\mathcal{G}(x_1, x_2) = \frac{\langle 0|\, T\Phi_1\Phi_2 \exp^{-\frac{i\lambda_0}{4!}\int d^4x\, \Phi_x^4}\, |0\rangle}{\langle 0|\exp^{-\frac{i\lambda_0}{4!}\int d^4x\, \Phi_x^4}\, |0\rangle} \tag{2.151}$$

and expand up to order λ_0.

We have to clarify how to use the Wick theorem for operators at the same space-time points, here in the Φ^4 interaction term. The prescription is to separate first the four operators from each other and contract:

$$\Phi_x^4 \to \Phi_{x+\epsilon_1}\Phi_{x+\epsilon_2}\Phi_{x+\epsilon_3}\Phi_{x+\epsilon_4} \xrightarrow{\text{Wick contractions}} \Phi_{x+\epsilon_1}\overbrace{\Phi_{x+\epsilon_2}}\Phi_{x+\epsilon_3}\overbrace{\Phi_{x+\epsilon_4}} + (2 \leftrightarrow 3) + (2 \leftrightarrow 4)\,. \tag{2.152}$$

Next, we take the limit when the point are coincident, to get

$$\Phi_{x+\epsilon_1}\overbrace{\Phi_{x+\epsilon_2}}\Phi_{x+\epsilon_3}\overbrace{\Phi_{x+\epsilon_4}} + (2 \leftrightarrow 3) + (2 \leftrightarrow 4) \xrightarrow{|\epsilon_i| \to 0} 3\Phi_x\overbrace{\Phi_x}\Phi_x\overbrace{\Phi_x} = 3D(0)^2\,. \tag{2.153}$$

As you may have noticed, this expression is severely divergent!

Expanding first the numerator of the expression (2.151) up to order λ_0, we find the following contributions:

$$\begin{aligned}\langle 0|\, T\,\Phi(x_1)\Phi(x_2) e^{-\frac{i\lambda_0}{4!}\int d^4x\, \Phi_x}\, |0\rangle = {} & \Phi_1\overbrace{\Phi_2} \\ & - \frac{i\lambda_0}{4!} \times 3 \times \Phi_1\overbrace{\Phi_2}\int d^4x\, \Phi_x\overbrace{\Phi_x}\Phi_x\overbrace{\Phi_x} \\ & - \frac{i\lambda_0}{4!} \times 4 \times 3 \int d^4x\, \Phi_1\overbrace{\Phi_x}\Phi_2\overbrace{\Phi_x}\Phi_x\overbrace{\Phi_x} + \mathcal{O}(\lambda_0^2)\end{aligned} \tag{2.154}$$

The combinatorial factor 4×3 in the last term can be understood very simply as follows: we have 4 choices to connect Φ_1 to one of the Φ_x's, then 3 choices to connect Φ_2 to the remaining Φ_x's.
From the denominator of (2.151), we will get at this order:

$$\frac{1}{\langle 0|\, T\, e^{\frac{-i\lambda_0}{4!}\int d^4x\, \Phi_x^4}\, |0\rangle} = 1 + \frac{i\lambda_0}{4!} \times 3 \times \int d^4x\, \overbrace{\Phi_x\Phi_x}\overbrace{\Phi_x\Phi_x} + \mathcal{O}(\lambda_0^2)\,, \tag{2.155}$$

We see in passing that, since the integrand does not depend on x at all, the infinite volume of space-time $\mathrm{Vol}(\mathbb{R}^{1,3})$ factorizes formally; we will come back to this later. Overall, the contribution from the numerator and denominator combine into:

$$\begin{aligned}\mathcal{G}(x_1,x_2) = \overbrace{\Phi_1\Phi_2}\Big(1 + \cancel{\frac{i\lambda_0}{8} \times \int d^4x\, \overbrace{\Phi_x\Phi_x}\overbrace{\Phi_x\Phi_x}\Big)} \\ \cancel{- \frac{i\lambda_0}{8}\overbrace{\Phi_1\Phi_2}\int d^4x\, \overbrace{\Phi_x\Phi_x}\overbrace{\Phi_x\Phi_x}} \\ - \frac{i\lambda_0}{2}\int d^4x\, \overbrace{\Phi_1\Phi_x}\overbrace{\Phi_2\Phi_x}\overbrace{\Phi_x\Phi_x} + \mathcal{O}(\lambda_0^2)\end{aligned} \tag{2.156}$$

We see that the order λ_0 term proportional to the propagator $\overbrace{\Phi_1\Phi_2}$ from the numerator cancels against the only contribution from the denominator of the same order; we will provide below a systematic understanding of this feature. More explictly, the full result reads:

$$\mathcal{G}(x_1,x_2) = D(x_1-x_2) - \frac{i\lambda_0}{2}\, D(0)\int d^4x\, D(x_1-x)D(x_2-x) + \mathcal{O}(\lambda_0^2)\,. \tag{2.157}$$

One can compute also the Fourier transform easily. One gets:

$$\begin{aligned}\mathcal{G}(p_1,p_2) = \int dx_1^4 dx_2^4\, e^{i(p_1\cdot x_1 + p_2\cdot x_2)}\Bigg(\int \frac{d^4p}{(2\pi)^4}\frac{ie^{-ip\cdot(x_1-x_2)}}{p^2-m_0^2+i\epsilon} \\ - \frac{i\lambda_0}{2}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m_0^2+i\epsilon}\int d^4x \int \frac{d^4p}{(2\pi)^4}\frac{ie^{-ip\cdot(x_1-x)}}{p^2-m_0^2+i\epsilon}\int \frac{d^4q}{(2\pi)^4}\frac{ie^{-ip\cdot(x_2-x)}}{q^2-m_0^2+i\epsilon}\Bigg)\end{aligned} \tag{2.158}$$

Doing the integrals over positions, one obtains first as expected – see eqn. (2.77) – the overall momentum conservation constraint, multiplied by the corrected momentum space Feynman propagator:

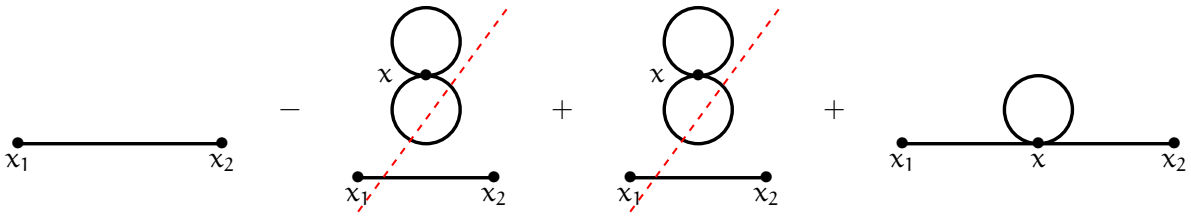
$$\begin{aligned}\mathcal{G}(p_1,p_2) = (2\pi)^4\delta^{(4)}(p_1+p_2)\,\widehat{\mathcal{G}}(p_1,p_2) = (2\pi)^4\delta^{(4)}(p_1+p_2) \\ \times \left(\frac{i}{p_1^2-m_0^2+i\epsilon} - \frac{i\lambda_0}{2}\frac{i}{p_1^2-m_0^2+i\epsilon}\left(\int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m_0^2+i\epsilon}\right)\frac{i}{p_2^2-m_0^2+i\epsilon}\right)\end{aligned} \tag{2.159}$$

In the first term, we recognize of course the Feynman propagator for a free scalar field of mass m_0; the second term is the lowest order correction due to the interactions.

As was noticed before, the integral between parentheses in the second term diverges; for $|p| \to \infty$ it exhibits a quadratic divergence, *i.e.* as $\int p dp$. It may look like a catastrophe (and in early days of QFT people thought it was) but we set this issue aside for the moment, and will come back to it in due time.

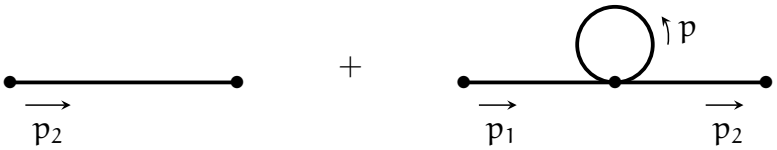
The external legs are associated with worldlines of particles satisfying the dispersion relation $p^2 = m^2$; they are said to be "on-shell". On the contrary, the loops are associated with worldlines of *virtual particles*, since generically $p^2 \neq 0$; they are said to be *off-shell*. This is allowed in quantum mechanics on sufficiently short time scales, thanks to the uncertainty on the energy.

The position space Feynman diagrams corresponding to the expression (2.156) are as follows:



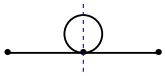
The second diagram (coming from the denominator) and the third diagram (coming from the numerator), that cancel each other are disconnected diagrams: they contain an ordinary propagator as well as one (divergent) piece that involves neither $\Phi(x_1)$ nor $\Phi(x_2)$. Sub-diagrams of this type, connected to none of the external fields are said to be *fully disconnected*. They are also known also as "vacuum bubbles", and can be interpreted as quantum fluctuations of the vacuum in the interacting theory. As we will develop below, the cancellation of vacuum bubbles between the numerator and the denominator of (2.151) is systematic.

In momentum space, the Feynman diagrams look like:



Let us make few observations about these calculations that will be useful for generalizing the rules of the game later:

- The points where the fields $\Phi(x_1)$, $\Phi(x_2)$ are attached are called *external points*.
- The "internal" points whose are integrated over coming from the expansion of the interaction Hamtilonian at called *vertices*.
  - the vertex comes with a factor $-i\lambda_0$.
  - In the momentum space, the integral $\int d^4x$ enforces momentum conservation at the vertex: $\int d^4 e^{i(p_1+\cdots p_4)\cdot x} = (2\pi)^4 \delta^{(4)}(p_1 + \cdots + p_4)$.

- Overall, one sees that the momentum space Feynman diagram exhibits manifestly momentum conservation: the momentum that "flows in" momentum that "flows out" (here $-p_1 = p_2$).
- For each of the term at order λ_0 we obtain a certain combinatorial factor since several Wick pairings give the same contribution. A convenient shortcut to get this coefficient correctly without counting all inequivalent contractions is to consider the following rule:[12] the combinatorial factor is given by the *symmetry factor*, which is the inverse of the order of the discrete group of symmetry G of the Feynman diagram, leaving the external points fixed. In the present case:
  - For the diagram  the loop has one axis of symmetry so $G = \mathbb{Z}_2$ and the symmetry factor is $1/|G| = 1/2$.
  - For the diagram  the symmetry group is $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$: one axis of symmetry for each loop (red and orange) and a third axial symmetry between the two loops (blue). The symmetry factor is then $1/|G| = 1/8$.

As a next example, we consider the similar one-loop correction to the propagator, this time in the ϕ^4 complex scalar theory. The Lagrangian density is given by:

$$\mathcal{L} \;=\; \partial_\mu \Phi \partial^\mu \Phi^\dagger - m_0^2 \Phi\Phi^\dagger \;- \frac{\lambda_0}{4}\left(\Phi\Phi^\dagger\right)^2 . \tag{2.160}$$

As you can see, the interaction term has been normalized differently compared to the case of the real scalar – see eqn. (2.87) for comparison – in order to get "natural" Feynman rules below. Regarding the computation of the correction, the main difference with the real scalar is that the only non-vanishing Wick contraction is $\overbrace{\Phi\Phi^\dagger}$, see corollary 2. For the correction to the propagator at order λ_0 one gets:

$$\begin{aligned}\mathcal{G}(x_1|x_2) = \frac{\langle 0|\,\mathsf{T}\,\Phi(x_1)\Phi^\dagger(x_2) e^{-i\frac{\lambda_0}{4}\int d^4x \left(\Phi_x\Phi_x^\dagger\right)^2}|0\rangle}{\langle 0|\,\mathsf{T}\, e^{-i\frac{\lambda_0}{4}\int d^4x \left(\Phi_x\Phi_x^\dagger\right)}|0\rangle} &= \overbrace{\Phi_1\Phi_2^\dagger} \\ &\quad - \frac{i\lambda_0}{4}\times 2\times 2\times \int d^4x\, \overbrace{\Phi_1\Phi_x^\dagger}\overbrace{\Phi_2^\dagger\Phi_x}\overbrace{\Phi_x\Phi_x^\dagger} + \mathcal{O}(\lambda_0^2)\end{aligned} \tag{2.161}$$

The combinatorial factor 2×2 is simple to understand: there are two choices to connect Φ_1 to one of the two $\Phi_x^\dagger$'s at the vertex, and two choices to connect $\Phi_2^\dagger$ to one of the two Φ_x's at the vertex. Vacuum bubbles cancel as before.

In order to represent graphically the possible connections between the charged field operators, the convention is to represent the propagators by *oriented lines* from Φ to $\Phi^\dagger$.[13].

[12] It assumes implicitly that the interaction term had been normalized accordingly, here as $\lambda_0/4!$.

[13] It can be represented as representing the flow of the $U(1)$ charge associated with the $U(1)$ global symmetry.

Likewise, we draw arrows on the legs connected to a vertex to express charge conservation at the vertex:

Figure 2.8: *Propagator and vertices for* $(\Phi\Phi^\dagger)^2$ *theory.*

The position space and momentum space Feynman diagrams corresponding to the one-loop correction to the propagator are:

One understands from these diagrams that there is no symmetry group acting on the diagram (since the loop is oriented) hence the symmetry factor is equal to one.

As a last example, that will illustrate a point that we did not yet cover, let's consider the ϕ^3 theory, *i.e.* a real scalar field with Lagrangian density:

$$\mathcal{L} = \underbrace{\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}m_0^2\Phi^2}_{\mathcal{L}_0}\ \underbrace{-\frac{\lambda_0}{3!}\Phi^3}_{\mathcal{L}_{\text{INT}}}\,. \tag{2.162}$$

with again a coefficient in front of the interaction term taylored to give simple calculation rules. As a side note, since the potential of the Φ^3 theory is not bounded from below, one may question the consistency of this quantum field theory beyond perturbation theory (which is, by definition, blind to these non-perturbative issues).

We will consider the connected four-point function, whose lowest-order contribution is at order λ_0^2 and is given by:

$$\begin{aligned}\mathcal{G}_C(x_1,x_2,x_3,x_4) &= \frac{\langle 0|\,T\Phi_1\Phi_2\Phi_3\Phi_4\exp^{-\frac{i\lambda_0}{3!}\int d^4x\,\Phi_x^3}\,|0\rangle}{\langle 0|\exp^{-\frac{i\lambda_0}{3!}\int d^4x\,\Phi_x^3}\,|0\rangle}\Big|_{\text{con.}}\\ &= \frac{1}{2!}\left(-\frac{i\lambda_0}{3!}\right)^2\times 2!\times(3\times 2)^2\int d^4x\int d^4y\,\Big(\overbrace{\Phi_1\Phi_x}\overbrace{\Phi_2\Phi_x}\overbrace{\Phi_x\Phi_y}\overbrace{\Phi_y\Phi_3}\overbrace{\Phi_y\Phi_4}\\ &\qquad + \overbrace{\Phi_1\Phi_x}\overbrace{\Phi_3\Phi_x}\overbrace{\Phi_x\Phi_y}\overbrace{\Phi_y\Phi_2}\overbrace{\Phi_y\Phi_4} + \overbrace{\Phi_1\Phi_x}\overbrace{\Phi_4\Phi_x}\overbrace{\Phi_x\Phi_y}\overbrace{\Phi_y\Phi_3}\overbrace{\Phi_y\Phi_2}\Big)\\ &\qquad + \mathcal{O}(\lambda_0^4)\,.\end{aligned} \tag{2.163}$$

The combinatorial factor is understood as follows:

- The $2!$ corresponds to the permutation of the two vertices, compensating the $1/2!$ from the expansion of the exponential; it is easy to see that this patterns persists at any order in λ_0.

- In the first term for instance, in order to connect Φ_1 and Φ_2 to the x-vertex we have 3×2 choices; same for connecting Φ_3 and Φ_4 to the y-vertex.

We will consider below that this type of 4-point correlation function is associated with a scattering process $(1,2) \to (3,4)$, in the context of the LSZ reduction. It is natural then to consider that the momenta of the particles 1 and 2 are incoming rather than outgoing, and consider the Fourier transform with reversed signs, see eqn. (2.78):

$$\begin{aligned}\mathcal{G}_c(-p_1,-p_2,p_3,p_4) &= (2\pi)^4\delta^{(4)}\big(p_1+p_2-p_3-p_4\big)\,\widehat{\mathcal{G}}(-p_1,-p_2,p_3,p_4)\\ &= \int \mathrm{d}^4x_1\cdots\mathrm{d}^4x_4\, e^{i(-p_1\cdot x_1-p_2\cdot x_2+p_3\cdot x_3+p_4\cdot x_4)}\mathcal{G}(x_1,x_2,x_3,x_4)\Big|_{\text{con.}}\,. \end{aligned} \tag{2.164}$$

With this convention for the momenta in mind, the momentum space Feynman diagrams corresponding to these three contributions to the connected 4-point function, see eqn. (2.163), are represented on fig. 2.9.

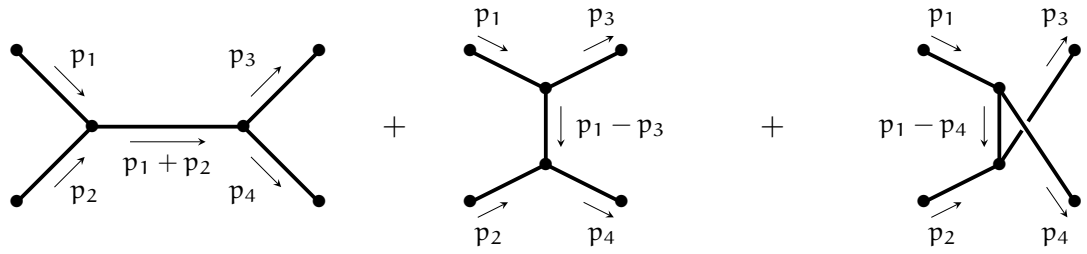


Figure 2.9: *s-,t- and u-channels contributions to* $2 \to 2$ *scattering at tree level.*

These three diagrams have a nice physical interpretation in terms of two-particle scattering processes:

- the left panel is known as an *s-channel* diagram. The corresponding physical process is the annihilation of particle (1) with momentum p_1 and particle (2) with momentum p_2, creating an intermediate unstable particle of momentum p_1+p_2, known as a *resonance*, that will eventually desintegrate into particle (3) of momentum p_3 and particle (4) of momentum p_4.

- the middle panel is known as a *t-channel* diagram. The corresponding physical process is the scattering between particle (1) with ingoing momentum p_1 and particle (2) with ingoing momentum p_2 through the exchange of a *virtual particle* of momentum p_1-p_3,

where the outgoing momenta of particles (1) and (2) after scattering are respectively p_3 and p_4 (indeed $t < 0$ for elastic scattering processes).

- the right panel is known as a *u-channel* diagram. It is almost the same as the t-channel diagram, except that the outgoing momenta of particles (1) and (2) after scattering are respectively p_4 and p_3.

The names are associated with the three Lorentz-invariant kinematic quantities known as the *Mandelstam invariants*:

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2 \tag{2.165a}$$
$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2 \tag{2.165b}$$
$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2 \tag{2.165c}$$

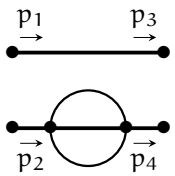
They are not independent since, calling m_i the masses of the particles,

$$s + t + u = 3p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2p_1 \cdot (\underbrace{p_2 - p_3 - p_4}_{-p_1}) = \sum_{i=1}^{4} p_i^2 = \sum_{i=1}^{4} m_i^2 \,, \tag{2.166}$$

where we have used momentum conservation. One can then express the momentum space connected 4-point function in terms of those. From eqn. (2.163) we get after Fourier transform:

$$\begin{aligned}\widehat{\mathcal{G}}(-p_1,-p_2,p_3,p_4) &= -\lambda_0^2 \frac{i}{p_1^2 - m_0^2 + i\epsilon} \frac{i}{p_2^2 - m_0^2 + i\epsilon} \left(\frac{i}{(p_1+p_2)^2 - m_0^2 + i\epsilon} \right. \\ &\left. + \frac{i}{(p_1-p_3)^2 - m_0^2 + i\epsilon} + \frac{i}{(p_1-p_4)^2 - m_0^2 + i\epsilon} \right) \frac{i}{p_3^2 - m_0^2 + i\epsilon} \frac{i}{p_4^2 - m_0^2 + i\epsilon} \\ &= -\lambda_0^2 \prod_{i=1}^{4} \frac{i}{p_i^2 - m_0^2 + i\epsilon} \left(\frac{i}{s - m_0^2 + i\epsilon} + \frac{i}{t - m_0^2 + i\epsilon} + \frac{i}{u - m_0^2 + i\epsilon} \right) \end{aligned} \tag{2.167}$$

Besides the connected diagrams of fig. 2.9, the 4-point correlation function at order λ_0^2 contains several non-connected contributions, like:



Perturbation theory for fermions. All the methods exposed so far can be extended to cover interacting QFTs with fermions. The main difference occurs with the Wick theorem,

because we have to deal with the fact that fermionic creation and annihilation operators are anti-commuting, rather than commuting, variables.

As for a scalar field, one can separate the free Dirac field Ψ and its conjugate $\bar{\Psi}$ in terms of positive- and negative-frequency parts:

$$\Psi(x) = \Psi^+(x) + \Psi^-(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} u_s(\vec{p}) b^s_{\vec{p}} e^{-ip\cdot x} + \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} v_s(\vec{p}) c^{s\,\dagger}_{\vec{p}} e^{ip\cdot x} \quad (2.168a)$$

$$\bar{\Psi}(x) = \bar{\Psi}^+(x) + \bar{\Psi}^-(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \bar{v}_s(\vec{p}) c^s_{\vec{p}} e^{-ip\cdot x} + \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \bar{u}_s(\vec{p}) b^{s\,\dagger}_{\vec{p}} e^{ip\cdot x} \quad (2.168b)$$

The free vacuum is annihilated by the positive frequency parts of the fields, as in the scalar theory:

$$\Psi^+(x)\,|0\rangle = 0\,, \quad \bar{\Psi}^+(x)\,|0\rangle = 0\,. \quad (2.169)$$

When normal-ordering fermionic operators, signs may arise because they are anticommuting variables, as in this very basic example:

$$:\! b^s_{\vec{p}} c^{s\,\dagger}_{\vec{q}} \!: \;= -\, c^{s\,\dagger}_{\vec{q}} b^s_{\vec{p}}\,. \quad (2.170)$$

As we had done for the scalar fields, let us decompose the time-ordered product of two field operators in terms of positive- and negative-frequency contributions. First for $x_0 > y_0$ we get, according to eqn. (1.352),

$$\begin{aligned} T\,\Psi^a(x)\bar{\Psi}^b(y) &= \left(\Psi^{+\,a}(x) + \Psi^{-\,a}(x)\right)\left(\bar{\Psi}^{+\,b}(y) + \bar{\Psi}^{-\,b}(y)\right) \\ &= \Psi^{+\,a}(x)\bar{\Psi}^{+\,b}(y) + \Psi^{-\,a}(x)\bar{\Psi}^{-\,b}(y) + \Psi^{-\,a}(x)\bar{\Psi}^{+\,b}(y) + \Psi^{+\,a}(x)\bar{\Psi}^{-\,b}(y) \\ &= \Psi^{+\,a}(x)\bar{\Psi}^{+\,b}(y) + \Psi^{-\,a}(x)\bar{\Psi}^{-\,b}(y) + \Psi^{-\,a}(x)\bar{\Psi}^{+\,b}(y) - \bar{\Psi}^{-\,b}(y)\Psi^{+\,a}(x) + \left\{\,\Psi^{+\,a}(x),\,\bar{\Psi}^{-\,b}(y)\,\right\} \\ &= :\!\Psi^a(x)\bar{\Psi}^b(y)\!: + \left\{\,\Psi^{+\,a}(x),\,\bar{\Psi}^{-\,b}(y)\,\right\} \end{aligned} \quad (2.171)$$

Then, for $x_0 < y_0$,

$$\begin{aligned} T\,\Psi^a(x)\bar{\Psi}^b(y) &= -\left(\bar{\Psi}^{+\,b}(y) + \bar{\Psi}^{-\,b}(y)\right)\left(\Psi^{+\,a}(x) + \Psi^{-\,a}(x)\right) \\ &= -\bar{\Psi}^{+\,b}(y)\Psi^{+\,a}(x) - \bar{\Psi}^{-\,b}(y)\Psi^{-\,a}(x) - \bar{\Psi}^{-\,b}(y)\Psi^{+\,a}(x) - \bar{\Psi}^{+\,b}(y)\Psi^{-\,a}(x) \\ &= -:\!\bar{\Psi}^b(y)\Psi^a(x)\!: - \left\{\,\bar{\Psi}^{+\,b}(y),\,\Psi^{-\,a}(x)\,\right\} = :\!\Psi^a(x)\bar{\Psi}^b(y)\!: - \left\{\,\bar{\Psi}^{+\,b}(y),\,\Psi^{-\,a}(x)\,\right\} \end{aligned} \quad (2.172)$$

So overall we have:

$$\boxed{T\,\Psi^a(x)\bar{\Psi}^b(y) \;=\; :\!\Psi^a(x)\bar{\Psi}^b(y)\!: + \overbrace{\Psi^a(x)\bar{\Psi}^b}(y)\,,} \quad (2.173)$$

the Wick contraction giving as expected the Dirac propagator (1.358):

$$\overbrace{\Psi^a(x)\bar{\Psi}^b}(y) = \begin{cases} +\left\{\,\Psi^{+\,a}(x),\,\bar{\Psi}^{-\,b}(y)\,\right\} & ,\quad x_0 > y_0 \\ -\left\{\,\bar{\Psi}^{+\,b}(y),\,\Psi^{-\,a}(x)\,\right\} & ,\quad x_0 < y_0 \end{cases} = D_F^{ab}(x-y)\,, \quad (2.174)$$

as can be seem explictely from eqn. (1.354,1.355).

The most important difference between the computations of Wick contractions involving fermions and those involving bosons is a possible minus sign coming from the anti-commutation of fermionic operators, as was already mentionned. More specifically, while contracting a fermionic operator Ψ with a non-adjacent operator $\bar{\Psi}$ one has to count how many other operators the first one has to "step over" in order to reach the second one of the pair. Le us define the $\mathbb{Z}_2$-valued symbol ϵ as:

$$\vdots \cdots \overbrace{\Psi \, \Psi_1 \cdots \Psi_n \bar{\Psi}} \cdots \vdots \; = \underbrace{(-1)^n}_{\epsilon} \; \overbrace{\Psi\bar{\Psi}} \; \vdots \cdots \Psi_1 \cdots \Psi_n \cdots \vdots \tag{2.175}$$

With this prescription in mind, the Wick theorem is essentially demonstrated similarly as in the bosonic case. It leads to the following result for vacuum expectation values.

Corollary 3 (Correlation functions of free Dirac fermions). *Let Ψ_i the free Dirac spinor field operator $\Psi(x)$ evaluated at the space-time coordinates $x = x_i$ and $\bar{\Psi}_j$ its Dirac conjugate evaluated at the space-time coordinates $y = y_j$. The only non-zero time-ordered correlators in the vacuum are the $2n$-point functions given by:*

$$\langle 0| \, T \, \Psi_1 \cdots \Psi_n \bar{\Psi}_1 \cdots \bar{\Psi}_n \, |0\rangle \; = \; \left(\prod_{i=1}^{n} \epsilon_i \right) \; \overbrace{\Psi_1\bar{\Psi}_1} \cdots \overbrace{\Psi_n\bar{\Psi}_n} \; + \; \text{inequivalent } \overbrace{\Psi\bar{\Psi}} \text{ pairings.} \tag{2.176}$$

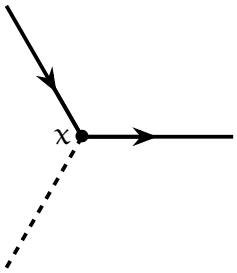
Let us consider an example using the *Yukawa theory*. This model was initially proposed by Hideki Yukawa to describe strong interactions between hadrons since, as we shall see later, the associated interaction potential is exponentially decreasing, thus the force is felt only at short distances. In the Standard Model, interactions between the Higgs field and fermions are of the Yukawa type. More modestly, we will consider here a single real scalar field ϕ of mass m_0, coupled to a Dirac fermion of mass M_0 coupled through the interaction Lagrangian density:

$$\mathcal{L}_{\text{INT}} = -g_0 \Phi(x) \bar{\Psi}(x) \Psi(x) \, . \tag{2.177}$$

This interaction is manifestly Lorentz- and parity-invariant, and the corresponding interaction Hamiltonian is Hermitian. In terms of dimensional analysis, since Φ has dimension $L^{1-d/2}$ and Ψ dimension $L^{(1-d)/2}$, the Yukawa coupling g_0 has dimension $L^{d/2-2}$ and is therefore dimension-less in four space-time dimensions.

In order to distinguish fermionic propagators from scalar propagators in the Feynman diagrams, we will use (oriented) solid lines for the former and dashed lines for the latter. The

vertex corresponding to the Yukawa interaction is then of the form:



Let us consider the one-loop correction to the propagator of the scalar field in Yukawa theory, which contains a fermionic loop. The corresponding momentum space Feynman diagram is represented on fig. 2.10.

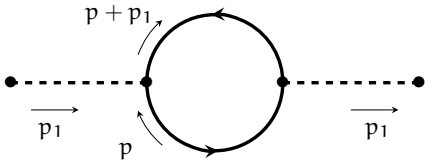


Figure 2.10: *One-loop correction to the scalar propagator in Yukawa theory.*

As one can check, both momentum conversation and charge conservation at each of the two vertices are satisfied. The scalar propagator at one loop is given by:

$$\mathcal{G}(x_1,x_2) = \overbrace{\Phi_1\Phi_2} + \frac{(-ig_0)^2}{2!}\times 2!\times \overbrace{\Phi_1\int d^4x\,\Phi_x}\overbrace{\bar\Psi_x\Psi_x}\int d^4y\,\overbrace{\Phi_y\bar\Psi_y}\overbrace{\Psi_y\Phi_2} + \mathcal{O}(g^4)\,. \quad (2.178)$$

There a very important point regarding the fermionic loops that needs to be clarified. Stripping out all the other details but restoring the spinor indices, we have to compute:

$$\begin{aligned}\overbrace{\bar\Psi^a_x\Psi^a_x}\overbrace{\bar\Psi^b_y\Psi^b_y} &= -\overbrace{\Psi^a_x\bar\Psi^b_y}\overbrace{\Psi^b_y\bar\Psi^a_x} = -D_F^{ab}(x-y)D_F^{ba}(y-x)\\ &= -\,\mathrm{Tr}\left(D_F(x-y)D_F(y-x)\right).\end{aligned} \quad (2.179)$$

Thus we have learn two important rules:

1. there is an overall minus sign in front of the loop.

2. in terms of fermionic indices, one gets the trace of the product of Dirac propagators.

This understood, one can come back to Yukawa theory and write directly the expression for the one-loop correction to the scalar propagator in momentum space. One reads from the Feynman diagram that:

$$\widehat{\mathcal{G}}(-p_1,p_2) = +g_0^2\frac{i}{p_1^2-m_0^2+i\epsilon}\int\frac{d^4p}{(2\pi)^4}\frac{\mathrm{Tr}\left(i(\not p+m)i(\not p+\not p_1+m)\right)}{\left(p^2-M_0^2+i\epsilon\right)\left((p+p_1)^2-M_0^2+i\epsilon\right)}\frac{i}{p_1^2-m_0^2+i\epsilon} \quad (2.180)$$

The numerator can be simplified a bit using $\mathrm{Tr}\,\gamma^\mu = 0$ and the identity:

$$\mathrm{Tr}\ (\gamma^\mu\gamma^\nu) = \tfrac{1}{2}\mathrm{Tr}\,(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) = 4\eta^{\mu\nu}\,, \tag{2.181}$$

such that

$$\mathrm{Tr}\left((\not{p} + m)(\not{p} + \not{p}_1 + m)\right) = 4\left(m^2 + p\cdot(p+p_1)\right)\,. \tag{2.182}$$

As in the case of a bosonic loop encountered previously, the fermion loop gives a quadratically divergent result, however with an overall relative minus sign. One may wonder whether this minus sign could give rise to some cancellations between bosonic and fermion loops; this is indeed the case in supersymmetric QFTs.

Hopefully, these examples have illustrated most important aspects of perturbation theory: position space and momentum space diagrams and correlation functions, bosonic and fermionic loops, symmetry factors. In the next subsection we will provide a broader overview of perturbation theory, building upon the intuition that we have gained.

2.2.3 General rules of perturbation theory

Evaluating a given perturbative correction to a correlation function can be done in a systematic way following few simple rules. We will give a summary of what we have understood so far through examples, and provide some generalization of those principles to arbirary interacting QFTs with spin 0 and 1/2 fields. The glaring omission in this discussion is the case of vector fields, which will be discussed in the next part of this course.

Perturbation theory in position space. In the computation of scattering amplitude through one of the LSZ formulæ, the starting point is a position space ordered n-point function is some quantum field theory expressed in the interaction picture.

For simplicity of the presentation, we will consider a real scalar field QFT but all important points in the discussion apply to other examples as well:

$$\mathcal{G}(x_1, x_2, \cdots, x_N) = \frac{\langle 0|\,\Phi(x_1)\cdots\Phi(x_N)e^{-i\int dt\, H_I}\,|0\rangle}{\langle 0|\,e^{-i\int dt\, H_I}\,|0\rangle}\,, \tag{2.183}$$

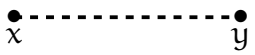
We will consider for illustrative purposes an interaction of the form:

$$H_I = \frac{\lambda_0}{n!}\Phi^n\,. \tag{2.184}$$

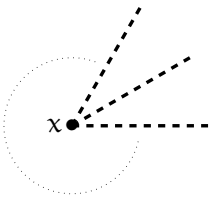
The correlation function (2.183) is then expressed as a power series expansion in the coupling constant λ_0.

At a given order in λ_0, every term in the expansion can be represented as a position space Feynman diagram made of two ingredients:

- Propagators $\overbrace{\Phi(x)\Phi(y)} = D(x-y)$ represented as



- Vertices from insertions of powers of the interaction term $-i\lambda_0/n!\int d^4x\,\Phi^n(x)$ in the expectation value represented as



Let us give some terminology:

- Points associated with the fixed coordinates $x_1, x_2, \ldots$ are called *external points*.
- Points corresponding to the positions of the vertices, that are integrated over, are called *internal points*.
- Propagators attached to an external point are called *external legs* of the diagram.
- Propagators between vertices are called *internal lines*.

The evaluation of perturbative corrections to a correlation function comes with some combinatorics considerations:

1. There are $n!$ equivalent ways to attach n distinct points to the Φ^n vertex (2.184) through propagators. This cancels exactly the $1/n!$ in the normalization of the interaction.
2. At order ℓ in perturbation theory one gets contributions with k vertices from the expansion of the integral:

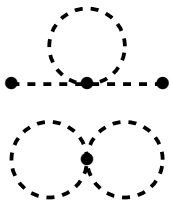
$$\frac{(-i\lambda_0/n!)^\ell}{\ell!}\int dy_1\,\Phi(y_1)^n\cdots\int dy_\ell\,\Phi(y_\ell)^n\,.$$

   All the vertices being identical, permuting them will give exactly the same contribution to the correlation function. This redundancy by an $\ell!$ factor exactly cancels the $1/\ell!$ from the exponential.
3. The counting of equivalent contributions is not correct whenever a diagram presents symmetry group G, *i.e.* symmetries of the Feynman diagram that keep the external points invariant. In those cases the correction should by multiplied by the *symmetry factor* $1/|G|$

From these considerations, the pre-factor of a given diagram with ℓ vertices is $(-i\lambda_0)^\ell/|G|$.

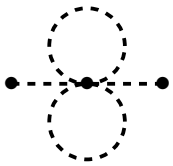
Finally, Feynman diagrams diagrams can be classified in two categories:

- Disconnected diagrams made of separate pieces. The corresponding contributions to the correlator factorizes into a product.



  Some pieces of the diagram can be *fully disconnected* if they are not connected to any external point.

- Fully-connected diagrams are made of one piece and connected to all external points. The corresponding contributions to the correlator do not factorize.



The fully connected diagrams are more fundamental, and furthermore these are the only ones that appear in the LSZ reduction formula derived in subsection 2.1.2. They will be our main focus from now.

Perturbation theory in momentum space. Formulating perturbation theory in momentum space is simpler, since the propagators are obtained as a Fourier transform, and is more useful since the LSZ formulæ for scattering amplitudes are formulated in position space.

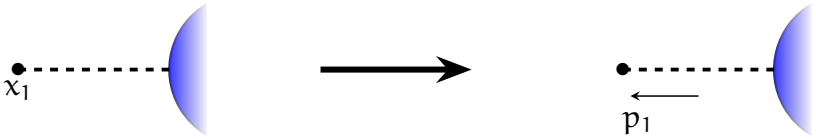
We recall that one can factorize out of the Fourier transform of an n-point correlation function the factor enforcing momentum conservation:

$$\int d^4x_1 \cdots \int d^4x_n \, e^{i(p_1\cdot x_1+\cdots p_n\cdot x_n)} \, \mathcal{G}(x_1,\ldots,x_n) = (2\pi)^4\delta^{(4)}\big(p_1+\cdots+p_N\big)\,\widehat{\mathcal{G}}(p_1,\ldots,p_n)\,. \tag{2.185}$$

The integral over each of those external momentum variables will act on a propagator associated to an external leg:

$$\int d^4x_1 \, e^{ip_1\cdot x_1} \, \cdots \int \frac{d^4p}{(2\pi)^4}\frac{ie^{-ip\cdot(x_1-x)}}{p^2-m_0^2+i\epsilon}\cdots \;=\; \cdots \frac{ie^{ip_1\cdot x}}{p_1^2-m_0^2+i\epsilon}\cdots \tag{2.186}$$

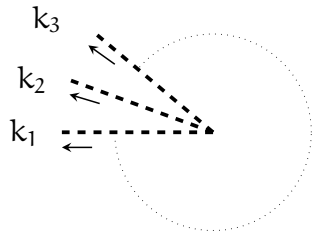
thereby assigning to it the external momentum p_1, which is *on-shell*. Graphically external legs are then labelled by external momenta rather than labelling external points with positions:



Notice that the sign convention for the Fourier transform, $\int dx\, exp(+ipx)f(x)$, is such that the momentum direction is that of an outgoing particle.

After the integration over all the external position variables $x_1, x_2, \ldots$ is done as in eqn. (2.185), the remaining position variables are those corresponding to the position of the vertices of the theory. Their integration enforce momentum conservation at each vertex:

$$\cdots \int d^4x\, e^{-i(k_1+\cdots+k_n)\cdot x} \cdots = \cdots (2\pi)^4 \delta^{(4)}(k_1 + \cdots + k_n) \cdots$$



(2.187)

Let us consider the contribution of a given connected diagram, with E external legs, V vertices and I internal lines. The Fourier transform of the amplitude looks like:

$$\int d^4x_1 \cdots \int d^4x_E\, e^{i(p_1\cdot x_1+\cdots p_E\cdot x_E)} \mathcal{G}(x_1,\ldots,x_E) \sim (2\pi)^4 \delta^{(4)}\big(p_1 + \cdots p_E\big)$$
$$\times (-i\lambda_0)^V \left(\prod_{e=1}^{E} \frac{i}{p_e^2 - m_0^2 + i\epsilon}\right) \int \prod_{i=1}^{I} \frac{d^4k_i}{(2\pi)^4} \frac{i}{k_i^2 - m_0^2 + i\epsilon} \prod_{v=1}^{V} (2\pi)^4 \delta^{(4)}(\text{k's and p's at } v) \tag{2.188}$$

At each vertex v momentum conservation will kill one the integral over internal momenta; the number of leftover momentum integrals is called the number of loops L of the diagram. It is easy to convince ourselves that the number of loops is given by:[14]

$$L = I - V + 1\,, \tag{2.189}$$

After all the delta-functions have been integrated over, the only remaining momentum integrals are then the loop integrals. Schematically we have:

$$\widehat{\mathcal{G}}(p_1,\ldots,p_E) \sim (-i\lambda_0)^V \left(\prod_{e=1}^{E} \frac{i}{p^2 - m_0^2 + i\epsilon}\right) \int \prod_{\ell=1}^{L} \frac{d^4k_\ell}{(2\pi)^4} \prod_{j=1}^{I} \frac{i}{q_j(p_e, k_\ell)^2 - m_0^2 + i\epsilon}\,, \tag{2.190}$$

where the momenta q_j in the loop propagators are in general combinations of external momenta p_i and loop momenta k_ℓ, as in the case of the one-loop correction to the scalar propagator in Yukawa theory represented on fig. 2.10.

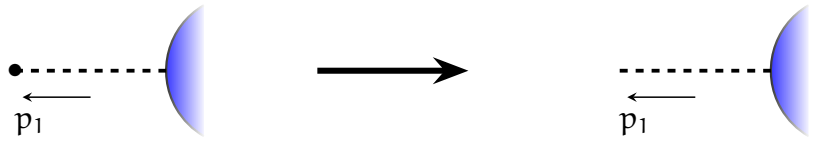
We notice in eqn. (2.190) that the first factor, the product of propagator for the external legs, is common to all perturbative contributions to a given momentum-space correlation

[14] The $+1$ can be understood as follows: for a diagram with no loops ($L = 0$), you should have one more vertex than internal lines in order to get the expected momentum conservation constraint for the external momenta: $\sum p_i = 0$.

function. This motivates the definition of the *amputated* diagram, where this factor is removed:

$$\widehat{\mathcal{G}}(p_1,\dots,p_n)\Big|_{\text{AMP.}} \overset{\text{def.}}{=} \widehat{\mathcal{G}}(p_1,\dots,p_n) \Big/ \left(\prod_{e=1}^{E} \frac{i}{p^2-m_0^2+i\epsilon}\right). \tag{2.191}$$

Graphically, we just remove the end dots from the external legs:



Loop expansion In perturbation theory, all the N-point correlation functions are expressed as power series in the coupling constant. More precisely, a diagram with V vertices has a weight proportionnal to λ_0^V, where λ_0 is the coupling constant of the interaction. On the one hand, the number of vertices in the diagram is related to the number of loops, *i.e.* of integrated internal momenta, through eqn. (2.189). With Φ^n interaction, it is also obvious that for any connected diagram:

$$nV = 2I + E\,, \tag{2.192}$$

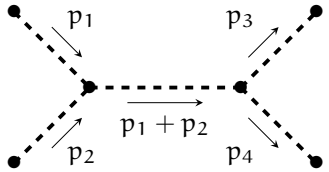
the left of side being the number of internal legs available. Combined together it allows to reorganize the perturbative expansion schematically as a loop expansion:

$$\widehat{\mathcal{G}}(p_1,\dots,p_N) \sim \sum_V \lambda_0^V(\#) \sim \lambda_0^{\frac{E-2}{n-2}} \sum_{L=0}^{\infty} \left(\lambda_0^{\frac{2}{n-2}}\right)^L (\#)\,. \tag{2.193}$$

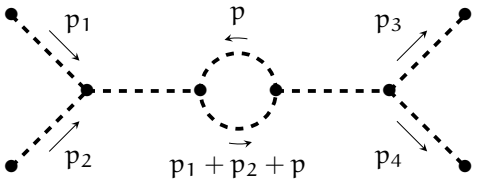
Presented in this form, it natural to use as a coupling constant, rather than λ_0,

$$\bar{g}_0 \overset{\text{def.}}{=} \lambda_0^{\frac{1}{n-2}}\,, \tag{2.194}$$

such that the expansion organises itself in $(\bar{g}_0^2)^L$. The leading contribution, at order $\lambda_0^{\frac{E-2}{n-2}} = \bar{g}^{E-2}$, does not contain loops and is called a *tree-level diagram*. For instance, the leading order contribution to the 4-point amplitude in ϕ^3 already considered, of order $\lambda_0^2 = \bar{g}_0^2$, is made of three diagrams like:



Next we have loops corrections to the same amplitude like this one-loop correction:



Restoring momentarily $\hbar$, one has first to modify the propagators as $D(x-y) \to \hbar D(x-y)$, according to the canonical commutation relations, and the vertices as $\lambda_0 \to \lambda_0/\hbar$, as we are now expanding $\exp(-\frac{i}{\hbar}\int H_{\text{INT}})$ to get the perturbative expansion. Then, from formula (2.189), an L-loop scales like $(\hbar\bar{g}^2)^L$.

Thus, the loop expansion of correlation functions in perturbation theory should be thought as an expansion in quantum corrections as, so to speak, it is an expansion in positive powers of $\hbar$.

Charged fields. An extra rule applies whenever there exists some global symmetry in the theory, for instance a $U(1)$ symmetry acting on complex valued fields as $\Phi \mapsto \exp(i\theta)\Phi$, or more generally if the fields transform in some non-trivial representation of a group G which is a symmetry of the action.

In those cases, as we have explained before, due to G-invariance of the vacuum any correlator should transform in the trivial representation of the symmetry group; for a $U(1)$ symmetry, it means that the sum of the charges of the fields should add up to zero, see eqn (1.350). The propagators of charged fields are oriented to express the flow of the charge:

$$\overbrace{\Phi_1\Phi_2^\dagger} \quad \longleftrightarrow \quad$$

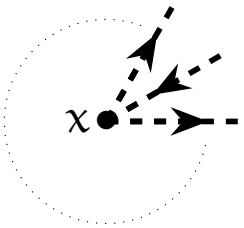
In order for the symmetry to be preserved by the interactions, $\mathcal{L}_{\text{INT}}$ should transform naturally in the trivial representation. For the case of a complex scalar field, it means that $\mathcal{L}_{\text{INT}}$ is a function of $\Phi\Phi^\dagger$ only.

A representative example is the $(\Phi\Phi^\dagger)^n$ theory, whose (properly normalized) interaction term in the Lagrangian density is given by:

$$\mathcal{L}_{\text{INT}} = -\frac{\lambda_0}{(n!)^2}\left(\Phi\Phi^\dagger\right)^n \tag{2.195}$$

As for the propagator, all the legs attached to the vertex are oriented, in order to distinguish "Φ-legs," against which a $\Phi^\dagger$ can be contracted, from "$\Phi^\dagger$-legs," against which a Φ can be contracted:

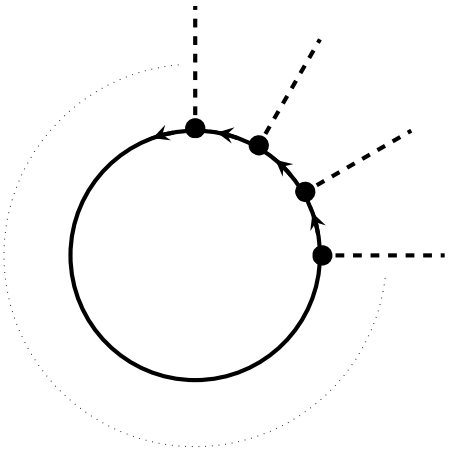
$$\int d^4x\,(\Phi_x\Phi_x^\dagger)^n \qquad \longleftrightarrow$$



The same principles apply to theories with charged fermions, *i.e.* Dirac or Weyl spinor fields; Majorana fields, being neutral, are not concerned by these rules.

Fermion loops. Another important point, that we have already touched upon in the context of Yukawa theory, is how to deal with loops of Dirac fields. The building blocs of Lorenz-invariant fermionic interactions is the Lorentz scalar bilinear $\bar{\Psi}\Psi$.[15]

Consider a given diagram with a fermion loop involving a chain of n vertices, each of them constructed with the bilinear $\bar{\Psi}\Psi$:



Leaving all the other details of the correlation function aside, the starting point is a chain of fermionic fields whose basic building block is $\bar{\Psi}\Psi$. Restoring the fermionic indices for clarity, it reads:

$$L = \overbrace{(\bar{\Psi}^{a_1}_{x_1}\Psi^{a_1}_{x_1})(\bar{\Psi}^{a_2}_{x_2}\Psi^{a_2}_{x_2})\cdots(\bar{\Psi}^{a_n}_{x_n}\Psi^{a_n}_{x_n})}^{} \tag{2.196}$$

Each Ψ is contracted with the $\bar{\Psi}$ immediatly to its right, except that, in order to close the loop, one needs to transport the first fermionic field $\bar{\Psi}^{a_1}$ to the end of the chain. For this it has to step over $(2n-1)$ fermions, giving:

$$L = -\overbrace{\Psi^{a_1}_{x_1}\bar{\Psi}^{a_2}_{x_2}}\cdots\overbrace{\Psi^{a_{n-1}}_{x_{n-1}}\bar{\Psi}^{a_n}_{x_n}}\overbrace{\Psi^{a_n}_{x_n}\bar{\Psi}^{a_1}_{x_1}} = -\mathrm{Tr}\left(D_F(x_1-x_2)\cdots D_F(x_{n-1}-x_n)D_F(x_n-x_1)\right). \tag{2.197}$$

The two general lessons to take away from this little exercise are that, whenever there is a loop of fermions in a Feynman diagram:

1. There is an overall minus sign in front of the amplitude, compared to a similar loop made of bosonic fields.

2. The evaluation of the loop gives the trace of the product of Dirac propagators, viewed as 4×4 matrices.

In a momentum space correlation function, a fermion loop will therefore give expressions of the form:

$$-\mathrm{Tr}\left(\frac{i(\not{q}_1+M_0)}{q_1^2-M_0^2+i\epsilon}\,\frac{i(\not{q}_2+M_0)}{q_2^2-M_0^2+i\epsilon}\cdots\frac{i(\not{q}_n+M_0)}{q_n^2-M_0^2+i\epsilon}\right).$$

[15] It is also possible to use the Lorentz pseudo-scalar $\bar{\Psi}\gamma^5\Psi$, like in the so-called "pseudo-Yukawa" interaction in $\Phi\bar{\Psi}\gamma^5\Psi$.

The evaluation of such traces will be possible using a machinery of gamma-matrices identities that we will present latter when needed.

Feynman rules. The general principles of perturbative QFT discussed above apply to any theory. The extra information needed in the context of a specific QFT is the expression of its propagators and vertices. This can be summarized in the form of the *Feynman rules* of the theory, which assign a certain "weight" to a given element of the diagram, propagator or vertex. They are, for examples covered so far:

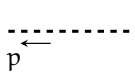
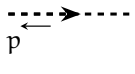
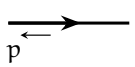
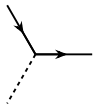
Real scalar propagator	$\overbrace{\Phi\Phi}$		$\frac{i}{p^2-m_0^2+i\epsilon}$
Φ^n vertex	$-\frac{\lambda_0}{n!}\int \Phi^n$		$-i\lambda_0$
Complex scalar propagator	$\overbrace{\Phi\Phi^\dagger}$		$\frac{i}{p^2-m_0^2+i\epsilon}$
$(\Phi\Phi^\dagger)^n$ vertex	$-\frac{\lambda_0}{(n!)^2}\int \left(\Phi\Phi^\dagger\right)^n$		$-i\lambda_0$
Dirac fermion propagator	$\overbrace{\Psi\bar{\Psi}}$		$\frac{i(\not{p}+M_0)}{p^2-M_0^2+i\epsilon}$
Yukawa interaction	$-g\int \Phi\bar{\Psi}\Psi$		$-ig$

Figure 2.11: *Feynman rules for scalar and spinor fields.*

Evaluating a perturbative correction to a correlation function in momentum space amounts to:

1. Draw the corresponding Feynman diagram, respecting the general rules (momentum and charge conservation at every vertex, etc.).
2. Assign accordingly their momenta to all propagators.
3. Multiply the various factors (propagators and vertices) as given by the Feynman rules.
4. Integrate over the loop momenta.
5. Multiply by the symmetry factor of the diagram.
6. Add an overall minus sign for every fermion loop.

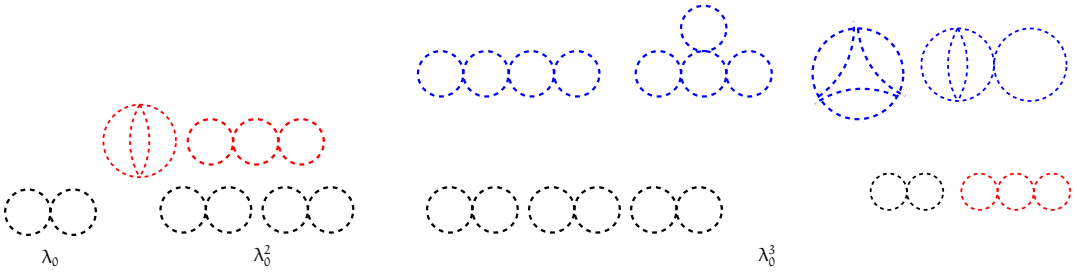
7. With fermions, relative signs between diagrams associated with the same process may occur because of the Wick contraction; we will see example of those in QED.

Vacuum bubbles As we have noticed previously, see the discussion below eqn. (2.159), the perturbative expansion generates diagrams containing a fully disconnected piece, without any connection to any external point through propagators. These sub-diagrams, or *vacuum bubbles*, are due to quantum fluctuations of the vacuum and, thanks to the normalization of correlators by the vacuum amplitude in the formula (2.126), their contribution cancels out from the final result.

Let us see in more detail how it works from a diagramatic point of view, to all orders in perturbation theory. We will still use the Φ^4 theory for definiteness, but the argument is completely general. Let us start with the denominator of eqn. (2.126), *i.e.* the vacuum amplitude,

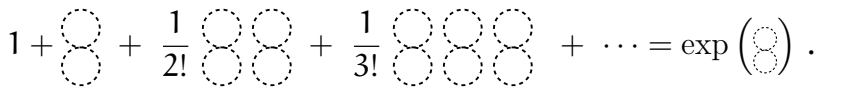
$$\mathcal{Z} \overset{\text{def.}}{=} \langle 0| \exp^{-\frac{i\lambda_0}{4!}\int d^4x\, \Phi_x^4} |0\rangle \ . \tag{2.198}$$

Schematically, the perturbative expansion in λ_0 contains the following type of vacuum bubbles order by order:



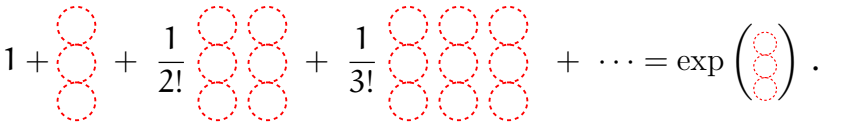
We discover some pattern: at a given order we get both new diagrams, with various topologies, and mutiple copies of diagrams that appeared first at a lower order.

Using this observation, the infinite perturbative expansion can be organized at follows. One first recognizes a sub-series made of terms with an increasing number of copies of "infinity diagrams" ◌◌. This sub-series can independently resummed as follows:



(2.199)

The combinatorial factors correspond to the permutation symmetries between the identical sub-diagrams. The same occurs for the other types of sub-diagrams with more vertices, starting at higher order:

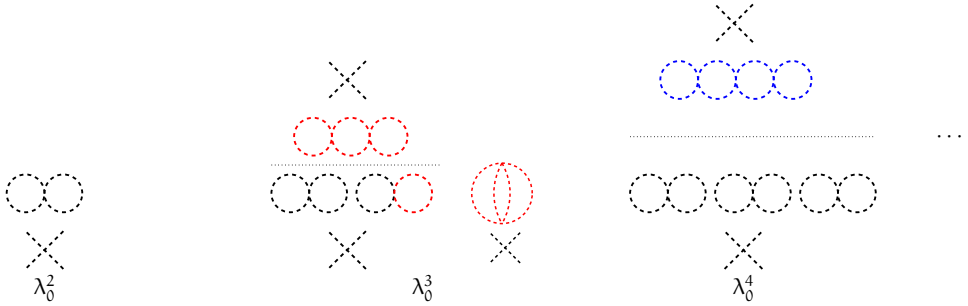


(2.200)

Putting all these factors together, the vacuum amplitude reads:[16]

$$\mathcal{Z} = \exp\left(\quad + \quad + \quad + \cdots \right) \tag{2.201}$$

Now, if we consider the expansion of the numerator in eqn. (2.126), for any partially connected diagram, higher orders in the power expansion will contain disconnected diagrams made of the same piece plus fully disconnected pieces, in an arbitrary number. For instance, for the 4-point correlation function in ϕ^4, we have, among others, the diagrams:



The contribution from all this class of diagrams can be resummed, leading to

$$\quad \exp\left(\quad + \quad + \quad + \cdots \right)$$

The same applies to every connected or partially connected diagram appearing in the perturbative expansion of the 4-point function. For instance, we have an identical vacuum bubble series "attached" to the one-loop correction to the vertex:

$$\quad \exp\left(\quad + \quad + \quad + \cdots \right)$$

In all cases, the contribution from the vacuum bubbles completely factorizes and cancels against the similar term from the denominator.

The vacuum amplitude $\mathcal{Z}$, known also as partition function, has an interesting physical interpretation. From eqn. (2.121) we deduce that

$$\mathcal{Z} = \lim_{\tau\to\infty(1-i\epsilon)} \left|\langle 0|\Omega\rangle\right|^{-2} e^{-2iE_\Omega\tau} = \exp\left(\quad + \quad + \cdots \right) \tag{2.202}$$

From which we deduce, by taking the logarithm on both sides of the equation, that the energy density of the interacting vacuum is given formally by the sum of all vacuum bubble diagrams:

$$\frac{E_\Omega}{\mathrm{Vol}(\mathbb{R}^3)} = \frac{1}{\mathrm{Vol}(\mathbb{R}^3)} \lim_{\tau\to\infty(1-i\epsilon)} \frac{i}{2\tau}\left(\quad + \quad + \cdots \right) = \frac{i\left(\quad + \quad + \cdots \right)}{\mathrm{Vol}(\mathbb{R}^{1,3})}\,. \tag{2.203}$$

[16] Cross-terms in the expansion of the exponential of the sum take care of the product of distinct vacuum bubbles, see the term at order λ_0^3, at lower right position.

This expression is of course rather formal. The infinite volume of space-time in the denominator is canceled against a similar term in the numerator coming from invariance of the vacuum bubbles diagrams under translations, not being attached to any point, see the comment below (2.155).

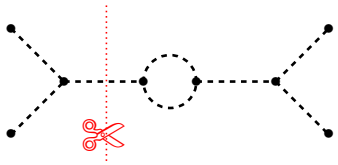
These large distance or *infrared* effects having been taken into account, the large momenta or *ultra-violet* divergences of the vacuum bubbles remain. As we will see when we will dig deeper into the renormalization program, this can be corrected order by order in the coupling constant λ by adding suitable terms to the Lagrangian.

2.2.4 A first look at renormalization

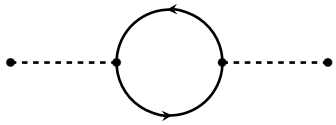
In this subsection, we will provide a first example of the renormalization program at work, by looking more closely at the properties of perturbative corrections to the propagator.

Definition 5 (one-particle irreducible diagram). *A Feynman diagram is said to be one-particle irreducible (1PI) if it cannot be separated into two disconnected diagrams by cutting a single internal line.*

Among the diagrams considered so far, we had examples of both:



Non-1PI diagram

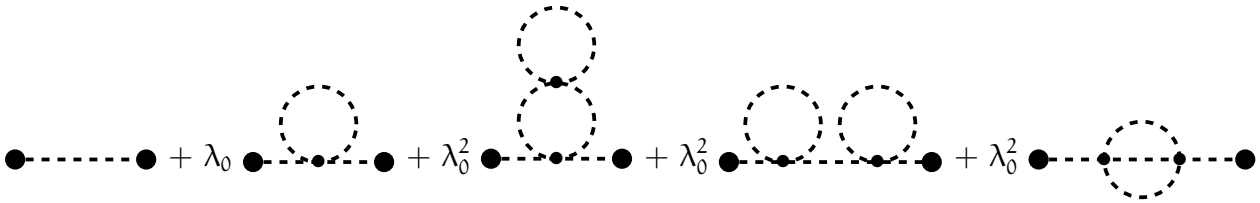


1PI diagram

This notion is useful in order to understand better the typology of Feynman diagrams, by dividing them in simpler pieces. It will allow in particular to resum partially the perturbative corrections associated with irreducible parts of more complicated diagrams.

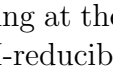
Self-energy. On the one hand, propagators in interacting QFTs receive an infinite number of perturbative corrections, whenever such a perturbative expansion exists. On the other hand, the Källén–Lehmann representation, see eqn. (2.26), predicts a very precise analytic structure for the non-perturbative propagator, in particular with simple poles in $\mathfrak{p}^2$ at one-particle states.

To understand how to organize the perturbative corrections in order to connect these two approaches, we will consider our all-time favorite Φ^4 theory for simplicity, but the ideas developed here are very general. The first few corrections to the propagator are given by the connected diagrams:



The first order correction to the propagator contains just a one-loop diagram known as the "tadpole"  . We have already computed the corresponding correction to the propagator, see (2.159):

$$\widehat{D}_{1-L}(p) = \frac{i}{p^2 - m_0^2 + i\epsilon}\left(-\frac{i\lambda_0}{2}\underbrace{\int \frac{d^4q}{(2\pi)^4}\frac{i}{q^2 - m_0^2 + i\epsilon}}_{D(0)}\right)\frac{i}{p^2 - m_0^2 + i\epsilon} \tag{2.204}$$

Looking at the corrections at order λ_0^2, one finds two new types of diagrams, as well as  , a 1PI-reducible diagram with two tadpoles. Likewise, at order λ_0^k one would find, among many diagrams, a diagram with k tadpoles. All these contributions can be resummed as:

$$\frac{i}{p^2 - m_0^2 + i\epsilon}\sum_{k=0}^{\infty}\left(-\frac{i\lambda_0}{2}D(0)\frac{i}{p^2 - m_0^2 + i\epsilon}\right)^k = \frac{i}{p^2 - m_0^2 + i\epsilon} \times \frac{1}{1 - \frac{\lambda_0 D(0)/2}{p^2 - m_0^2 + i\epsilon}}$$
$$= \frac{i}{p^2 - (m_0^2 + \lambda_0 D(0)/2) + i\epsilon} \tag{2.205}$$

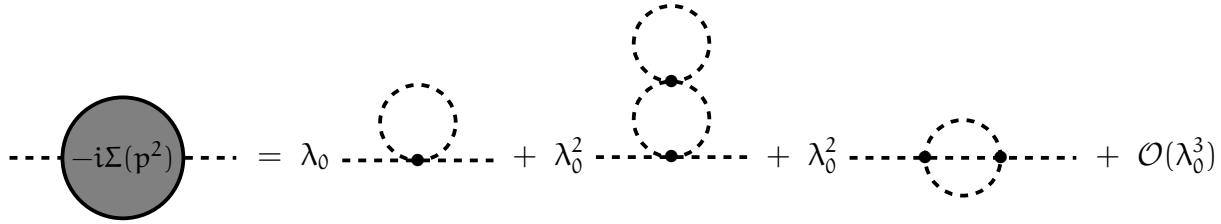
Next is the crucial point. When we had, rather artificially, separated the Lagrangian of the theory into a free part and interactions, we had assumed that the parameter m_0 was the mass of the particle. However, thanks to the Källén–Lehmann analysis, the actual mass of the particle, which is a measurable quantity, is rather the value of p^2 giving a single pole of the momentum space propagator. We would define then the *physical mass* to be:

$$m^2 = m_0^2 + \lambda_0 D(0)/2 \tag{2.206}$$

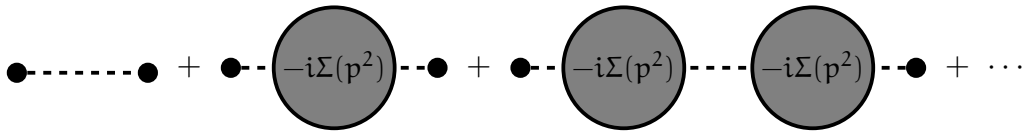
The quantity $D(0)$ is divergent (as we have seen, the integral diverges quadratically) but the only quantity which needs to be finite is m^2, not m_0^2 which has not direct physical signifiance, being just an artifact of perturbation theory.

This analysis gives an idea of how to treat the problem, however is incomplete because we have considered only corrections built out of the the tadpole diagram. However there is an infinite number of other diagram typologies, starting at order λ_0^2 with the "cactus diagram"  , as well as the "sunset diagram"  .

To take all of these diagrams into account, we define the *self-energy* $-i\Sigma(p^2)$, as the amputated sum of all 1-particle irreducible contributions to the propagator:



In term of the self-energy, the whole propagator looks like:



With exactly the same calculation as before, the whole series can be resummed in what is known – surprisingly! – as the *resummed propagator* containing all perturbative corrections to the propagator:

$$\widehat{D}(p) \;=\; \frac{i}{p^2 - \left(m_0^2 + \Sigma(p^2)\right) + i\epsilon}\,. \tag{2.207}$$

According to our previous analyis, the expansion of self-energy in powers of the coupling constant λ_0 starts with:

$$\Sigma(p^2) = \lambda_0 D(0)/2 + \mathcal{O}(\lambda_0^2)\,, \tag{2.208}$$

which does not depend on p^2. The dependence on p^2 appears at next order, through the "sunset diagram"  .

Mass renormalization. We expect that the resummed propagator (2.207), viewed as a function of p^2, has a first order pole at $p^2 = m^2$, where m^2 is the physical mass of the particle.[17] From eqn. (2.207), it means first that we should impose the following *renormalization condition*:

$$\Sigma(p^2)\big|_{p^2=m^2} \;=\; m^2 - m_0^2\,. \tag{2.209}$$

Physically, the difference between the bare mass m_0 and the physical mass m is due to self-interactions of the field through the ϕ^4 coupling. In the context of quantum electrodynamics, this is related to the question of the self-energy of the electron, hence the name chosen for $\Sigma(p^2)$.

At first order in λ_0, to make sense of the relation (2.209), one starts by regularizing the one-loop correction,

$$D(0) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m_0^2 + i\epsilon}\,.$$

It is convenient first to take advantage of the position of the poles in the p_0 complex plane, see fig. 1.5, to perform a change of contour.

Let us consider the closed contour $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_R \cup \mathcal{C}_2 \cup \widetilde{\mathcal{C}}_R$ on fig 2.12, where R is the radius of the arcs. Since this contour does not contain any pole,

$$\oint_{\mathcal{C}} \frac{dp^0}{2\pi} \frac{i}{(p^0)^2 - \vec{p}^2 - m_0^2 + i\epsilon} = 0\,. \tag{2.210}$$

In the limit $R \to \infty$, the contribution from the arcs goes to zero, because the integrand goes like $1/R^2$. It allows to relate the integral over the real axis to an integral over the imaginary axis:

$$\int_{-\infty}^{\infty} \frac{dp^0}{2\pi} \frac{i}{(p^0)^2 - \vec{p}^2 - m_0^2 + i\epsilon} = \int_{-\infty}^{\infty} \frac{i dp_E^0}{2\pi} \frac{i}{(ip_E^0)^2 - \vec{p}^2 - m_0^2}\,. \tag{2.211}$$

[17] More precisely, this is the lowest-lying pole on the real axis, which main contain other poles associated with resonances.

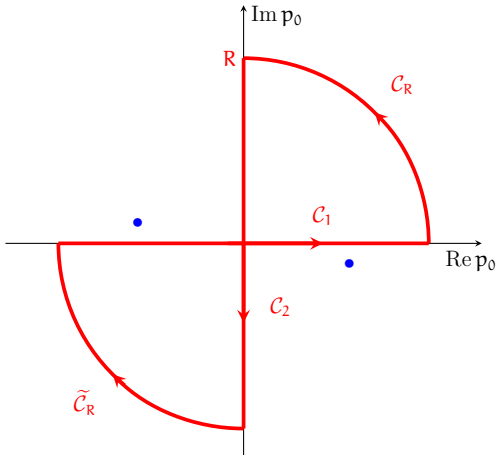


Figure 2.12: *Change of contour in* p_0*-plane for Wick rotation.*

Thus the integrand can be written in terms of the Euclidean four-vector $p_E = (p_E^0, \vec{p})$ and the propagator expressed as an integral over $\mathbb{R}^4$:

$$D(0) = \int \frac{d^4 p_E}{(2\pi)^4} \frac{1}{p_E^2 + m_0^2} \,. \tag{2.212}$$

We have dropped the $i\epsilon$ here, since there are no singularities along the imaginary axis. We can then express the propagator in 4D hyper-spherical coordinates. Recall first that

$$\pi^{d/2} = \int d^d x \, e^{-\vec{x}^2} = \int d\Omega \int_0^\infty r^{d-1} dr \, e^{-r^2} \overset{u=r^2}{=} \Omega^d \frac{1}{2} \int_0^\infty u^{d/2-1} e^{-u} du = \frac{\Omega_d \Gamma(d/2)}{2} \,. \tag{2.213}$$

We have then:

$$D(0) = \frac{1}{8\pi^2} \int_0^\infty \frac{p^3 dp}{p^2 + m_0^2} \,. \tag{2.214}$$

In summary, we have mapped an integral in Minkowski (momentum) space $\mathbb{R}^{1,3}$ to an integral in Euclidian (momentum) space $\mathbb{R}^4$. This procedure is known as *Wick rotation*.

It is actually much more than a trick used to compute integrals; there are deep relations between QFTs on Lorentzian manifolds and QFTs on Euclidian manifolds used to describe critical phenomena, that you will appreciate later in your theoretical physics curriculum.

In the Euclidian form of eqn. (2.214), the divergence of the propagator at coincident points is more manifest. This form also suggests a simple way to regularize this integral, as an intermediate step in the computations, by introducing some high energy scale Λ, that we call a *UV cutoff scale*:

$$D(0) \longmapsto D_\Lambda(0) \overset{\text{def.}}{=} \frac{1}{8\pi^2} \int_0^\Lambda \frac{p^3 dp}{p^2 + m_0^2} \,. \tag{2.215}$$

At first order in the coupling λ, we can interpret the renormalization condition (2.209) as an equation m_0, the unphysical *bare mass* parameter that appears in the free part of the Lagrangian:

$$m_0^2(\Lambda) = m^2 - \frac{\lambda_0}{16\pi^2}\int_0^\Lambda \frac{p^3 dp}{p^2+m_0^2} + \mathcal{O}(\lambda_0^2)\,. \tag{2.216}$$

In this way, the parameter m_0 is tuned such that the resummed propagator has a pole corresponding to the physical mass m. This relation is corrected order by order in perturbation theory. By definition, the physical quantities should depend only on the physical mass m and not on the bare mass m_0 (which is part of the internal machinery of perturbative QFT). In particular, they do not depend on the cut-off Λ which can be removed at the end of the computation; it should also not depend on the way the divergence was regularized.

It is important though that the regularization procedure used in the computation is compatible with the fundamental symmetries of the theory. Anticipating a bit, one notices that:

- the cutoff regularization preserves Euclidian Lorentz invariance but neither gauge invariance (see next section) nor scale invariance (by definition);
- the Pauli–Vilars regularization (that we may discuss later) preserves Euclidian Lorentz invariance and gauge invariance of abelian theories, but neither non-abelian gauge invariance nor scale invariance.
- the dimensional regularization (see below) preserves Euclidian Lorentz invariance and gauge invariance, but not scale invariance.

As a matter of fact, no regularization preserves scale invariance, so one may wonder whether it can be a "good" symmetry in quantum field theory.

A popular and efficient way to regularize the divergent integrals – albeit less intuitive – is to use *dimensional regularization*. The starting point is the Euclidian propagator (2.212). In generic dimensions d, we would have:

$$\int \frac{d^4p_E}{(2\pi)^4}\frac{1}{p_E^2+m_0^2} \;\longrightarrow\; \mu^{4-d}\int \frac{d^dp_E}{(2\pi)^d}\frac{1}{p_E^2+m_0^2}\,, \tag{2.217}$$

where we have introduced an arbitrary mass scale μ to preserve the dimension-less nature of the coupling λ_0 in eqn. (2.208). This can be first simplified using eqn. (2.213):

$$\mu^{4-d}\int \frac{d^dp_E}{(2\pi)^d}\frac{1}{p_E^2+m_0^2} = \frac{\mu^{4-d}}{(4\pi)^{d/2}}\frac{2}{\Gamma(d/2)}\int_0^\infty \frac{p^{d-1}dp}{p^2+m_0^2} = \frac{(\mu/m_0)^{4-d}}{(4\pi)^{d/2}}\frac{2}{\Gamma(d/2)}m_0^2\int_0^\infty \frac{u^{d-1}du}{u^2+1} \tag{2.218}$$

One can do the change of variables $x = 1/(u^2+1)$ giving:

$$\int_0^\infty u^{d-2}\frac{1}{u^2+1}u du = \int_0^1 \left(\frac{1-x}{x}\right)^{d/2-1} x\,\frac{dx}{x^2} = \frac{1}{2}\int_0^1 dx\,(1-x)^{d/2-1}\,x^{-d/2}\,. \tag{2.219}$$

One recognizes the beta-function:

$$B(a,b) = \int_0^1 dx\, x^{a-1}(1-x)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\,, \tag{2.220}$$

so

$$\mu^{4-d}\int \frac{d^d p_E}{(2\pi)^d}\frac{1}{p_E^2+m_0^2} = m_0^2\frac{(\mu/m_0)^{4-d}}{(4\pi)^{d/2}}\Gamma(1-d/2)\,. \tag{2.221}$$

Let us set $d = 4-\epsilon$. The singularity at $\epsilon = 0$ occurs from the expansion:

$$\Gamma(z) = z^{-1} - \gamma + \mathcal{O}(z)\,, \tag{2.222}$$

where $\gamma \simeq 0.577216$ is the Euler–Mascheroni constant. One has then, using $\Gamma(x) = \Gamma(x+1)/x$,

$$\Gamma(-1+\epsilon/2) = -2/\epsilon + \gamma - 1 + \mathcal{O}(\epsilon)\,, \tag{2.223}$$

From which one obtains the dimensionally-regularized contribution to the self-energy:

$$\begin{aligned}\Sigma_\epsilon(p^2) &= \frac{\lambda_0}{2}\frac{m_0^2}{16\pi^2}\left(\frac{\mu}{m_0}\right)^\epsilon (4\pi)^{\epsilon/2}\Big(-2/\epsilon+\gamma-1+\mathcal{O}(\epsilon)\Big) + \mathcal{O}(\lambda_0^2)\\ &\overset{\epsilon\to 0^+}{\sim} \frac{\lambda_0}{2}\frac{m_0^2}{16\pi^2}\left(-\frac{2}{\epsilon}+\gamma-1-\ln 4\pi - \ln\left(\frac{\mu}{m_0}\right)^2 + \mathcal{O}(\epsilon)\right) + \mathcal{O}(\lambda_0^2)\,.\end{aligned} \tag{2.224}$$

A major advantage of this regularization method is that, unlike the UV cutoff, it is compatible with the gauge invariance of electrodynamics (and more generally, of gauge theories).

Bare vs. renormalized perturbation theory. Perturbation theory for the Φ^4 theory was developped by splitting the Lagrangian into a free part and interactions,

$$\mathcal{L} = \tfrac{1}{2}(\partial\Phi)^2 - \tfrac{1}{2}m_0^2\Phi^2 - \tfrac{\lambda_0}{4!}\Phi^4\,, \tag{2.225}$$

written in terms of a mass parameter m_0 that does not correspond to the actual physical mass of one-particle states, due to the self-interactions of the field.

It is possible to present the problem in a different, but completely equivalent way, by insisting that the physical mass, rather than the bare mass, appears in the free part of the Lagrangian. Let us write $m_0^2 = m^2 + \delta m^2$, such that:

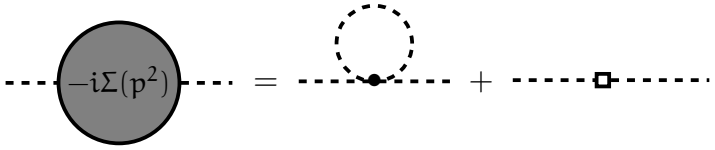
$$\mathcal{L} = \Big(\tfrac{1}{2}(\partial\Phi)^2 - \tfrac{1}{2}m^2\Phi^2\Big) + \Big(\tfrac{1}{2}\delta m^2\Phi^2 - \tfrac{\lambda_0}{4!}\Phi^4\Big)\,. \tag{2.226}$$

In this way the propagator computed with the free Lagrangian would have a pole at $p^2 = m^2$ as it should. The price to pay is that we needed to add a new term in the interactions, known as a *counter-term*. It has to be included in the Feynman rules as:

$$-i\delta m^2 \qquad \text{----□----} \tag{2.227}$$

The counter-term δm^2 admits also a perturbative expansion. By construction, the first possibly non-zero contribution occurs at order λ_0.

In terms of this reformulated perturbation theory, the expansion of the self-energy looks like:



(2.228)

One has to tune the value of the counterterm δm^2, order by order in perturbation theory, such that the position of the pole is not modified. At order λ it means that, in the regularized theory:

$$\delta m^2(\Lambda) = -\lambda_0 D_\Lambda(0)/2 + \mathcal{O}(\lambda_0^2)\,, \tag{2.229}$$

where here

$$D_\Lambda(0) = \frac{1}{8\pi^2}\int_0^\Lambda \frac{p^3 dp}{p^2+m^2}\,, \tag{2.230}$$

i.e. expressed in terms of the physical mass m. It is of course equivalent to eqn. (2.209) at order λ_0. We remark that the mass renormalization (2.229) is not proportional to the mass itself, hence does not vanish in the limit $m \to 0$.

Wave-function and coupling renormalization. According to the Källén–Lehmann decomposition of the exact propagator, the residue at the physical mass $p^2 = m^2$ is Z, the "wave-function renormalization" factor that appears in spectral density, see eqn. (2.27).

Comparing the KL form of the propagator with the resummed pertubative propagator (2.207), we have the identification:

$$Z = \frac{1}{\left(p^2 - \left(m_0^2 + \Sigma(p^2)\right)'\right.}\Bigg|_{p^2=m^2} \quad\Longrightarrow\quad \frac{d\Sigma(p^2)}{dp^2}\Big|_{p^2=m^2} = 1 - 1/Z\,. \tag{2.231}$$

There is no wavefunction renormalization at order λ_0 in perturbation theory, as the first term in the expansion of the self-energy is constant, see eqn. (2.208). The first contribution occurs at order λ_0^2, with the sunset diagram . We will study this correction later in detail.

Finally, as we have done before for the mass, it is possible to elimitate the wave-function renormalization factor Z from the propagator by expressing the Lagrangian not in terms of the *bare field* Φ but in terms of the *renormalized field*

$$\Phi_R(x) \stackrel{\text{def.}}{=} \frac{1}{\sqrt{Z}}\Phi(x)\,. \tag{2.232}$$

Likewise, assuming that we can measure the coupling constant of the theory with some physical experiment, one can introduce the physical coupling λ (to be contrasted with the *bare coupling* λ_0). We define then

$$\delta Z = Z - 1 \quad , \qquad \delta m^2 = Z m_0^2 - m^2 \qquad \delta\lambda = Z^2\lambda_0 - \lambda\,. \tag{2.233}$$

We split accordingly the Lagrangian (2.225) as:

$$\mathcal{L} = \Big(\tfrac{1}{2}(\partial\Phi_{\rm R})^2 - \tfrac{1}{2}m^2\Phi_{\rm R}^2\Big) - \frac{\lambda}{4!}\Phi_{\rm R}^4 + \Big(-\tfrac{1}{2}\delta m^2\Phi_{\rm R}^2 + \tfrac{1}{2}\delta Z(\partial\Phi_{\rm R})^2 - \tfrac{\delta\lambda}{4!}\Phi_{\rm R}^4\Big)\,. \tag{2.234}$$

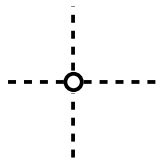
The counter-term in δZ starts to contribute at order λ^2. It should be included as well in the Feynman rules:

$i\delta Z\,p^2$ (2.235)

We have then to tune the mass and wave-function counter-terms, order by order in λ, such that the propagator for the renormalized field $\phi_{\rm R}$ behave near the single-particle pole like:

$$\hat{D}(p) \overset{p^2\to m^2}{\sim} \frac{i}{p^2 - m^2 + i\epsilon}\,, \tag{2.236}$$

i.e. such that propagator has a simple pole at $p^2 = m^2$ with residue equals to one. Likewise, the counter-term in $\delta\lambda$ is associated with the new vertex:

$-i\delta\lambda$  (2.237)

which starts at order λ^2 with the diagram  .

One can show that these three counter-terms (mass, wave-function and vertex) are enough to absorb the divergences at every order in perturbation theory; the theory is then said to be *renormalizable*.

2.2.5 Final form of LSZ formulæ

Let us summarize what we have learnt in this section. Through the LSZ formalism, see eqn. (2.78) in particular, we derived a relation between scattering amplitudes and momentum space (truncated) n-point correlation functions.

We have developed after that perturbative methods to evaluate those correlation functions, order by order in perturbation theory in the coupling constant, and scratched the surface of the renormalization program, which allows to deal with high-energy divergences of loop integrals.

Let us reproduce the LSZ formula (2.78) for complex scalar fields:

$$\begin{aligned}{}_{\rm OUT}\langle p_1,\dots,p_\ell|q_1,\dots,q_n\rangle_{\rm IN} &= \text{disc.} + \\ \prod_{k=1}^{\ell}\frac{p_k^2 - m^2}{i\sqrt{Z}}\prod_{j=1}^{n}\frac{q_j^2 - m^2}{i\sqrt{Z}}&(2\pi)^4\delta^{(4)}\Big(\sum p_i - \sum q_j\Big)\;\widehat{\mathcal{G}}(p_1,\dots,p_\ell|-q_1,\dots,-q_n)\Big|_{\text{on-shell}}\end{aligned}$$

The important point is that the inverse propagators that multiply the momentum space correlator involve the *physical mass* $\mathfrak{m}$, rather than the bare bass $\mathfrak{m}_0$. Therefore, they extract from $\widehat{\mathcal{G}}$, the (lowest lying) simple poles of the Källén–Lehmann propagators, rather than of the free propagators. However the residue at the poles in eqn. (2.26) is different from what we have here ($\mathrm{i}\sqrt{\mathsf{Z}}$ instead of $\mathrm{i}\mathsf{Z}$), hence will need to be corrected in the final LSZ formulæ.

We stress also, as noticed earlier, that only diagrams with as many propagators as external in/out states, *i.e.* connected diagrams, can contribute to the last term on the RHS of (2.238), since otherwise multiplying with the product of propagators and going on-shell gives zero.

Definition 6 (fully amputated correlation functions.) *A correlation function is said to be fully amputated if the full propagators associated with the external legs have been removed, including all quantum corrections.*

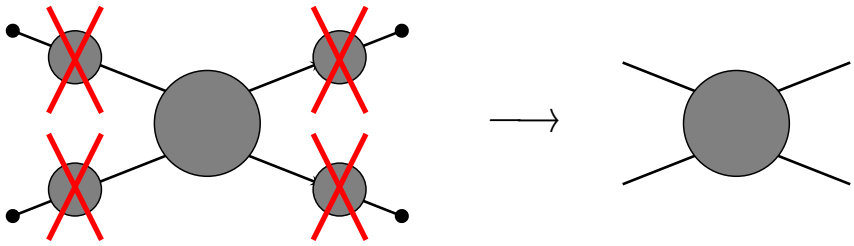


Figure 2.13: *Fully amputated momentum space correlator*

This is represented pictorially on fig. 2.13.

One can write the (almost!) final form of the LSZ formula for a complex scalar as relation between scattering amplitudes, as defined in eqn. (2.42) an fully amputated and connected momentum space correlation functions:

$$\mathrm{i}\mathcal{M}(\mathsf{p}_1,\dots,\mathsf{p}_\ell;\mathsf{q}_1,\dots,\mathsf{q}_n) = \mathsf{Z}^{\frac{\ell+n}{2}}\widehat{\mathcal{G}}(\mathsf{p}_1,\dots,\mathsf{p}_\ell|-\mathsf{q}_1,\dots,-\mathsf{q}_n)\Big|_{\substack{\text{fully amput.}\\ \text{connected.}\\ \text{on-shell}}} \tag{2.238}$$

Let us stress again that *on-shell* here means that, for every external momentum p (or q), $\mathsf{p}^2 = \mathfrak{m}^2$, where $\mathfrak{m}$ is the *physical* mass of the corresponding particle, *i.e.* giving the position of the pole of the Källén–Lehmann propagator in the p^2 plane associated with this one-particle state.

LSZ in renormalized perturbation theory In renormalized perturbation theory (as opposed to bare perturbation theory) the fields are rescaled to absorb the $\sqrt{\mathsf{Z}}$ wave-function renormalization factor see eqn. (2.232). As a consequence, the wave-function renormalization prefactor in the LSZ formula above disappears. We get just

$$\mathrm{i}\mathcal{M} = \widehat{\mathcal{G}}|_{\text{ful. amp., connect., on-shell}} \tag{2.239}$$

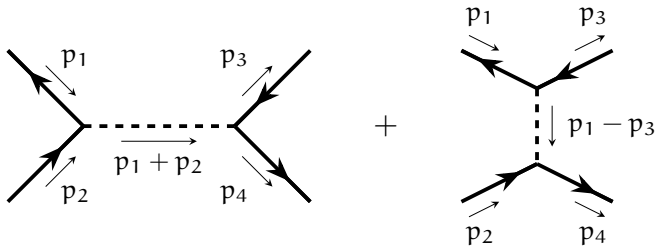
Final form of LSZ for Dirac spinors. Naturally, an analoguous LSZ formula can be obtained for amplitude with in/out spinors. The main difference is that, according to the LSZ rules summarized on page 87, the polarization spinors of the in/out states remain. In summary we have:

- incoming particle: $\widehat{\mathcal{G}}(\cdots|\cdots - p\cdots)|_{\text{ful. amp.},\ldots}u_s$
- incoming anti-particle: $\bar{v}_s\widehat{\mathcal{G}}(\cdots - p\cdots|\cdots)|_{\text{ful. amp.},\ldots}$
- outgoing particle: $\bar{u}_s\widehat{\mathcal{G}}(\cdots p\cdots|\cdots)|_{\text{ful. amp.},\ldots}$
- outgoing anti-particle: $\widehat{\mathcal{G}}((\cdots|\cdots p\cdots)|_{\text{ful. amp.},\ldots}v_s$

To show how it works, let consider electron-positron scattering at order g^2 in Yukawa theory. The corresponding momentum space connected Feynman diagrams are represented on fig. 2.14. Notice that there is no u-channel diagram, because of charge conservation. At this order, $Z = 1$, $g = g_0$ and $m = m_0$ so one has:

$$\begin{aligned} i\mathcal{M}(p_1, p_2 \to p_3, p_4) \; = -g^2\Big(&\bar{v}_{s_2}(\vec{p}_2)u_{s_1}(\vec{p}_1)\frac{i}{s - m^2 + i\epsilon}\bar{u}_{s_3}(\vec{p}_3)v_{s_4}(\vec{p}_4) \\ &+ \bar{u}_{s_3}(\vec{p}_3)u_{s_1}(\vec{p}_1)\frac{i}{t - m^2 + i\epsilon}\bar{v}_{s_2}(\vec{p}_2)v_{s_4}(\vec{p}_4)\Big)\,. \end{aligned} \quad (2.240)$$

Importantly, according to the LSZ rules for spinors, the spin polarizations for in/out states attached to the same vertex are contracted with each other.



s-channel *t-channel*

Figure 2.14: *Electron/positron scattering in Yukawa theory.*

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Part 3

Quantum electrodynamics

Now that we have some idea about what a quantum field theory is, and how to handle interactions, we will use this knowledge to study quantum electrodynamics (QED), which describes the interactions between photons and charged elementary particles. We need first to understand how to obtain a quantum description of the electromagnetic field.

3.1 Electromagnetic field, classical and quantum

We start by recalling elementary facts about classical electrodynamics. The starting point is evidently the Maxwell equations:

$$\vec{\nabla}\cdot\vec{E}=\rho\,,\quad \vec{\nabla}\wedge B=\vec{\jmath}+\partial_t\vec{E}\,,\quad \vec{\nabla}\cdot\vec{B}=0\,,\quad \vec{\nabla}\wedge\vec{E}=-\partial_t\vec{B}\,, \tag{3.1}$$

where the charge density ρ and the current density $\vec{\jmath}$ satisfy the conservation relation:

$$\partial_t\rho+\vec{\nabla}\cdot\vec{\jmath}=0\,. \tag{3.2}$$

The relativistic formulation of electrodynamics is achieved by packaging $\vec{E}$ and $\vec{B}$ into an anti-symmetric two-index tensor, the *field strength*

$$F_{\mu\nu}=\begin{pmatrix} 0 & E_x & E_y & E_z\\ -E_x & 0 & -B_z & B_y\\ -E_y & B_z & 0 & -B_x\\ -E_z & -B_y & B_x & 0\end{pmatrix}\,, \tag{3.3}$$

and the charge and current density into the 4-current $j^\mu=(\rho,\vec{\jmath})$. The first two Maxwell equations read:

$$\partial_\mu F^{\mu\nu}=j^\nu\,. \tag{3.4}$$

The last two Maxwell equations become the *Bianchi identity* (the $[\cdots]$ notation meaning antisymmetrization):

$$\partial_{[\mu}F_{\nu\rho]}=0\,. \tag{3.5}$$

At least locally, the Bianchi is solved by expressing $F_{\mu\nu}$ in terms of a 4-vector $A^\mu(x)$, known as the 4-potential, as:

$$F_{\mu\nu} = \partial_{[\mu} A_{\nu]} = \partial_\mu A_\nu - \partial_\nu A_\mu \,. \tag{3.6}$$

thanks to total antisymmetrization on the left-hand side of eqn. (3.5). In terms of the electric and magnetic fields, writting $A^\mu = (\phi, \vec{A})$,

$$\vec{E} = -\vec{\nabla}\phi - \partial_t \vec{A} \,, \quad \vec{B} = \vec{\nabla} \wedge \vec{A} \,. \tag{3.7}$$

Lagrangian formulation of electrodynamics. The Lagrangian density of a Maxwell two-form field $F^{\mu\nu}$ coupled to an external current j^μ, associated with some charged matter, is given by:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu \,, \tag{3.8}$$

from which we get the Euler–Lagrange equation:

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu \left(\partial_\mu A^\mu\right) = j^\nu \,, \tag{3.9}$$

which are naturally the same as the Maxwell equations (3.4).

It is a well-known fact that, by antisymmetry, the right-hand side of eqn. (3.6) is invariant under a *gauge transformation*

$$A_\mu \;\longmapsto\; A_\mu + \partial_\mu \Gamma \,. \tag{3.10}$$

Since the field strength (3.6) that contains the physical fields is invariant, no measurable quantity should change under the transformation (3.10), or in other words should depend on the choice of 4-potential in its equivalence class. On a more mathematical side, the equation of motion (3.9) cannot fully determine the evolution of the gauge potential A^μ from some initial data on a time-like slice, since the differential operator $D_{\mu\nu} = (\eta_{\mu\nu} \partial_\rho \partial^\rho - \partial_\mu \partial_\nu)$ has a non-trivial kernel, containing all 4-vectors A_μ of the form $\partial_\mu \Gamma$.

Gauge transformations are often said to correspond to a symmetry, called *local symmetry*, but the meaning is rather different. A symmetry maps a system to a system with equivalent properties. For instance, invariance under rotations means that if one rotates a whole experimental setup, the outcome of the experiment would be the same; it does not mean though that that one cannot measure how the experiment is oriented in space! The relation between gauge invariance and symmetries is as follows. The equation of motion (3.4) tells us that, by antisymmetry:

$$\partial_\nu j^\nu = \partial_\nu \partial_\mu F^{\mu\nu} = 0 \,, \tag{3.11}$$

hence the Maxwell field couples necessarily to a conserved 4-current. This process is called the *gauging* of a continuous symmetry but, in some sense, it makes the symmetry disappear.

One may be tempted to get rid of this redundancy in the description of electrodynamics and formulate electrodynamics only in terms of the physical fields $\vec{E}$ and $\vec{B}$; however, as you have learnt in your quantum mechanics course last year, the Hamiltonian of charged particles involve rather the gauge potentials ϕ and $\vec{A}$. Besides this, while $\vec{A}$ by itself is not physical, the Aharonov–Bohm experiment teaches us that the integral of the vector potential over non-contractible closed loops, $\oint \vec{A} \cdot d\vec{\ell}$, is gauge-invariant and has measurable effects.

Formulated in terms of the gauge 4-potential A_μ, the electro-magnetic field seems to have too many degrees of freedom. It is a well-known fact that electromagnetic waves are described by two transverse polarizations components; in the corresponding quantum theory, we expect the photon to have only two physical degrees of freedom. This problem is addressed by choosing a particular representative in the orbit of the gauge transformation (3.10). Among the infinite number of possibilities, which should all be equivalent to each other, two are particularly common:

- The *Coulomb gauge*
$$\vec{\nabla} \cdot \vec{A} = 0 \,. \tag{3.12}$$
The equation of motion for the electric field $\vec{E}$ in the Coulomb gauge reads $\Delta\phi = -\rho$. Thus, in the absence of electric charges, one can set $\phi = 0$. Electromagnetic plane-waves are described by a 3-potential $\vec{A} = \vec{a}\exp(i\vec{k} \cdot \vec{x} - i\omega_k t)$, with $\vec{k} \cdot \vec{a} = 0$ thanks to the Coulomb gauge condition (3.12). This gauge isolates conveniently the physical degrees of freedom of the electromagnetic field, at the expense of manifest Lorentz invariance.[1]

- The *Lorenz gauge*
$$\partial_\mu A^\mu = 0 \,, \tag{3.13}$$
which is Lorentz-invariant.[2] The Lorenz condition does not completely fix the gauge, as there is still the freedom to shift
$$A_\mu \longmapsto A_\mu + \partial_\mu V \,, \quad \Box V = 0 \,, \tag{3.14}$$
without spoiling the condition (3.13). In the following, we will work in the Lorenz gauge.

We will also consider for now the pure electromagnetic theory, without charged matter, thus we will set $j^\mu = 0$ everywhere.

Quantizing the electromagnetic field is a little bit delicate. First, even at classical level, one notices that the time-like component A_0 of the 4-potential is not a dynamical field, since it has no time-derivative term in the Lagrangian density (3.158); in consequence, its conjugate momentum vanishes:
$$\Pi_0(t, \vec{x}) = \frac{\partial \mathcal{L}}{\partial \dot{A}^0(t, \vec{x})} = 0 \,. \tag{3.15}$$

[1] Of course the theory is still Lorentz-invariant, since the physics is independent of the gauge choice!

[2] There is no misprint in the name. This gauge is due to the Danish physicist Ludvig Lorenz, not to Hendrik Lorentz, the father of Lorentz transformations!

In order to overcome this problem, we modify the Lagrangian density (3.158) as follows:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\beta}{2} \left(\partial_\mu A^\mu\right)^2 . \tag{3.16}$$

Modifying the Maxwell Lagrangian seems a rather bold move! β, rather than an extra parameter, should be thought of as a *Lagrange multiplier*. A Lagrange multiplier in field theory is a field appearing in the action without any derivatives, thus its equation of motion just enforces some constraint. In this case, varying the action w.r.t. β enforces the Lorenz gauge $\partial \cdot A = 0$. In the theory defined by the Lagrangian density (3.16), the field Π_μ canonically conjugated to A^μ reads:

$$\Pi_\mu(t, \vec{x}) = \frac{\partial \mathcal{L}}{\partial \dot{A}^\mu(t, \vec{x})} = F_{\mu 0} - \beta \eta_{\mu 0} \partial_\nu A^\nu . \tag{3.17}$$

The idea is to quantize the theory based on the modified Lagrangian (3.16), imposing in a consistent way the Lorenz-gauge constraint $\partial_\mu A^\mu$ enforced by the Lagrange multiplier. The Lagrangian being linear in β, any value can be chosen for convenience without consequences on the physics of the system. The choices are refered to, rather confusingly, as "gauges". We will make the popular choice:

$$\text{Feynman "gauge"} : \quad \beta = 1 . \tag{3.18}$$

The equations of motion based on the Lagrangian density (3.16) are:

$$\Box A^\nu - (1 - \beta) \partial^\nu \left(\partial_\mu A^\mu\right) = 0 \quad \xrightarrow{\beta=1} \quad \Box A^\nu = 0 . \tag{3.19}$$

It seems that the 4-potential A^μ is just a collection of four massless Klein–Gordon fields. Indeed the Lagrangian density in the Lorenz gauge simplifies to

$$\begin{aligned} -2\mathcal{L} &= \partial_\mu A_\nu (\partial^\mu A^\nu - \partial^\nu A^\mu) + (\partial_\mu A^\mu)^2 = \partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu + (\partial_\mu A^\mu)^2 \\ &= \partial_\mu A_\nu \partial^\mu A^\nu + A_\nu \partial^\nu (\partial_\mu A^\mu) - A_\nu \partial^\nu (\partial_\mu A^\mu) + \text{ total derivative} \\ &= \eta_{\rho\sigma} \partial_\mu A^\rho \partial^\mu A^\sigma + \text{ t.d.} \end{aligned} \tag{3.20}$$

Notice though that the $(0,0)$ component has a "wrong" sign in front of the kinetic term:

$$\mathcal{L} = -\frac{1}{2} \partial_\mu A^0 \partial^\mu A^0 + \sum_{i=1}^{3} \frac{1}{2} \partial_\mu A^i \partial^\mu A^i . \tag{3.21}$$

The corresponding canonical momentum is simply

$$\Pi_\mu(x) = -\dot{A}_\mu(x) . \tag{3.22}$$

We shall not forget to impose the constraint $\partial \cdot A = 0$, in the appropriate way.

3.1.1 Gupta–Bleuler quantization

The canonical quantization of the theory is obtained in the Heisenberg picture by imposing, as usual, the equal time commutation relations:

$$[\,A^\mu(t,\vec{x}),\,\Pi_\nu(t,\vec{y})\,] = i\delta^\mu{}_\nu\delta^{(3)}(\vec{x}-\vec{y})\;,\quad [\,A^\mu(t,\vec{x}),\,A^\nu(t,\vec{y})\,] = 0\,,\quad [\,\Pi_\mu(t,\vec{x}),\,\Pi_\nu(t,\vec{y})\,] = 0\,. \tag{3.23}$$

Importantly, the canonical commutation relations prevent the Lorenz gauge condition $\partial\cdot A = 0$ from being imposed as an operator constraint. We would have then, from eqn. (3.22):

$$0 \stackrel{?}{=} \left[\,A^0(t,\vec{x}),\,\partial_\rho A^\rho(t,\vec{y})\,\right] = \left[\,A^0(t,\vec{x}),\,\dot{A}^0(t,\vec{y})\,\right] = -\left[\,A^0(t,\vec{x}),\,\Pi_0(t,\vec{y})\,\right] = -i\delta^{(3)}(\vec{x}-\vec{y})\,. \tag{3.24}$$

Before dealing with this issue, one can expand the fields A^μ and Π^μ into oscillators.

It is convenient to construct, for a given 3-momentum $\vec{k}$, an orthonormal basis of polarization states $\{\varepsilon^\mu_\lambda(\vec{k}), \lambda = 0,1,2,3\}$, in the sense that:

$$\eta_{\mu\nu}\varepsilon^{\mu\,*}_\lambda(\vec{k})\,\varepsilon^\nu_{\lambda'}(\vec{k}) \;=\; \eta_{\lambda\lambda'}\,, \tag{3.25a}$$

$$\eta^{\lambda\lambda'}\varepsilon^{\mu\,*}_\lambda(\vec{k})\,\varepsilon^\nu_{\lambda'}(\vec{k}) \;=\; \eta^{\mu\nu}\,. \tag{3.25b}$$

A real basis of states is obtained as follows.[3]

- Let n^μ be a normalized time-like vector in the future light-cone, *i.e.* with $n_\mu n^\mu = 1$ and $n^0 > 0$. First, we identify a *time-like polarization* vector with n,

$$\varepsilon^\mu_0(\vec{k}) = n^\mu \quad n^2 = 1\,. \tag{3.26}$$

- We consider then two *transverse* polarization vectors $\varepsilon^\mu_\lambda(\vec{k})$, $\lambda = 1,2$, with are orthogonal to the (k,n) plane:

$$n\cdot\varepsilon_\lambda(\vec{k}) = k\cdot\varepsilon_\lambda(\vec{k}) = 0\;,\quad \varepsilon_\lambda(\vec{k})\cdot\varepsilon_\lambda(\vec{k}) = -\delta_{\lambda\lambda'}\;,\quad \lambda = 1,2\,. \tag{3.27}$$

- We define finally a *longitudinal* polarization vector $\varepsilon^\mu_3(\vec{k})$, belonging to the (k,n) plane and orthogonal to n:

$$n\cdot\varepsilon_3(\vec{k}) = 0\;,\quad \varepsilon_3(\vec{k})\cdot\varepsilon_3(\vec{k}) = -1\,. \tag{3.28}$$

  Explicitly, because it lies in the (k,n) plane,

$$\varepsilon_3(\vec{k}) = \frac{k}{k\cdot n} - n\,. \tag{3.29}$$

  Indeed,

$$n\cdot\left(\frac{k}{k\cdot n} - n\right) = \frac{n\cdot k}{k\cdot n} - n^2 = 0\;,\quad \left\|\frac{k}{k\cdot n} - n\right\|^2 = \overbrace{\frac{k^2}{(k\cdot n)^2}}^{=0} - \frac{2k\cdot n}{k\cdot n} + n^2 = -1\,.$$

[3] A complex basis of circular polarization vectors is often useful as well.

By construction, the longitudinal polarization vector is orthogonal to the pair of transverse polarization vectors:

$$\varepsilon_3(\vec{k}) \cdot \varepsilon_\lambda(\vec{k}) = 0 \ , \quad \lambda = 1, 2 \, . \tag{3.30}$$

One notices also that, thanks to eqn. (3.29),

$$\varepsilon_0^\mu + \varepsilon_3^\mu = \tfrac{1}{k\cdot n}\, k^\mu \, . \tag{3.31}$$

For instance, for $k^\mu = (\omega, 0, 0, \omega)$, one can choose:

$$\varepsilon^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \varepsilon^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \varepsilon^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \varepsilon^4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \, . \tag{3.32}$$

In terms of the basis $\{\varepsilon_\lambda, \lambda = 0, 1, 2, 3\}$ of polarization vectors and of the creation and annihilation operators, one can decompose the field operator A^μ and its conjugate Π_μ in the Heisenberg picture in the usual way:

$$A^\mu(x) = \sum_{\lambda=1}^{4} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \left(\varepsilon_\lambda^\mu(\vec{k}) a_{\vec{k}}^\lambda e^{-ik\cdot x} + \varepsilon_\lambda^{\mu\,*}(\vec{k}) a_{\vec{k}}^{\lambda\,\dagger} e^{ik\cdot x} \right) , \tag{3.33a}$$

$$\Pi^\mu(x) = +i \sum_{\lambda=1}^{4} \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{\omega_k}{2}} \left(\varepsilon_\lambda^\mu(\vec{p}) a_{\vec{k}}^\lambda e^{-ik\cdot x} - \varepsilon_\lambda^{\mu\,*}(\vec{p}) a_{\vec{k}}^{\lambda\,\dagger} e^{ik\cdot x} \right) . \tag{3.33b}$$

with $\omega_k = \|\vec{k}\|$. We have allowed also the polarization basis vectors to be complex in order to be more general.

Notice again the opposite sign in (3.33b) compared to eqn. (1.210). We can the express the canonical commutation relations (3.23) in terms of the commutators of the modes, for instance:

$$\begin{aligned} [\, A^\mu(t, \vec{x}), \Pi_\nu(t, \vec{y}) \,] = \frac{i}{2} \sum_{\lambda,\lambda'} \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \frac{\omega_{k'}}{\omega_k} \varepsilon_\lambda^\mu(\vec{k}) \varepsilon_{\nu\,\lambda'}(\vec{k}') \Bigg(\left[a_{\vec{k}}^\lambda, a_{\vec{k}'}^{\lambda'} \right] e^{-i(k\cdot x + k'\cdot y)} \\ - \left[a_{\vec{k}}^{\lambda\,\dagger}, a_{\vec{k}'}^{\lambda'\,\dagger} \right] e^{i(k\cdot x + k'\cdot y)} + \left[a_{\vec{k}}^{\lambda\,\dagger}, a_{\vec{k}'}^{\lambda'} \right] e^{i(k\cdot x - k'\cdot y)} - \left[a_{\vec{k}}^{\lambda\,\dagger}, a_{\vec{k}'}^{\lambda'\,\dagger} \right] e^{-i(k\cdot x - k'\cdot y)} \Bigg) \end{aligned} \tag{3.34}$$

Exercise: do the same for the other commutations relation appearing in (3.23).

We then deduce that:

$$\left[\, a^{\lambda}_{\vec{k}},\, a^{\lambda'}_{\vec{k}'} \,\right] = 0\,, \quad \left[\, a^{\lambda\dagger}_{\vec{k}},\, a^{\lambda'\dagger}_{\vec{k}'} \,\right] = 0\,, \quad \left[\, a^{\lambda}_{\vec{k}},\, a^{\lambda'\dagger}_{\vec{k}'} \,\right] = -(2\pi)^3 \eta^{\lambda\lambda'} \delta^{(3)}\big(\vec{k} - \vec{k}'\big)\,. \tag{3.35}$$

One can easily express the Hamiltonian in terms of the modes, starting from eqns. (3.21) and (3.22). The computation is essentially the same as in the massless Klein–Gordon case – except the malign minus sign – see eqn. (1.216) and below. One gets, up to the normal ordering additive constant:

$$H = \int \frac{d^3k}{(2\pi)^3} \omega_k \left(\sum_{\lambda=1}^{3} a^{\lambda\dagger}_{\vec{k}} a^{\lambda}_{\vec{k}} - a^{0\dagger}_{\vec{k}} a^{0}_{\vec{k}} \right)\,. \tag{3.36}$$

At first sight, the one-particle states of the theory are given as usual by:

$$\Big|\vec{k},\, \lambda\Big\rangle = \sqrt{2\omega_k}\, a^{\lambda\dagger}_{\vec{k}}\, |0\rangle\,. \tag{3.37}$$

where $|0\rangle$ is the Poincaré-invariant vacuum annihilated by all the operators $a^{\lambda}_{\vec{k}}$.

This would lead to four polarization states for the photon instead of the expected two. Even worse, one of those states has negative norm squared:

$$\Big\langle \vec{k},\, \lambda = 0 \Big| \vec{k}',\, \lambda = 0 \Big\rangle = 2\sqrt{\omega_k \omega_{k'}}\, \langle 0|\, a^{0}_{\vec{k}}\, a^{0\dagger}_{\vec{k}'}\, |0\rangle = -2\omega_k (2\pi)^3 \delta^{(3)}\big(\vec{k} - \vec{k}'\big)\,. \tag{3.38}$$

Crucially, one still has to implement the gauge-fixing constraint in some form. Since it cannot be enforced as the operator equation $\partial \cdot A = 0$, see eqn. (3.24), one could try to enforce that this operator annihilate every physical state:

$$\forall\, |\phi\rangle \in \mathcal{H}_{\text{phys}}\,, \quad \partial \cdot A\, |\phi\rangle \stackrel{?}{=} 0\,.$$

This is unfortunately much too strong, as it will remove the vacuum $|0\rangle$ from the physical Hilbert space $\mathcal{H}_{\text{phys}}$! In order to formulate a milder constraint, we first separate the field operator A^μ into positive- and negative-frequency components:

$$A^\mu(x) = \underbrace{\sum_\lambda \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \varepsilon^\mu_\lambda(\vec{k})\, a^{\lambda}_{\vec{k}}\, e^{-ik\cdot x}}_{A^{\mu\,+}(x)} + \underbrace{\sum_\lambda \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \varepsilon^{\mu\,*}_\lambda(\vec{k})\, a^{\lambda\dagger}_{\vec{k}}\, e^{ik\cdot x}}_{A^{\mu\,-}(x)} \tag{3.39}$$

We will impose that every physical state satisfies the *Gupta–Bleuler condition*

$$\forall\, |\phi\rangle \in \mathcal{H}_{\text{phys}}\,, \quad \partial \cdot A^+\, |\phi\rangle \,=\, 0\,. \tag{3.40}$$

This condition ensures that the Lorenz gauge condition is satisfied, not as an operator equation – which was too much to ask – but as an equation for the matrix elements between any pair of states satisfying (3.40):

$$\langle \psi'|\, \partial \cdot A\, |\psi\rangle = \langle \psi'|\, \underbrace{\partial \cdot A^+\, |\psi\rangle}_{=0} + \underbrace{\langle \psi'|\, \partial \cdot A^-}_{=0} |\psi\rangle = 0\,. \tag{3.41}$$

As we shall see, the Gupta–Bleuler condition removes the negative-norm states from the physical Hilbert space $\mathcal{H}_{\rm phys}$ and eventually gives only two physical transverse polarizations, as expected.

To see how this works, let us consider first a generic one-particle state, with an arbitrary polarization vector χ:

$$\chi = \sum_{\lambda=0}^{3} \eta_{\lambda\lambda'}\chi^\lambda \varepsilon^{\lambda'}(\vec{k}) \ , \quad \left|\vec{k},\chi\right\rangle = \eta_{\lambda\lambda'}\chi^\lambda \left|\vec{k},\,\lambda'\right\rangle = \sqrt{2\omega_k}\,\chi_\lambda\, a^{\lambda\,\dagger}_{\vec{k}}\,|0\rangle \tag{3.42}$$

We have

$$\begin{aligned}\partial_\mu A^{\mu+}\left|\vec{k},\chi\right\rangle &= -i\sum_{\lambda,\lambda'}\int\frac{d^3k'}{(2\pi)^3}\sqrt{\frac{\omega_k}{\omega_{k'}}}k'_\mu\varepsilon^\mu_{\lambda'}(\vec{k}')e^{-ik'\cdot x}\chi_\lambda\,\left[\,a^{\lambda'}_{\vec{k}'},\,a^{\lambda\,\dagger}_{\vec{k}}\,\right]|0\rangle \\ &= i\sum_{\lambda,\lambda'}\eta^{\lambda\lambda'}k_\mu\varepsilon^\mu_{\lambda'}(\vec{k})e^{-ik\cdot x}\chi_\lambda\,|0\rangle = ik_\mu\chi^\mu e^{-ik\cdot x}\,|0\rangle\ .\end{aligned} \tag{3.43}$$

We conclude that the polarization vector χ of any physical one-particle state should satisfy:

$$k\cdot\chi = 0 \implies (k\cdot n)\left(\chi^0-\chi^3\right) = 0\,, \tag{3.44}$$

where we have used the decomposition (3.42). As a consequence, a generic one-particle state can be decomposed as:

$$\left|\vec{k},\chi\right\rangle = \sqrt{2\omega_k}\overbrace{\sum_{\lambda=1}^{2}\chi_\lambda\,a^{\lambda\,\dagger}_{\vec{k}}\,|0\rangle}^{\text{transverse}} + \sqrt{2\omega_k}\,\chi^0\left(a^{0\,\dagger}_{\vec{k}} - a^{3\,\dagger}_{\vec{k}}\right)|0\rangle\ . \tag{3.45}$$

The first term contains the two physical degrees of freedom associated with transverse polarizations:

$$\left|\vec{k},\chi\right\rangle_{\rm T} = \sqrt{2\omega_k}\sum_{\lambda=1}^{2}\chi_\lambda\,a^{\lambda\,\dagger}_{\vec{k}}\,|0\rangle \tag{3.46}$$

Such state has a positive norm:

$${}_{\rm T}\left\langle\vec{k}',\chi\middle|\vec{k},\chi\right\rangle_{\rm T} = \sqrt{2\omega_{k'}2\omega_k}\sum_{\lambda,\lambda'=1}^{2}\chi^*_{\lambda'}\chi_\lambda\,\langle 0|\underbrace{\left[\,a^{\lambda'}_{\vec{k}'},\,a^{\lambda\,\dagger}_{\vec{k}}\,\right]}_{-(2\pi)^3\delta^{(3)}(\vec{k}'-\vec{k})\eta^{\lambda\lambda'}}|0\rangle = 2\omega_k(2\pi)^3\delta^{(3)}(\vec{k}'-\vec{k})\sum_{\lambda=1}^{2}|\chi_\lambda|^2\,. \tag{3.47}$$

The second term on the RHS of eqn. (3.45) contains a specific combination of longitudinal and time-like polarizations, which is usually referred to as a *spurious state*:

$$\left|\vec{k},\chi\right\rangle_{\rm S} = \sqrt{2\omega_k}\chi_0\left(a^{0\,\dagger}_{\vec{k}} - a^{3\,\dagger}_{\vec{k}}\right)|0\rangle\ . \tag{3.48}$$

Such state has, by construction, vanishing inner product with the transverse polarization states, see eqn. (3.27). It has also zero norm:

$$ {}_{\rm s}\Big\langle \vec{k}'\Big|\vec{k}\Big\rangle_{\rm s} \propto \langle 0|\left(a^0_{\vec{k}'} - a^3_{\vec{k}'}\right)\left(a^{0\dagger}_{\vec{k}} - a^{3\dagger}_{\vec{k}}\right)|0\rangle = \langle 0|\Big(\underbrace{\left[a^0_{\vec{k}'},\, a^{0\dagger}_{\vec{k}}\right]}_{-(2\pi)^3\delta^{(3)}(\vec{k}'-\vec{k})} + \underbrace{\left[a^3_{\vec{k}'},\, a^{3\dagger}_{\vec{k}}\right]}_{+(2\pi)^3\delta^{(3)}(\vec{k}'-\vec{k})}\Big)|0\rangle = 0\,. \tag{3.49} $$

Using eqn. (3.31) one notices finally that the polarization vector of the spurious state is parallel to k^μ:

$$ \chi^\mu = \chi^0\varepsilon^\mu_0 + \chi^3\varepsilon^\mu_3 = \chi^0\big(\varepsilon^\mu_0 + \varepsilon^\mu_3\big) = \tfrac{\chi^0}{k\cdot n}\,k^\mu\,. \tag{3.50} $$

Thus we have succeded in obtaining a one-particle Hilbert space endowed with a non-negative norm.

A generic n-particle state satisfying the Gupta–Bleuler condition (3.40) can be similarly decomposed on Fock state basis vectors of the form (suppressing momentarily the momentum labels for clarity):

$$ \left(a^{1\dagger}\right)^{n_1}\left(a^{2\dagger}\right)^{n_2}\left(a^{0\dagger} - a^{3\dagger}\right)^{n_{\rm s}}|0\rangle\,. $$

All the states with $n_{\rm s} \neq 0$ have zero norm, since

$$ \begin{aligned} &\langle 0|\left(a^0_{\vec{k}'} - a^3_{\vec{k}'}\right)^{n_{\rm s}}\left(a^{0\dagger}_{\vec{k}} - a^{3\dagger}_{\vec{k}}\right)^{n_{\rm s}}|0\rangle \\ &\qquad = n_{\rm s}\,\langle 0|\left(a^0_{\vec{k}'} - a^3_{\vec{k}'}\right)^{n_{\rm s}-1}\left(a^{0\dagger}_{\vec{k}} - a^{3\dagger}_{\vec{k}}\right)^{n_{\rm s}-1}\Big(\underbrace{\left[a^0_{\vec{k}'},\, a^{0\dagger}_{\vec{k}}\right]}_{-(2\pi)^3\delta^{(3)}(\vec{k}'-\vec{k})} + \underbrace{\left[a^3_{\vec{k}'},\, a^{3\dagger}_{\vec{k}}\right]}_{+(2\pi)^3\delta^{(3)}(\vec{k}'-\vec{k})}\Big)|0\rangle = 0\,. \end{aligned} $$

and are by construction orthogonal to states with only transverse polarization modes. Crucially, *spurious states do not contribute to physical observables*.

This decoupling means not only that those spurious states are orthogonal to physical states (built on transverse polarizations), but have also vanishing matrix elements for all the observables in the theory. In the free theory, one can check in particular that, as far as the Hamiltonian is concerned,

$$ \begin{aligned} H\left(a^{0\dagger}_{\vec{k}} - a^{3\dagger}_{\vec{k}}\right)|0\rangle = \int\frac{d^3k'}{(2\pi)^3}\omega_{k'}\left(a^{3\dagger}_{\vec{k}'}a^3_{\vec{k}'} - a^{0\dagger}_{\vec{k}'}a^0_{\vec{k}'}\right)\left(a^{0\dagger}_{\vec{k}} - a^{3\dagger}_{\vec{k}}\right)|0\rangle = \omega_k\left(a^{0\dagger}_{\vec{k}} - a^{3\dagger}_{\vec{k}}\right)|0\rangle \\ \Longrightarrow\quad {}_{\rm s}\Big\langle \vec{k}',\chi'\Big|\,H\,\Big|\vec{k},\chi\Big\rangle_{\rm s} = 0 \end{aligned} \tag{3.51} $$

using eqn. (3.49). The same holds for the same reason for the momentum operator. This decoupling means that any physical state built with $n_{\rm s} \neq 0$ time-like/longitudinal photon pairs is gauge-equivalent to a state with $n_{\rm s} = 0$, *i.e.* with only transverse excitations. Proving that the decoupling of spurious states holds in the interacting theory is non trivial, and will be studied in section 3.3.

Since the spurious states do not affect the physical properties of the states of the theory, one can endow the space of physical states $\mathcal{H}_{\text{PHYS}}$ with an equivalence relation:

$$|\phi_1\rangle \sim |\phi_2\rangle \;\Leftrightarrow\; \exists\, |\phi\rangle_{\text{s}} \;, \quad |\phi_1\rangle = |\phi_2\rangle + |\phi\rangle_{\text{s}} \;. \tag{3.52}$$

where $|\phi\rangle_{\text{s}}$ is a spurious state.[4]

This equivalence relation is the quantum version of the residual gauge freedom (3.14) in the Lorenz gauge. To see this, let us a consider a generic spurious state in the Fock space of the theory:

$$|\phi\rangle_{\text{s}} = |0\rangle + \int \frac{d^3k}{(2\pi)^3} f(k)\sqrt{2\omega_k}\big(a^{0\,\dagger}_{\vec{k}} - a^{3\,\dagger}_{\vec{k}}\big)\,|0\rangle + (\text{2-particle state}) + \cdots \tag{3.53}$$

The one-particule contribution to the expectation value ${}_{\text{s}}\langle\phi|\, A^\mu\, |\phi\rangle_{\text{s}}$ reads, using eqn. (3.29):

$$\begin{aligned}
&\sum_\lambda \int \frac{d^3k'}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} f(k) \sqrt{\frac{\omega_k}{\omega_{k'}}}\, \varepsilon^\mu_\lambda(\vec{k}')\, e^{-ik'\cdot x}\, \langle 0|\, \Big[\, a^\lambda_{\vec{k}'},\, a^{0\,\dagger}_{\vec{k}} - a^{3\,\dagger}_{\vec{k}}\,\Big]\, |0\rangle + \text{c.c.} \\
&\qquad = \int \frac{d^3k}{(2\pi)^3} f(k) e^{-ik\cdot x}\big(\varepsilon^\mu_3 + \varepsilon^\mu_0\big) + \text{c.c} = \int \frac{d^3k}{(2\pi)^3} f(k) e^{-ik\cdot x} \frac{k^\mu}{k\cdot n} + \text{c.c.} \\
&\qquad\qquad = \partial_\mu \left(i \int \frac{d^3k}{(2\pi)^3} f(k) \frac{e^{-ik\cdot x}}{k\cdot n} + \text{c.c.} \right)
\end{aligned} \tag{3.54}$$

and similarly for multi-particle contributions. Because the 4-momentum k^μ in eqn. (3.54) is on-shell, shifting a physical state with a spurious state induces the transformation

$$\langle A^\mu\rangle \;\longmapsto\; \langle A^\mu\rangle + \partial_\mu V, \quad \Box V = 0\,. \tag{3.55}$$

It is important to stress that the time-like and longitudinal polarization cannot be simply removed from the description in the theory, since they will appear for instance as intermediate states in scattering processes.

Propagator. In the Feynman "gauge" the propagator is easy to write as the 4-potential reduces to a collection of four real and massless Klein–Gordon fields with an indefinite metric. One gets for the Feynman propagator:

$$D^{\mu\nu}(x-y) = \langle 0|\, T\, A^\mu(x) A^\nu(y)\, |0\rangle = \int \frac{d^4k}{(2\pi)^4} \frac{-i\eta^{\mu\nu} e^{-ik\cdot(x-y)}}{k^2 + i\epsilon} \tag{3.56}$$

Exercise: Show that this can be also obtained directly using the completeness relation (3.25b).

[4]Some of you may notice that it starts to look like a cohomology problem. This is exactly the case, however to show it one needs to introduce Faddeev–Popov ghosts and the BRST charge, that you will study during the second semester (QFT II course).

More generally, from a Lagrangian of the form (3.16) with arbitary β, one can show with a bit of work that:

$$D^{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{-i\left(\eta^{\mu\nu} + (1/\beta - 1)\frac{k^\mu k^\nu}{k^2+i\epsilon}\right) e^{-ik\cdot(x-y)}}{k^2 + i\epsilon} \,. \tag{3.57}$$

We stress again than any physical quantity should be independent of the particular value of β that was chosen. This is a non-trivial – but crucial – statement that will be proven later on.

3.1.2 Massive vector fields

The most general Lorentz-invariant Lagrangian density for a vector field $A^\mu(x)$ is made of three terms: a kinetic term, a mass term and a divergence term:

$$\mathcal{L} = -\frac{1}{2}\big(\partial_\mu A_\nu\big)\big(\partial^\mu A^\nu\big) + \frac{m^2}{2} A_\mu A^\mu + \frac{\alpha}{2}\big(\partial_\mu A^\mu\big)^2 \,. \tag{3.58}$$

The equations of motion read:

$$0 = \frac{\partial\mathcal{L}}{\partial A_\mu} - \partial_\nu \frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\mu)} = (\Box + m^2)A^\mu - \alpha\partial^\mu\big(\partial_\rho A^\rho\big) \,. \tag{3.59}$$

Taking the divergence of this expression gives:

$$\big((1-\alpha)\Box + m^2\big)\big(\partial_\mu A^\mu\big) = 0 \,. \tag{3.60}$$

Then, whenever $\alpha \neq 1$, $\partial_\mu A^\mu$ should be understood as a scalar field, creating massive particles of spin zero from the vacuum.

This is more easily understood in a rest frame of the particle. For any on-shell 4-momentum p, *i.e.* such that $p^2 = m^2$, one reaches a rest frame using a Lorentz tranformation Λ_p^{-1} such that:

$$p = \Lambda_p \bar{p} \;, \quad \bar{p} = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} \,. \tag{3.61}$$

Space rotation naturally maps the rest frame to another one, as it keeps $\bar{p}$ invariant: $R(\theta)\bar{p} = \bar{p}$. It acts on the 4-vector A^μ as

$$A^\mu = (A^0, \vec{A}) \mapsto (A^0, R(\theta)\vec{A}) \,. \tag{3.62}$$

In other words, A^0 is a spin-zero field and $\vec{A}$ a spin-one field. One-particle states in the rest frame can be labeled as $|\bar{p}, \lambda\rangle$, where $\lambda = 0$ refers to the spin-zero state and $\lambda = 1, 2, 3$ are associated with the spin-one states. By the Wigner–Eckart theorem, one should have then:

$$\langle 0|\, A^0(0)\, |\bar{p}, \lambda\rangle = \sqrt{Z_s}\delta_{\lambda,0} \;, \quad \langle 0|\, A^i(0)\, |\bar{p}, \lambda\rangle = \sqrt{Z_s}\delta_{\lambda,i} \;, \quad i = 1, 2, 3 \,. \tag{3.63}$$

In a generic frame, one has:

$$\begin{aligned}\langle 0|\, A^\mu(0)\,|p,0\rangle &= \langle 0|\, A^\mu(0) U(\Lambda_p)\,|\bar{p},0\rangle = \langle 0|\, U(\Lambda_p)^{-1} A^\mu(0) U(\Lambda_p)\,|\bar{p},0\rangle \\ &= \sqrt{Z_s}\,(\Lambda_p)^\mu{}_\nu \underbrace{\delta^\nu{}_0}_{\bar{p}^\nu/m} = \frac{p^\mu}{m}\sqrt{Z_s}\,. \end{aligned} \tag{3.64}$$

Thus one finds that:

$$\langle 0|\, \partial_\mu A^\mu(x)\,|p,0\rangle = \partial_\mu \langle 0|\, A^\mu(0)\,|p,0\rangle\, e^{-ip\cdot x} = \frac{\sqrt{Z_s}}{m} p^\mu \partial_\mu e^{-ip\cdot x} = -im\sqrt{Z_s} e^{-ip\cdot x}\,, \tag{3.65}$$

exactly as the wave-function (2.10) for a scalar particle introduced in the derivation of the Källén–Lehmann representation.

This scalar degree of freedom can be avoided by choosing $\alpha = 1$ in the Lagrangian (3.58), leading after two integration by parts to the *Proca Lagrangian*:

$$\boxed{\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m}{2} A_\mu A^\mu\,.} \tag{3.66}$$

The associated QFT describes a massive spin-one particle, with three physical degrees of freedom. The divergence-free condition $\partial_\mu A^\mu = 0$ is imposed by the equations of motion, see eqn. (3.60). In the massless theory, this was imposed as a gauge-fixing condition, but in the massive case the mass term explicitly breaks gauge invariance.

The mode expansion for the massive vector field A^μ can be written in a polarization basis $\{\varepsilon^\mu_\lambda(\vec{p}), \lambda = 1,2,3\}$ with $k_\mu \epsilon^\mu_\lambda = 0$, as:

$$A^\mu(x) = \sum_{\lambda=1}^{3} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \left(\varepsilon^\mu_\lambda(\vec{k}) a^\lambda_{\vec{k}} e^{-ik\cdot x} + \varepsilon^{\mu\,*}_\lambda(\vec{k}) a^{\lambda\,\dagger}_{\vec{k}} e^{ik\cdot x} \right) \Big|_{E_p = \sqrt{\vec{p}^2 + m^2}} \tag{3.67}$$

Starting from the rest frame one can choose the real basis:

$$\varepsilon_1 = \Lambda_p \bar{\varepsilon}_1 = \Lambda_p \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\,, \quad \varepsilon_2 = \Lambda_p \bar{\varepsilon}_2 = \Lambda_p \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\,, \quad \varepsilon_3 = \Lambda_p \bar{\varepsilon}_3 = \Lambda_p \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\,. \tag{3.68}$$

We have then the inner product:

$$\varepsilon_\lambda \cdot \varepsilon_{\lambda'} = \bar{\varepsilon}_\lambda \cdot \bar{\varepsilon}_{\lambda'} = -\delta_{\lambda\lambda'}\,. \tag{3.69}$$

Consider the following spin sum, in the rest frame:

$$\sum_\lambda \epsilon^\mu_\lambda(\bar{p}) \epsilon^{\nu\,*}_\lambda(\bar{p}) = \sum_{\lambda=1}^{3} \delta^\mu_\lambda \delta^\nu_\lambda = \left(-\eta^{\mu\nu} + \frac{\bar{p}^\mu \bar{p}^\nu}{m^2} \right)\,. \tag{3.70}$$

Using the Lorentz transform Λ_p, one gets:

$$\sum_\lambda \epsilon^\mu_\lambda(p) \epsilon^{\nu\,*}_\lambda(p) = (\Lambda_p)^\mu{}_\rho\ (\Lambda_p)^\nu{}_\sigma \sum_\lambda \epsilon^\rho_\lambda(\bar{p}) \epsilon^{\sigma\,*}_\lambda(\bar{p}) = -\eta^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}\,. \tag{3.71}$$

3.1.3 Coupling of the electromagnetic field to matter

As was noticed before, the coupling between matter fields and the electromagnetic field involves a conserved current $j^\mu(x)$ for consistency of the Maxwell equations, see eqn. (3.11), *i.e.* satisfying $\partial_\mu j^\mu = 0$ on-shell.[5]

Charged Dirac fermion. Let us describe first the coupling of a *Dirac fermion*. As we have already discussed this theory possesses a $U(1)$ symmetry, acting as (choosing a convenient normalization):

$$\Psi(x) \;\mapsto\; e^{-ie\theta}\,\Psi(x)\,, \quad \bar\Psi(x) \;\mapsto\; e^{ie\theta}\bar\Psi(x)\,. \tag{3.72}$$

The associated Noether current reads:

$$j^\mu = e\bar\Psi\gamma^\mu\Psi\,. \tag{3.73}$$

We consider the following Lagrangian density for the electromagnetic field interacting with the Dirac field:

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar\Psi\big(i\not\partial - m\big)\Psi - eA_\mu\bar\Psi\gamma^\mu\Psi \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar\Psi\Big(i\big(\not\partial + ie\not A\big) - m\Big)\Psi\end{aligned} \tag{3.74}$$

Crucially, under a gauge transformation $A_\mu \mapsto A_\mu + \partial_\mu\Gamma$, this Lagrangian density is invariant provided that Ψ transforms as

$$\Psi(x) \;\overset{\Gamma}{\longmapsto}\; e^{-ie\Gamma(x)}\Psi(x)\,. \tag{3.75}$$

Indeed,

$$\not\partial e^{-ie\Gamma}\Psi = e^{-ie\Gamma}\big(-ie\not\partial\Gamma + \not\partial\big)\Psi\,. \tag{3.76}$$

Phrased differently, one introduces the *covariant derivative*

$$\not D \;\overset{\text{def.}}{=}\; \not\partial + ie\not A\,, \tag{3.77}$$

such that

$$\not D\Psi(x) \;\overset{\Gamma}{\longmapsto}\; e^{-ie\Gamma(x)}\not D\Psi(x)\,. \tag{3.78}$$

In terms of this operator, the Lagrangian density of the theory of a Dirac fermion interacting with the electormagnetic field reads:

$$\boxed{\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar\Psi\big(i\not D - m\big)\Psi\,.} \tag{3.79}$$

[5] On-shell means here that this equation needs only to hold if the matter fields are solutions to their equations of motion.

This construction, where the coupling of the gauge field is accomplished by replacing the ordinary space-time derivatives with covariant derivatives, is known as *minimal coupling*.

In more physical terms, what we have done here is "promoting" the $U(1)$ symmetry of the free Dirac theory to invariance under gauge transformations in the combined gauge/matter theory. One usually calls this process *gauging* a global symmetry into a *local symmetry*. As we have already discussed, gauge invariance is not a symmetry in the usual sense but rather a redundancy.

In terms of dimensional analysis, the 4-potential A^μ has dimension $L^{1-d/2}$, like scalar fields. Since the Dirac fermion has dimension $L^{(1-d)/2}$, the electric charge e has dimension $L^{d/2-2}$ so is dimensionless in 4 dimensions (in natural units).

General construction Let us consider a matter field theory coupled to the electromagnetic field. Schematically, one writes:

$$S = S[A] + S_{\text{INT.}}[A, \Phi] + S_{\text{MAT.}}[\Phi]\,, \tag{3.80}$$

where $S[A] = -\frac{1}{4}F^2$ is the free action for the Maxwell field and S_{INT} contains the interaction terms between the field Φ (that represents collectively the matter fields) and the eletromagnetic field. As we will justify below, a necessary and sufficient condition to get a consistent electromagnetic interaction *at the classical level* is:

$$\frac{\delta S_{\text{INT.}}}{\delta A_\mu(x)} = -j^\mu(x)\,, \tag{3.81}$$

where j^μ is a conserved current of the *full* theory with action S associated with a $U(1)$ symmetry.

Let us show that it implies that the matter field Φ transforms under a gauge transformation. First, we consider an infinitesimal $U(1)$ transformation associated with the aforementioned symmetry, but with a *space-time dependent* parameter:

$$\Phi(x) \;\mapsto\; \Phi(x) + \alpha(x)\delta\Phi(x)\,. \tag{3.82}$$

If α were a constant, the action would be invariant (up to a total derivative). Here we have:

$$\delta S = \int d^4x \left(\frac{\partial\mathcal{L}}{\partial\Phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\right)\alpha\delta\Phi + \int d^4x\, \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\alpha\delta\Phi\right) = \int d^4x\, j^\mu\partial_\mu\alpha + \text{t.d.} \tag{3.83}$$

where j^μ is the Noether current for the $U(1)$ symmetry.

Thus under a combined transformation of the form (3.82) for the matter fields and a gauge transformation $A_\mu \mapsto A_\mu + \partial_\mu\Gamma$, the action (3.80) transforms as:

$$\delta_{\alpha,\Gamma} S = \int d^4x \left(j^\mu\partial_\mu\alpha + \frac{\delta S_{\text{INT.}}}{\delta A_\mu(x)}\partial_\mu\Gamma\right)\,, \tag{3.84}$$

since the action $S[A]$ for the Maxwell field is gauge invariant. Thus, as claimed the action is invariant provided that:

1. matter fields are charged under gauge transformations, *i.e.* at the infinitesimal level

$$\Phi(x) \xrightarrow{\Gamma} \Phi(x) + \Gamma(x)\delta\Phi(x) + \mathcal{O}(\Gamma^2)\,. \tag{3.85}$$

2. the interaction terms in the action satisfy $\delta S_{\text{INT.}}/\delta A_\mu(x) = -j^\mu(x)$.

Conversely, if Φ transforms under gauge variations as in (3.85), gauge invariance of the full action (3.80) implies the existence of a conserved current j^μ.

Complex scalar field. As we have discussed in the first part of these lectures, the Lagrangian density for a free complex scalar field,

$$\mathcal{L} = \partial_\mu\Phi^*\partial^\mu\Phi - m^2\Phi^*\Phi \tag{3.86}$$

is invariant under the $U(1)$ symmetry:

$$\Phi \;\mapsto\; e^{-ie\theta}\Phi\;, \quad \Phi^* \;\mapsto\; e^{+ie\theta}\Phi^*\,. \tag{3.87}$$

with Noether current:

$$j^\mu_{\text{free}} = ie\big(\Phi^*\partial^\mu\Phi - \Phi\partial^\mu\Phi^*\big)\,. \tag{3.88}$$

In order to minimally couple this theory to the electromagnetic field, inspired by the Dirac fermion theory we introduce the covariant derivative:

$$D_\mu\Phi \overset{\text{def.}}{=} \partial_\mu\Phi + ieA_\mu\Phi\,, \tag{3.89}$$

in terms of which the Lagrangian density reads:

$$\boxed{\mathcal{L} = D_\mu\Phi^* D^\mu\Phi - m^2\Phi^*\Phi = (\partial_\mu - ieA_\mu)\,\Phi^*\,(\partial^\mu + ieA^\mu)\,\Phi - m^2\Phi^*\Phi\,.} \tag{3.90}$$

This action is invariant under the gauge transformation:

$$A_\mu \mapsto A_\mu + \partial_\mu\Gamma\;, \quad \Phi \;\mapsto\; e^{-ie\Gamma}\Phi\,. \tag{3.91}$$

The interaction term $\mathcal{L}_{\text{INT.}}$ is by definition the set of terms containing the gauge field A_μ:

$$\mathcal{L}_{\text{INT.}} = -\underbrace{ie\,(\Phi^*\partial^\mu\Phi - \Phi\partial^\mu\Phi^*)}_{j^\mu_{\text{free}}}A_\mu + e^2A_\mu A^\mu\Phi^*\Phi\,. \tag{3.92}$$

The first term looks like the coupling of the conserved current (3.88) of the free theory with the gauge field. The Noether current associated with the symmetry (3.87) in the theory defined by the Lagrangian density (3.90) reads:

$$j^\mu = ie\,(\Phi^*D^\mu\Phi - \Phi D^\mu\Phi^*) = ie\,(\Phi^*(\partial^\mu + ieA^\mu)\Phi - \Phi(\partial^\mu - ieA^\mu\Phi^*)\;, \tag{3.93}$$

such that as expected

$$\frac{\delta S_{\text{INT.}}}{\delta A^\mu} = -j^\mu\,, \tag{3.94}$$

where j^μ is the conserved current in the *full theory*, which involves the gauge field A_μ itself.

3.2 Quantum electrodynamics processes

Quantum electrodynamics, in the historical sense, is the quantum theory describing the interactions between electrons/positrons and the electromagnetic field. As described above, the fundamental property of this theory is gauge invariance, which consists of the combined transformations

$$\Psi \mapsto \ e^{-ie\Gamma}\Psi\,, \tag{3.95a}$$

$$A_\mu \mapsto \ A_\mu + \partial_\mu \Gamma\,. \tag{3.95b}$$

Based on the symmetries of the problem, this theory is described by the Lagrangian density (3.79), *i.e.* the minimal coupling between a massive Dirac spinor field and a massless spin one field. The free parameter e, the electric charge of the electron, is the coupling constant of the problem, measuring the strength of the interactions between the electrons and the electromagnetic field. Of course the charge e is dimension-full (in non-natural units). The correct dimension-less coupling constant of the theory is the well-known *fine-structure constant*,

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \simeq \frac{1}{137}\,. \tag{3.96}$$

In natural units, one has $\alpha = e^2/4\pi$ as the expansion parameter of perturbation theory. Using the Noether current (3.73) and the mode expansion (1.317), one finds that the electric charge operator is given by:

$$Q = e\int d^3x\,\bar\Psi(x)\gamma^0\Psi(x) = e\int \frac{d^3p}{(2\pi)^3}\sum_{s=1}^{2}\left(b^{s\,\dagger}_{\vec p}b^{s}_{\vec p} - c^{s\,\dagger}_{\vec p}c^{s}_{\vec p}\right)\,. \tag{3.97}$$

As expected, the charges of electron and positron one-particle states are opposite to each other:

$$Q\sqrt{2E_{\vec p}}b^{s\,\dagger}_{\vec p}\,|0\rangle = e\sqrt{2E_{\vec p}}\int\frac{d^3q}{(2\pi)^3}\sum_{r=1}^{2} b^{r\,\dagger}_{\vec q}\left\{\,b^{r}_{\vec q},\,b^{s\,\dagger}_{\vec p}\,\right\}|0\rangle = e\sqrt{2E_{\vec p}}b^{s\,\dagger}_{\vec p}\,|0\rangle\ , \tag{3.98a}$$

$$Q\sqrt{2E_{\vec p}}c^{s\,\dagger}_{\vec p}\,|0\rangle = -e\sqrt{2E_{\vec p}}\int\frac{d^3q}{(2\pi)^3}\sum_{r=1}^{2} c^{r\,\dagger}_{\vec q}\left\{\,c^{r}_{\vec q},\,c^{s\,\dagger}_{\vec p}\,\right\}|0\rangle = -e\sqrt{2E_{\vec p}}c^{s\,\dagger}_{\vec p}\,|0\rangle\ . \tag{3.98b}$$

3.2.1 Perturbation theory

The rules for computing QED scattering amplitudes in perturbation theory follow the general logic presented in the previous section. The only important difference, gauge invariance, will be discussed in detail, because the consistency of the whole theory, in particular unitarity, depends on it.

LSZ reduction for the electromagnetic field. The derivation of the LSZ reduction formulæ is very similar to the case to of a scalar field, except that we need to take care of the photon polarization vectors.

The starting point are inversion formulæ analoguous to eqns. (1.315) obeyed by the scalar field modes. Using the decomposition (3.33a) of the gauge field into creation and annihilation modes, and using the completeness relation (3.25b) obeyed by any basis of photon polarization vectors, one finds that:

$$a^{\lambda}_{\vec{k}} = \eta^{\lambda\lambda'} \varepsilon^{\mu}_{\lambda'}(\vec{k}) \left\langle \tfrac{e^{-ik\cdot x}}{\sqrt{2\omega_k}} \middle| A_\mu(x) \right\rangle \tag{3.99a}$$

$$a^{\dagger}_{\vec{k}} = -\eta^{\lambda\lambda'} \varepsilon^{\mu *}_{\lambda'}(\vec{k}) \left\langle \tfrac{e^{ik\cdot x}}{\sqrt{2\omega_k}} \middle| A_\mu(x) \right\rangle \tag{3.99b}$$

With the same reasoning as in the previous section, we reach the following pair of rules in position space:

LSZ reduction for the electromagnetic field

- Incoming photon of momentum k and polarization λ:
$$\tfrac{i}{\sqrt{Z_3}} \int d^4x \; e^{-ik\cdot x} \Box_x \langle T \cdots A_\mu(x) \cdots \rangle \epsilon^{\mu}_{\lambda}(\vec{k})$$
- Outgoing photon of momentum k and polarization λ:
$$\tfrac{i}{\sqrt{Z_3}} \int d^4x \; e^{+ik\cdot x} \Box_x \langle T \cdots A_\mu(x) \cdots \rangle \epsilon^{\mu *}_{\lambda}(\vec{k})$$

In this expression, as you may guess, Z_3 is the wave-function renormalization of the photon, *i.e.* such that the Källén–Lehmann propagator of the photon has residue Z_3 at the simple pole $p^2 = 0$ corresponding to a one-photon state.

Feynman rules. Let us summarize how to compute a given scattering amplitude in QED. By convention, the photon propagator is represented by a wiggled line.

In the following we will use renormalized perturbation theory, as opposed to bare perturbation theory, since we have convinced ourselves that this is a more convenient way to organize the perturbative corrections. Setting aside momentarily the various counter-terms, one obtains the rules given on fig. 3.1.

Using these rules and the general principles of perturbation theory (momentum and charge conservation, symmetry factors) one can evaluate the contribution of a given (fully amputated) Feynman diagram by dressing the result with the following factors associated with the external particles:

- for an incoming electron, $u_s(\vec{p})$
- for an incoming positron, $\bar{v}_s(\vec{p})$
- for an outgoing electron, $\bar{u}_s(\vec{p})$

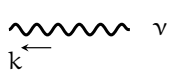
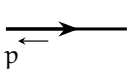
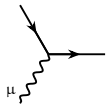
Photon propagator	$\overbrace{A^\mu A^\nu}$		$\frac{-i\eta^{\mu\nu}}{k^2+i\epsilon}$
Dirac fermion propagator	$\overbrace{\Psi\bar{\Psi}}$		$\frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}$
QED vertex	$-e\int A^\mu\bar{\Psi}\gamma_\mu\Psi$		$-ie\gamma_\mu$

Figure 3.1: *Feynman rules for QED.*

- for an outgoing positron, $v_s(\vec{p})$
- for an incoming photon, $\varepsilon^\mu_\lambda(\vec{k})$
- for an outgoing photon, $\varepsilon^{\mu\,*}_\lambda(\vec{k})$

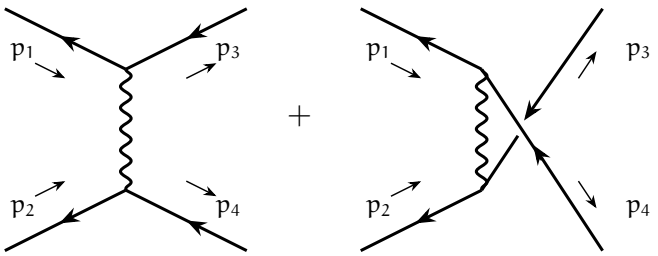
Notice that the wave-function renormalization factors for the electron and the photon, respectively $\sqrt{Z_2}$ and $\sqrt{Z_3}$, do not appear as we have chosen to work in renormalized perturbation theory; the structure of the counter-terms will be discussed shortly.

Let us start by getting to grips with few examples of tree-level processes in quantum electrodynamics, *i.e.* without loop corrections. We will also relate them to the corresponding classical quantities.

3.2.2 Tree-level scattering processes

The study of tree-level scattering amplitudes involving electrons, positrons and photons allows to make contact with well-known results of classical electrodynamics. We will also make important consistency checks, related to gauge invariance of the theory, that will be generalized later on, and introduce useful technology like gamma-matrix traces and polarization averages.

Electron-electron scattering. Scattering of two electrons of incoming momenta p_1 and p_2 at tree-level receives, by charge conservation, contributions from a t-channel and a u-channel diagram:



Denoting by s_1 and s_2 (resp. s_3 and s_4) the polarization of the incoming (resp. outgoing) electrons, the amputated scattering amplitude corresponding to these Feynman diagrams reads:

$$\begin{aligned} i\mathcal{M}(p_1,p_2\to p_3,p_4) &= (-ie)^2\Bigg(\bar u_{s_3}(\vec p_3)\gamma_\mu u_{s_1}(\vec p_1)\frac{-i\eta^{\mu\nu}}{(p_1-p_3)^2+i\epsilon}\bar u_{s_4}(\vec p_4)\gamma_\nu u_{s_2}(\vec p_2) \\ &\qquad - \bar u_{s_4}(\vec p_4)\gamma_\mu u_{s_1}(\vec p_1)\frac{-i\eta^{\mu\nu}}{(p_1-p_4)^2+i\epsilon}\bar u_{s_3}(\vec p_3)\gamma_\nu u_{s_2}(\vec p_2)\Bigg) \\ &= ie^2\left(\frac{\bar u_{s_3}(\vec p_3)\gamma_\mu u_{s_1}(\vec p_1)\bar u_{s_4}(\vec p_4)\gamma^\mu u_{s_2}(\vec p_2)}{t+i\epsilon} - \frac{\bar u_{s_4}(\vec p_4)\gamma_\mu u_{s_1}(\vec p_1)\bar u_{s_3}(\vec p_3)\gamma^\mu u_{s_2}(\vec p_2)}{u+i\epsilon}\right) \end{aligned} \tag{3.100}$$

Both terms are interpreted as the exchange of a virtual photon between the incoming electrons. The relative minus sign is due to the crossing of fermionic lines; indeed we find a relative minus sign if we compare the Wick contractions:

$$\overbrace{\Psi_3\Psi_4\int\bar\Psi_x\Psi_x\int\bar\Psi_y\Psi_y} \quad\text{and}\quad \overbrace{\Psi_3\Psi_4\int\bar\Psi_x\Psi_x\int\bar\Psi_y\Psi_y}\,.$$

Let us take this opportunity to check that, as claimed before, the same result can be obtained using any photon propagator of the form (3.57), with arbitrary parameter β. Looking for instance at the effect of the extra term in $k^\mu k^\nu$ to the t-channel contribution to eqn. (3.100), one notices that:

$$\bar u_{s_3}(\vec p_3)(\not p_1-\not p_3)u_{s_1}(\vec p_1)\bar u_{s_4}(\vec p_4)(\not p_1-\not p_3)u_{s_2}(\vec p_2)=0\,,$$

because, using eqns. (1.179,1.181),

$$\not p_1 u_{s_1}(\vec p_1)=m u_{s_1}(\vec p_1)\quad,\qquad \bar u_{s_3}(\vec p_3)\not p_3 = m\bar u_{s_3}(\vec p_3)\,.$$

The extra contribution from the u-channel vanishes as well, using a similar argument (that external momenta are on-shell).

Recovering the Coulomb potential. The scattering amplitude for electron-electron scattering should be compatible, in the non-relativistic limit, with the repulsive Coulomb potential between two identical electric charges.

First, let us examine the non-relativistic limit limit of Dirac spinors, *i.e.* the limit $m\gg\|\vec p\|$. Starting from eqns. (1.192,1.195), one finds that

$$u_s(\vec p)\overset{m\gg\|\vec p\|}{\sim}\sqrt{m}\begin{pmatrix}\eta_s\\ \eta_s\end{pmatrix}\quad,\quad v_s(\vec p)\overset{m\gg\|\vec p\|}{\sim}\sqrt{m}\begin{pmatrix}\eta_s\\ -\eta_s\end{pmatrix}\,. \tag{3.101}$$

In this limit we have the contractions

$$\bar{u}_r \gamma^0 u_s = m \begin{pmatrix} \eta_r^\dagger & \eta_r^\dagger \end{pmatrix} \begin{pmatrix} \eta_s \\ \eta_s \end{pmatrix} = 2m \eta_r^\dagger \eta_s = 2m \delta_{rs} \,, \tag{3.102a}$$

$$\bar{u}_r \vec{\gamma} u_s = m \begin{pmatrix} \eta_r^\dagger & \eta_r^\dagger \end{pmatrix} \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \begin{pmatrix} \eta_s \\ \eta_s \end{pmatrix} = 0 \,. \tag{3.102b}$$

We also have in the non-relativistic limit

$$t = (p_1^0 - p_3^0)^2 - \|\vec{p}_1 - \vec{p}_3\|^2 \simeq -\|\vec{p}_1 - \vec{p}_3\|^2 \,.$$

In the non-relativistic computation of Coulomb scattering one considers an electron moving in a classical Coulomb potential created by another, stationnary, charge, so it makes sense to consider that the electrons appearing in the relativistic scattering process as *distinguishable* particles, so we keep only the t-channel contribution. The scattering amplitude (3.100) becomes then:

$$\langle \vec{p}_3, \vec{p}_4 | \, i\mathcal{T} \, | \vec{p}_1, \vec{p_2} \rangle \, \big|_{\text{con.}} \simeq (2\pi)^3 \delta^{(3)}(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4) 2\pi\delta\big(E_{\vec{p}_3} - E_{\vec{p}_1}\big) \\ \frac{-ie^2}{\|\vec{p}_1 - \vec{p}_3\|^2 - i\epsilon} \big(2m\delta_{s_1,s_3}\big)\big(2m\delta_{s_2,s_4}\big) \,. \tag{3.103}$$

The $2m$ factors are due to the choice of relativistic normalization of the on-particle states, and will be ignored in the following. If the stationnary electron is at $\vec{x}_0$, one should consider the Fourier transform:

$$\int \frac{d^3 p_2}{(2\pi)^3} e^{i\vec{p}_2 \cdot \vec{x}_0} \, \langle \vec{p}_3, \vec{p}_4 | \, i\mathcal{T} \, | \vec{p}_1, \vec{p_2} \rangle \, \big|_{\text{con.}} \,.$$

One can naturally choose to put the particle at the origin, *i.e.* $\vec{x}_0$, thus it just amounts to remove the momentum conservation factor.

Recall that, in non-relativistic quantum mechanics, the wave-function corresponding to the scattering of a plane wave of momentum $\vec{k}$ by a potential $V(\vec{r})$ is of the form

$$\psi_{\vec{k}}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} + f(\vec{k}', \vec{k}) \frac{e^{ikr}}{r} \,, \tag{3.104}$$

where $\vec{k}' = \|\vec{k}\| \vec{e}_r$. The scattering amplitude $f(\vec{k}', \vec{k})$ is given, as in QFT, in terms of the components of the transition matrix $\mathcal{T}$,

$$\left\langle \vec{k}' \right| i\mathcal{T} \left| \vec{k} \right\rangle = 2\pi\delta\big(E_{\vec{k}'} - E_{\vec{k}}\big) \, \frac{2i\pi}{m} \, f(\vec{k}, \vec{k}') \,. \tag{3.105}$$

which are given in the Born approximation by:

$$\left\langle \vec{k}' \right| \mathcal{T} \left| \vec{k} \right\rangle = -2\pi\delta\big(E_{\vec{k}'} - E_{\vec{k}}\big) \widehat{V}\big(\vec{k}' - \vec{k}\big) \,. \tag{3.106}$$

where $\widehat{V}$ is the Fourier transform of the potential, see Sakurai [1] for more details.[6]

[6]The $\mathcal{T}$ matrix defined in Sakurai has an opposite sign convention to ours, *i.e.* he has chosen $S = 1 - i\mathcal{T}$ for the S-matrix.

Comparing both expressions, one gets the scattering potential by an inverse Fourier transform (with $\vec{q} = \vec{p}_1 - \vec{p}_3$):

$$V(\vec{r}) = e^2 \int \frac{d^3q}{(2\pi)^3} \frac{e^{i\vec{q}\cdot\vec{r}}}{q^2 - i\epsilon} \tag{3.107}$$

Let us choose spherical coordinates with $\vec{q}\cdot\vec{r} = qr\cos\theta$. One has then:

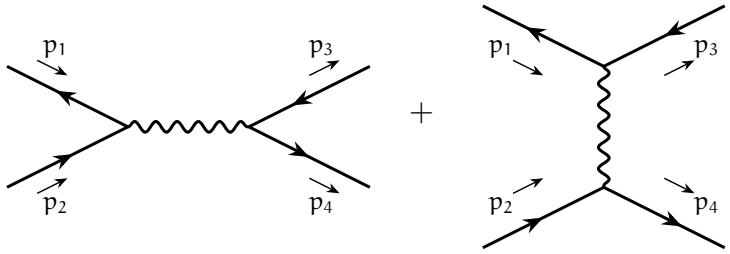
$$\begin{aligned} V(\vec{r}) &= \frac{e^2}{4\pi^2} \int_0^\infty \frac{q^2 dq}{q^2 - i\epsilon} \int_0^\pi \sin\theta d\theta \, e^{iqr\cos\theta} = \frac{e^2}{4\pi^2 ir} \int_0^\infty \frac{q dq}{q^2 - i\epsilon} \left(e^{iqr} - e^{-iqr}\right) \\ &= \frac{e^2}{4\pi^2 ir} \int_{-\infty}^\infty \frac{q\, e^{iqr} dq}{(q - \sqrt{i\epsilon})(q + \sqrt{i\epsilon})} \,. \end{aligned}$$

Since $r > 0$ by construction, one can close the contour in the upper half-plane, picking the residue at $q = \sqrt{i\epsilon}$. It gives:

$$V(\vec{r}) = \frac{e^2}{4\pi^2 ir} \times 2i\pi \operatorname*{Res}_{q=\sqrt{i\epsilon}} \left(\frac{q\, e^{iqr} dq}{(q - \sqrt{i\epsilon})(q + \sqrt{i\epsilon})} \right) = \frac{e^2}{2\pi} \frac{\sqrt{i\epsilon} e^{i\sqrt{i\epsilon}r}}{2\sqrt{i\epsilon}} \overset{\epsilon\to 0^+}{\longmapsto} \frac{e^2}{4\pi r} \,, \tag{3.108}$$

as expected.

Electron-positron scattering. Scattering of an electron of incoming momenta p_1 and a positron p_2 at tree-level receives, by charge conservation, contributions from an s-channel and a t-channel diagram:

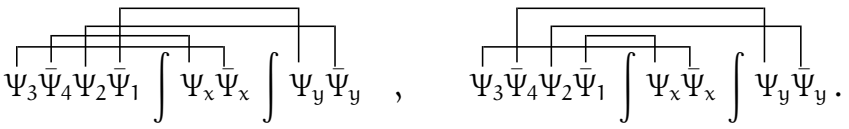


The s-channel diagram is interpreted as a process where the incoming electron and positron annihilate into a photon, which eventually decays into an electron-positron pair of outgoing particles. The t-channel diagram is interpreted as before as the exchange of a virtual photon between the incoming particles. The scattering amplitude corresponding to the sum of these two diagrams is given by:

$$\begin{aligned} &i\mathcal{M}(p_1, p_2 \to p_3, p_4) = \\ &\quad -ie^2 \left(\frac{\bar{v}_{s_2}(\vec{p}_2)\gamma_\mu u_{s_1}(\vec{p}_1)\bar{u}_{s_3}(\vec{p}_3)\gamma^\mu v_{s_4}(\vec{p}_4)}{s + i\epsilon} - \frac{\bar{u}_{s_3}(\vec{p}_3)\gamma_\mu u_{s_1}(\vec{p}_1)\bar{v}_{s_2}(\vec{p}_2)\gamma^\mu v_{s_4}(\vec{p}_4)}{t + i\epsilon} \right) . \end{aligned} \tag{3.109}$$

Independence on the choice of photon propagator can be again verified, using eqn. (1.194) and its Dirac conjugate. The relative minus sign between both terms can be obtained by

comparing the two corresponding Wick contractions:



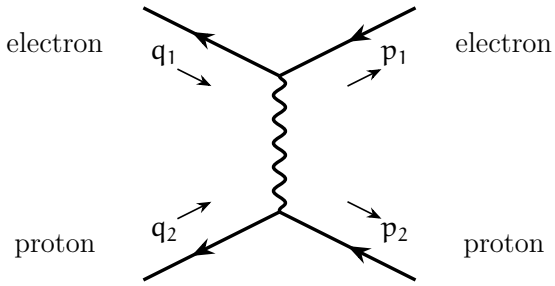
Rutherford scattering. A variant of this process is given by the scattering of an electron with a proton, which is known as *Rutherford scattering*. It is of major historical importance as the corresponding experiment led to the modern views on the atom structure, with a nucleus carrying positive electric charge. A standard result of classical electrodynamics is the Rutherford scattering formula

$$\mathrm{d}\sigma = \frac{\alpha^2}{4m_e^2 v^4 \sin^4\frac{\theta}{2}}\mathrm{d}\Omega\,, \tag{3.110}$$

which represents the differential cross section for the elastic scattering of an electron by the Coulomb potential created by the proton, in terms of the scattering angle θ and of the electron speed v.

Let us recall that, for an incoming flux Φ of particles (*i.e.* the number of incoming particles per unit time and per unit area), the number of scattering events per unit time is $\mathrm{d}N = \Phi\sigma$. The differential cross section $\mathrm{d}\sigma$ gives the fraction of the scattering events producing scattered particles within the solid angle $\mathrm{d}\Omega$.

Let us consider this problem from the point of view of QED. Compared to the previous case of electron/positron scattering, since we have two distinct vertices, $e^-e^-\gamma$ and $p^+p^+\gamma$. Said differently, it is obvious that an electron and a proton cannot annihilate into a photon!



Therefore, the corresponding scattering amplitude has only a contribution from the t-channel diagram:

$$i\mathcal{M}(q_1, q_2 \to p_1, p_2) = ie^2\,\frac{\bar{u}_{r_1}(\vec{p}_1)\gamma_\mu u_{s_1}(\vec{q}_1)\bar{v}_{s_2}(\vec{q}_2)\gamma^\mu v_{r_2}(\vec{p}_2)}{t + i\epsilon}\,, \tag{3.111}$$

where s_1 (resp. r_1) is the polarization of the incoming (resp. outgoing) electron, s_2 (resp. r_2) is the polarization of the incoming (resp. outgoing) proton.

The quantum mechanical probability for a scattering event with two incoming particles of momenta q_1 and q_2 and F outgoing particles of momenta $p_1, \ldots, p_F$ in the range $\mathrm{d}^3p_1 \cdots \mathrm{d}^3p_F$

is in principle given excluding the case where no scattering occurs, by:[7]

$$p\big(q_1, q_2 \to \{p_f\}\big) = \prod_{f=1}^{F} \frac{d^3 p_f}{(2\pi)^3 2E_f} \big| \langle p_1, \ldots, p_f | \overbrace{S - 1}^{i\mathcal{T}} | q_1, q_2 \rangle \big|^2 , \tag{3.112}$$

where we have used the Lorentz-invariant measure, that we will denote by $d\Pi_F$, for the outgoing particles. This expression as it stands is ill-defined as the incoming and outgoing states are unphysical plane waves rather than physical wave-packets. A proper treatment of the problem can be found in Peskin and Schroeder (p.102-108) and will be also studied during the tutorials. Let us use here a slightly sloppy shortcut. We have

$$\langle p_1, \ldots, p_f | \mathcal{T} | q_1, q_2 \rangle = (2\pi)^4 \delta^{(4)}\big(q_1 + q_2 - \textstyle\sum p_f\big) \mathcal{M}\big(q_1, q_2 \to \{p_f\}\big) , \tag{3.113}$$

where the Dirac distribution, as discussed before, enforces momentum conservation. Squaring this matrix element is certainly dubious; let us attempt to interpret it as:

$$\big| \langle p_1, \ldots, p_f | \mathcal{T} | q_1, q_2 \rangle \big|^2 \overset{?}{=} (2\pi)^4 \delta^{(4)}\big(q_1 + q_2 - \textstyle\sum p_f\big) \underbrace{(2\pi)^4 \delta^{(4)}(0)}_{\mathrm{Vol}\left(\mathbb{R}^{1,3}\right)} \big| \mathcal{M}\big(q_1, q_2 \to \{p_f\}\big) \big|^2 . \tag{3.114}$$

This divergence due to the infinite "time volume" factor is the same as in the derivation of Fermi's golden rule in quantum mechanics. Inspired by this example we define a *transition rate* per unit time and volume:

$$\omega\big(q_1, q_2 \to \{p_f\}\big) = (2\pi)^4 \delta^{(4)}\big(q_1 + q_2 - \textstyle\sum p_f\big) \big| \mathcal{M}\big(q_1, q_2 \to \{p_f\}\big) \big|^2 d\Pi_F . \tag{3.115}$$

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Differential cross-section. In a scattering experiment, the measurable quantity is the *cross section*. Imagine an idealized classical scattering experiment as represented on figure 3.2.

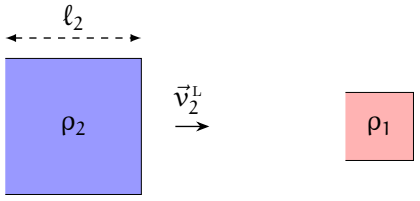

Figure 3.2: *Idealized scattering experiment*

A beam of particles of type 2, with constant density ρ_2 and length ℓ_2, is sent towards a target made of particles of type 1 with constant density ρ_1 occupying a volume $\mathcal{V}_1$. We

[7] We have considered that the outgoing particles are distinguishable, otherwise the appropriate factor should be added, for instance $1/F!$ if they are all undistinguishable.

assume that the transverse area of the incoming beam overlaps completely the transverse area of the target, and we work in the *lab frame* in which the target 1 is motionless and the incoming beam moves at a constant and uniform velocity $\vec{v}_2^{\,L}$. Let N be the number of scattering events. The cross-section σ is defined by the relation:

$$N = \sigma \ell_2 \rho_2 \times \mathcal{V}_1 \rho_1 \,. \tag{3.116}$$

Thus σ can be interpreted as the effective area of the incoming beam of particles that scatters off the target. Let n be the number of events by unit volume and unit time. It is given by:

$$n = \sigma \, \rho_2 \rho_1 \big\| \vec{v}_2^{\,L} \big\| \,, \tag{3.117}$$

because the effective interaction time is given by $\ell_2 / \big\| \vec{v}_2^{\,L} \big\|$. In a quantum mechanical context, the densities should be replaced by probability densities. With the same sloppy approach as above we have for momentum states:

$$\langle \vec{p} | \vec{p} \rangle = (2\pi)^2 2E_p \delta^{(3)}(\vec{0}) \;\to\; \overbrace{2E_p}^{\rho} \mathrm{Vol}\big(\mathbb{R}^3\big) \tag{3.118}$$

Thus one can relate the transition rate (3.115) to the cross section as:

$$\omega = 2E_1 2E_2 \big\| \vec{v}_2^{\,L} \big\| \sigma \tag{3.119}$$

This was established in the lab frame. A more general expression invariant under boosts along the direction of the beam is given by:

$$\omega = 2E_1 2E_2 \big\| \vec{v}_1 - \vec{v}_2 \big\| \sigma \,, \tag{3.120}$$

involving the relative velocity $\|\vec{v}_1 - \vec{v}_2\|$ between the two sets of particles.

Again, a proper derivation using wave-packets is given in Peskin and Schroeder. The final result is as follows. The contribution to the cross section of an outgoing state made of F particles, with momenta $\vec{p}_1, \ldots, \vec{p}_F$ in the range $d^3p_1 \cdots d^3p_F$ is given by:

$$\boxed{d\sigma = \frac{1}{2E_{q_1} 2E_{q_2} \|\vec{v}_1 - \vec{v}_2\|} \prod_{f=1}^{F} \frac{d^3p_f}{(2\pi)^3 2E_f} (2\pi)^4 \delta^{(4)}\left(q_1 + q_2 - \sum p_f\right) |\mathcal{M}(q_1, q_2 \to p_1, \ldots, p_F)|^2} \tag{3.121}$$

The Lorentz-invariant measure over the final state momenta $d\Pi_F$ has to be multiplied by $1/F!$ if the final state particles are undistinguishable.

Whenever the final state contains two particle one can reduce the integral over final momenta to an angular integral. In the center-of-mass frame, calling the total energy E_{cm},

one has

$$\begin{aligned}\int \mathrm{d}\Pi_2 &= \frac{1}{16\pi^2}\int \frac{\mathrm{d}^3p_1}{\sqrt{\vec{p}_1^{\,2}+m_1^2}}\int \frac{\mathrm{d}^3p_2}{\sqrt{\vec{p}_2^{\,2}+m_2^2}}\delta^{(3)}(\vec{p}_1-\vec{p}_2)\,\delta\Big(\sqrt{\vec{p}_1^{\,2}+m_1^2}+\sqrt{\vec{p}_2^{\,2}+m_2^2}-E_{cm}\Big)\\ &= \frac{1}{16\pi^2}\int \frac{\mathrm{d}^3p_1}{\sqrt{\vec{p}_1^{\,2}+m_1^2}\sqrt{\vec{p}_1^{\,2}+m_2^2}}\delta\Big(\sqrt{\vec{p}_1^{\,2}+m_1^2}+\sqrt{\vec{p}_1^{\,2}+m_2^2}-E_{cm}\Big)\\ &= \frac{1}{16\pi^2}\int \mathrm{d}\Omega\int \frac{p^2\mathrm{d}p}{\sqrt{(p^2+m_1^2)(p^2+m_2)}}\delta\Big(\sqrt{p^2+m_1^2}+\sqrt{p^2+m_2^2}-E_{cm}\Big)\\ &= \frac{1}{16\pi^2}\int \mathrm{d}\Omega\,\frac{p}{\sqrt{p^2+m_1^2}+\sqrt{p^2+m_2^2}}\Big|_{\sqrt{p^2+m_1^2}+\sqrt{p^2+m_2^2}=E_{cm}}\end{aligned}$$

Thus one finds the following result for the differential cross-section of $2\to 2$ scattering:

$$\mathrm{d}\sigma = \frac{1}{2E_{q_1}2E_{q_2}\|\vec{v}_1-\vec{v}_2\|}\frac{\|\vec{p}_1\|}{16\pi^2 E_{cm}}\Big|\mathcal{M}(q_1,q_2\to p_1,p_2)\Big|^2\mathrm{d}\Omega\,. \tag{3.122}$$

Total cross-section and the optical theorem. From the expression of the differential cross-section (3.121), one can deduce the *total cross-section* for the scattering of two incoming particles with any number of outgoing particles of arbitary momenta:

$$\begin{aligned}&\sigma_T(q_1,q_2)\overset{\text{def.}}{=}\sigma(q_1,q_2\to\text{anything})=\\ &\frac{1}{2E_{q_1}2E_{q_2}\|\vec{v}_1-\vec{v}_2\|}\sum_F\int\prod_{f=1}^{F}\frac{\mathrm{d}^3p_f}{(2\pi)^3 2E_f}(2\pi)^4\delta^{(4)}\left(q_1+q_2-\textstyle\sum p_f\right)|\mathcal{M}(q_1,q_2\to p_1,\ldots,p_F)|^2\end{aligned} \tag{3.123}$$

The right-hand side of this equation is very similar to the right-hand side of the optical theorem, see eqn. (2.47) in chapter 2. The only difference is the kinematic prefactor. In the center-of-mass frame, it can be simplified a bit:

$$E_{q_1}E_{q_2}\|\vec{v}_1-\vec{v}_2\| = E_{q_1}E_{q_2}\|\vec{v}_1\|(1+E_{q_1}/E_{q_2}) = \|\vec{q}_1\|E_{CM}\,, \tag{3.124}$$

where $E_{CM}=E_{q_1}+E_{q_2}$ is the center-of-mass energy. We also introduce $q_{CM}=\|\vec{q}_1\|=\|\vec{q}_2\|$ and reach the classical form of the optical theorem:

$$\mathrm{Im}\,\mathcal{M}(q_1,q_2\to q_1,q_2) = 2E_{CM}q_{CM}\,\sigma_T(q_1,q_2) \tag{3.125}$$

The LHS of this equation is the imaginary part of the *forward* $2\to 2$ *scattering amplitude*, the scattering amplitude for a $2\to 2$ process with identical incoming and outgoing states.

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After this long detour, let us come back to our Rutherford scattering process in QED. In a typical experiment, the initial beams are not polarized, *i.e.* one should take an average of the polarizations of the incoming spinors. Likewise, if the detector is not sensitive to the spin polarizations, one should sum over the contributions coming all polarizations of the outgoing spinors. One should replace then in the expression of the cross-section:

$$|\mathcal{M}(q_1, q_2 \to p_1, p_2)|^2 \;\longrightarrow\; \sum_{r_1}\sum_{r_2}\frac{1}{2}\sum_{s_1}\frac{1}{2}\sum_{s_2}|\mathcal{M}(q_1, q_2 \to p_1, p_2)|^2 \,.$$

In the case of Rutherford scattering, one considers the following expression from eqn. (3.111):

$$\begin{aligned}&\frac{1}{4}\sum_{r_1,r_2,s_1,s_2}\left|\bar{u}_{r_1}(\vec{p}_1)\gamma_\mu u_{s_1}(\vec{q}_1)\bar{v}_{s_2}(\vec{p}_2)\gamma^\mu v_{r_2}(\vec{q}_2)\right|^2\\ &=\frac{1}{4}\sum_{r_1,r_2,s_1,s_2}\bar{u}_{r_1}(\vec{p}_1)\gamma_\mu u_{s_1}(\vec{q}_1)\bar{u}_{s_1}(\vec{q}_1)\gamma_\nu u_{r_1}(\vec{p}_1)\bar{v}_{s_2}(\vec{p}_2)\gamma^\mu v_{r_2}(\vec{q}_2)\bar{v}_{r_2}(\vec{q}_2)\gamma^\nu v_{s_2}(\vec{p}_2)\end{aligned} \quad (3.126)$$

Now one can use the spinor identities:

$$\sum_{s_1} u_{s_1}(\vec{q}_1)\bar{u}_{s_1}(\vec{q}_1) = \not{q}_1 + m_e \,, \quad \sum_{r_2} v_{r_2}(\vec{q}_2)\bar{v}_{r_2}(\vec{q}_2) = \not{q}_2 - m_p \,, \quad (3.127)$$

so what remains to compute is:

$$\frac{1}{4}\left(\sum_{r_1}\bar{u}_{r_1}(\vec{p}_1)\gamma_\mu\left(\not{q}_1 + m_e\right)\gamma_\nu u_{r_1}(\vec{p}_1)\right)\left(\sum_{s_2}\bar{v}_{s_2}(\vec{p}_2)\gamma^\mu\left(\not{q}_2 - m_p\right)\gamma^\nu v_{s_2}(\vec{p}_2)\right)$$

Looking for instance at the first term in parenthesis, one realizes that it has the structure of a trace over spinor indices $a, b = 1, \cdots 4$:

$$\sum_{r_1}\sum_{a,b}\bar{u}^a_{r_1}M_{ab}u^b_{r_1} = \sum_{a,b}\left(\sum_{r_1}u^b_{r_1}\bar{u}^a_{r_1}M_{ab}\right) = \left(\not{p}_1 + m_e\right)^{ba}M_{ab} = \mathrm{Tr}\left(\left(\not{p}_1 + m_e\right)M\right),$$

with $M = \gamma_\mu\left(\not{q}_1 + m_e\right)$. The same applies of course to the second term. One obtains then a simplier expression for the summed/averaged scattering probability:

$$\frac{1}{4}\sum|\mathcal{M}|^2 = \frac{e^4}{4t^2}\mathrm{Tr}\left(\left(\not{p}_1 + m_e\right)\gamma_\mu\left(\not{q}_1 + m_e\right)\gamma_\nu\right)\mathrm{Tr}\left(\left(\not{p}_2 - m_p\right)\gamma^\mu\left(\not{q}_2 - m_p\right)\gamma^\nu\right) \quad (3.128)$$

In order to simplify this expression further, one needs few gamma-matrix identities.

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Gamma-matrix identities

$$\mathrm{Tr}\,(\gamma^{\mu_1}\cdots\gamma^{\mu_p}) = 0 \quad \text{for any odd number } p \text{ of gamma-matrices} \tag{3.129a}$$
$$\mathrm{Tr}\,(\gamma^{\mu}\gamma^{\nu}) = 4\eta^{\mu\nu} \tag{3.129b}$$
$$\mathrm{Tr}\,(\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}) = 4\,(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}) \tag{3.129c}$$
$$\gamma^{\mu}\gamma_{\mu} = 4 \tag{3.129d}$$
$$\gamma^{\mu}\gamma^{\nu}\gamma_{\mu} = -2\gamma^{\nu} \tag{3.129e}$$
$$\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma_{\mu} = 4\eta^{\nu\rho} \tag{3.129f}$$
$$\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\gamma_{\mu} = -2\gamma^{\sigma}\gamma^{\rho}\gamma^{\nu} \tag{3.129g}$$

In order to prove property (3.129a), one uses the cyclicity of the trace and the fact that γ^5 anticommutes with all gamma-matrices:

$$\mathrm{Tr}\,(\gamma^{\mu_1}\cdots\gamma^{\mu_p}) = \mathrm{Tr}\,\big((\gamma^5)^2\gamma^{\mu_1}\cdots\gamma^{\mu_p}\big) = \mathrm{Tr}\,\big(\gamma^5\gamma^{\mu_1}\cdots\gamma^{\mu_p}\gamma^5\big) = (-1)^p\mathrm{Tr}\,(\gamma^{\mu_1}\cdots\gamma^{\mu_p})\,. \tag{3.130}$$

Property (3.129b) was already proven, see eqn. (2.181). Let us prove property (3.129c), one uses similar arguments:

$$\begin{aligned}\mathrm{Tr}\,(\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}) &= \mathrm{Tr}\,\big((2\eta^{\mu\nu} - \gamma^{\nu}\gamma^{\mu})\gamma^{\rho}\gamma^{\sigma}\big) = 8\eta^{\mu\nu}\eta^{\rho\sigma} - \mathrm{Tr}\,(\gamma^{\nu}\gamma^{\mu}\gamma^{\rho}\gamma^{\sigma})\\ &= 8\eta^{\mu\nu}\eta^{\rho\sigma} - \mathrm{Tr}\,\big(\gamma^{\nu}(2\eta^{\mu\rho} - \gamma^{\rho}\gamma^{\mu})\gamma^{\sigma}\big) = 8\,(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma}) + \mathrm{Tr}\,(\gamma^{\nu}\gamma^{\rho}\gamma^{\mu}\gamma^{\sigma})\\ &= 8\,(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma}) + \mathrm{Tr}\,\big(\gamma^{\nu}\gamma^{\rho}(2\eta^{\mu\sigma} - \gamma^{\sigma}\gamma^{\mu})\big)\\ &= 8\,(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}) - \mathrm{Tr}\,(\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma})\end{aligned} \tag{3.131}$$

For property (3.129d) we use

$$\gamma^{\mu}\gamma_{\mu} = \eta_{\mu\nu}\frac{\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu}}{2} = \eta_{\mu\nu}\eta^{\mu\nu} = 4\,. \tag{3.132}$$

Exercise: prove properties (3.129e,3.129f,3.129g).

* * *

After this second detour, we can finish the computation of the differential cross-section for Rutherford scattering. Let us simplify:

$$\begin{aligned}&\mathrm{Tr}\,\Big((\not{p}_1 + m_e)\gamma_{\mu}\,(\not{q}_1 + m_e)\,\gamma_{\nu}\Big) = \mathrm{Tr}\,\Big((p_1^{\rho}\gamma_{\rho} + m_e)\gamma_{\mu}\,(q_1^{\sigma}\gamma_{\sigma} + m_e)\,\gamma_{\nu}\Big)\\ &= m_e^2\mathrm{Tr}\,\Big(\gamma_{\mu}\gamma_{\nu}\Big) + p_1^{\rho}q_1^{\sigma}\mathrm{Tr}\,\Big(\gamma_{\rho}\gamma_{\mu}\gamma_{\sigma}\gamma_{\nu}\Big) = 4\,\big((m_e^2 - p_1\cdot q_1)\eta_{\mu\nu} + p_{1\,\mu}q_{1\,\nu} + p_{1\,\nu}q_{1\,\mu}\big)\end{aligned} \tag{3.133}$$

In the same way, one gets that

$$\mathrm{Tr}\,\Big((\not{p}_2 - m_p)\gamma^{\mu}\,(\not{q}_2 - m_p)\,\gamma^{\nu}\Big) = 4\,\big((m_p^2 - q_2\cdot p_2)\eta^{\mu\nu} + p_2^{\mu}q_2^{\nu} + p_2^{\nu}q_2^{\mu}\big) \tag{3.134}$$

Putting everything together, one finds that

$$\frac{1}{4}\sum|\mathcal{M}|^2 = \frac{8e^4}{(p_1-p_3)^4}\Big(2(m_e^2 - q_1\cdot p_1)(m_p^2 - q_2\cdot p_2) + q_1\cdot p_2\, p_1\cdot q_2 \\ + q_1\cdot q_2\, p_1\cdot p_2 + q_1\cdot p_1(m_e^2 - q_1\cdot p_1) + p_2\cdot q_2(m_p^2 - q_2\cdot p_2)\Big)\,. \quad (3.135)$$

A more compact formula can be obtained using the Mandelstam invariants. One has

$$s = (q_1+q_2)^2 = (p_1+p_2)^2 = m_e^2 + m_p^2 + 2q_1\cdot q_2 = m_e^2 + m_p^2 + 2p_1\cdot p_2 \quad (3.136)$$
$$t = (q_1-p_1)^2 = (q_2-p_2)^2 = 2m_e^2 - 2q_1\cdot p_1 = 2m_p^2 - 2q_2\cdot p_2 \quad (3.137)$$
$$u = (q_1-p_2)^2 = (q_2-p_1)^2 = m_e^2 + m_p^2 - 2q_1\cdot p_2 = m_p^2 + m_e^2 - 2q_2\cdot p_1 \quad (3.138)$$

We can then replace

$$\begin{aligned} &2(m_e^2 - q_1\cdot p_1)(m_p^2 - q_2\cdot p_2) + q_1\cdot p_2\, p_1\cdot q_2 \\ &\qquad + q_1\cdot q_2\, p_1\cdot p_2 + q_1\cdot p_1(m_e^2 - q_1\cdot p_1) + p_2\cdot q_2(m_p^2 - q_2\cdot p_2) \\ &= \tfrac{1}{2}t^2 + \tfrac{1}{4}\left(m_p^2 + m_e^2 - u\right)^2 + \tfrac{1}{4}\left(m_p^2 + m_e^2 - s\right)^2 + \tfrac{1}{2}t\left(m_e^2 - \tfrac{1}{2}t\right) + \tfrac{1}{2}t\left(m_p^2 - \tfrac{1}{2}t\right) \\ &= \tfrac{1}{2}\left(m_p^2 + m_e^2\right)^2 - \tfrac{1}{2}\left(m_p^2 + m_e^2\right)\big(\underbrace{s+u}_{2m_e^2+2m_p^2-t}\big) + \tfrac{1}{4}(s^2+u^2) + \tfrac{1}{2}t\left(m_p^2 + m_e^2\right) \\ &= \left(m_p^2 + m_e^2\right)t - \tfrac{1}{2}\left(m_p^2 + m_e^2\right)^2 + \tfrac{1}{4}(s^2+u^2) \end{aligned}$$

So we get finally:

$$\frac{1}{4}\sum|\mathcal{M}|^2 = \frac{2e^4}{t^2}\left(4\left(m_p^2 + m_e^2\right)t - 2\left(m_p^2 + m_e^2\right)^2 + s^2 + u^2\right)\,. \quad (3.139)$$

We can now estimate the leading correction to the Rutherford formula (3.110) as follows. The 4-momenta of the particles in the center-of-mass frame are given, in the limit where the proton is non-relativistic, by

$$q_1^\mu = (E, p\vec{e}_z)\,,\quad q_2^\mu \simeq (m_p, -p\vec{e}_z)\,,\quad p_1^\mu = (E, p\vec{e}_r)\,,\quad p_2^\mu \simeq (m_p, -p\vec{e}_r)\,, \quad (3.140)$$

with the scattering angle θ determined by $\cos\theta = \vec{e}_r\cdot\vec{e}_z$. We have then

$$t = 2m_e^2 - 2q_1\cdot p_1 = -2p^2\big(1-\cos\theta\big) = -4p^2\sin^2\tfrac{\theta}{2} \quad (3.141a)$$
$$u = m_e^2 + m_p^2 - 2q_1\cdot p_2 = m_e^2 + m_p^2 - 2(Em_p + p^2\cos\theta) \quad (3.141b)$$
$$s = m_e^2 + m_p^2 + 2q_1\cdot q_2 = m_e^2 + m_p^2 + 2(m_pE + p^2) \quad (3.141c)$$

Let us extract the leading terms in the expansion in the proton mass. We have

$$u^2 + s^2 = 2m_p^4 + 4m_p^2\left(m_e^2 + 2p^2\sin^2\tfrac{\theta}{2} + 2E^2\right) + \mathcal{O}(m_p)\,, \quad (3.142)$$

so that

$$4m_p^2 t + s^2 + u^2 - 2m_p^4 - 4m_p^2 m_e^2$$
$$= 4m_p^2\big(-4p^2\sin^2\tfrac{\theta}{2} + m_e^2 + 2p^2\sin^2\tfrac{\theta}{2} + 2\overbrace{E^2}^{m_e^2+p^2} - m_e^2\big) = 8m_p^2\left(m_e^2 + p^2\cos^2\tfrac{\theta}{2}\right) \quad (3.143)$$

we have then at leading order:

$$\frac{1}{4}\sum\left|\mathcal{M}\right|^2 = \frac{e^4 m_p^2}{p^4\sin^4\frac{\theta}{2}}\left(m_e^2 + p^2\cos^2\tfrac{\theta}{2}\right) \quad (3.144)$$

What remains to be done is to evaluate the prefactor of eqn. (3.122. We have, in the limit $\|\vec{v}_1\| \gg \|\vec{v}_2\|$ and $E_{cm} \simeq m_p$:

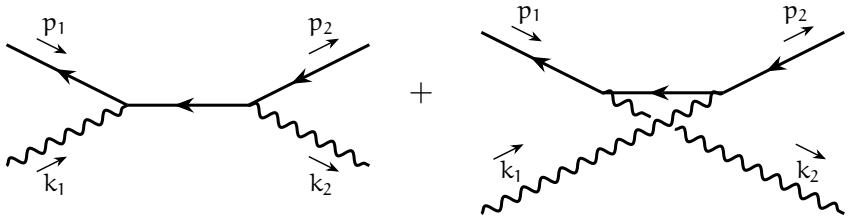
$$\frac{1}{2E_{q_1}2E_{q_2}\|\vec{v}_1 - \vec{v}_2\|}\frac{\|\vec{p}_1\|}{16\pi^2 E_{cm}} = \frac{1}{4}\frac{1}{m_p E\|\vec{v}_1\|}\frac{\|\vec{p}_1\|}{16\pi^2 m_p} = \frac{1}{64\pi^2 m_p^2}\,.$$

In terms of the fine structure constant $\alpha = e^2/4\pi$, one has then:

$$d\sigma = \frac{\alpha^2}{4p^4\sin^4\frac{\theta}{2}}\left(m_e^2 + p^2\cos^2\tfrac{\theta}{2}\right)d\Omega \quad (3.145)$$

This formula, which is a relativistic correction to the Rutherford formula (3.110), is known at the *Mott formula*. As you can see, even in this basic example, going from a scattering amplitude to a cross section is very tedious!

Compton scattering. Compton scattering is the scattering of an electron, of momentum p_1 and polarization s, with a photon of momentum k_1 and polarization λ. The scattering amplitude contains contributions from the following s-channel and u-channel diagrams:



The corresponding scattering amplitude reads:[8]

$$i\mathcal{M}(p_1, k_1 \to p_2, k_2) = -ie^2\,\varepsilon^{\mu *}_{\lambda_2}(\vec{k}_2)\,\varepsilon^{\nu}_{\lambda_1}(\vec{k}_1)\left(\bar{u}_{s_2}(\vec{p}_2)\gamma_\mu\frac{\not{p}_1 + \not{k}_1 + m}{s - m^2 + i\epsilon}\gamma_\nu u_{s_1}(\vec{p}_1)\right.$$
$$\left. + \bar{u}_{s_2}(\vec{p}_2)\gamma_\nu\frac{\not{p}_1 - \not{k}_2 + m}{u - m^2 + i\epsilon}\gamma_\mu u_{s_1}(\vec{p}_1)\right) \quad (3.146)$$

[8]Pay attention that, between the two terms, the photon polarizations are contracted with different vertices.

Let us make another consistency check. The amplitude should be invariant if we shift a photon polarization, say the polarization of the incoming photon, by a term proportional the its 4-momentum:

$$\varepsilon_{\lambda_1}(\vec{k}_1) \mapsto \varepsilon_{\lambda_1}(\vec{k}_1) + \alpha k_1 \,,$$

as the extra term corresponds to a spurious external state. Ignoring the $i\epsilon$ prescription, the s-channel contribution to eqn. (3.146) is shifted by the term:

$$\frac{\varepsilon^{\mu\,*}_{\lambda_2}(\vec{k}_2)\,\bar{u}_{s_2}(\vec{p}_2)\gamma_\mu\,\left(\not{p}_1 + \not{k}_1 + m\right)\,\not{k}_1 u_{s_1}(\vec{p}_1)}{s - m^2} = \bar{u}_{s_2}(\vec{p}_2)\not{\varepsilon}^{\,*}_{\lambda_2}(\vec{k}_2)\,u_{s_1}(\vec{p}_1),$$

where we have used the identities

$$\not{k}\not{k} = \gamma_\mu\gamma_\nu k^\mu k^\nu = \frac{1}{2}\{\gamma_\mu,\,\gamma_\nu\}\,k^\mu k^\nu = k^2 = 0\,, \tag{3.147}$$

as well as

$$\left(\not{p}\not{k} + m\not{k}\right)u_s(\vec{p}) = \big(\underbrace{\gamma_\mu\gamma_\nu}_{-\gamma_\nu\gamma_\mu + 2\eta_{\mu\nu}}\,p^\mu k^\nu + m\not{k}\big)u_s(\vec{p}) = \not{k}\,\underbrace{\left(-\not{p} + m\right)u_s\left(\vec{p}\right)}_{=0} + 2k\cdot p\,u_s\left(\vec{p}\right), \tag{3.148}$$

and finally $s - m^2 = (p_1 + k_1)^2 - m^2 = p_1^2 - m^2 + k_1^2 + 2p_1\cdot k_2 = +2p_1\cdot k_1$.

In a similar way, the u-channel contribution to eqn. (3.146) is shifted by the same quantity but with an opposite sign:

$$\frac{\varepsilon^{\mu\,*}_{\lambda_2}(\vec{k}_2)\,\bar{u}_{s_2}(\vec{p}_2)\not{k}_1\,\left(\not{p}_1 - \not{k}_2 + m\right)\,\gamma_\mu u_{s_1}(\vec{p}_1)}{u - m^2} = -\bar{u}_{s_2}(\vec{p}_2)\not{\varepsilon}^{\,*}_{\lambda_2}(\vec{k}_2)\,u_{s_1}(\vec{p}_1),$$

where we have used that, thanks to momentum conservation,

$$\bar{u}_{s_2}(\vec{p}_2)\not{k}_1(\not{p}_1 - \not{k}_2 + m) = \bar{u}_{s_2}(\vec{p}_2)\not{k}_1(\not{p}_2 - \not{k}_1 + m) = 2k_1\cdot p_2\,\bar{u}_{s_2}(\vec{p}_2)\,,$$

and that $u - m^2 = (p_2 - k_1)^2 - m^2 = -2k_1\cdot p_2$. Of course, we would find the same behavior under the shift $\epsilon_{\lambda_2}(\vec{k}_2) \mapsto \epsilon_{\lambda_2}(\vec{k}_2) + \alpha k_2$.

As in the previous study of Rutherford scattering, in most experimental conditions *(i)* the beams of incoming particles are not polarized and *(ii)* the polarization of the beams of outgoing particles are not measured. In this case, one should average over the former and sum over the latter. The technology to do so for spinor polarization sums have been discussed above; let us explain how to deal with photon polarization sums.

The scattering amplitude (3.146) is contracted with one polarization vector for the incoming photon and one polarization vector for the outgoing photon, so we are schematically considering matrix elements in polarization space of the form:

$$\mathcal{M}_{\lambda_2\lambda_1} = \epsilon^{\mu *}_{\lambda_2}(\vec{k}_2)\epsilon^{\nu}_{\lambda_1}(\vec{k}_1)\mathcal{M}_{\mu\nu}\,. \tag{3.149}$$

As we have seen just above (and will demonstrate in full generality below), $\mathcal{M}_{\mu\nu}$ is necessarily transverse, ensuring that unphysical polarizations decouple from the amplitudes:

$$k_2^\mu\mathcal{M}_{\mu\nu} = 0\,, \quad k_1^\nu\mathcal{M}_{\mu\nu} = 0\,. \tag{3.150}$$

In principle, one should sum/average the modulus squared of the scattering amplitude (3.149) over *physical polarizations*, which means $\lambda_{1,2} \in \{1, 2\}$ only:

$$\left|\mathcal{M}_{\lambda_2\lambda_1}\right|^2 \longrightarrow \frac{1}{2}\sum_{\lambda_2=1}^{2}\sum_{\lambda_1=1}^{2}\left|\mathcal{M}_{\lambda_2\lambda_1}\right|^2 = \frac{1}{2}\sum_{\lambda_2=1}^{2}\epsilon^{\mu*}_{\lambda_2}(\vec{k}_2)\epsilon^{\rho}_{\lambda_2}(\vec{k}_2)\sum_{\lambda_1=1}^{2}\epsilon^{\sigma*}_{\lambda_1}(\vec{k}_1)\epsilon^{\nu}_{\lambda_1}(\vec{k}_1)\mathcal{M}_{\mu\nu}\mathcal{M}^*_{\rho\sigma}\,. \tag{3.151}$$

To simplify this expression, one notices that, thanks to eqn. (3.31),

$$\left(\varepsilon^{\nu}_0(\vec{k}_1) + \varepsilon^{\nu}_3(\vec{k}_1)\right)\mathcal{M}_{\mu\nu} \propto k_1^{\nu}\mathcal{M}_{\mu\nu} = 0\,, \tag{3.152}$$

and similarly for the polarization of the outgoing photon. Hence one can add to the unphysical polarizations to the photon polarization sums in (3.151) without changing the result:

$$\frac{1}{2}\sum_{\lambda_2=1}^{2}\sum_{\lambda_1=1}^{2}\left|\mathcal{M}_{\lambda_2\lambda_1}\right|^2 = \frac{1}{2}\sum_{\lambda_2,\kappa_2=0}^{3}\epsilon^{\mu*}_{\lambda_2}(\vec{k}_2)\epsilon^{\rho}_{\kappa_2}(\vec{k}_2)\left(-\eta^{\kappa_2\lambda_2}\right)\sum_{\lambda_1,\kappa_1=0}^{3}\epsilon^{\sigma*}_{\lambda_1}(\vec{k}_1)\epsilon^{\nu}_{\kappa_1}(\vec{k}_1)\left(-\eta^{\kappa_1\lambda_1}\right)\mathcal{M}_{\mu\nu}\mathcal{M}^*_{\rho\sigma}\,. \tag{3.153}$$

Using the completeness relation (3.25b), it amounts to make the following substitution into each photon polarization sum:

$$\sum_{\lambda=1}^{2}\epsilon^{\mu*}_{\lambda}(\vec{k})\epsilon^{\rho}_{\lambda}(\vec{k}) \longrightarrow -\sum_{\lambda,\kappa=0}^{3}\epsilon^{\mu*}_{\lambda}(\vec{k})\epsilon^{\rho}_{\kappa}(\vec{k})\eta^{\kappa_2\lambda_2} = -\eta^{\mu\rho}\,. \tag{3.154}$$

This will always be possible for scattering processes involving incoming and/or outgoing photon states. As a result, we get in the present case that

$$\sum_{\lambda_2=1}^{2}\sum_{\lambda_1=1}^{2}\left|\mathcal{M}_{\lambda_2\lambda_1}\right|^2 = \eta^{\mu\rho}\eta^{\nu\sigma}\mathcal{M}_{\mu\nu}\mathcal{M}^*_{\rho\sigma}\,. \tag{3.155}$$

Finishing the computation of the differential cross-section for Compton scattering from eqn. (3.146) involves a few more steps, including performing the spinor polarization sums, and some tedious algebra, giving the following spin summed/averaged squared amplitude:

$$\frac{1}{4}\sum_{s_1,s_2,\lambda_1,\lambda_2}\left|\mathcal{M}\right|^2 = 2e^4\left(\frac{p_1\cdot k_2}{p_1\cdot k_1}+\frac{p_1\cdot k_1}{p_1\cdot k_2}+2m^2\left(\frac{1}{p_1\cdot k_1}-\frac{1}{p_1\cdot k_2}\right)+m^4\left(\frac{1}{p_1\cdot k_1}-\frac{1}{p_1\cdot k_2}\right)^2\right) \tag{3.156}$$

The resulting differential cross-section is known as the *Klein–Nishina formula*. This was one of the first results of QED, published in 1928 [2].

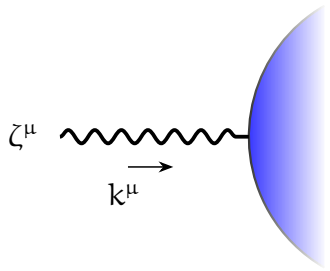
3.3 Ward–Takahashi identity

In the previous example we have checked and used a very important property of the quantum theory of electrodynamics: the scattering amplitude associated with any physical process is

unchanged if we replace the polarization vector ζ^μ of any incoming or outgoing photon state of 4-momentum k^μ by $\zeta^\mu + \alpha k^\mu$.

This looks naturally like a statement about invariance under gauge transformations (3.10) in momentum space. More precisely, since external states are definition on-shell, such a transformation is associated with the residual gauge invariance present in the Lorenz gauge, see eqn. (3.14) as well as eqn. (3.55) and the discussion above it for its quantum version.

It is crucial for the consistency of the theory that this statement holds in the fully interacting theory, beyond tree-level, otherwise the decoupling of unphysical photon states would not hold. Let us consider *any* contribution to the S-matrix associated with a QED scattering process involving some external photon with polarization vector ζ^μ *i.e.* a scattering amplitude having an external photon leg:



According to the rules of LSZ reduction for the electromagnetic field, this scattering amplitude is necessarily of the form:

$$\mathcal{M}(k, \ldots) = \zeta^\mu \mathcal{M}_\mu(k, \ldots) \tag{3.157}$$

Stating that unphysical polarizations of all external states decouple from scattering amplitudes is known as the *Ward–Takahashi identity*:

$$k^\mu \mathcal{M}_\mu(k, \ldots) = 0\,. \tag{3.158}$$

Related identities appear actually for any continuous global symmetry of a quantum field theory. Several types of proofs of the Ward–Takahashi identities in QED can be given, as the sophisticated diagrammatic proof given in Peskin–Schroeder (p. 238-242). We will give here a simpler – but non-perturbative – proof, which will be revisited near the end of this course, using path integral methods.

Let us consider a field theory having a global symmetry $\phi(x) \mapsto \phi(x) + \delta\phi(x)$ (here ϕ refers to any type of field). According to Noether theorem, one associates to such symmetry a conserved current, see eqn. (1.106):

$$j^\mu(x) = \frac{\partial \mathcal{L}}{\partial_\mu \phi} \delta\phi - \Lambda^\mu\,, \quad \partial_\mu j^\mu = 0 \tag{3.159}$$

Let us consider a time-ordered correlation function involving the current j^μ and a field ϕ and

take its divergence. One has:

$$\begin{aligned}\partial_{x^\mu}\langle\Omega|\,\mathsf{T}\,j^\mu(x)\phi(y)\,|\Omega\rangle &= \partial_{x^\mu}\langle\Omega|\left(\Theta(x^0-y^0)j^\mu(x)\phi(y)+\Theta(y^0-x^0)\phi(y)j^\mu(x)\right)|\Omega\rangle\\ &= \langle\Omega|\,\mathsf{T}\big(\partial_\mu j^\mu\big)(x)\phi(y)\,|\Omega\rangle + \langle\Omega|\left[\,j^0(x^0,\vec{x}),\,\phi(x^0,\vec{y})\,\right]|\Omega\rangle\,\delta(x^0-y^0)\,.\end{aligned} \tag{3.160}$$

To simplify the discussion, we consider that neither $\delta\phi$ nor Λ^μ involve time derivatives of the fields. One has then the equal-time commutator:

$$\left[\,j^0(x^0,\vec{x}),\,\phi(x^0,\vec{y})\,\right] = \left[\,\pi(x^0,\vec{x}),\,\phi(x^0,\vec{y})\,\right]\delta\phi(x) = -i\delta^{(3)}(\vec{x}-\vec{y})\delta\phi(x)\,. \tag{3.161}$$

Assuming that current conservation $\partial_\mu j^\mu = 0$ holds in the interacting QFT as an operator equation (in the Heisenberg picture), one has:

$$\partial_{x^\mu}\langle\Omega|\,\mathsf{T}\,j^\mu(x)\phi(y)\,|\Omega\rangle = -i\delta^{(4)}(x-y)\,\langle\Omega|\,\delta\phi(x)\,|\Omega\rangle\ . \tag{3.162}$$

A similar statement holds for a correlation function involving more fields. For instance for one current plus two fields, the time-ordered product looks like:

$$\mathsf{T}\,j^\mu(x_1)\phi(x_2)\phi(x_3) = \Theta(x_1^0-x_2^0)\Theta(x_2^0-x_3^0)j^\mu(x_1)\phi(x_2)\phi(x_3) + \text{ permut.}\,, \tag{3.163}$$

such that acting with $\partial_{x_1^\mu}$ will similarly produce Dirac distributions in the time variable multiplied by equal-time canonical commutation relations. The final result looks like:

$$\begin{aligned}&\partial_{y^\mu}\langle\Omega|\,\mathsf{T}\,j^\mu(y)\phi(x_1)\cdots\phi(x_n)\,|\Omega\rangle\\ &\qquad = -i\sum_{\ell=1}^{n}\langle\Omega|\,\mathsf{T}\,\phi(x_1)\cdots\cancel{\phi(x_\ell)}\delta\phi(x_\ell)\cdots\phi(x_n)\,|\Omega\rangle\,\delta^{(4)}\big(y-x_\ell\big)\end{aligned} \tag{3.164}$$

Crucially, this statement does not hold if current conservation is not satisfied in the interacting QFT. Typically, the computation of correlation functions in perturbation theory will exhibit divergences that needs to be regularized. One should choose a regularization method that preserves the symmetry, or in other words conservation of the Noether current; if no such regularization method exists, the symmetry is said to be *anomalous*.

Similar relations involve the equations of motion of the fields. Let us consider for instance an interacting QFT of a scalar field Φ with arbitrary potential $V(\Phi)$. The classical equation of motion, $\Box\Phi + V'(\Phi) = 0$, should hold in the quantum theory as an operator equation for

the Heisenberg field. Consider now:

$$\begin{aligned}\Box_y \langle\Omega|\,\mathsf{T}\,\Phi(y)\Phi(x_1)\cdots\Phi(x_n)\,|\Omega\rangle &= -\langle\Omega|\,\mathsf{T}\,\Delta_y\Phi(y)\Phi(x_1)\cdots\Phi(x_n)\,|\Omega\rangle \\ &+ \partial_{y^0}\sum_{\ell=1}^{n}\delta(y^0-x_\ell^0)\,\langle\Omega|\,\mathsf{T}\,\underbrace{[\,\Phi(y_0,\vec{y}),\,\Phi(y_0,\vec{x}_\ell)\,]}_{=0}\Phi(x_1)\cdots\cancel{\Phi(x_\ell)}\cdots\Phi(x_n)\,|\Omega\rangle \\ &+ \partial_{y^0}\,\langle\Omega|\,\mathsf{T}\,\dot{\Phi}(y)\Phi(x_1)\cdots\Phi(x_n)\,|\Omega\rangle \\ \\ &= \langle\Omega|\,\mathsf{T}\,\Box_y\Phi(y)\Phi(x_1)\cdots\Phi(x_n)\,|\Omega\rangle \\ &+ \partial_{y^0}\sum_{\ell=1}^{n}\delta(y^0-x_\ell^0)\,\langle\Omega|\,\mathsf{T}\,\underbrace{\Big[\,\dot{\Phi}(y_0,\vec{y}),\,\Phi(y_0,\vec{x}_\ell)\,\Big]}_{-i\delta^{(3)}(\vec{y}-\vec{y}_\ell)}\Phi(x_1)\cdots\cancel{\Phi(x_\ell)}\cdots\Phi(x_n)\,|\Omega\rangle\end{aligned}$$

So in conclusion we have proven that:

$$\Box_y \langle\Omega|\,\mathsf{T}\,\Phi(y)\Phi(x_1)\cdots\Phi(x_n)\,|\Omega\rangle + \langle\Omega|\,\mathsf{T}\,V'\big(\Phi\big)(y)\Phi(x_1)\cdots\Phi(x_n)\,|\Omega\rangle \tag{3.165}$$

$$= -i\sum_{\ell=1}^{n}\langle\Omega|\,\mathsf{T}\,\Phi(x_1)\cdots\cancel{\Phi(x_\ell)}\cdots\Phi(x_n)\,|\Omega\rangle\,\delta^{(4)}(y-x_\ell) \tag{3.166}$$

The contributions to the right-hand side are known as *contact terms* as they arise whenever the field at y collides with one of the other fields. Said differently, the equations of motion for a quantum field inside a correlation function hold up to contact terms. Relations like (3.166) are known as *Schwinger–Dyson equations*. We will provide later a better derivation using path integral methods.

As we have explained before, consistent coupling of a massless vector field to matter fields is tied with the existence of a conserved current associated with a continuous $\mathsf{U}(1)$ symmetry, see eqn. (3.80) and below. The 4-potential A^μ obeys, for an arbitrary choice of "gauge" β, the equation of motion:

$$\Box_{\mu\nu}A^\nu \overset{\text{def.}}{=} \Big(\Box\eta_{\mu\nu} - (1-\beta)\partial_\mu\partial_\nu\Big)A^\nu = j_\mu\,. \tag{3.167}$$

Let us consider a time-ordered correlation function involving a 4-potential field together with other fields:

$$\mathcal{G}^\mu(y,x_1,\ldots,x_n) = \langle\Omega|\,\mathsf{T}\,A^\mu(y)\phi(x_1)\cdots\phi(x_n)\,|\Omega\rangle\ . \tag{3.168}$$

A scattering amplitude $\zeta^\mu\mathcal{M}_\mu$ with an incoming photon extracted from this correlation function by LSZ reduction involves, among other integrals,

$$\frac{i}{\sqrt{Z_3}}\zeta^\mu\int d^4y e^{-ik\cdot y}\Box^y_{\mu\nu}\mathcal{G}^\nu(y,x_1,\ldots,x_n)\Big|_{k^2=0}\,. \tag{3.169}$$

According to the reasoning before, one has:

$$\Box^y_{\mu\nu}\mathcal{G}^\nu(y,x_1,\ldots,x_n) = \langle\Omega|\,\mathsf{T}\,\left(\Box_{\mu\nu}A^\nu\right)(y)\,\phi(x_1)\cdots\phi(x_n)\,|\Omega\rangle + \text{ contact terms}\,. \tag{3.170}$$

Contact terms cannot occur unless there is at least another gauge field A^ρ in the correlation function (since $[\dot{A}^\nu, \phi] = 0$). We assume this to be the case, otherwise the proof gets a bit more complicated, especially if not using the Feynman "gauge".[9]

Therefore, using the equation of motion (3.167) for the gauge field, one can make the following substitution within the scattering amplitude, *i.e.* within the fully LSZ-reduced correlation function:

$$\zeta^\mu \int d^4y e^{-ik\cdot y} \Box^y_{\mu\nu} \mathcal{G}^\nu(y, x_1, \ldots, x_n) \xrightarrow{\text{LSZ}} \zeta^\mu \int d^4y e^{-ik\cdot y} \langle\Omega| T\, j_\mu(y)\phi(x_1)\cdots\phi(x_n)|\Omega\rangle \ . \tag{3.171}$$

Consider now shifting the polarization vector ζ^μ by k^μ, where k^μ is the *on-shell* momentum of the incoming photon. The extra term added to the scattering amplitude is of the form:

$$\int d^4y\, e^{-ik\cdot y} k^\mu \langle\Omega| T\, j_\mu(y)\cdots|\Omega\rangle \overset{\text{IBP}}{=} -i\int d^4y\, e^{-ik\cdot y} \partial_\mu \langle\Omega| T\, j^\mu(y)\cdots|\Omega\rangle \tag{3.172}$$

Now, using eqn. (3.164), this gives:

$$-i\int d^4y\, e^{-ik\cdot y} \left(\langle\Omega| T\, \underbrace{(\partial_\mu j^\mu)(y)}_{=0} \cdots|\Omega\rangle + \text{ contact terms.} \right) .$$

The contact terms will not contribute to the full LSZ reduction of the correlation function. To show this, assume for simplicity that the variation under the symmetry is linear in the field: $\delta\phi(x) = c\phi(x)$ (which is the case both in spinor and scalar QED). For an outgoing scalar particle associated with $\phi(x_1)$, one would have a contribution to the scattering amplitude of the form:

$$\int d^4x_1\, e^{ip_1\cdot x_1} \left(\Box_{x_1} + m^2\right) \int d^4y\, e^{-ik\cdot y} \langle\Omega| T\, \delta\phi(x_1)\cdots|\Omega\rangle\, \delta^{(4)}(y - x_1)$$
$$\overset{\text{IBP}}{=} -c\left(p_1^2 - m^2\right) \int d^4x_1\, e^{i(p_1-k)\cdot x_1} \langle\Omega| T\, \phi(x_1)\cdots|\Omega\rangle \tag{3.173}$$

This expression has formally the same structure as a step of the LSZ reduction involving an outgoing state, but the momentum in the Fourier transform is shifted from the on-shell value p_1 to the off-shell value $p_1 - k$. Therefore this term does not have the right pole structure

[9] Essentially it goes as follows. If we start with $\langle T\, A^\nu(y) A^\rho(x_1)\cdots\rangle$ without other gauge fields, the contact term gives $i\zeta^\rho \int d^4y e^{-ik\cdot x} \langle T\, \phi(x_2)\cdots\rangle \delta(y - x_1) = \zeta^\rho e^{-ik\cdot x_1} \langle T\, \phi(x_2)\cdots\rangle$. Next we LSZ reduce w.r.t. $A^\rho(x_1)$:

$$\langle T\, \phi(x_2)\cdots\rangle \tilde{\zeta}^\sigma \zeta^\rho \int d^4x_1 e^{-i\tilde{k}\cdot x_1} \Box^{x_1}_{\sigma\rho} e^{-ik\cdot x_1} = -\langle T\, \phi(x_2)\cdots\rangle \tilde{\zeta}^\sigma \zeta^\rho \int d^4x_1 e^{-i(k+\tilde{k})\cdot x_1} (k^2\eta_{\sigma\rho} - (1-\beta)k_\sigma k_\rho)$$
$$\sim \langle T\, \phi(x_2)\cdots\rangle \delta^{(4)}(k + \tilde{k})(1-\beta)(\zeta\cdot k)(\tilde{\zeta}\cdot\tilde{k})$$

This is a disconnected component of the S-matrix hence it will not contribute to the scattering amplitude after we reduce w.r.t. the other particles (whose on-shell momenta were chosen in compliance with momentum conservation in the connected diagram).

to cancel the inverse propagator and will not contribute to the scattering amplitude through LSZ reduction.

In conclusion, replacing the polarization ζ^μ of an external incoming photon by $\zeta^\mu + \alpha k^\mu$, where k^μ is the on-shell momentum of the photon, does not change the value of the scattering amplitude $\mathcal{M} = \zeta^\mu \mathcal{M}_\mu$, because $k^\mu \mathcal{M}_\mu = 0$. Of course the same applies for an outgoing photon. □

3.4 Radiative corrections

Tree-level processes like those which we considered above receive an infinite number of quantum corrections, involving loops of photons and electrons. As in the ϕ^4 theory, those corrections generally suffer from *UV divergences* at large loop momenta, thus should be regularized, leading ultimately to a renormalization of the fields and of the couplings as we have sketched in section. 2.2. The loop corrections to QED are historically called *radiative corrections.*

Compared to the ϕ^4 theory, there are two important novel aspects in theories with gauge interactions like quantum electrodynamics:

1. Gauge invariance, through the Ward–Takahashi identity, will impose exact relations between the various divergences appearing in perturbation theory, thus corresponding relations between the values of the counter-terms in the renormalized theory.

2. Besides UV divergences, the loop corrections suffer from *IR divergences* at low momenta, because the photon is an exactly massless particle. These divergences are of completely different nature. IR divergences associated with a given scattering process are ultimately cancelled by other Feynman diagrams corresponding to scattering processes indistinguishable physically from the former one.

QED counterterms. First, let us define, in the usual way, the parameters Z_2 and Z_3 as the normalization of the one-particle states, respectively for the electron and for the photon, in the interacting theory, *i.e.* the residue near the one-particle pole:

$$\int d^4x\, e^{ip\cdot x} \langle \Omega |\, T\, \Psi(x) \bar{\Psi}(0) \,| \Omega \rangle \overset{p^2 \to m^2_{\text{PHYS}}}{\sim} \frac{i Z_2 (\not{p} + m_{\text{PHYS}})}{p^2 - m^2_{\text{PHYS}}}\,, \tag{3.174a}$$

$$\int d^4x\, e^{ik\cdot x} \langle \Omega |\, T\, A^\mu(x) A^\nu(0) \,| \Omega \rangle \overset{k^2 \to 0}{\sim} \frac{-i Z_3 \eta^{\mu\nu}}{k^2}\,. \tag{3.174b}$$

The factors $Z_{2,3}$ are not physical, but appear nonetheless in the computation of the scattering amplitudes through the LSZ formulæ: every external leg for an electron (resp. for a photon) multiplies the fully amputated diagrams by a factor of $\sqrt{Z_2}$ (resp. by a factor of $\sqrt{Z_3}$). They may (and will!) contain UV divergences.

Instead of the fields Ψ and A^μ, one can use renormalized fields Ψ_R and A^μ_R, defined by the redefinitions

$$\Psi = \sqrt{\tilde{Z}_2}\, \Psi_R\,, \quad A^\mu = \sqrt{\tilde{Z}_3}\, A^\mu_R\,, \tag{3.175}$$

in terms of which:

$$\int d^4x\, e^{ip\cdot x} \langle\Omega|\, T\,\Psi_R(x)\bar\Psi_R(0)\,|\Omega\rangle \overset{p^2\to m^2_{PHYS}}{\sim} \frac{i(Z_2/\tilde Z_2)(\not p + m_{PHYS})}{p^2 - m^2_{PHYS}}\,, \tag{3.176a}$$

$$\int d^4x\, e^{ik\cdot x} \langle\Omega|\, T\,A^\mu_R(x)A^\nu_R(0)\,|\Omega\rangle \overset{k^2\to 0}{\sim} \frac{-i(Z_3/\tilde Z_3)\eta^{\mu\nu}}{k^2}\,. \tag{3.176b}$$

One can choose:

$$Z_2/\tilde Z_2 = 1\,, \quad Z_3/\tilde Z_3 = 1\,, \tag{3.177}$$

in order to remove the need of the $\sqrt{Z}$ factors in the LSZ formulæ. Choosing a different normalization of the renormalized fields is perfectly fine, as long as (i) the divergences present in $\sqrt{Z_2}$ and $\sqrt{Z_3}$ are absorbed in the redefinition of the fields and *(ii)* the LSZ formulæ are modified accordingly.

Let us look at the kinetic term for the Dirac fermion first. Starting from the bare Lagrangian density, we write

$$\mathcal{L}_D = \bar\Psi\big(i\not\partial - m_0\big)\Psi = \bar\Psi_R\big(i\not\partial - m\big)\Psi_R + \delta_2\bar\Psi_R i\not\partial\Psi_R - \delta m\bar\Psi_R\Psi_R \tag{3.178}$$

where

$$\delta_2 = \tilde Z_2 - 1 \quad , \qquad \delta m = \tilde Z_2 m_0 - m\,. \tag{3.179}$$

Hence we have two counter-terms, one for the wave-function renormalization discussed above and one for the mass. We will impose that m is the physical mass of the electron – or at least differs from it by a finite quantity.

Next, let us consider the bare QED vertex. We decompose the interaction Lagrangian density of QED as follows:

$$\mathcal{L}_{INT} \;=\; -e_0 A^\mu\bar\Psi\gamma_\mu\Psi = -eA^\mu_R\bar\Psi_R\gamma_\mu\Psi_R - e\,\delta_1\,A^\mu_R\bar\Psi_R\gamma_\mu\Psi_R\,, \tag{3.180}$$

where

$$e\,\delta_1 = \sqrt{\tilde Z_3}\tilde Z_2 e_0 - e = e(Z_1 - 1)\,. \tag{3.181}$$

Intuitively, we would like to impose that e is equal to the actual value of the electron electric charge, as measured in experiments. However, as we shall see, the value of the electric charge is dependent on the energy scale, thus we will have to set its value at some given scale.

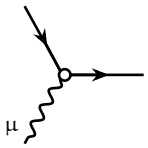
No mass counter-term should be considered for the photon, since such term would destroy gauge invariance. Ultimately, the presence of counter-terms is dictated by the structure of the divergences, which will have to be compatible with the absence of such counter-term. Finally, to write the wave-function renormalization counter-term it is useful to rewrite first the kinetic term as:

$$\begin{aligned}
\mathcal{L}_A &= -\tfrac14 F_{\mu\nu}F^{\mu\nu} = -\frac12\partial_\mu A_\nu\big(\partial^\mu A^\nu - \partial^\nu A^\mu\big) = -\frac12\partial_\mu A_\nu\big(\eta^{\nu\rho}\partial^\mu - \eta^{\mu\rho}\partial^\nu\big)A_\rho \\
&\overset{IBP}{=} -\frac12 A_\nu\big(\partial^\nu\partial^\rho - \eta^{\nu\rho}\partial_\mu\partial^\mu\big)A_\rho + \partial_\mu\big(\cdots\big)\,, \\
&= -\frac12 A^\nu_R\big(\partial_\nu\partial_\rho - \eta_{\nu\rho}\partial_\mu\partial^\mu\big)A^\rho_R \;-\; \frac12\delta_3\,A^\nu_R\big(\partial_\nu\partial_\rho - \eta_{\nu\rho}\partial_\mu\partial^\mu\big)A^\rho_R\,,
\end{aligned} \tag{3.182}$$

with

$$\delta_3 = \tilde{Z}_3 - 1\,. \tag{3.183}$$

The four new vertices associated with the QED counter-terms and the associated Feynman rules are summarized on figure 3.3.

Electron mass counterterm		$-i\delta m$
Electron wave-fct counterterm		$i\not{p}\,\delta_2$
Photon wave-fct. counterterm		$-i\delta_3\left(\eta^{\mu\nu}k^2 - k^\mu k^\nu\right)$
QED vertex counter-term		$-ie\delta_1\,\gamma^\mu$

(3.184)

Figure 3.3: *Counter-terms for renormalized QED.*

To summarize, we have split the Lagrangian density in three parts:

$$\mathcal{L} = \underbrace{\bar{\Psi}_R\left(i\not{\partial} - m\right)\Psi_R - \frac{1}{4}F_R^2}_{\text{free}} + \underbrace{-e\,A_R^\mu\bar{\Psi}_R\gamma_\mu\Psi_R}_{\text{vertex}} + \underbrace{\delta_2\bar{\Psi}_R i\not{\partial}\Psi_R - \delta m\bar{\Psi}_R\Psi_R - \frac{1}{4}\delta_3 F_R^2 - e\,\delta_1\,A_R^\mu\bar{\Psi}_R\gamma_\mu\Psi_R}_{\text{counter-terms.}} \tag{3.185}$$

The value of the four parameters δ_1, δ_2, δ_3 and δm are determined order by order in perturbation theory by the renormalization conditions. The overall strategy is:

1. consider all the 1PI, resummed and amputated diagrams containing UV divergences.
2. use the renormalization of the fields by Z_2 and Z_3 in order to absorb two linear combinations of the diverging factors.
3. impose renormalization conditions on e and m in order to absorb two more linear combinations of divergences.

In QED there are just three such diagrams (at least at order e^2 in perturbation theory) that we will examine in turn:

1. the self-energy of the electron,

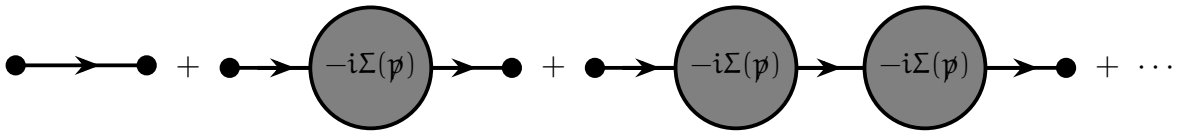
2. the self-energy of the photon,

3. the QED vertex.

It seems an embarrassment of riches as we have four counter-terms available for only three divergences! As we shall see, two of the counterterms are actually tied to each other.

3.4.1 Electron self-energy

Let us start by studying the loop correction to the electron propagator. As we did for the scalar field, we define the *self-energy* $-i\Sigma(\not{p})$ as the amputated sum of all 1-particle irreducible contributions to the electron propagator, such that the full propagator is of the form:



This Dyson series can be resummed as:

$$\hat{D}_F(p) = \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}\sum_{\ell=0}^{\infty}\left(-i\Sigma(\not{p})\frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}\right)^{\ell} = \frac{i(\not{p}+m)}{p^2-m^2-\Sigma(\not{p})(\not{p}+m)+i\epsilon} \tag{3.186}$$

Using the identity:

$$(\not{p}+m)(\not{p}-m) = \not{p}\not{p}-m^2 = p^2-m^2\,, \tag{3.187}$$

the propagator (3.186), without the $i\epsilon$ prescription explicit, can be rewritten as an inverse matrix:

$$\hat{D}_F(p) = \frac{i}{\not{p}-\big(m+\Sigma(\not{p})\big)}\,. \tag{3.188}$$

Let us understand in more detail the structure of this expression before stating what the renormalization conditions are. By Lorentz invariance, $\Sigma(\not{p})$ has to be of the form:

$$\Sigma(\not{p}) = f(p^2)+g(p^2)\not{p}\,, \tag{3.189}$$

such that the propagator (3.188) can be rewritten as:

$$\hat{D}_F(p) = i\frac{\big(1-g(p^2)\big)\not{p}+m+f(p^2)}{\big(1-g(p^2)\big)^2p^2-\big(m+f(p^2)\big)^2}\,. \tag{3.190}$$

According to LSZ, we expect a simple pole in the variable p^2 at $p^2 = m^2_{\text{PHYS}}$, where m_{PHYS} is the physical mass of the electron.

It is natural (but not compulsory) to choose to identify the parameter m in the renormalized Lagrangian with the physical mass m_{PHYS}, and impose then the *on-shell renormalization condition*:

$$g(m^2) = f(m^2) = 0\,. \tag{3.191}$$

Formally, since $p^2 = \not{p}\not{p}$, this could be summarized as:

$$\Sigma(\not{p})\big|_{\not{p}=m} = 0\,. \tag{3.192}$$

This notation, which is used in most textbooks, is rather sloppy as the matrix $\not{p}$ is traceless! The proper way to understand this expression is that $\not{p} = m$ does not mean exactly going on shell, but rather substituting m for $\not{p}$ in the expression of $\Sigma(\not{p})$. A more correct notation would be:

$$\Sigma(\not{p})\big|_{\not{p}\to m} = 0\,. \tag{3.193}$$

Next, according to the Källén–Lehmann representation for Dirac fermions (see the tutorials), we expect that the propagator of the *renormalized* electron field behaves near the single-particle pole like:

$$\hat{D}_F(p) \overset{p^2\sim m^2}{\sim} \frac{i(Z_2/\tilde{Z}_2)(\not{p}+m)}{p^2-m^2}\,, \tag{3.194}$$

where $\Psi = \sqrt{\tilde{Z}_2}\,\Psi_R$. Starting from (3.190) it means that:

$$\frac{d}{dp^2}\Big(\big(1-g(p^2)\big)^2 p^2 - \big(m+f(p^2)\big)^2\Big)\Big|_{p^2=m^2} \overset{!}{=} \frac{\tilde{Z}_2}{Z_2}\,, \tag{3.195}$$

which implies that

$$1 - 2m^2 g'(m^2) - 2m f'(m^2) = \tilde{Z}_2/Z_2\,. \tag{3.196}$$

The natural (but by no means unique) renormalization condition we would like impose is that

$$Z_2/\tilde{Z}_2 = 1\,, \tag{3.197}$$

such that there are no $z_2 = Z_2/\tilde{Z}_2$ factors for external legs in scattering amplitudes, *i.e.* in/out electrons come just with factors u or $\bar{u}$ rather than $\sqrt{z_2}u$ or $\sqrt{z_2}\bar{u}$. Schematically, one has

$$Z_2 = 1 + \alpha(\text{1-loop}) + \cdots \quad , \qquad \tilde{Z}_2 = 1 + \alpha\delta_2^{(1)} + \cdots\,, \tag{3.198}$$

such that $\delta_2^{(1)}$ is chosen to absorb the one-loop correction, and so on. The condition (3.197) translates into:

$$m g'(m^2) + f'(m^2) = 0\,. \tag{3.199}$$

Since

$$\frac{d\Sigma(\not{p})}{d\not{p}} = \frac{df}{dp^2}\frac{dp^2}{d\not{p}} + \frac{dg}{dp^2}\frac{dp^2}{d\not{p}}\not{p} + g(p^2) = 2\not{p}\Big(f'(p^2) + \not{p}\,g'(p^2)\Big) + g(p^2) \tag{3.200}$$

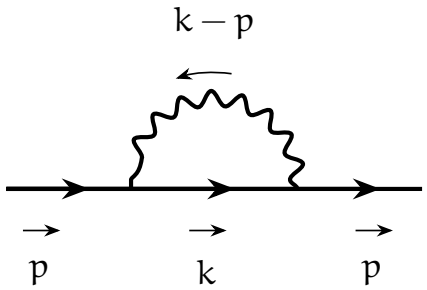
Since in renormalized perturbation theory $\Sigma(\not{p})$ containing *both* the contributions of the loop and of the counter-term, the condition (3.199) can be rewritten (with the same reservations as before) as:

$$\frac{d\Sigma(\not{p})}{d\not{p}}\Big|_{\not{p}\to m} = 0\,. \tag{3.201}$$

To summarize, the on-shell renormalization conditions that we choose to impose on the electron self-energy are:

$$\Sigma(\not{p})\big|_{\not{p}\to m} = 0 \quad , \qquad \frac{d\Sigma(\not{p})}{d\not{p}}\Big|_{\not{p}\to m} = 0\,, \tag{3.202}$$

The leading contribution to the self-energy of the electron appears at order $e^2 = 4\pi\alpha$. It corresponds to the Feynman diagram:



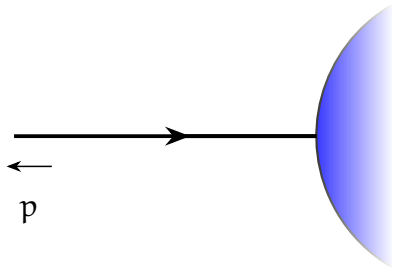
We have then:

$$-i\Sigma(\not{p}) = \big(-ie\big)^2 \int \frac{d^4k}{(2\pi)^4}\, \frac{i\gamma^\mu(\not{k}+m)\gamma^\nu}{k^2-m^2+i\epsilon}\, \frac{-i\eta_{\mu\nu}}{(k-p)^2+i\epsilon} \;+\; \mathcal{O}(e^4) \tag{3.203}$$

$$= -e^2 \int \frac{d^4k}{(2\pi)^4}\, \frac{\gamma^\mu(\not{k}+m)\gamma_\mu}{\big(k^2-m^2+i\epsilon\big)\big((k-p)^2+i\epsilon\big)} \;+\; \mathcal{O}(e^4) \tag{3.204}$$

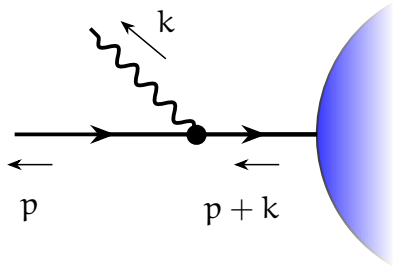
At large loop momentum, the integral behaves like $\int dk$, leading at first sight to a linear divergence that will be taken care of by regularizing the integral, as was done in subsection 2.2.4 for the Klein–Gordon self-energy.

Infrared divergences. The integrand of eqn. (3.204) has also a non-integrable divergence for $k - p \to 0$, because of the massless propagator of the virtual photon in the loop. This type of divergence, which occurs at the low momentum/energy end of the loop integrals, is known as an *infrared (IR) divergence*. The origin of this type of divergence, is completely different from the origin of UV divergences. It is not due to our lack of knowledge of the laws of physics at very high energy, but rather due to the physical limitations of any apparatus used in an experimental setup, thereby to limitations in the measurement process. Consider a QED process involving an outgoing electron leg:



$$\mathcal{M}_1\big(\{\cdots\} \to \{p, \cdots\}\big) \tag{3.205}$$

Imagine now a similar process, where the outgoing electron emits a photon after having interacted with all the other particles:



$$\mathcal{M}_2\big(\{\cdots\}\to\{p,k,\cdots\}\big) \tag{3.206}$$

Compared to the previous type, this type of amputated diagrams has an extra propagator with a singularity structure

$$D\sim\frac{1}{(p+k)^2-m^2}\,,\quad (p+k)^2-m^2=p^2+k^2+2k\cdot p-m^2=2k\cdot p\,,$$

since both 4-momenta p and k are on-shell. In the experimental context, outgoing particles are detected with some apparatus that cannot detect events below a certain threshold of energy E_{MIN}. In practice it means that, if we would like to compute the transition rate associated with some process (3.205), we should add the contributions for all associated processes like (3.206) involving extra low-energy *soft photons*, with $k^0 < E_{\text{MIN}}$, since one cannot distinguish them experimentally one from the other. While the former contributes to the transition rate with $|\mathcal{M}_1|^2$ (dressed with the suitable phase space factor), the latter contributes with terms like $|\mathcal{M}_2|^2$. As we have seen, processes with soft photons have extra factors in $1/k\cdot p$, that will diverge in the soft photon limit $k^0\to 0^+$.

One can show that these divergences cancel precisely the infrared divergences of loop corrections. Since each extra soft photon leg adds a factor of e^2 to the probabilty, divergences from diagrams with different number of loops can actually cancel each other. For more details, see for instance Peskin–Schroeder (p.199-209).

Electron self-energy at one loop. Since we understand the origin of IR divergences, we can regularize them without fearing of missing some important physical property. We add then a fictious mass parameter ν to the photon, giving

$$\Sigma_{\text{1-LOOP}}(\not{p})=-ie^2\int\frac{d^4k}{(2\pi)^4}\,\frac{\gamma^\mu(\not{k}+m)\gamma_\mu}{\big(k^2-m^2+i\epsilon\big)\big((k-p)^2-\nu^2+i\epsilon\big)} \tag{3.207}$$

To handle this integral, one can first simplify the denominator using the *Feynman parametrization*. It starts from the elementary identity:[10]

$$\frac{1}{AB} = \int_0^1 \frac{dx}{(xA + (1-x)B)^2}\,. \tag{3.208}$$

More generally, one can show that:

$$\boxed{\frac{1}{A_1 \cdots A_n} = (n-1)! \int_0^1 \frac{dx_1 \cdots dx_n}{(x_1 A_1 + \cdots + x_n A_n)^n}\, \delta(x_1 + \cdots + x_n - 1)} \tag{3.209}$$

In the present case, starting from eqn. (3.207) one obtains:

$$\frac{1}{\big(k^2 - m^2 + i\epsilon\big)\big((k-p)^2 - \nu^2 + i\epsilon\big)} = \int_0^1 \frac{dx}{\big(x(k-p)^2 + (1-x)k^2 - x\nu^2 - (1-x)m^2 + i\epsilon\big)^2} \tag{3.210}$$

$$= \int_0^1 \frac{dx}{\big((k - xp)^2 - \Delta + i\epsilon\big)^2}\,, \tag{3.211}$$

where we have completed the square and defined

$$\Delta = x\nu^2 + (1-x)m^2 - x(1-x)p^2\,. \tag{3.212}$$

Thus, the expression (3.210) looks like the square of a propagator with shifted momentum $k - xp$ and mass squared Δ. With the change of variables $\ell = k - xp$ the integral eqn. (3.207) becomes:

$$\Sigma_{\text{1-LOOP}}(\not p) = -ie^2 \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4}\, \frac{\gamma^\mu(\not\ell + x\not p + m)\gamma_\mu}{\big(\ell^2 - \Delta + i\epsilon\big)^2} \tag{3.213}$$

The numerator can be simplified using gamma-matrix identities of the kind considered before. Since we will ultimately use dimensional regularization, let us consider that the dimension d of space-time is arbitrary. We have

$$\gamma^\mu\gamma_\mu = \gamma^\mu\gamma^\nu\eta_{\mu\nu} = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\eta_{\mu\nu} = \eta^{\mu\nu}\eta_{\mu\nu} = d\,, \tag{3.214a}$$

$$\gamma^\mu\gamma^\nu\gamma_\mu = (2\eta^{\mu\nu} - \gamma^\nu\gamma^\mu)\,\gamma_\mu = (2-d)\gamma^\nu\,. \tag{3.214b}$$

So we have, in $d = 4$,

$$\Sigma_{\text{1-LOOP}}(\not p) = -ie^2 \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4}\, \frac{-2\,(\not\ell + x\not p) + 4\,m}{\big(\ell^2 - \Delta + i\epsilon\big)^2} \tag{3.215}$$

[10] To show this, one writes

$$\frac{1}{AB} = \frac{1}{A-B}\left(\frac{1}{B} - \frac{1}{A}\right) = \frac{1}{A-B}\int_B^A \frac{dz}{z^2} \overset{x=\frac{z-B}{A-B}}{=} \int_0^1 \frac{dx}{(xA + (1-x)B)^2}\,.$$

By isotropy, $\int d^4\ell f(\ell^2)\ell^\mu = 0$, so the linear term in the numerator can be dropped, giving a loop integral with a softer UV behavior (logarithmic rather than linear divergence):

$$\Sigma_{\text{1-LOOP}}(\not{p}) = -ie^2 \int_0^1 dx \, \big(4m - 2x\not{p}\big) \int \frac{d^4\ell}{(2\pi)^4} \, \frac{1}{\big(\ell^2 - \Delta + i\epsilon\big)^2} \tag{3.216}$$

Since this integral has a similar pole structure in the complex ℓ^0-plane as the loop integral with a single propagator, one can perform Wick rotation by defining $\ell^0 = i\ell^0_E$, giving the Euclidian integral:

$$\Sigma_{\text{1-LOOP}}(\not{p}) = e^2 \int_0^1 dx \, \big(4m - 2x\not{p}\big) \int \frac{d^4\ell_E}{(2\pi)^4} \, \frac{1}{\big(\ell_E^2 + \Delta\big)^2} \tag{3.217}$$

This integral is still divergent in the UV. We will regularize it using dimensional regularization. Let us remark first that:

- As in ϕ^4, one needs to introduce an arbitrary mass scale μ to keep the coupling e dimensionless.
- The gamma-matrix identities (3.214) used at the previous step depend on space-time dimension d, so the formula (3.217) has to be modified accordingly.

This understood, one has

$$\Sigma^{4-d}_{\text{1-LOOP}}(\not{p}) = e^2\mu^{4-d} \int_0^1 dx \, \big(dm + (2-d)x\not{p}\big) \int \frac{d^d\ell_E}{(2\pi)^d} \, \frac{1}{\big(\ell_E^2 + \Delta\big)^2} \tag{3.218}$$

The integral over the Euclidian momentum ℓ_E is done in a similar way as below eqn. (2.217). One has

$$I_d = \int \frac{d^d\ell_E}{\big(\ell_E^2 + \Delta\big)^2} = \frac{2\pi^{d/2}}{\Gamma(d/2)} \Delta^{\frac{d-4}{2}} \int_0^\infty \frac{u^{d-1} du}{\big(u^2+1\big)^2} \tag{3.219}$$

Next we make the change of variables $t = 1/(1+u^2)$ giving

$$I_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \Delta^{\frac{d-4}{2}} \frac{1}{2} \int_0^1 dt \, t^{1-d/2} (1-t)^{d/2-1} = \pi^{d/2} \Delta^{\frac{d-4}{2}} \Gamma\big(2 - d/2\big) \tag{3.220}$$

Overall, in terms of $\epsilon = 4-d$, one obtains the following expression for the one-loop correction to the electron self-energy:

$$\Sigma^{\epsilon}_{\text{1-LOOP}}(\not{p}) = \frac{e^2}{16\pi^2} \big(4\pi\big)^{\epsilon/2} \mu^\epsilon \int_0^1 dx \, \big((4-\epsilon)m - (2-\epsilon)x\not{p}\big) \, \Delta^{-\epsilon/2} \Gamma\big(\epsilon/2\big) \tag{3.221}$$

Using eqn. (2.222) this can be expanded in powers of ϵ as:

$$\boxed{\Sigma^{\epsilon}_{\text{1-LOOP}}(\not{p}) = \frac{\alpha}{2\pi} \int_0^1 dx \left(\big(2 - \tfrac{\epsilon}{2}\big)m - \big(1 - \tfrac{\epsilon}{2}\big)x\not{p}\right) \left(\frac{2}{\epsilon} + \ln\left(\frac{4\pi e^{-\gamma}\mu^2}{\Delta}\right) + \mathcal{O}(\epsilon)\right)} \tag{3.222}$$

In the $\epsilon \to 0^+$ limit, one gets a divergence from the $2/\epsilon$ factor as well as several finite terms.[11] Unlike in ϕ^4 theory, the one-loop contribution to the self-energy is a (rather complicated) function of the momentum p, which appears in (3.222) both directly and in Δ, see eqn. (3.212).

Pauli–Villars regularization. Dimensional regularization is today the most common regularization method used, but it is by no means the only one. Any regularization that preserves the symmetries of the theory is admissible, and will not affect the physical outcomes of the theory. Another common method is called *Pauli–Villars regularization*: the idea is to substract to the photon propagator a fictious propagator with a very large mass parameter M. In the original expression (3.207) for the one-loop correction we replace:

$$\frac{1}{(k-p)^2-\nu^2+i\epsilon} \longrightarrow \frac{1}{(k-p)^2-\nu^2+i\epsilon} - \frac{1}{(k-p)^2-M^2+i\epsilon}\,. \tag{3.223}$$

Following the same step as in the previous computation, we replace eqn. (3.217) by:

$$\begin{aligned}\Sigma_{\text{1-LOOP}}(\not p) &= \frac{2e^2}{(4\pi)^2}\int_0^1 dx\,\left(4m-2x\not p\,\right)\int_0^\infty \ell_E^3 d\ell_E\left(\frac{1}{(\ell_E^2+\Delta)^2}-\frac{1}{(\ell_E^2+\Delta_M)^2}\right)\\ &= \frac{e^2}{(4\pi)^2}\int_0^1 dx\,\left(4m-2x\not p\right)\ln\left(\frac{\Delta_M}{\Delta}\right)\,.\end{aligned} \tag{3.224}$$

where

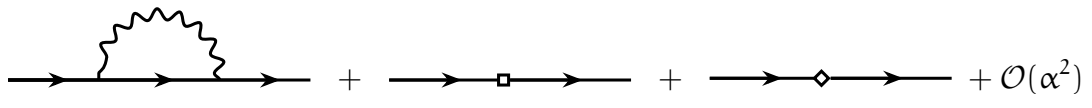
$$\Delta_M = xM^2+(1-x)m^2-x(1-x)p^2\,. \tag{3.225}$$

In the large M limit, one can then re-express the result as:

$$\Sigma_{\text{1-LOOP}}(\not p)\big|_{\text{PV}} = \frac{e^2}{(4\pi)^2}\int_0^1 dx\,\left(4m-2x\not p\right)\left(\ln\left(\frac{xM^2}{\mu^2}\right)+\ln\left(\frac{\mu^2}{\Delta}\right)\right)\,, \tag{3.226}$$

where μ is an arbitrary mass scale. Compared to the computation of the self-energy using dimensional regularization, see eqn. (3.222), the divergence occurs as a $\ln M^2/\mu^2$ factor rather than a $1/\epsilon$ factor.

Renormalization conditions. Let us go back to expression of the one-loop correction obtained using dimensional regularization. The full correction to the self-energy at order α is given by the sum of the one-loop correction and the counter-terms (electron mass and wave-function renormalization):

$-i\Sigma(\not p) =$  $+\,\mathcal{O}(\alpha^2)$

[11] The factors $(2-\epsilon/2)$ and $(1-\epsilon/2)$ should be kept, as they give finite terms in this limit.

We impose first on this sum of three diagrams that:

$$\frac{\mathrm{d}}{\mathrm{d}\not{p}}\Sigma(\not{p})\Big|_{\not{p}\to m} = -\delta_2 + \frac{\mathrm{d}}{\mathrm{d}\not{p}}\Sigma^{\epsilon}_{\text{1-LOOP}}(\not{p})\Big|_{\not{p}\to m} + \mathcal{O}(\alpha^2) = 0\,, \tag{3.227}$$

ensuring that no $\sqrt{Z_2}$ factors appear in the computation of the scattering amplitudes at this order. At least in principle, this equation can be solved for δ_2 at order α using the expression (3.222). Given the structure of this expression, δ_2 will be a function of regulator ϵ, of the physical mass m, of the mass scale μ and of the physical coupling constant α.

The second renormalization condition we can impose at order α in perturbation theory is that the parameter m in the renormalized Lagrangian density (3.178) is the actual mass of the electron, corresponding to the one-particle pole of the LSZ propagator. It leads to the condition:

$$\Sigma(\not{p})\Big|_{\not{p}\to m} = \delta m - m\delta_2 + \Sigma^{\epsilon}_{\text{1-LOOP}}(\not{p})\Big|_{\not{p}\to m} + \mathcal{O}(\alpha^2) = 0\,. \tag{3.228}$$

As before, this condition can be solved in principle for the mass counter-term parameter δm at order α from eqn. (3.222). Then δm would be given as a function of regulator ϵ, of the physical parameters m and α and of the scale μ. Notice that, unlike the mass renormalization in ϕ^4 theory, see eqn. (2.229), the one-loop correction to the mass term is proportional to the mass parameter m, hence vanishes in the massless limit $m \to 0$; one says that the massless nature of the field is *protected* against quantum corrections. This can be understood by the existence of a new symmetry in the massless limit. A massless Dirac fermion is equivalent to a pair of Weyl fermions of opposite chirality. The $U(1)$ global symmetry of the Dirac Lagrangian is then enhanced to $U(1) \times U(1)$, the second $U(1)$ factor acting, in the Dirac representation, as $\Psi \mapsto \exp(i\gamma^5\theta)\Psi$.

Crucially, the singular part of the one-loop correction to the electron self-energy (3.222) has exactly the right form to be absorbed into the coefficients of the electron mass and wave-function counterterms (which are *constant* coefficients, not functions of p):

$$\Sigma^{\epsilon}_{\text{1-LOOP}}(\not{p})\Big|_{\text{SING.}} = -\frac{\alpha}{4\pi}(\not{p} - 4m)\frac{2}{\epsilon} \quad\Longrightarrow\quad \delta_2\Big|_{\text{SING.}} = -\frac{\alpha}{4\pi}\frac{2}{\epsilon} \quad , \qquad \delta m\Big|_{\text{SING.}} = -\frac{m\alpha}{\pi}\frac{2}{\epsilon}\,. \tag{3.229}$$

It is important that this structure persists to all orders: the parameters of the four counter-terms that we have defined, see fig. 3.3, should be able to absorb all UV divergences, appearing in any diagram, at any order in α. If this is the case, the theory is said to be *renormalizable*. Beyond the divergent parts, the finite parts of the counter-terms are an arbitrary choice, that should not impact the physical quantities. We will provide a more thoughtful discussion at the end of the chapter.

Minimal substraction and $\overline{\text{MS}}$ renormalization schemes. If we demand that the parameter m in the renormalized Lagrangian density (3.178) is the physical mass of the electron (and that the residue of the pole is one), we have to solve eqn. (3.228) for δm and δZ in the regularized theory, where $\Sigma^\epsilon(\not{p} \to m)$ is a complicated function of the mass, whose leading one-loop contribution is given by eqn. (3.222).

If we want to understand the structure of the singularities of perturbation theory, or if we want to see how the couplings depend on the momenta through loop effects, it is much simpler to use renormalization corrections that just make sure that the mass parameter m is finite.

Instead of just absorbing the singularity, which is called the *minimal substraction scheme*, one can also absorb the "universal" factors of dimensonal regularization, leading to what is known as $\overline{\text{MS}}$ *scheme*:

$$\delta_2\big|_{\overline{\text{MS}}} = -\frac{\alpha}{4\pi}\left(\frac{2}{\epsilon} + \ln\left(4\pi e^{-\gamma}\right)\right) \quad , \qquad \delta m\big|_{\overline{\text{MS}}} = -\frac{m\alpha}{\pi}\left(\frac{2}{\epsilon} + \ln\left(4\pi e^{-\gamma}\right)\right) . \quad (3.230)$$

The price to pay for this simplification is two-fold: *(i)* there is a finite renormalization of the external states contributions in the Feynman rules and *(ii)*, more importantly, the parameter m, although finite, is not the physical mass of the electron.

The mimimal substraction scheme is also very useful whenever we consider that the electron is a massless particle. In this case the on-shell renormalization conditions are ill-defined, because there is a continuum of states starting at $p^2 = 0$. While it is not completely transparent that applying the renormalization conditions (3.227) with $m = 0$ to the expression (3.222) leads to a problem, because of the regularization of the photon IR divergence, it is more obvious *e.g.* in massless ϕ^4 theory.
In the $\overline{\text{MS}}$ scheme, the parameter m is the solution of the implict equation:

$$\Sigma(\not{p})\Big|_{\not{p}\to m_{\text{PHYS}}} = m_{\text{PHYS}} - m\,, \quad (3.231)$$

at a given order in the perturbative expansion. Crucially, with the mass counter-term coefficient δm set by the $\overline{\text{MS}}$ renormalization condition, this equation has a smooth $\epsilon \to 0$ limit.

Let us evaluate this expression at order α. At this order, one can replace m_{PHYS} by m in Σ, since the error made is next order. We have then:

$$m_{\text{PHYS}} = m\left(1 + \frac{\alpha}{2\pi}\left(-\frac{1}{2} + \int_0^1 dx\,(2-x)\ln\left(\mu^2/\Delta\right)\right)\right) , \quad \Delta = m^2(1-x)^2 + \nu^2 . \quad (3.232)$$

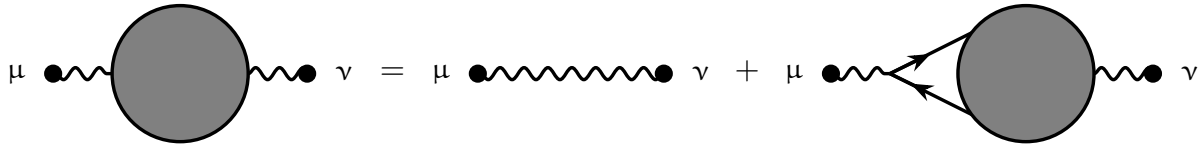
Of course, the physical mass should not depend on the arbitary scale μ, so m and α should be ajusted accordingly. Assuming that the variation of the coupling e is next order (we will prove it) We arrive to the equation:

$$0 = \frac{dm_{\text{PHYS}}}{d\ln\mu} = \frac{dm}{d\ln\mu} + \frac{3\alpha}{2\pi}m + \mathcal{O}(\alpha^2) \quad \Longrightarrow \quad \gamma(m) \overset{\text{def.}}{=} \frac{1}{m}\frac{dm}{d\ln\mu} = -\frac{3\alpha}{2\pi} + \mathcal{O}(\alpha^2) . \quad (3.233)$$

3.4.2 Photon self-energy

Next we consider the corrections to the photon propagator. Before computing the one-loop corrections one can derive general results using the properties of the theory.

Gauge-invariance, through the Ward–Takahashi identities, constrains indeed severely the form of the corrections to the photon propagator. Let $\mathcal{D}^{\mu\nu}(x-y) = \langle\Omega|\,\mathsf{T}\,A^{\mu}(x)A^{\mu}(y)\,|\Omega\rangle$ be the full Feynman propagator of the interacting theory. Schematically, it can be decomposed as:



Because any perturbative correction to the propagator should begin with a QED vertex, *i.e.* the propagating photon creating a virtual electron/positron pair. Let $D_{\mu\nu}$ be the free photon propagator. The diagrammatic identity above amounts to the equation:

$$\mathcal{D}^{\mu\nu}(x-y) = D^{\mu\nu}(x-y) - i\int d^4z\, D^{\mu\rho}(x-z)\ \langle\Omega|\,\mathsf{T}\,j_\rho(z)A^{\nu}(y)\,|\Omega\rangle\ , \tag{3.234}$$

in terms of the QED current $j^{\rho}(x) = e\bar{\Psi}\gamma^{\rho}\Psi(x)$. Taking the divergence of this equation gives:

$$\begin{aligned}\partial_{x^\mu}\mathcal{D}^{\mu\nu}(x-y) &= \partial_{x^\mu}D^{\mu\nu}(x-y) - i\int d^4z\,\partial_{x^\mu}D^{\mu\rho}(x-z)\ \langle\Omega|\,\mathsf{T}\,j_\rho(z)A^{\nu}(y)\,|\Omega\rangle \\ &\overset{\text{IBP}}{=} \partial_{x^\mu}D^{\mu\nu}(x-y) - i\int d^4z\, D^{\mu\rho}(x-z)\,\partial_{z^\mu}\,\langle\Omega|\,\mathsf{T}\,j_\rho(z)A^{\nu}(y)\,|\Omega\rangle\end{aligned} \tag{3.235}$$

In the Feynman "gauge", since $D^{\mu\rho} \propto \eta^{\mu\rho}$, see eqn. (3.56), the integrand of the second term contains

$$\partial_{z^\rho}\,\langle\Omega|\,\mathsf{T}\,j^{\rho}(z)A^{\nu}(y)\,|\Omega\rangle = \langle\Omega|\,\mathsf{T}\,\underbrace{\big(\partial_\rho j^{\rho}(z)\big)}_{=0}A^{\nu}(y)\,|\Omega\rangle - i\,\langle\Omega|\,\delta A^{\nu}(y)\,|\Omega\rangle\,\delta^{(4)}(z-y) = 0\,,$$

using the Ward–Takahashi identity (3.164) for the QED current and the fact that the photon does not carry an electromagnetic charge. Thus we have:

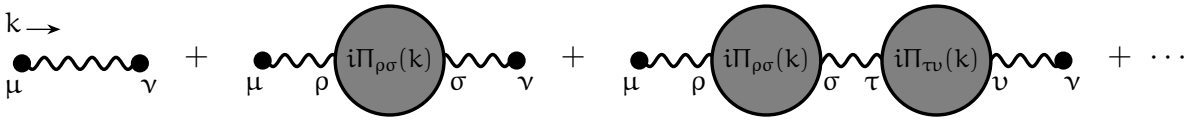
$$\partial_\mu\left(\mathcal{D}^{\mu\nu}(x) - D^{\mu\nu}(x)\right) = 0\,. \tag{3.236}$$

In other words, the sum of all corrections to the photon propagator is purely transverse.[12]

[12]The same holds in any other "gauge" β; indeed, leaving aside the $i\epsilon$ prescription, the contributions of the extra terms involve:

$$\begin{aligned}\int d^4z\,\partial_{x^\mu}\int d^4k\,\frac{k^\mu k^\rho}{k^4}e^{-ik\cdot(x-z)}\,\langle j_\rho(z)\cdots\rangle &= -i\int d^4z\int d^4k\,\frac{k^\rho}{k^2}e^{-ik\cdot(x-z)}\,\langle j_\rho(z)\cdots\rangle \\ &= -\int d^4z\int d^4k\,\frac{1}{k^2}\partial_{z^\rho}e^{-ik\cdot(x-z)}\,\langle j_\rho(z)\cdots\rangle \overset{\text{IBP}}{=} \int d^4z\int d^4k\,\frac{1}{k^2}e^{-ik\cdot(x-z)}\,\partial^\rho\langle j_\rho(z)\cdots\rangle\,.\end{aligned}$$

This important point understood, we introduce the photon self-energy $i\Pi^{\mu\nu}$ as the amputated 1PI contribution to the full propagator, which is known as the *vacuum polarization* of the photon, due to virtual electron/positron pairs. As in the previous case, the latter can be expanded as in momentum space as:



Crucially, the transversality of the full correction to the photon propagator, see eqn. (3.236), enforces that:

$$k_\mu \Pi^{\mu\nu}(k) = 0\,. \tag{3.237}$$

Indeed, one has schematically:

$$k_\mu \left(\hat{\mathcal{D}}^{\mu\nu} - \hat{D}^{\mu\nu}\right) \propto k_\mu \hat{D}^{\mu\rho}\Pi_{\rho\sigma}\left(\hat{D}^{\sigma\nu} + \hat{D}^{\sigma\tau}\Pi_{\tau\upsilon}\hat{D}^{\upsilon\nu} + \cdots\right) \tag{3.238}$$

From the expression of the free photon propagator in momentum space:

$$\hat{D}^{\mu\nu}(k) = \frac{-i\left(\eta^{\mu\nu} + (1/\beta - 1)\frac{k^\mu k^\nu}{k^2}\right)}{k^2}\,, \tag{3.239}$$

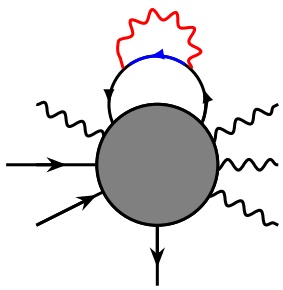
one deduces that:

$$k_\mu \hat{D}^{\mu\rho}\Pi_{\rho\sigma}(\cdots) = \frac{1}{\beta}\frac{-i}{k^2}\, k^\rho \Pi_{\rho\sigma}(\cdots)\,, \tag{3.240}$$

So, the expression between parenthesis being non-trivial, $k^\rho \Pi_{\rho\sigma}(k) = 0$. By Lorentz covariance, $\Pi^{\mu\nu}$ can only involve the tensors $\eta^{\mu\nu}$ and $k^\mu k^\nu$. The transversality condition implies that it is necessarily of the form:

$$\Pi^{\mu\nu}(k) = \left(k^2\eta^{\mu\nu} - k^\mu k^\nu\right)\Pi(k^2)\,. \tag{3.241}$$

We expect $\Pi(k^2)$ to be regular at $k^2 = 0$. For instance, a simple a pole would correspond to a intermediate one-particle massless state, but such that is not expected to appear in QED. The only massless particle, the photon, is only created together with an electron or positron due to the structure of the QED vertex, as in this example:



Let us introduce a projector on transverse 4-vectors:

$$P^{\mu}{}_{\nu}(k) \stackrel{\text{def.}}{=} \delta^{\mu}{}_{\nu} - k^{\mu}k_{\nu}/k^2 \,. \tag{3.242}$$

It satisfies naturally $P^{\mu}{}_{\rho}P^{\rho}{}_{\nu} = P^{\mu}{}_{\nu}$, as well as $P^{\mu}{}_{\rho}k^{\rho} = 0$. In term of this projector, one has:

$$\Pi^{\mu}{}_{\nu}(k) = k^2\Pi(k^2)P^{\mu}{}_{\nu} \,. \tag{3.243}$$

The full propagator can be then written as:

$$\begin{aligned}\hat{\mathcal{D}}_{\mu\nu}(k) = \frac{-i\left(\eta_{\mu\nu} + (\frac{1}{\beta} - 1)\frac{k_\mu k_\nu}{k^2}\right)}{k^2} &+ \frac{-i\eta_{\mu\rho}}{k^2}k^2 i\Pi(k^2)P^{\rho\sigma}\frac{-i\eta_{\sigma\nu}}{k^2} \\ &+ \frac{-i\eta_{\mu\rho}}{k^2}k^2 i\Pi(k^2)P^{\rho\sigma}\frac{-i\eta_{\sigma\tau}}{k^2}k^2 i\Pi(k^2)P^{\tau\upsilon}\frac{-i\eta_{\upsilon\nu}}{k^2} + \cdots \\ = \frac{-iP_{\mu\nu}}{k^2} + \frac{-i\eta_{\mu\rho}}{k^2}\Big(\Pi(k^2)P^{\rho}{}_{\nu} + \Pi(k^2)^2 \underbrace{P^{\rho}{}_{\tau}P^{\tau}{}_{\nu}}_{P^{\rho}{}_{\nu}} + \cdots\Big) &- \frac{i}{\beta}\frac{k_\mu k_\nu}{k^2}\end{aligned} \tag{3.244}$$

In this form, in it easy to resum the whole Dyson series, giving:

$$\hat{\mathcal{D}}_{\mu\nu}(k) = -i\,\frac{\eta_{\mu\nu} - k_\mu k_\nu/k^2}{\big(1 - \Pi(k^2)\big)k^2} - \frac{i}{\beta}\frac{k_\mu k_\nu}{k^4} \,. \tag{3.245}$$

The propagator takes a simpler form for $\beta \to \infty$, known as the *Landau "gauge"*.

From eqn. (3.245) we learn that, as expected by gauge-invariance, the photon remains exactly massless, thus no mass counter-term for the photon – which would break gauge invariance – is needed to renormalize the theory. The full photon propagator as a pole at $k^2 = 0$, with residue:

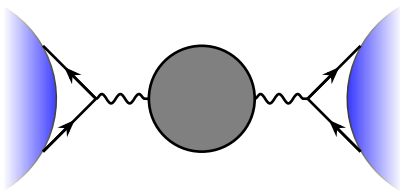
$$\frac{1}{1 - \Pi(0)} = Z_3/\tilde{Z}_3 \,, \tag{3.246}$$

in the renormalized theory. We would like to impose then for convenience the renormalization condition:

$$\Pi(0) = 0 \,, \tag{3.247}$$

to avoid $\mathfrak{z}_3 = Z_3/\tilde{Z}_3$ factors in the Feynman rules.

Finally, thanks to the Ward–Takahashi identity one can also check that the terms in $k^{\mu}k^{\nu}$ in the propagators of internal photon lines do not contribute to the scattering amplitudes. To prove this one has to notice that, within any perturbative contribution to the amplitude, the fully corrected photon propagator is necessarily sandwiched between two QED vertices:



As a consequence, the correlation functions contributing to the scattering amplitudes are schematically of the form:

$$\mathcal{G} \sim \int d^4x d^4y\, \mathcal{D}^{\mu\nu}(x-y)\, \langle\Omega|\, T\, j_\mu(x) j_\nu(y) \cdots |\Omega\rangle\ . \tag{3.248}$$

Let us consider a shift:

$$\mathcal{D}^{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} \hat{\mathcal{D}}^{\mu\nu}(k) e^{-ik\cdot(x-y)} \ \mapsto\ \mathcal{D}^{\mu\nu}(x-y) + \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} f(k^2) k^\mu k^\nu\ . \tag{3.249}$$

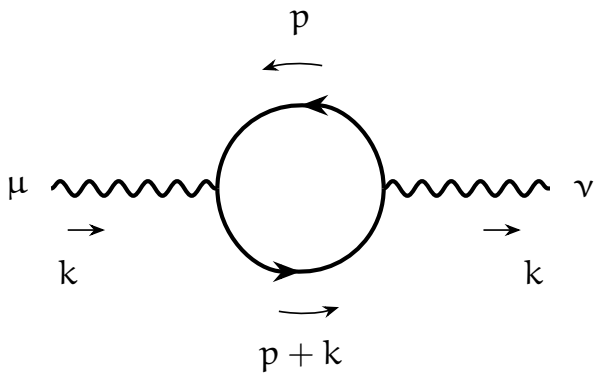
where $f(k^2)$ is arbitary. The contribution of the new term involves:

$$k^\mu \int d^4x\, e^{-ik\cdot x}\, \langle\Omega|\, T\, j_\mu(x) \cdots |\Omega\rangle \overset{\text{IBP}}{=} -i \int d^4x\, e^{-ik\cdot x} \partial^\mu\, \langle\Omega|\, T\, j_\mu(x) \cdots |\Omega\rangle \tag{3.250}$$

By the Ward–Takahashi identity, the RHS will only give contact terms, that will not contribute to scattering amplitudes as in the previous cases. Therefore, for the purpose of computing scattering amplitudes one can replace for the internal photon lines:

$$\hat{\mathcal{D}}_{\mu\nu}(k) \ \longrightarrow\ \frac{-i\eta_{\mu\nu}}{\big(1-\Pi(k^2)\big)k^2}\,. \tag{3.251}$$

Photon self-energy at one-loop. At order α in perturbation theory, the photon propagator receives a fermion loop correction associated with the amputated diagram:



Taking into account the extra minus sign for the fermion loop, we get:

$$i\Pi^{\mu\nu}_{\text{1-LOOP}} = -(-ie)^2 \int \frac{d^4p}{(2\pi)^4} \frac{\text{Tr}\,\big(\gamma^\mu i(\not{p}+m)\gamma^\nu i(\not{p}+\not{k}+m)\big)}{(p^2-m^2+i\epsilon)((p+k)^2-m^2+i\epsilon)}\,. \tag{3.252}$$

The numerator can be simplified using the gamma-matrix trace identities. One has:

$$\begin{aligned}\text{Tr}\,\big(\gamma^\mu(\not{p}+m)\gamma^\nu(\not{p}+\not{k}+m)\big) &= \text{Tr}\,(\gamma^\mu\gamma^\nu)\, m^2 + \text{Tr}\,(\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma)\, p_\rho (p+k)_\sigma \\ &= 4\, m^2\eta^{\mu\nu} + 4\,(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\nu}\eta^{\rho\sigma} + \eta^{\mu\sigma}\eta^{\rho\nu})\, p_\rho (p+k)_\sigma \\ &= 4\,\eta^{\mu\nu}\big(m^2 - p\cdot(p+k)\big) + 4\,\big(p^\mu(p+k)^\nu + p^\nu(p+k)^\mu\big)\,.\end{aligned} \tag{3.253}$$

Regarding the denominator, we proceed as in the previous calculation:

$$\begin{aligned}\frac{1}{(p^2-m^2+i\epsilon)((p+k)^2-m^2+i\epsilon)} &= \int_0^1 \frac{dx}{\big(xp^2+(1-x)(p+k)^2-m^2+i\epsilon\big)^2} \\ &= \int_0^1 \frac{dx}{\Big(\big(p+(1-x)k\big)^2+x(1-x)k^2-m^2+i\epsilon\Big)^2}\end{aligned} \tag{3.254}$$

So that one can definie

$$\Delta \stackrel{\text{def.}}{=} -x(1-x)k^2+m^2\,, \tag{3.255}$$

and shift the loop momentum in (3.252) to get:

$$\begin{aligned}i\Pi^{\mu\nu}_{\text{1-LOOP}} = -4e^2\int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2-\Delta+i\epsilon)^2}\Bigg(&\eta^{\mu\nu}\big(m^2-\ (\ell+(x-1)k)\cdot(\ell+xk)\big) \\ &+\ \big((\ell-(1-x)\,k)^\mu(\ell+x\,k)^\nu+(\ell-(1-x)\,k)^\nu(\ell+x\,k)^\mu\big)\Bigg)\end{aligned} \tag{3.256}$$

The expression can be simplified using that, by isotropy:

$$\int d^d\ell\, f(\ell^2)\ell^\mu = 0 \quad , \qquad \int d^d\ell\, f(\ell^2)\ell^\mu\ell^\nu = \int d^d\ell\, f(\ell^2)\frac{\ell^2}{d}\eta^{\mu\nu}\,. \tag{3.257}$$

The coefficient of proportionnality in the second identity is obtained by contracting with the metric on both sides. After a Wick rotation and going to d dimensions, one gets:

$$\begin{aligned}\Pi^{\mu\nu}_{\text{1-LOOP}} &= -4e^2\mu^{4-d}\int_0^1 dx\int\frac{d^d\ell_E}{(2\pi)^d}\frac{1}{(\ell_E^2+\Delta)^2}\Bigg(\eta^{\mu\nu}\big(m^2+\ell_E^2+\ (1-x)x\,k^2\big) \\ &\qquad\qquad +2\Big(-\eta^{\mu\nu}\ell_E^2/d-x(1-x)\,k^\mu k^\nu\Big)\Bigg) \\ &= -4e^2\mu^{4-d}\int_0^1 dx\int\frac{d^d\ell_E}{(2\pi)^d}\frac{1}{(\ell_E^2+\Delta)^2}\Big(\eta^{\mu\nu}\big(m^2+(1-2/d)\,\ell_E^2+(1-x)x\,k^2\big)-2x(1-x)\,k^\mu k^\nu\Big)\end{aligned} \tag{3.258}$$

We have two type of integrals over Euclidian momenta. One was already computed, see eqns. (3.219,3.220), the other one gives:

$$\begin{aligned}\int\frac{d^d\ell_E}{(2\pi)^d}\frac{\ell_E^2}{(\ell_E^2+\Delta)^2} &= \frac{2}{\Gamma(d/2)}\frac{\Delta^{d/2-2}}{(4\pi^2)^{d/2}}\int_0^\infty\frac{u^{d+1}\,du}{(u^2+1)^2} = \frac{1}{\Gamma(d/2)}\frac{\Delta^{d/2-1}}{(4\pi^2)^{d/2}}\int_0^1 x^{-d/2}(1-x)^{d/2}dx \\ &= \frac{1}{\Gamma(d/2)}\frac{\Delta^{d/2-1}}{(4\pi^2)^{d/2}}\Gamma(1+d/2)\Gamma(1-d/2) = \frac{d}{2}\frac{\Delta^{d/2-1}}{(4\pi^2)^{d/2}}\Gamma(1-d/2)\,.\end{aligned} \tag{3.259}$$

So overall one gets:

$$\Pi^{\mu\nu}_{\text{1-LOOP}} = -\frac{4e^2\mu^{4-d}}{(4\pi)^{d/2}}$$
$$\int_0^1 dx\,\Delta^{d/2-2}\Big(\big(\eta^{\mu\nu}\big(m^2+(1-x)x\,k^2\big)-2x(1-x)\,k^\mu k^\nu\big)\,\Gamma\big(2-\tfrac{d}{2}\big)+\underbrace{\big(\tfrac{d}{2}-1\big)\Gamma\big(1-\tfrac{d}{2}\big)}_{-\Gamma\left(2-\frac{d}{2}\right)}\eta^{\mu\nu}\Delta\Big)$$
$$= -\frac{4e^2\mu^{4-d}}{(4\pi)^{d/2}}\Gamma\big(2-\tfrac{d}{2}\big)\int_0^1 dx\,\Delta^{d/2-2}\,\big(\eta^{\mu\nu}\big(m^2+(1-x)x\,k^2\big)-2x(1-x)\,k^\mu k^\nu-\eta^{\mu\nu}\Delta\big) \tag{3.260}$$

Now let us plug in the expression of Δ, eqn. (3.255); then eqn. (3.260) simplifies significantly to (with $\alpha = e^2/4\pi$):

$$\begin{aligned}\Pi^{\mu\nu}_{\text{1-LOOP}} &= \left(-\frac{8e^2\mu^{4-d}}{(4\pi)^{d/2}}\Gamma\big(2-\tfrac{d}{2}\big)\int_0^1 dx\,(1-x)x\,\Delta^{d/2-2}\right)\big(k^2\eta^{\mu\nu}-k^\mu k^\nu\big)\\ &= \underbrace{\left(-\frac{2\alpha}{\pi}\big(4\pi\mu^2\big)^{\epsilon/2}\Gamma(\epsilon/2)\int_0^1 dx\,(1-x)x\,\Delta^{-\epsilon/2}\right)}_{\Pi(k^2)}\big(k^2\eta^{\mu\nu}-k^\mu k^\nu\big)\end{aligned} \tag{3.261}$$

Thus one sees that $\Pi^{\mu\nu}_{\text{1-LOOP}}$ is of the expected form (3.241).[13] In particular it satisfies:

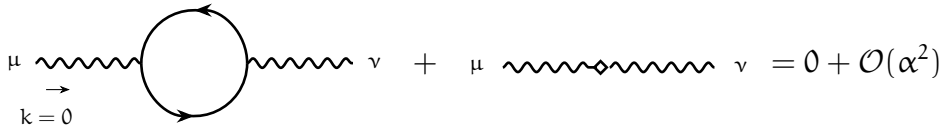
$$k_\mu\Pi^{\mu\nu}_{\text{1-LOOP}} = 0\,. \tag{3.262}$$

The ϵ expansion of the prefactor $\Pi(k^2)$ in eqn. (3.261) gives, after adding the photon wave-function counter-term, the final result:

$$\begin{aligned}\Pi^{\mu\nu} &= \Pi^{\epsilon}_{\text{1-LOOP}}(k^2)\,\big(k^2\eta^{\mu\nu}-k^\mu k^\nu\big)-\delta_3(\epsilon)\big(k^\mu k^\nu-\eta^{\mu\nu}k^2\big)+\mathcal{O}(\alpha^2)\,,\\ \Pi^{\epsilon}_{\text{1-LOOP}}(k^2) &= -\frac{2\alpha}{\pi}\int_0^1 dx\,(1-x)x\left(\frac{2}{\epsilon}+\ln\left(\frac{4\pi e^{-\gamma}\mu^2}{\Delta}\right)+\mathcal{O}(\epsilon)\right)\end{aligned} \tag{3.263}$$

We deduce from this expression and eqn. (3.246) the leading contribution to the renormalization condition (3.247):

$$\Pi^{\epsilon}_{\text{1-LOOP}}(0)-\delta_3(\epsilon) = 0+\mathcal{O}(\alpha^2)\,. \tag{3.264}$$



[13] If we had used a UV cutoff Λ instead, one can show that we would have found that: $\Pi^{\mu\nu}\sim\eta^{\mu\nu}\Lambda+\cdots$, thereby giving a mass to the photons and breaking gauge-invariance of the theory, which is forbidden!

We can then extract the momentum dependence of the vacuum polarization of the photon:

$$\Pi_{1\text{-L}}(k^2)-\delta_3(\epsilon) = \Pi^{\epsilon}_{1\text{-L}}(k^2)-\Pi^{\epsilon}_{1\text{-L}}(0)+\mathcal{O}(\alpha) = \frac{2\alpha}{\pi}\int_0^1 dx\,(1-x)x\ln\left(1-x(1-x)\frac{k^2}{m^2}\right)+\mathcal{O}(\alpha) \tag{3.265}$$

which is finite and independent of the dimensional regularization parameter ϵ. For large and space-like photon momentum, it has a logarithmic running:

$$\Pi_{1\text{-L}}(k^2)+\delta_3(\epsilon) \overset{-k^2/m^2\gg 1}{\sim} \frac{\alpha}{3\pi}\ln\left(-k^2/m^2\right)\,. \tag{3.266}$$

Analytic structure of the photon propagator. The remaining integral over x in (3.265) can also done analytically in order to probe the analytic structure of the photon self-energy in the k^2 complex plane. It gives:

$$\Pi_{1\text{-L}}(k^2)-\delta_3 = \frac{\alpha}{9\pi}+\frac{2\alpha}{3\pi}\left(1+\frac{2m^2}{k^2}\right)\left(\left(\frac{4m^2}{k^2}-1\right)^{1/2}\text{Arccot}\left(\frac{4m^2}{k^2}-1\right)^{1/2}-1\right)\,. \tag{3.267}$$

The interesting feature of this non-trivial expression is a branch cut for $\Pi(k^2)$ (thereby for the loop-corrected photon propagator) along the positive k^2 real axis starting at $k^2 = 4m^2$. It is rather straightforward to understand from the Källén–Lehmann perspective; it corresponds to the appearance of a two-particle state made of one electron and one positron.

As we have seen in the general analysis of the analytic properties of the KL propagator, the discontinuity of the propagator, or in other words the imaginary part, is related to the density of states in the continuum, see eqn. (2.29). Here we have :

$$\rho_c(k^2) = \frac{1}{\pi}\,\text{Im}\,\Pi(k^2) = \frac{\alpha}{3\pi}\left(1-\frac{4m^2}{k^2}\right)^{1/2}\left(1+\frac{2m^2}{k^2}\right)+\mathcal{O}(\alpha^2)\,. \tag{3.268}$$

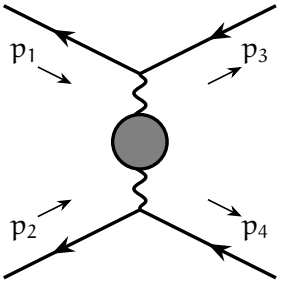
One can show that the rate of electron/positron pair production in an electromagnetic field can be related to this expression as:

$$\Gamma = \int d^4k\,\text{Im}\,\Pi(k^2)\,\left(\left|\vec{E}(k)\right|^2-\left|\vec{B}(k)\right|^2\right)^2\,, \tag{3.269}$$

see e.g. Itzykson–Zuber (p.191-195).

Running of the electric charge In subsection 3.2.2, we have learnt how to extract the Coulomb potential from tree-level electron-electron scattering, consisting in the virtual exchange of a photon. Taking into account the quantum corrections to the photon propagator,

the previous calculation should be corrected as:



It means that we could take into account the one-loop quantum effects by replacing the electric charge e in the tree-level amplitude (3.100) by an effective, momentum-dependent charge:

$$e \longmapsto e(k^2) \stackrel{\text{def.}}{=} \frac{e}{\sqrt{1-\Pi(k^2)}}\,, \quad k = p_1 - p_3\,. \tag{3.270}$$

In other words, the electric charge is not a constant, but depends on the energy of the virtual photon responsible for the Coulomb interaction, which is equal to the momentum transfer between the electrons, see fig. 3.4.

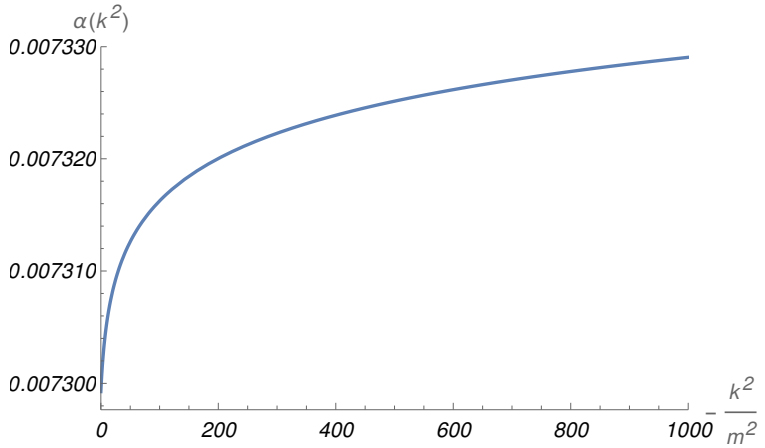


Figure 3.4: *One-loop running of the QED coupling constant* $\alpha(k^2)$.

In the limit of vanishing momentum transfer, *i.e.* the electrostatic limit, the renormalization condition (3.247) gives:

$$\frac{e}{\sqrt{1-\Pi(k^2)}} \xrightarrow{k^2/m^2 \to 0} \frac{e}{\sqrt{1-\Pi(0)}} = e\,. \tag{3.271}$$

This allows to make more precise the statement about the electric charge: the parameter e in the renormalized Lagrangian density (3.180) can be identified with the electron electric charge, but only *in the limit of vanishing momentum transfer*. The decrease of the electric charge at small momenta, *i.e.* large distances, is interpreted as an effect of the screening of the charges by electron/positron virtual pairs.

In the opposite limit, let us examine the behavior in the high-energy limit $k^2/m^2 \to -\infty$, at one-loop order. One has:

$$\Pi_{1\text{-L}}(k^2) + \delta_3 \overset{k^2/m^2 \to -\infty}{\sim} \frac{\alpha}{3\pi} \ln(-k^2/m^2)\,. \tag{3.272}$$

So it seems that the effective charge of the electron, thereby the coupling constant of QED, diverges for k^2 such that $\frac{\alpha}{3\pi}\ln(-k^2/m^2) = 1$, signaling a breakdown of the quantum theory at high energies, known as a *Landau pole*. Numerically, it occurs at energies far larger than any energy scale in physics, including the Planck energy. While one cannot trust the perturbative one-loop computation in this regime obviously, it suggests that the electromagnetic interaction should be somehow completed to a larger framework at high energies, which is actually the case in the standard model...

One may wonder why the loop corrections to the QED vertices were not taken into account in establishing eqn. (3.270), the running of the electric charge. This will be the topic covered in the next subsection.

Lamb shift. Let us end this subsection by discussing an effect of vacuum polarization that has been observed in atomic physics by Willis Lamb and Robert Rutherford in 1947. First, the expansion of eqn. (3.267) for small and space-like momentum, $-k^2/m^2 = \vec{k}^2/m^2 \ll 1$, gives:

$$\Pi(-\vec{k}^2) = \frac{\alpha}{15\pi}\frac{\vec{k}^2}{m^2} + \mathcal{O}(k^4/m^4)\,. \tag{3.273}$$

Therefore in the computation of the Coulomb potential done in eqn. (3.107) and below, one should replace:

$$\frac{e^2}{\vec{k}^2 + i\epsilon} \;\longmapsto\; \frac{e^2}{\vec{k}^2 + i\epsilon}\left(1 + \frac{\alpha}{15\pi}\frac{\vec{k}^2}{m^2}\right) \tag{3.274}$$

After Fourier transform, we get in position space:

$$V(r) = -\frac{e^2}{4\pi r} \;\longmapsto\; \left(1 - \frac{\alpha}{15\pi m^2}\Delta\right)\frac{-e^2}{4\pi r} = -\frac{e^2}{4\pi r} - \frac{\alpha}{15\pi}\frac{e^2}{m^2}\delta^{(3)}(\vec{r})\,. \tag{3.275}$$

This leads to an energy correction for the s-wave atomic levels:

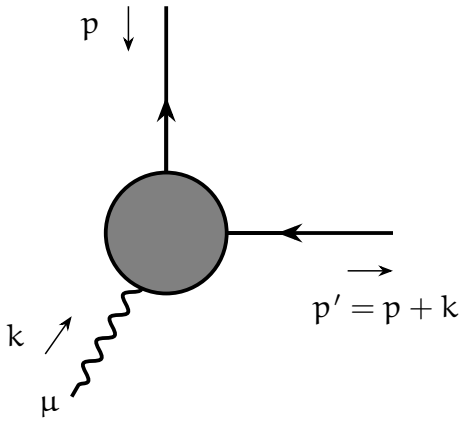
$$\delta E_{n,0} = -\frac{\alpha e^2}{15\pi m^2}\int d^3r\;, \Psi^*_{n,0}(\vec{r})\delta^{(3)}(\vec{r})\Psi_{n,0}(\vec{r}) = -\frac{\alpha e^2}{15\pi m^2}\left|\Psi_{n,0}(\vec{0})\right|^2 = -\frac{4}{15\pi}\frac{\alpha^5}{n^3}m\,, \tag{3.276}$$

where we have used $\Psi_{n,0}(\vec{0}) = (\pi n^3 a^3)^{-1/2}$ with $a = (m\alpha)^{-1}$. This actually just a small contribution to the splitting between the $2s_{1/2}$ and $2p_{1/2}$ levels; for a complete treatement of the problem, see Itzykson–Zuber (p.358-365). Overall there's an excellent agreement between QED and experiment.

3.4.3 Renormalized QED vertex

We first define the *QED vertex function* as the fully amputated 1PI correlation function in momentum space involving two electrons and one photon, including all quantum corrections:

$-ie\Gamma^\mu(p,p') \;=$



(3.277)

As we shall see below, in the limit $k = p' - p \to 0$, $\Gamma^\mu(p,p')$ depends only on $k = p' - p$ (besides its tensorial structure imposed by Lorentz covariance). The renormalization condition that we will impose, as discussed in the previous subsection, is that the parameter e is identified with the electric charge as measured in electrostatics, *i.e.* in the limit when the momentum transfer $p' - p$ goes to zero. It leads, *in the on-shell scheme*, to the renormalization condition:

$$\lim_{p' \to p} \Gamma^\mu(p,p') = \gamma^\mu \,. \tag{3.278}$$

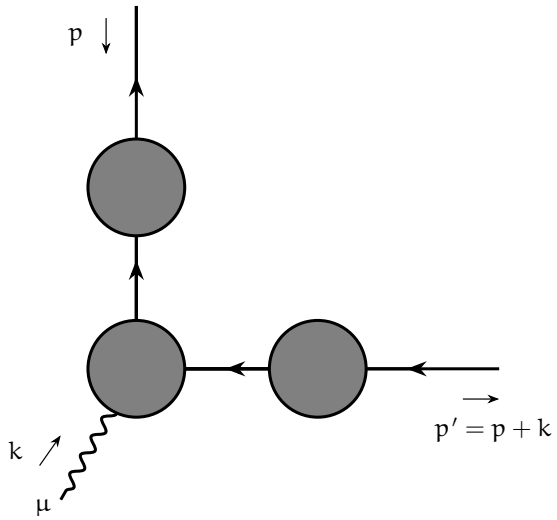
Electron wave-function vs. vertex renormalization. The expression of $\Gamma^\mu(p,p')$ is strongly related to the electron wave-function renormalization by the Ward–Takahashi identities. Let us consider the *regularized theory before renormalization*, defined by the first two parts in eqn. (3.185) without the counter-terms, and regularized by dimensional regularization or another method, which crucially *should preserve gauge invariance*.

We consider in this framework the following momentum space connected correlation function:

$$\mathcal{G}^\mu(-p,p',-k) = \int d^4x d^4y d^4z \, e^{i(p'\cdot y - p\cdot z - k\cdot x)} \, \langle\Omega|\, T\, j^\mu(x)\Psi(y)\bar\Psi(z)\,|\Omega\rangle\,\big|_{\text{CONNECT.}}\,, \tag{3.279}$$

where as usual $j^\mu(x)$ is the composite operator $j^\mu(x) = e\bar\Psi\gamma^\mu\Psi(x)$, *i.e.* the QED current.

Diagrammatically, it corresponds to :



in other words the fully corrected vertex attached to fully corrected Dirac propagators. On the one hand one can then write:

$$\mathcal{G}^{\mu}(-p,p',-k) = (2\pi)^4\delta^{(4)}(p-p'+k)\frac{i}{\not{p}'-m-\Sigma(\not{p}')}e\Gamma^{\mu}(p,p')\frac{i}{\not{p}-m-\Sigma(\not{p})}\,, \quad (3.280)$$

in terms of the vertex function Γ^{μ} and the electron self-energy $\Sigma(\not{p})$.[14] On the other hand, the Ward–Takahashi identity tells us that:

$$\begin{aligned} k_{\mu}\mathcal{G}^{\mu}(-p,p',-k) &= i\int d^4x d^4y d^4z\, \partial_{x^{\mu}} e^{i(p'\cdot y - p\cdot z - k\cdot x)} \langle\Omega|\, T\, j^{\mu}(x)\Psi(y)\bar{\Psi}(z)\,|\Omega\rangle \\ &\overset{\text{IBP}}{=} -i\int d^4x d^4y d^4z\, e^{i(p'\cdot z - p\cdot y - k\cdot x)}\partial_{x^{\mu}}\, \langle\Omega|\, T\, j_{\mu}(x)\Psi(y)\bar{\Psi}(z)\,|\Omega\rangle \\ &= -\int d^4x d^4y d^4z\, e^{i(p'\cdot y - p\cdot z - k\cdot x)}\Big(\, \langle\Omega|\, T\, \delta\Psi(y)\bar{\Psi}(z)\,|\Omega\rangle\, \delta^{(4)}(x-y) \\ &\qquad + \langle\Omega|\, T\, \Psi(y)\delta\bar{\Psi}(z)\,|\Omega\rangle\, \delta^{(4)}(x-z)\Big)\,. \end{aligned} \quad (3.281)$$

The infinitesimal $U(1)$ transformations are $\delta\Psi = -ie\Psi$ and $\delta\bar{\Psi} = +ie\Psi$, thus we have:

$$\begin{aligned} k_{\mu}\mathcal{G}^{\mu}(-p,p',-k) &= +ie\int d^4y d^4z\, e^{i((p'-k)\cdot y - p\cdot z)}\, \langle\Omega|\, T\, \Psi(y)\bar{\Psi}(z)\,|\Omega\rangle \\ &\quad - ie\int d^4y d^4z\, e^{i(p'\cdot y - (p+k)\cdot z)}\, \langle\Omega|\, T\, \Psi(y)\bar{\Psi}(z)\,|\Omega\rangle \\ &= ie(2\pi)^4\,\delta^{(4)}(p-p'+k)\left(\frac{i}{\not{p}-m-\Sigma(\not{p})} - \frac{i}{\not{p}'-m-\Sigma(\not{p}')}\right) \end{aligned} \quad (3.282)$$

[14] Indeed, in the perturbative expansion one would have an insertion of $-ie\bar{\Psi}\gamma^{\mu}\Psi$ instead of $e\bar{\Psi}\gamma^{\mu}\Psi$.

Multiplying expressions (3.280) and (3.282) by the inverse of the fully corrected Dirac propagators on the left and and the right, we arrive to:

$$\begin{aligned} k_\mu \Gamma^\mu(p,p') &= \left(\not{p}' - m - \Sigma(\not{p}')\right)\left(\frac{1}{\not{p} - m - \Sigma(\not{p})} - \frac{1}{\not{p}' - m - \Sigma(\not{p}')}\right)\left(\not{p} - m - \Sigma(\not{p})\right) \\ &= \not{p}' - \not{p} - \Sigma(\not{p}') + \Sigma(\not{p}) \end{aligned} \tag{3.283}$$

Let us take the limit of zero momentum-transfer, $k^\mu = (p' - p)^\mu \to 0$, and put the electron on-shell. We obtain:

$$\Gamma^\mu_{\text{REG}}(p,p) - \gamma^\mu = -\frac{\partial}{\partial p^\mu}\Sigma_{\text{REG}}(\not{p})\Big|_{\not{p}\to m}, \tag{3.284}$$

where we have added the suffix REG to remind ourselves that we work in the regularized theory, as opposed to the renormalization theory.

The left-hand side contains all the perturbative corrections to the vertex function (which is equal to γ^μ at zeroth order). At one-loop order, we'll see shortly that we get an order α correction $\delta\Gamma^{\mu\,\epsilon}_{\text{1-LOOP}}(p,p')$ containing a divergent piece. To renormalize the theory, one adds the counter-terms, in particular the vertex counter-term δ_1. In the *on-shell renormalization scheme*, we impose the renormalization condition (3.278):

$$\delta_1(\epsilon) + \Gamma^{\mu\,\epsilon}_{\text{1-LOOP}}(p,p) = 0 + \mathcal{O}(\alpha^2)\,. \tag{3.285}$$

In the same way, the right-hand side of (3.284) starts at order α with the derivative of $\Sigma^\epsilon_{\text{1-LOOP}}(\not{p})$ computed before, see eqn. (3.222). The renormalization condition, in the on-shell scheme, see eqn. (3.227), gives then:

$$\frac{\partial}{\partial p^\mu}\Sigma_{\text{REG}}(\not{p})\Big|_{\not{p}\to m} - \delta_2 = 0 + \mathcal{O}(\alpha^2)\,. \tag{3.286}$$

Therefore, one finds at one loop order the following relation between the coefficients of the counter-terms for the vertex and for the electron wave-function:

$$\delta_1 = \delta_2 + \mathcal{O}(\alpha^2)\,. \tag{3.287}$$

This relation actually persists to all order in perturbation theory. At order α^2, the regularized vertex function $\Gamma^\mu_{\text{REG}}(p,p)$ will contain a new two-loop divergence (in the limit where the regulator is removed), that is cancelled by an order α^2 correction to δ_1. The same occurs for $\frac{\partial}{\partial p^\mu}\Sigma_{\text{REG}}(\not{p})$, leading to an order α^2 correction to δ_2. Then equation (3.284) imposes that those correction are identical (up to order α^3 corrections). The same holds at higher order.

Since $Z_2 = \tilde{Z}_2$ in the on-shell renormalization scheme, we arrive then to an important relation, valid to all orders in perturbation theory:

$$\boxed{Z_1 = Z_2\,.} \tag{3.288}$$

Using the minimal substraction or $\overline{MS}$ scheme gives also the relation $Z_1 = \tilde{Z}_2$, as one extracts the same "universal" diverging part from both loop corrections. However in a less common renormalization scheme Z_1 and $\tilde{Z}_2$ may differ by a *finite* quantity.

In retrospect, we could have anticipated this result. Whenever this equality holds, the electron wave-function and vertex counter-terms in the Lagrangian density (3.185) can be put together in terms of a covariant derivative:

$$\delta_2 \bar{\Psi}_R i\partial\!\!\!/ \Psi_R - e\,\delta_1\, A^\mu_R \bar{\Psi}_R \gamma_\mu \Psi_R \;\longrightarrow\; \delta_1 \bar{\Psi}_R i\big(\partial\!\!\!/ + ieA\!\!\!/\big)\Psi_R = \delta_1 \bar{\Psi}_R i D\!\!\!/ \Psi_R\,, \tag{3.289}$$

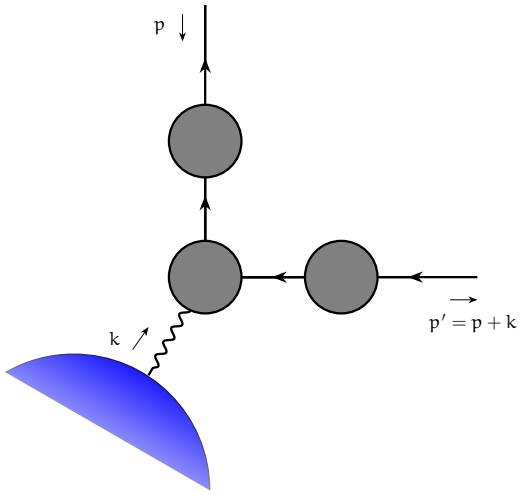
ensuring gauge-invariance of the counter-term contributions.

The relation (3.288) has another important consequence. Starting from the relation between the bare charge e_0 and the physical charge e we obtain:

$$\sqrt{\tilde{Z}_3}\tilde{Z}_2 e_0 = e Z_1 \overset{Z_1=\tilde{Z}_2}{\Longrightarrow} \boxed{e = \sqrt{\tilde{Z}_3} e_0} \tag{3.290}$$

In other words, the renormalization of the electric charge is only due to the wave-function renormalization of the photon; it depends neither on the wave-function renormalization of the electron, nor on the vertex renormalization. This justifies that we have evaluated the running of the electric charge, see eqn. (3.270), whithout taking into account the perturbative corrections to the vertex and to the electron propagator.

Vertex function and form factors. To simplify the computations, let us consider cases where the fully corrected QED vertex is part of a scattering amplitude where it is connected to the external legs of an incoming and an outgoing electron:



In this situation, the scattering amplitude involves the expression:

$$\mathcal{M}(p, p', \cdots) \;\sim \underbrace{\bar{u}_{s'}(\vec{p}\,')\Gamma^\mu(p,p')u_s(\vec{p})}_{F^\mu(p,p')}\Big|_{p^2=(p')^2=m^2}. \tag{3.291}$$

By Lorentz invariance, F^μ is necessarily of the form:

$$F^\mu = \bar{u}(p')\gamma^\mu u(p) A \; + (p+p')^\mu B + (p-p')^\mu C\,, \tag{3.292}$$

where A, B and C are functions of Lorentz scalars.

Since p and p' are on-shell, the only possible non trivial momentum dependence of the vertex function is in $k^2 = p^2 + (p')^2 - 2p \cdot p' = 2m^2 - 2p \cdot p'$. However, the photon 4-momentum k^μ is not on-shell, as we would like to think of the vertex function as attached to a bigger diagram.

Due to the Ward–Takahashi identity, more precisely to equation (3.283) derived before, we have that

$$k_\mu F^\mu = -\bar{u}_{s'}(\vec{p}\,')\big(\not{p}\,' - \not{p}\big)u_s(\vec{p}) = 0\,, \tag{3.293}$$

using $\not{p}\,u_s(\vec{p}) = m u_s(\vec{p})$. Therefore we should set $C = 0$.

In order to trade the dependence in $(p+p')$ in eqn. (3.292) for a dependence in $k = p'-p$, *i.e.* in the momentum transfer, we can use the *Gordon identity*:

$$\bar{u}_{s'}(\vec{p}\,')\gamma^\mu u_s(\vec{p}) = \bar{u}_{s'}(\vec{p}\,')\left(\frac{(p+p')^\mu}{2m} + \frac{iS^{\mu\nu}k_\nu}{m}\right)u_s(\vec{p})\,, \tag{3.294}$$

in terms of the Lorentz generator $S^{\mu\nu} = \frac{i}{4}[\,\gamma^\mu, \gamma^\nu\,]$. This identity is easier to prove starting from the right-hand side. We have:

$$\begin{aligned}\bar{u}_{s'}(\vec{p}\,')(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)(p_\nu - p'_\nu)u_s(\vec{p}) &= \bar{u}_{s'}(\vec{p}\,')\gamma^\mu(-\not{p}\,' + \not{p})u_s(\vec{p}) - \bar{u}_{s'}(\vec{p}\,')(-\not{p}\,' + \not{p})\gamma^\mu u_s(\vec{p}) \\ &= \bar{u}_{s'}(\vec{p}\,')\gamma^\mu(-\not{p}\,' + m)u_s(\vec{p}) - \bar{u}_{s'}(\vec{p}\,')(-m + \not{p})\gamma^\mu u_s(\vec{p}) \\ &= 2m\,\bar{u}_{s'}(\vec{p}\,')\gamma^\mu u_s(\vec{p}) - \bar{u}_{s'}(\vec{p}\,')\big(\gamma^\mu\not{p}\,' + \not{p}\gamma^\mu\big)u_s(\vec{p}) \end{aligned} \tag{3.295}$$

Together with

$$\begin{aligned} 2(p+p')^\mu &= \{\,\gamma^\mu, \gamma^\nu\,\}(p+p')_\nu = (\not{p} + \not{p}\,')\gamma^\mu + \gamma^\mu(\not{p} + \not{p}\,') \\ \implies\quad \bar{u}_{s'}(\vec{p}\,')2(p+p')^\mu u_s(\vec{p}) &= \bar{u}_{s'}(\vec{p}\,')\left((\not{p} + m)\gamma^\mu + \gamma^\mu(m + \not{p}\,')\right)u_s(\vec{p}) \end{aligned} \tag{3.296}$$

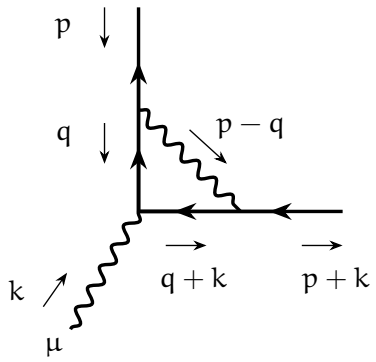
Adding the two pieces gives the desired result. □

Using this identity, the vertex function $\Gamma^\mu(p, p')$ can be put into its standard form:

$$\boxed{\Gamma^\mu(p, p') = \gamma^\mu F_1(k^2) + \frac{iS^{\mu\nu}k_\nu}{m}F_2(k^2)\,,} \tag{3.297}$$

where it is understood that Γ^μ is sandwiched between on-shell polarization vectors $\bar{u}(\vec{p}\,')$ and $u(\vec{p})$. The functions $F_1(k^2)$ and $F_2(k^2)$, which characterize the interaction between the electrons and electromagnetic field in the quantum theory, are known as *form factors*.

One-loop correction to the vertex function. The one-loop contribution to the vertex function is given by the amputated diagram:



The corresponding momentum space correlation function, with a fictious small photon mass ϕ as an infrared regulator, is given by:

$$-i\Gamma^{\mu}_{\text{1-LOOP}} = (-ie)^3 \int \frac{d^4q}{(2\pi)^4}\gamma^{\rho}\frac{i(\not{q}+\not{k}+m)}{(q+k)^2-m^2+i\epsilon}\gamma^{\mu}\frac{i(\not{q}+m)}{q^2-m^2+i\epsilon}\gamma^{\nu}\frac{-i\eta_{\rho\nu}}{(p-q)^2-\phi^2+i\epsilon} \tag{3.298}$$

For the computation of the one-loop correction to the form factors, it leads to:

$$\begin{aligned}\bar{u}(\vec{p}\,')\Gamma^{\mu}_{\text{1-LOOP}}(p,p+k)u(\vec{p}) = \\ -ie^3\int\frac{d^4q}{(2\pi)^4}\frac{\bar{u}(\vec{p}\,')\gamma^{\rho}(\not{q}+\not{k}+m)\gamma^{\mu}(\not{q}+m)\gamma_{\rho}u(\vec{p})}{\big((q+k)^2-m^2+i\epsilon\big)\big(q^2-m^2+i\epsilon\big)\big((p-q)^2-\phi^2+i\epsilon\big)}\end{aligned} \tag{3.299}$$

Considering the scattering amplitude with two external electrons rather than the vertex alone has two advantages: *(i)* the electrons are on-shell, *i.e.* satisfy $p^2=(p')^2=m^2$ and *(ii)* the Dirac equation $(\not{p}+m)u(\vec{p})=0$ will allow to make some further simplifications at the end of the computation.

The numerator in eqn. (3.298) can be simplified using the properties (3.129e-3.129g) generalized to dimension $d=4-\epsilon$:

$$\gamma^{\mu}\gamma^{\nu}\gamma_{\mu} = (2\eta^{\nu\mu}-\gamma^{\nu}\gamma^{\mu})\gamma_{\mu} = 2\gamma^{\nu}-\gamma^{\nu}\frac{\gamma^{\mu}\gamma^{\sigma}+\gamma^{\sigma}\gamma^{\mu}}{2}\eta_{\mu\sigma} = -(2-\epsilon)\gamma^{\nu} \tag{3.300a}$$

$$\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma_{\mu} = 4\eta^{\nu\rho}-\epsilon\gamma^{\nu}\gamma^{\rho} \tag{3.300b}$$

$$\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\gamma_{\mu} = -2\gamma^{\sigma}\gamma^{\rho}\gamma^{\nu}+\epsilon\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma} \tag{3.300c}$$

One has then:

$$\begin{aligned}\gamma^{\rho}(\not{q}+\not{k}+m)\gamma^{\mu}(\not{q}+m)\gamma_{\rho} = -2m^2\gamma^{\mu}+4m(2q+k)^{\mu}-2\not{q}\gamma^{\mu}(\not{q}+\not{k}) \\ +\epsilon\Big(m^2\gamma^{\mu}-m\big((\not{q}+\not{k})\gamma^{\mu}+\gamma^{\mu}\not{q}\big)+(\not{q}+\not{k})\gamma^{\mu}\not{q}\Big)\end{aligned} \tag{3.301}$$

The next step is to use the Feynman parametrization (3.209) to express the product of the denominators as:

$$\frac{1}{(q+k)^2-m^2+i\epsilon}\frac{1}{q^2-m^2+i\epsilon}\frac{1}{(p-q)^2-\phi^2+i\epsilon} = 2\int_0^1 dx\,dy\,dz\,\frac{\delta(x+y+z-1)}{\big(x(q+k)^2+yq^2+z(p-q)^2-(x+y)m^2-z\phi^2)+i\epsilon\big)^3} \quad (3.302)$$

The denominator can be factorized as:

$$(\underbrace{q+xk-zp}_{\ell})^2+x(1-x)k^2+z(1-z)p^2+2xz\,k\cdot p-(x+y)m^2-z\phi^2$$

And can be further simplified by using that the incoming and outgoing electrons are on-shell:

$$p^2=m^2\ ,\quad k\cdot p=(p'-p)\cdot p=p'\cdot p-m^2=-k^2/2$$

to

$$\ell^2+x(1-x)k^2+z(1-z)m^2-xz\,k^2-(x+y)m^2-z\phi^2=\ell^2-\big(\underbrace{z\phi^2+(1-z)^2m^2-xz\,k^2}_{\Delta}\big)$$

Under the change of variables $q=\ell+zp-xk$, as explained before the linear terms in the numerator drop by isotropy and the quadratic one can be simplified using eqn. (3.257).

Then, after few more pages of gamma-matrices algebra, using in particular the Dirac equation (*i.e.* $\not{p}u=mu$ and $\bar{u}\not{p}'=m\bar{u}$) and the Gordon identity (3.294), we reach a result of the requested form:

$$\Gamma^{\mu}_{\text{1-LOOP}}(p,p+k)\ =\ \gamma^{\mu}F_1^{\text{1-L}}(k^2)+\frac{iS^{\mu\nu}k_{\nu}}{m}F_2^{\text{1-L}}(k^2)\,, \quad (3.303)$$

with the integrals (after Wick rotation):

$$F_1^{\text{1-L}}(k^2)=2e^2\int_0^1 dxdydz\,\delta(x+y+z-1)\mu^{4-d}\int\frac{d^d\ell_E}{(2\pi)^d}\frac{\frac{(d-2)^2}{d}\ell_E^2+(d-2)\Delta-2z(2m^2-k^2)}{(\ell_E^2+\Delta)^3} \quad (3.304a)$$

$$F_2^{\text{1-L}}(k^2)=2e^2\int_0^1 dxdydz\,\delta(x+y+z-1)\mu^{4-d}\int\frac{d^d\ell_E}{(2\pi)^d}\frac{(1-z)m^2\big(4z+2(1-z)(4-d)\big)}{(\ell_E^2+\Delta)^3} \quad (3.304b)$$

where

$$\Delta=z\phi^2+(1-z)^2m^2-xz\,k^2\,. \quad (3.305)$$

3.4.4 The electron anomalous moment

We will first analyze the one-loop correction (3.304b) to the vertex function. Notice first that the large-momentum behavior of this integral is $F_2 \sim \int \ell^{d-7} d\ell$, so it is convergent in dimension four and no UV regularization of any sort if needed. As we will see, the integral is also convergent in the IR so one can set $\phi = 0$ consistently. The loop integral is then computed in $d = 4$ without any particular trick:

$$\begin{aligned}\int \frac{d^4\ell_E}{(2\pi)^4} \frac{4z(1-z)m^2}{(\ell_E^2+\Delta)^3} &= \frac{4z(1-z)m^2}{8\pi^2} \int_0^\infty \frac{\ell^3 d\ell}{(\ell^2+\Delta)^3} = \frac{4z(1-z)m^2}{16\pi^2} \int_0^\infty \frac{u du}{(u+\Delta)^3} \\ &= \frac{4z(1-z)m^2}{16\pi^2} \left[-\frac{\Delta+2u}{2(\Delta+u)^2}\right]_0^\infty = \frac{4z(1-z)m^2}{32\pi^2}\frac{1}{\Delta} \end{aligned} \tag{3.306}$$

We will be in particular interested in the value of F_2 at $k^2 = 0$. We have then:

$$F_2^{1\text{-L}}(0) = \frac{e^2}{4\pi^2} \int_0^1 dx dy dz\, \delta(x+y+z-1) \frac{z}{1-z} \tag{3.307}$$

Since nothing depends on x and y, we have

$$\int_0^1 dx dy\, \delta(x+y+z-1) = \int_0^{1-z} dx = 1-z\,, \tag{3.308}$$

such that:

$$F_2^{1\text{-L}}(0) = \frac{e^2}{4\pi^2} \int_0^1 z dz = \frac{e^2}{8\pi^2} = \frac{\alpha}{2\pi}\,. \tag{3.309}$$

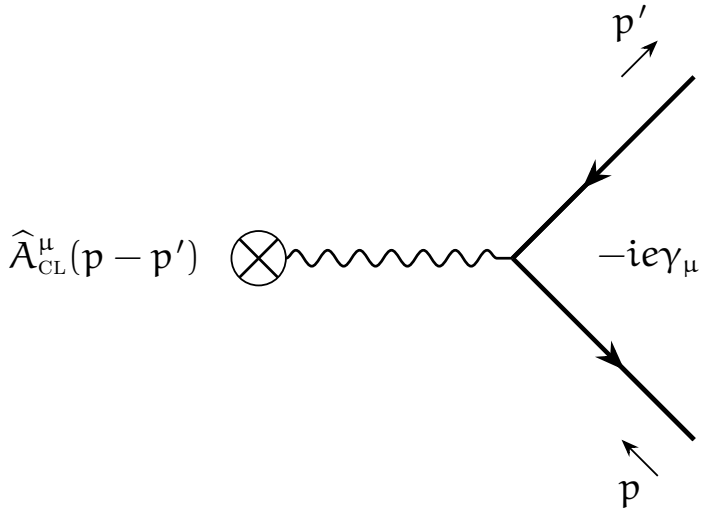
Coupling to a classical electromagnetic field. To understand the significance of this result, it is useful to consider that the quantum electron field, besides interacting with the quantum photon field, is subject to an external, *classical* electromagnetic field represented by the 4-potential $A^\mu_{CL}(x)$; this is similar to what you've done in the past in your non-relativistic quantum mechanics course.

The interaction Hamiltonian of perturbation theory is then modified by adding a new term corresponding to the coupling to the external field:

$$H_{INT} \longrightarrow H_{INT} + \int d^3y\, A^\mu_{CL}(t,\vec{y}) j_\mu(t,\vec{y})\,, \tag{3.310}$$

where as usual $j^\mu = e\bar{\Psi}\gamma^\mu\Psi$, *i.e.* the operator associated with the QED Noether current. This term adds a two-legged vertex to perturbation theory, where the classical electromagnetic field

is conventionally represented by a symbol $\otimes$:



Correspondingly, the tree-level scattering amplitude for an electron in the classical electric field is obtained from the Wick contraction

$$-ie\overbrace{\Psi(x_1)\bar{\Psi}(x_2)}\int d^4x\bar{\Psi}(x)\gamma_\mu\Psi(x)A^{\mu}_{CL}(x) \tag{3.311}$$

and the integral over x, after LSZ reduction, gives eventually the Fourier transform of A^{μ}_{CL}:

$$i\mathcal{M}_{ss'}(p,p') \;=\; -ie\,\bar{u}'_s(\vec{p}')\gamma_\mu u_s(\vec{p})\,\widehat{A}^{\mu}_{CL}(p'-p)\,. \tag{3.312}$$

Including all corrections to the QED vertex (but, importantly, not of the photon propagator since the electromagnetic field is classical), it gives:

$$\begin{aligned} i\mathcal{M}_{ss'}(p,p') \;&=\; -ie\,\bar{u}'_s(\vec{p}')\,\Gamma_\mu(p,p')\,u_s(\vec{p})\,\widehat{A}^{\mu}_{CL}(k) && (3.313)\\ &=\; -ie\,\bar{u}'_s(\vec{p}')\left(\gamma^\mu F_1(k^2)+\frac{iS^{\mu\nu}k_\nu}{m}F_2(k^2)\right)u_s(\vec{p})\,\widehat{A}^{\mu}_{CL}(k) && (3.314)\end{aligned}$$

with as before $k = p' - p$.

Let us consider a time-independent magnetic field characterized by its Fourier transform $\widehat{A}^{\mu}(0,\vec{k}) = \big(0,\widehat{\vec{A}}(\vec{k})\big)$. In the non-relativistic limit,

$$u(\vec{p}) = \begin{pmatrix}\sqrt{p\cdot\sigma}\,\eta\\ \sqrt{p\cdot\bar{\sigma}}\eta\end{pmatrix} \simeq \sqrt{m}\begin{pmatrix}(1-\vec{p}\cdot\vec{\sigma}/2m)\eta\\ (1+\vec{p}\cdot\vec{\sigma}/2m)\eta\end{pmatrix} \tag{3.315}$$

The contribution from the first form factor F_1 involves then

$$\bar{u}(p')\vec{\gamma}u(p) = m\begin{pmatrix}\eta'^{\dagger}(1-\vec{p}'\cdot\vec{\sigma}/2m) & \eta'^{\dagger}(1+\vec{p}'\cdot\vec{\sigma}/2m)\end{pmatrix}\begin{pmatrix}-\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix}\begin{pmatrix}(1-\vec{p}\cdot\vec{\sigma}/2m)\eta\\ (1+\vec{p}\cdot\vec{\sigma}/2m)\eta\end{pmatrix}$$
$$= \eta'^{\dagger}\big((\vec{p}'\cdot\vec{\sigma})\vec{\sigma}+\vec{\sigma}(\vec{p}\cdot\vec{\sigma})\big)\eta \quad (3.316)$$

Using the identity

$$\sigma^i\sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k\,, \quad (3.317)$$

it gives:

$$\bar{u}(p')\vec{\gamma}u(p)\cdot\widehat{\vec{A}}_{CL}(\vec{p}'-\vec{p}) = (\vec{p}'+\vec{p})\cdot\widehat{\vec{A}}_{CL}(\vec{p}'-\vec{p})\,\eta'^{\dagger}\eta$$
$$+ i\epsilon^{ijk}(p'-p)^i\widehat{A}^j_{CL}(\vec{p}'-\vec{p})\,\eta'^{\dagger}\sigma^k\eta \quad (3.318)$$

The term in the first line corresponds to the usual coupling to the electromagnetic vector potential in non-relativistic quantum mechanics.

The second term corresponds to the coupling between the spin and the magnetic field. This coupling receives also a contribution from the form factor F_2. At the same linear order in the momentum, it is given by

$$\widehat{A}^i_{CL}(p'-p)^j\begin{pmatrix}\eta'^{\dagger} & \eta'^{\dagger}\end{pmatrix}\underbrace{\frac{i}{4}\left[\gamma^i,\gamma^j\right]}_{\frac{1}{2}\epsilon^{ijk}\left(\begin{smallmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{smallmatrix}\right)}\begin{pmatrix}\eta\\ \eta\end{pmatrix} = \widehat{A}^i_{CL}(p'-p)^j\epsilon^{ijk}\eta'^{\dagger}\sigma^k\eta\,. \quad (3.319)$$

Putting together the contributions of both form factors to the scattering amplitude, the linear terms in the momentum $\vec{k}$ add up to:

$$\bar{u}\Gamma^\mu u \longrightarrow 2m\,\eta'^{\dagger}\left(-\frac{i}{2m}\epsilon^{ijk}k^j\sigma^k\left(F_1(0)+F_2(0)\right)\right)\eta\,. \quad (3.320)$$

It gives the following scattering amplitude:

$$i\mathcal{M} = 2m\,e\left(F_1(0)+F_2(0)\right)\left(\eta'^{\dagger}\sigma^k\eta\right)\frac{\epsilon^{ijk}}{2m}k^i\widehat{A}^j_{CL}(\vec{k}) \quad (3.321)$$

In terms of the classical external magnetic field $= -i\epsilon^{ijk}k^i\widehat{A}^j_{CL}$, we reach the final expression for the scattering amplitude:

$$\mathcal{M} = \frac{e}{2m}\left(F_1(0)+F_2(0)\right)\left(\eta'^{\dagger}\sigma^k\eta\right)\widehat{B}^k_{CL}\times\left(\sqrt{2m}\right)^2 \quad (3.322)$$

Leaving aside the $\sqrt{2m}$ relativistic renormalization factors, this scattering amplitude be interpreted in terms of non-relativistic scattering in the Born approximation, as we have done for the Coulomb potential, see eqn. (3.106). In the present case the potential reads:

$$V(\vec{x}) = -\langle\vec{\mu}\rangle\cdot\vec{B}_{CL}(\vec{x})\,, \quad (3.323)$$

in terms of the matrix element of the magnetic moment operator:

$$\langle\vec{\mu}\rangle = \frac{e}{2m}\,2\left(F_1(0)+F_2(0)\right)\,\left(\eta'^{\dagger}\,\vec{S}\,\eta\right) \quad , \qquad \vec{S}=\tfrac{1}{2}\vec{\sigma}\,. \tag{3.324}$$

The factor of proportionnality between the Bohr magneton $\mu_B = e/2m$ and the magnetic moment of the electron is known as the *Landé factor* g of the electron, a dimensionless quantity. If by e in formula (3.324) we really mean the physical charge of the electron at zero momentum transfer, then one should adopt the on-shell renormalization scheme $F_1(0)=1$, see below. Then the Landé factor is given by:

$$\boxed{g = 2 + 2F_2(0)} \tag{3.325}$$

The first factor is obtained directly from the expansion of the Dirac Lagrangian in the non-relativistic limit. The second factor is a genuine prediction of quantum electrodynamics, known as the *anomalous magnetic moment* of the electron. Experimentally was observed first in 1947 by studying the hyperfine structure of hydrogen.

Acccording to our analysis, $F_2 = 0$ at tree-level and the first contribution arises at one loop order (*i.e.* at order α):

$$g = 2 + \frac{\alpha}{\pi} + \mathcal{O}(\alpha^2)\,. \tag{3.326}$$

This is a celebrated result obtained by Julian Schwinger in 1948 [3]. Numerically,

$$g \simeq 2,002\,323\,4 \tag{3.327}$$

Precision experiments give today the value

$$g_{\text{EXP}} \simeq 2,002\,319\,304\,361\,82(52) \tag{3.328}$$

This was one of the first triumphs of quantum field theory.

3.4.5 Renormalization of the vertex function

We consider now the 1-loop correction to the form factor $F_1(k^2)$, whose UV divergence will be compensated by the δ_1 counter-term, see eqn. (3.304a). It is schematically of the form:

$$F_1^{1\text{-L}}(k^2) = 2e^2\int_0^1 dx dy dz\;\delta(x+y+z-1)\,\mu^{4-d}\int\frac{d^d\ell_E}{(2\pi)^d}\left(\frac{\frac{(d-2)^2}{d}\ell_E^2}{(\ell_E^2+\Delta)^3}+\frac{B}{(\ell_E^2+\Delta)^3}\right)\,. \tag{3.329}$$

The second term gives a UV-finite contribution, as was the case for F_2. For the first term, we consider the integral:

$$\begin{aligned} I_d &= \frac{(d-2)^2}{d}\mu^{4-d}\int\frac{d^d\ell_E}{(2\pi)^d}\frac{\ell_E^2}{(\ell_E^2+\Delta)^3} = \frac{(d-2)^2}{d}\frac{2\mu^{4-d}}{(4\pi)^{d/2}\Gamma(d/2)}\Delta^{d/2-2}\int_0^\infty\frac{u^{d+1}du}{(u^2+1)^3} \\ &= \frac{(d-2)^2}{d}\frac{\mu^{4-d}}{(4\pi)^{d/2}\Gamma(d/2)}\Delta^{d/2-2}\int_0^1 dx\,(1-x)^{d/2}x^{1-d/2} \\ &= \frac{(d-2)^2}{d}\frac{\mu^{4-d}}{(4\pi)^{d/2}\Gamma(d/2)}\Delta^{d/2-2}\frac{\Gamma(d/2+1)\Gamma(2-d/2)}{\Gamma(3)} \\ &= (d/2-1)^2\frac{\mu^{4-d}}{(4\pi)^{d/2}}\Delta^{d/2-2}\Gamma(2-d/2) \\ &= \frac{1}{16\pi^2}\left(\frac{2}{\epsilon}+\ln\left(\frac{4\pi\mu^2e^{-\gamma}}{\Delta}\right)+\mathcal{O}(\epsilon)\right). \end{aligned} \tag{3.330}$$

One can use the on-shell renormalization scheme to ensure that the parameter e matches the physical charge of the electron, in the non-relativistic electrostatics regime:

$$\delta_1 = -F_1^{1\text{-L}}(0)+\mathcal{O}(\alpha^2)\,. \tag{3.331}$$

In this scheme the counter-term absorbs also the finite terms that survive the $\epsilon\to 0$ limit in eqn. (3.304a). One can check by direct comparison between the integral (3.304a) evaluated at $k^2=0$ and the integral (3.222) that $\delta_1=\delta_2$ as expected.

* * *

Instead of the on-shell renormalization scheme one can use the $\overline{MS}$ scheme. In this case we absorb the divergence plus universal structure into the vertex counterterm:

$$\delta_1\big|_{\overline{MS}} = -\frac{\alpha}{4\pi}\left(\frac{2}{\epsilon}+\ln\left(4\pi e^{-\gamma}\right)\right). \tag{3.332}$$

Comparing with eqn. (3.230), we find that:

$$\delta_1\big|_{\overline{MS}} = \delta_2\big|_{\overline{MS}} \tag{3.333}$$

As discussed below eqn. (3.287), this equality was expected indeed to hold both in the on-shell and $\overline{MS}$ renormalization schemes.

Beta-function of QED. We now discuss how the coupling e in the theory depends on the arbitrary scale μ introduced in the $\overline{MS}$ renormalization scheme; in the following $\delta_{1,2,3}$ and δm will be assumed to be computed in this scheme. First, in the $\overline{MS}$ renormalization scheme, the LSZ formula still gives (finite) renormalization factors for the fields. Let us define:

$$Z_2/\tilde{Z}_2 = \mathfrak{z}_2\,,\quad Z_3/\tilde{Z}_3=\mathfrak{z}_3\,. \tag{3.334}$$

Then we consider the scattering amplitude of an electron by a photon in the limit of zero transfer momentum, which should give the physical electric charge $e_{\rm PHYS}$. At order α in perturbation theory, the parameter e is related to $e_{\rm PHYS}$ through:

$$e_{\rm PHYS} = e\Big(1 + \;\Big|_{k=0} + \delta_1 + \mathcal{O}(\alpha^2)\Big)\,\sqrt{\mathfrak{z}_3}\,\mathfrak{z}_2\,. \tag{3.335}$$

with

$$\mathfrak{z}_3 = \frac{1 + \;\Big|_{k=0} + \mathcal{O}(\alpha^2)}{1+\delta_3} \;=\; 1 + \;\Big|_{k=0} - \delta_3 + \mathcal{O}(\alpha^2) \tag{3.336}$$

and similarly:

$$\mathfrak{z}_2 = \frac{1 + \frac{\mathrm{d}}{\mathrm{d}\not{p}}\;\Big|_{\not{p}\to m} + \mathcal{O}(\alpha^2)}{1+\delta_2} \;=\; 1 + \frac{\mathrm{d}}{\mathrm{d}\not{p}}\;\Big|_{\not{p}\to m} - \delta_2 + \mathcal{O}(\alpha^2)\,. \tag{3.337}$$

According to eqn. (3.284) the loop corrections to the vertex function and to $\mathfrak{z}_2$ will compensate each other, thereby also the counter-terms δ_1 and δ_2, as in the minimal substraction scheme they merely remove the universal singular part of both. Therefore we have:

$$e_{\rm PHYS} = e\Big(1 + \frac{1}{2}\Big(\;\Big|_{k=0} - \delta_3\Big) + \mathcal{O}(\alpha^2)\Big)\,. \tag{3.338}$$

This allows to extract how the parameter e, which is not the physical charge, depends implicitely on the scale μ introduced in the $\overline{\rm MS}$ scheme. Since $e = e_{\rm PHYS}$ at leading order, we have:

$$\frac{\mathrm{d}e}{\mathrm{d}\ln\mu} = -\frac{e}{2}\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\Big(\;\Big|_{k=0} - \delta_3\Big) \tag{3.339}$$

where, according to eqn. (3.263), we have in the $\overline{\rm MS}$ scheme:

$$\;\Big|_{k=0} - \delta_3 = -\frac{2\alpha}{\pi}\underbrace{\int_0^1 \mathrm{d}x\,(1-x)x}_{1/6}\,\ln\mu^2 + (\mu\text{-independent terms}) \tag{3.340}$$

Hence we arrive to the important result:

$$\boxed{\beta(e) = \frac{\mathrm{d}e}{\mathrm{d}\ln\mu} = \frac{e^3}{12\pi^2}\,,} \tag{3.341}$$

giving the variation of the coupling e with respect to the arbitrary renormalization scale μ. This is known as the *beta function* of QED. The coupling grows with the scale, as was noticed, in a slightly different context before, see fig. 3.4.

One may argue that the scale μ is just an artifact of dimensional regularization and has no physical meaning. This is not really the case for the following reason. After the 1PI diagrams (electron and photon self-energy and vertex function) have been renormalized in the

$\overline{\text{MS}}$ scheme, they still depend on the momenta through logarithmic terms like $\ln(-p^2/\mu^2)$. For perturbation theory to make sense, those logarithmic factors should remain small for the momenta involved in the scattering process considered. Hence, one has to consider μ as the "typical" energy scale involved in the physical process under study.

In this context, it is useful to view QED as a theory of free photons and *massless* electrons, perturbed by two couplings, the mass term and the QED interaction.[15] Then the two coupling constants of the theory, the electric charge and the electron mass, depends on the scale according to eqns. (3.341,3.233):[16]

$$\beta(e) = \frac{\mathrm{d}e}{\mathrm{d}\ln\mu} = \frac{e^3}{12\pi^2} + \mathcal{O}(e^5)\,, \quad \gamma(m) = \frac{1}{m}\frac{\mathrm{d}m}{\mathrm{d}\ln\mu} = -\frac{3e^2}{8\pi^2} + \mathcal{O}(e^4)\,. \tag{3.342}$$

Callan–Symanzik equation. The observables of the theory, the scattering amplitudes, are obtained according to LSZ reduction from the *renormalized correlation functions*,[17] that we can define as:

$$\mathcal{G}^{\text{REN}} = Z_3^{\ell/2} Z_2^{n/2} \langle\Omega|\, \mathsf{T}\, A^{\mu_1}(x_1)\cdots A^{\mu_\ell}(x_\ell)\Psi(y_1)\cdots\bar{\Psi}(y_n)\,|\Omega\rangle_{\text{REN}}\,, \tag{3.343}$$

where the right-hand side is computed in the renormalized theory. The renormalized correlation functions can be expressed more naturally in terms of the renormalized field. In any renormalization scheme, one has:

$$\mathcal{G}^{\text{REN}} = \mathfrak{z}_3^{\ell/2}\mathfrak{z}_2^{n/2} \langle\Omega|\, \mathsf{T}\, A_{\text{R}}^{\mu_1}(x_1)\cdots A_{\text{R}}^{\mu_\ell}(x_\ell)\Psi_{\text{R}}(y_1)\cdots\bar{\Psi}_{\text{R}}(y_n)\,|\Omega\rangle_{\text{REN}}\,. \tag{3.344}$$

Crucially the scattering amplitudes, thereby the renormalized correlation functions, should be independent on the renormalization scale chosen. This dependence in the renormalization scale appears in $\mathcal{G}^{\text{REN}}$. In the RHS of (3.344) there is an implicit dependence in μ through the wave-function renormalization factors $\mathfrak{z}_{2,3}$, the electric charge parameter e a and the electron mass parameter m. This can be formulated as a differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\mathcal{G}^{\text{REN}} = \left(\frac{\partial}{\partial\ln\mu} + \frac{\ell}{2}\frac{1}{\mathfrak{z}_3}\frac{\mathrm{d}\mathfrak{z}_3}{\mathrm{d}\ln\mu} + \frac{n}{2}\frac{1}{\mathfrak{z}_2}\frac{\mathrm{d}\mathfrak{z}_2}{\mathrm{d}\ln\mu} + \frac{\mathrm{d}e}{\mathrm{d}\ln\mu}\frac{\partial}{\partial e} + \frac{\mathrm{d}m}{\mathrm{d}\ln\mu}\frac{\partial}{\partial m}\right)\mathcal{G}^{\text{REN}} = 0\,. \tag{3.345}$$

Let us introduce:

$$\gamma_2 \overset{\text{def.}}{=} \frac{1}{\mathfrak{z}_2}\frac{\mathrm{d}\mathfrak{z}_2}{\mathrm{d}\ln\mu}\,, \quad \gamma_3 \overset{\text{def.}}{=} \frac{1}{\mathfrak{z}_3}\frac{\mathrm{d}\mathfrak{z}_3}{\mathrm{d}\ln\mu}\,. \tag{3.346}$$

Then one can rephrase equation (3.345) as the famous *Callan–Symanzik equation*:

$$\left(\frac{\partial}{\partial\ln\mu} + \frac{\ell}{2}\gamma_2 + \frac{n}{2}\gamma_3 + \beta(e)\frac{\partial}{\partial e} + \gamma(m)m\frac{\partial}{\partial m}\right)\mathcal{G}^{\text{REN}} = 0 \tag{3.347}$$

[15] This point of view makes sense at high enough energy scales, see below.

[16] The function $\gamma(m)$ is defined a bit differently than $\beta(e)$ in order to absorb the *classical* dimensionality of the mass parameter m.

[17] Through the usual steps: *(i)* Fourier transform *(ii)* full amputation *(iii)* multiplication by the external polarizations and *(iv)* putting the external momenta on-shell.

To write it in a more symmetric form one can trade the dependence in m to the dependence of a dimensional coupling g_m defined as $m = g_m \mu$. Then $\mathcal{G}^{\text{REN}}$ aquires an extra explicit source of dependence in μ that needs to be compensated for. Then eqn. (3.347) can be rewritten as:

$$\left(\frac{\partial}{\partial \ln \mu} + \frac{\ell}{2}\gamma_2 + \frac{n}{2}\gamma_3 + \beta(e)\frac{\partial}{\partial e} + \big(\gamma(m) - 1\big) g_m \frac{\partial}{\partial g_m} \right) \mathcal{G}^{\text{REN}} = 0 \tag{3.348}$$

Defining

$$\beta(g_m) \stackrel{\text{def.}}{=} \frac{\mathrm{d} g_m}{\mathrm{d} \ln \mu} = \frac{\mathrm{d}}{\mathrm{d} \ln \mu} \frac{m}{\mu} = g_m \left(\frac{1}{m} \frac{\mathrm{d} m}{\mathrm{d} \ln \mu} - 1 \right) = g_m \big(\gamma(m) - 1\big) \,, \tag{3.349}$$

One reaches the more symmetric form of the Callan–Symanzik equation:

$$\boxed{\left(\frac{\partial}{\partial \ln \mu} + \frac{\ell}{2}\gamma_2 + \frac{n}{2}\gamma_3 + \beta(e)\frac{\partial}{\partial e} + \beta(g_m) \frac{\partial}{\partial g_m} \right) \mathcal{G}^{\text{REN}} = 0} \tag{3.350}$$

This equation gives the scaling behavior of the scattering amplitudes. Up to order e^2 one has:

$$\beta(e) = \frac{e^3}{12\pi^2} \,, \quad \beta(g_m) = g_m \left(-1 - \frac{3e^2}{8\pi^2} \right) \,, \quad \gamma_2 = \frac{e^2}{16\pi^2} \,, \quad \gamma_3 = \frac{e^2}{12\pi^2} \,. \tag{3.351}$$

The leading order term in $\beta(g_m)$ gives the classical scaling of the correlation function in terms of the mass scale.[18] The order e^2 and e^3 contributions gives the quantum corrections to the scaling dimensions. Those are called *anomalous dimensions*.

The meaning of the anomalous dimensions is clearer in the case of massless QED (*i.e.* with $m = 0$). This theory is classically scale-invariant, since its only parameter, the electric charge e, is dimensionless in natural units. The Callan–Symanzik equation indicates how this scale-invariance is broken by quantum effects, in the form of the loop corrections.

In an arbitrary quantum field theory a given interaction term in the Lagrangian density is said to be:

- ***relevant*** if its coupling constant grows at low energies;
- ***irrelevant*** if its coupling constant shrinks at low energies.
- ***marginal*** if its coupling constant does not depend on the energy.

Quantum field theories well-defined perturbatively at high energies have $\beta < 0$ for all their coupling constants, such that in the UV they can be considered free with a good approximation. Those theories are said to be *asymptotically free*. This is for instance the case for quantum chromodynamics (QCD), which describes strong interactions. Theories with $\beta = 0$

[18] It is easier to see from eqn. (3.345): $\mathcal{G}^{\text{REN}}$ is expected, classically, to depend on μ and m through the ratio μ/m, and we would have then $(\mu\partial_\mu + m\partial_m)\mathcal{G}^{\text{REN}} = 0$.

exactly for all their couplings do not depend on the scale; under mild assumptions they are invariant under conformal transformations, hence are called *conformal field theories*. Whenever $\beta > 0$, the coupling constant grows in the UV, hence is not well-defined at energies high enough, as we have already discovered in four-dimensional QED. In the deep infrared (*i.e.* at very low energies) *massless* QED in 4d tends to a free theory. Whenever the electron is massive however, there is a new energy scale in the problem associated with the new coupling g_m, and the low-energy behavior changes at energy scales $\mu \leqslant m$. One can show that the flow of the coupling constant effectively "stops", which is why electromagnetic interactions exist at very large spatial scales. A more precise and complete picture would be given in the Wilsonian approach to renormalization, that we unfortunately won't have time do discuss in this course.

3.5 Renormalization of QED: towards a general picture

In this last section we will sketch how renormalization of QED works, beyond the one-loop corrections that we have explicitly derived and studied.

3.5.1 Summary of one-loop renormalization of QED

Let us first summarize the renormalization process that we have conducted at one loop order, through several steps that we will summarize in turn:

(i) Reorganizing perturbation theory as renormalized perturbation theory.

(ii) Regularizing the loop integrals.

(iii) Imposing renormalization conditions.

(iv) Removing the regulator.

This sequence of steps applies to any quantum field theory that admits a perturbative expansion.

***(i)* Renormalized perturbation theory.** The original Lagrangian $\mathcal{L}(\Psi, A^\mu, m_0, e_0)$ is written in terms of the fields Ψ and A^μ, with a parameter m_0 that plays the role of a mass term for the electron, and an interaction term with a parameter e_0 that plays the role of the charge of the electron.

However the parameters (m_0, e_0) have no reason to correspond respectively to the physical mass m and the physical charge e, as measured in experiments. The parameters (m_0, e_0) are called *bare parameters* because they are "dressed" with the interacting fields and are not measurable as one cannot "turn off" interactions.

The fields Ψ and A^μ, being also dressed by the interactions, are not normalized as the corresponding free fields would be: their exact propagators have single-particle poles with residues $Z_2 \neq 1$ and $Z_3 \neq 1$. These coefficients, called *renormalization of the wave-functions*,

are not physical in sense that, through they appear in the LSZ formula, the scattering amplitudes should not depend on them.

In renormalized perturbation theory, the original Lagrangian $\mathcal{L}(\Psi, A^\mu, m_0, e_0)$ is decomposed as:

$$\mathcal{L}_R(\Psi, A^\mu, m_0, e_0) = \mathcal{L}_0(\Psi_R, A^\mu_R, m) + \mathcal{L}_{INT}(\Psi_R, A^\mu_{R,e}) + \mathcal{L}_{ct}(\Psi_R, A^\mu_R)\,, \tag{3.352}$$

where:

- the parameter m is either the actual electron mass, or at least differs from it by a finite quantity.
- the parameter e is either the physical charge of the electron, defined in a specific regime (for instance Coulomb scattering in the limit of vanishing momentum transfer), or differs from it by a finite quantity.
- The renormalized fields Ψ_R and A^μ_R are normalized to one, or at least the residue at the one-particle pole differs from one by a finite quantity.[19]

The counter-term Lagrangian $\mathcal{L}_{ct}(\Psi_R, A^\mu_R)$ is just equal to the difference $\mathcal{L}_R - \mathcal{L}_0$. The different terms it contains are considered as contributing to the overall interaction Lagrangian:

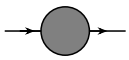
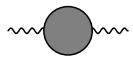
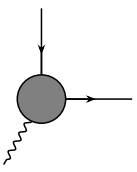
$$\mathcal{L}_{ct} = -\delta m\, \bar\Psi_R \Psi_R + \delta_2 \bar\Psi_R \partial\!\!\!/ \Psi_R \cdots .$$

Importantly, the parameters δm, δ_2, etc. are really parameters, in the sense that they do not depend on the momenta. Any dependence in the momenta should come from the derivatives of the fields they multiply; for instance the counter-term in δ_2 will contribute to momentum space amplitude with factors of $i\delta_2 p\!\!\!/$. Local counter-terms, *i.e.* involving a finite number of derivatives, can absorb at best divergences which are polynomial in the momenta.

Since the counter-terms are not needed at tree-level, the expansion of their coefficients in powers of the coupling constant α starts at first order:

$$\delta_i = \alpha \delta_i^{(1)} + \alpha^2 \delta_i^{(2)} + \cdots . \tag{3.353}$$

***(ii)* Regularizing the loop integrals.** At one-level, one identifies three divergent 1PI amputated diagrams. Each of them can be associated with the one-loop contribution to a fully resummed diagram:

1. one-loop contribution to the electron self-energy, 
2. one-loop contribution to the photon self-energy, 
3. one-loop contribution to the vertex function, 

[19] If the residues are equal to one, no $\sqrt{Z}$ factors for external legs are needed in the evaluation of scattering amplitudes. While convenient, it is not the only choice possible.

Integral over the loop momenta should be regularized in order to make sense of the computations. Regularization methods should be compatible with the symmetries of the theory. Indeed, we have made heavy use of the Ward–Takahashi identities in the regularized theory, assuming that the latter preserved the symmetry originally present.

We have mentionned in our study of one-loop corrections two types of commonly used regularization methods:

1. with *dimensional regularization*, the integrals over Euclidian 4-momenta are evaluated in dimension $d = 4 - \epsilon$. It is a rather efficient method, although there are some complications while using dimension-dependent gamma-matrix identities. The UV singularities manifest themselves as simple poles in the ϵ expansion of integrals, and the finite part of the integrals depend on momenta as $\ln(-p^2/\mu^2)$, where μ is an arbitrary mass scale that needs to be introduced, by dimensional analysis, when moving away from 4 dimensions.

2. with *Pauli–Villars regularization*, one substracts from a propagator in the loop a propagator with the same momentum but a very large fictious mass M. The UV singularities manifest themselves as terms diverging as $\ln(M^2/\mu^2)$ when $M \to \infty$, while the integrals depend on momenta as $\ln(-p^2/\mu^2)$. As before, μ is an arbitary mass scale.

***(iii)* Imposing renormalization conditions.** To each of the three 1PI diagrams, one adds the suitable contributions from the four counter-terms appearing in the decomposition (3.352). Their leading order coefficients $\delta_i^{(1)}$ in the expansion (3.353) in terms of the parameter α (the fine structure constant) are imposed by a choice of *renormalization conditions*.

The relation between the parameters e and m in the first term of the Lagrangian (3.352) can be any finite function of the corresponding physical parameters. The most natural choice (but not necessarily the simplest) is to use the *on-shell renormalization scheme* in which both are the same:

1. the electron self-energy $\Sigma(\not{p})$ vanishes for $p^2 = m^2$, identifying m with the physical mass. One linear combination of counter-terms coefficients at order α, $\delta m^{(1)} - m\delta_2^{(1)}$, is determined by this condition, as a function of m, ϵ and μ in dimensional regularization (or as a function of m and M in the Pauli–Villars regularization).

2. the vertex function $\Gamma^\mu(p, p')$ goes to one in the limit $p' \to p$, in order for the parameter e in the Lagrangian to coincide with the physical electric charge of the electron measured in low-energy processes. This determines the order α contribution to the counter-term coefficient $\delta_1^{(1)}$.

The other two renormalization conditions are of a slightly different nature, as they don't relate parameters in the Lagrangians to physical quantities, but rather allow to normalize the fields A^μ and Ψ of the theory in a convenient way, to make perturbation theory simpler. If we define:

$$\Psi = \sqrt{\tilde{Z}_2}\,\Psi_R\ , \quad A^\mu = \sqrt{\tilde{Z}_3}\,A_R^\mu\,,$$

The most natural renormalization conditions one can choose are:

$$Z_2/\tilde{Z}_2 = 1 \ , \quad Z_3/\tilde{Z}_3 = 1 \, ,$$

thereby determining the values of the order α contributions to the respective counter-term coefficients $\delta_2^{(1)}$ and $\delta_3^{(1)}$. It is perfectly fine to make another choice,

$$Z_2/\tilde{Z}_2 = \mathfrak{z}_2 \ , \quad Z_3/\tilde{Z}_3 = \mathfrak{z}_3 \, ,$$

where $\mathfrak{z}_{2,3}$ are *finite* coefficients, provided that the Feynman rules are modified accordingly: external electrons (resp. photons) come with factors of $\sqrt{\mathfrak{z}_2}$ (resp. of $\sqrt{\mathfrak{z}_3}$). At the end of the day it does not affect physical observables.

Thus overall we have four renormalization conditions involving four counter-terms coefficients, $\delta_{1,2,3}$ and δm, but only three UV-divergent one-loop diagrams. However, as we have seen, not all counter-terms coefficients are independent of each other. We have proven that, at any order in perturbation theory, $\delta_1 = \delta_2$, using the Ward–Takahashi identity.

In general, physical predictions of the theory should not depend on the choice of renormalization conditions, provided that it leads to well-defined observables when the regulator is removed. The choice of renormalization conditions is called the *renormalization scheme*, not to be confused with the regularization method.

One convenient renormalization scheme that we have presented is the $\overline{\text{MS}}$ *scheme*. If one wants, for instance, to study the renormalizabilty of a given QFT, it is enough to consider only the diverging part of the loop integrals. Using dimensional regularization, the divergence is always accompagnied by some universal finite factor that can be removed as well irrespectively of the particular diagram under consideration. One adjusts then the coefficient of the counter-terms to remove just these two contributions, for, instance for the mass counter-term:

$$\delta m^{(1)}\big|_{\overline{\text{MS}}} = -\frac{m}{\pi}\left(\frac{2}{\epsilon} + \ln\left(4\pi e^{-\gamma}\right)\right) \, .$$

In this renormalization scheme, the parameter m is not the physical mass m_{PHYS} of the electron.

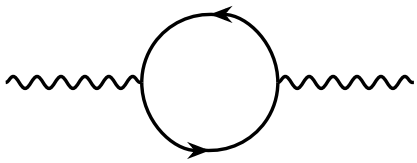
3.5.2 Degree of UV divergence and renormalizabilty

Let us consider a given Feynman diagram of QED in d dimensions at L loops. Each loop integral leads potentially to a UV divergence in $\int k^{d-1} dk$. However these divergences are tamed by propagators, which contribute to the amplitude with negative powers of the momenta. Let I_e and I_γ be respectively the number of electron and photon propagators. The electron propagators scale like $1/k$, while the photon propagators like $1/k^2$. We define then the *superficial degree of divergence* of the diagram as:

$$D = dL - I_e - 2I_\gamma \tag{3.354}$$

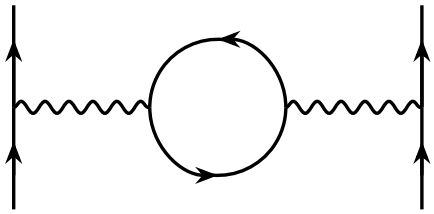
If we regularize the loop integrals with a UV cutoff Λ, we expect then a divergence in Λ^D when $D > 0$, a divergence in $\ln \Lambda$ when $D = 0$, and no divergence when $D < 0$. This quantity is called superficial for a good reason.

Sometimes the situation is better than expected, thanks to to the symmetries of the theory. For instance, in four-dimensional QED, the one-loop contribution to the self energy,



has a superficial degree of divergence $D = 2$ but, thanks to the Ward–Takahashi identities, it has only a logarithmic divergence, rather than a quadratic one.

In some other cases, the situation is worse than expected because of diverging sub-diagrams which are part of a bigger diagram, for instance the one-loop correction to electron/electron scattering studied before:



This diagram has $D = -2$ but has a logarithmic divergence coming from the one-loop correction to the photon propagator. We will see how to deal with these sub-diagram singularities on an example below.

The superficial degree of divergence can be usefully expressed in terms of the number of external legs. Using eqn. (2.189) adapted to the present situtation, the number of loop integrals is related to the number V of vertices as:

$$L = I_e + I_\gamma - V + 1\,. \tag{3.355}$$

Next, since each vertex is connected to two electron lines and one photon line, V is equal to the number of photon lines (internal and external), and to half of the number of electron lines (internal and external as well:

$$V = 2I_\gamma + E_\gamma = \frac{1}{2}(2I_e + E_e)\,. \tag{3.356}$$

Combining all these relations, the superficial degree of divergence of QED in dimension d is given by:

$$D = d(I_e + I_\gamma - V + 1) - I_e - 2I_\gamma = (d-1)(V - E_e/2) + (d-2)(V/2 - E_\gamma/2) - dV \tag{3.357}$$

Hence

$$D = d - \tfrac{d-1}{2} E_e - \tfrac{d-2}{2} E_\gamma + \tfrac{d-4}{2} V \qquad (3.358)$$

One notices that the high-energy behavior of QED depends very strongly on the dimension of space-time:

- In a space-time of dimension bigger than four, for a given type of diagram (*i.e.* a given number of external electron and photon legs) the superficial degree of divergence grows with the number of vertices. Therefore, any type of scattering amplitude becomes divergent at some order in perturbation theory. The theory is said to be ***non-renormalizable***.

- In a space-time of dimension less than four, adding vertices will reduce D, hence the theory has only a finite number of divergent Feynman diagrams: a quantum field theory with this property is said to be ***super-renormalizable***.

- Finally, in a space-time of dimension four, the superficial degree of divergence is independent of the number of vertices. Only a finite number of scattering amplitudes are superficially divergent, and their degree of divergence is expected to be the same at every order in perturbation theory. The theory is said to be ***renormalizable***.

One can get an intuitive picture of these different qualitative behaviors using dimensional analysis, in the context of effective field theory. Let us consider that QED gives a good description of physics under a certain high-energy scale E_0. As noticed before, the electric charge has dimension $L^{d/2-2}$, *i.e.* dimension $E^{2-d/2}$.

As a consequence, contributions to scattering amplitudes at a given order n in the perturbative expansion will be weighted by a dimension-less factor of order $\left(e/E_0^{2-d/2}\right)^n$. In $d < 4$, it implies that higher order correction are suppressed at low energies (they scale with a negative power of E_0), while in $d > 4$ it is exactly the opposite.

*
* *

Exercise: do the same analyis for the ϕ^n real scalar QFT in dimension d. Show that the superficial degree of divergence is given by:

$$D = d + \big(n(d-2)/2 - d\big)V - E(d-2)/2\,. \qquad (3.359)$$

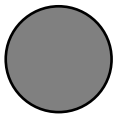
Determine in which dimensions ϕ^3 and ϕ^4 are non-renormalizable, renormalizable and super-renormalizable. Show in particular that ϕ^4 is renormalizable in four and ϕ^3 in six. Confirm with dimensional analysis.

Divergences in four-dimensional QED. According to this general analysis, only a finite number of scattering amplitudes in four-dimensional QED are superficially divergent.

We restrict ourselves to fully amputated 1PI diagrams in order to isolate the "elementary" types of potentially divergent diagrams. As noticed before, they will also appear as (potentially) diverging sub-diagrams in more complicated higher-order diagrams, a problem that we will discuss later on.

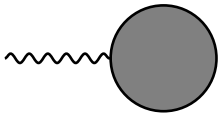
Let us examine those diagrams one by one:

1. the vacuum bubble with no external legs has $D = 4$.

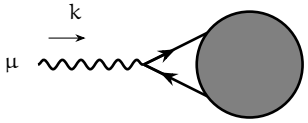
   

   As we have explained before, the vacuum bubbles play no role in perturbation theory and can be ignored.

2. The photon tadpole, *i.e.* with a single photon external leg, has $D = 3$.

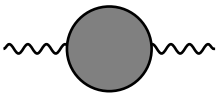
   

   This amplitude should vanish by Lorentz-invariance of the vacuum. Indeed since the photon line is necessarily attached to a QED vertex, it can be represented as a current one-point function, which vanishes by Lorentz invariance of the vacuum:

   $$= -i \int d^4x\, e^{-ik\cdot x} \langle \Omega | j^\mu(x) | \Omega \rangle = 0 .$$

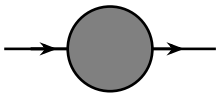
3. The photon self-energy has $D = 2$.

   We have shown before that, thanks to the Ward identities, it is necessarily of the form $\Pi_{\mu\nu} = (k^2\eta_{\mu\nu} - k_\mu k_\nu)\Pi(k^2)$. Therefore there is at most a logarithmic divergence in the constant term $\Pi(0)$ which is exactly what we have found at one-loop for the following reason.

   The particular tensorial structure of the self-energy implies that the Taylor expansion of $\Pi_{\mu\nu}$ in powers of k starts at order two. The coefficient of the term in k^2 in the expansion is proportionnal to $\Pi''_{\mu\nu}(k = 0)$, which has improved convergence of loop-integrals by two powers, *e.g.* $\frac{1}{\not{k}+\cdots-m} \to \frac{1}{(\not{k}+\cdots-m)^3}$.

4. The electron self-energy has superfical degree of divergence $D = 1$.



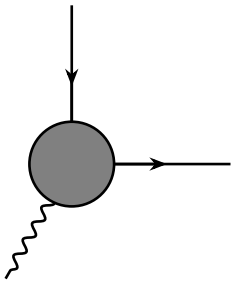
It can be expanded in powers of the electron momentum $\not{p}$:

$$\Sigma(\not{p}) = A_0 + A_1 \not{p} + A_2 p^2 + \cdots \tag{3.360}$$

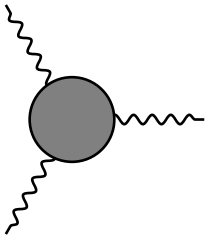
The coefficient A_0 has at most a linear divergence, while the coefficient A_1 at most a logarithmic divergence and all the other ones are finite, by the same type of arguments as above.

One can argue that the coefficient A_0, which contributes to the renormalization of the electron mass, should be itself proportional to the mass, hence should diverge at most like $m \ln(\Lambda/m)$. Indeed, a mass-independent contribution to the mass renormalization would survive the massless limit $m \to 0$, which is forbidden by the new symmetry appearing in this limit ($\Psi \to \exp(i\theta\gamma^5)\Psi$).

5. The vertex function Γ^μ has $D = 0$, hence is at most logarithmically divergent, *i.e.* of the form $\Gamma^\mu(p, p') \sim \gamma^\mu \ln(\Lambda/m) +$ finite..



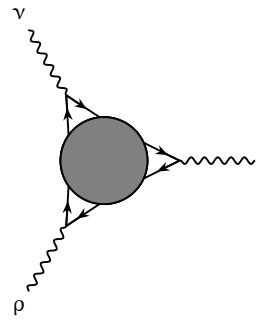
6. A potentially divergent amplitude is the photon 3-point scattering amplitude, with $D = 1$:



However, this amplitude vanishes thanks to charge conjugation symmetry of the vacuum, *i.e.* $C\,|\Omega\rangle = |\Omega\rangle$. Since charge conjugation exchanges particles with anti-particles, which have opposite charges, one should have

$$C j^\mu(x) C^\dagger = -j^\mu(x)\,, \tag{3.361}$$

which can be explicitely checked by plugging the transformation (1.174) into the current $j^\mu = \bar{\Psi}\gamma^\mu\Psi$. Using that the three-point function can be represented as:



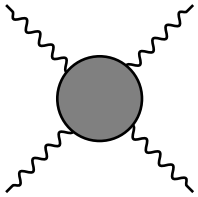
$$= (-ie)^3 \int d^4x\, d^4y\, d^4z\, e^{-i(k_1\cdot x + k_2\cdot y + k_3\cdot z)} \langle\Omega|\, j^\mu(x) j^\nu(y) j^\rho(z)\,|\Omega\rangle\ , \tag{3.362}$$

one can use:

$$\begin{aligned}\langle\Omega|\, j^\mu(x) j^\nu(y) j^\rho(z)\,|\Omega\rangle &= \langle\Omega|\, Cj^\mu(x)C^\dagger Cj^\nu(y)C^\dagger Cj^\rho(z)C^\dagger\,|\Omega\rangle \\ &= -\langle\Omega|\, j^\mu(x) j^\nu(y) j^\rho(z)\,|\Omega\rangle = 0\,. \end{aligned} \tag{3.363}$$

This property, which holds for any odd number of currents, is known as Furry's theorem and applies to a correlator of any odd number of currents.[20]

7. Finally, a last potentially divergent amplitude is the photon 4-point scattering amplitude, with $D = 0$:



Let us denote by $k_1, \dots k_4$ the photon external momenta. Using the Ward identity, one has:

$$k_1^\mu \mathcal{M}_{\mu\nu\rho\sigma} = k_2^\nu \mathcal{M}_{\mu\nu\rho\sigma} = \cdots = 0\,. \tag{3.364}$$

One can show that the tensor structure of $\mathcal{M}_{\mu\nu\rho\sigma}$ is constrained to be linear in each of the four momenta $k_1, \dots, k_4$. Therefore, by similar arguments as before, one expects this scattering amplitude to be finite, which is indeed the case.

To summarize, using the symmetries of the theory, we have argued that there are only three "primitive" type of scattering amplitudes that diverges in QED: the photon and electron self-energies and the QED vertex, which we have considered at the level of one-loop corrections. These three amplitudes are expected to have only logarithmic divergences in the cut-off (or equivalently simple poles in $1/\epsilon$) at any order in perturbation theory. Therefore, they can be absorbed, order by order in the expansion in α, by adjusting the coefficients of the counterterms accordingly.

[20] Naturally one could have used it to prove the vanishing of the photon tadpole.

3.5.3 Divergences of sub-diagrams

One problem was left unresolved in the previous analysis. Sometimes the degree of divergence of a given Feynman diagram can be worse than what the superficial degree of divergence predicts, because of UV divergences of loops in sub-diagrams.

To solve this problem, one needs to realize that, since the coefficients of the counter-terms have an expansion in the coupling, starting at order α,

$$\delta = \alpha\delta^{(1)} + \alpha^2\delta^{(2)} + \cdots ,$$

at a given order in perturbative expansion there will be in general three types of diagrams:

1. diagrams with only loops and no counter-terms,

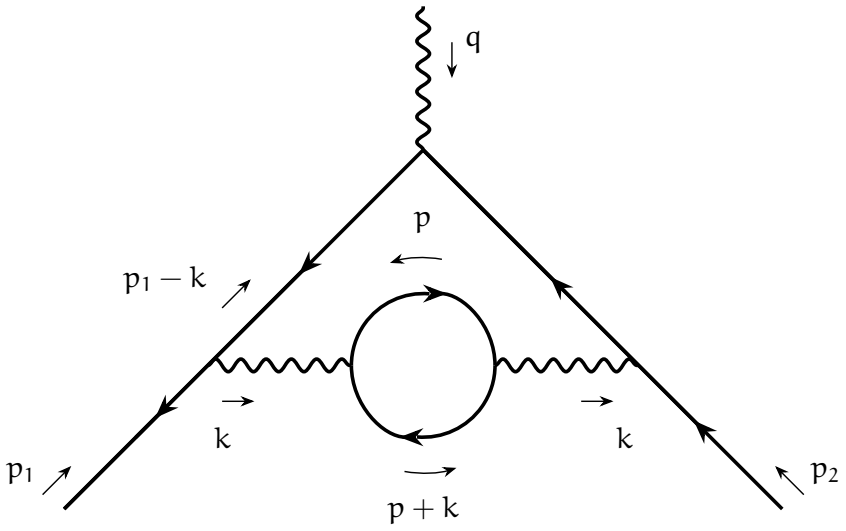
2. diagrams with counter-terms coefficients of the given order and no loops,

3. "hybrid" diagrams with both loops and counter-terms coefficients of lowest order.

Those hybrid diagrams will precisely cancel the divergences of the loops in sub-diagrams that they correspond to. Let us see how this works in a particular two-loop example.

QED vertex at two loops. We consider the two-loop renormalization of the form factors F_1 and F_2 in the QED vertex function, see eqn. (3.297). A detailed analysis can be found for instance in reference [4].

At order α^2 in perturbation theory, one finds the two-loops diagrams represented on figure 3.5. All the diagrams except diagram (b) contain the previously seen one-loop diagrams as sub-diagrams.
Let us have a closer look at diagram (g):



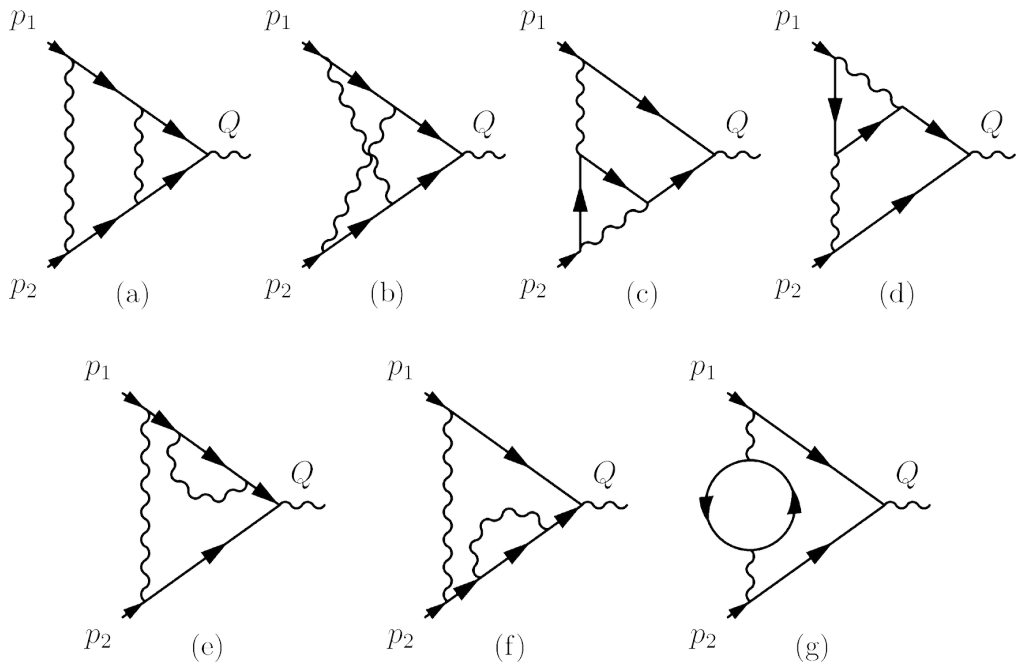


Figure 3.5: *Two-loop corrections to the QED vertex, from Bonciani, Mastrolia and Remiddi.*

There are two loop momenta in this diagram, p and k, that are integrated over. In the region $p \to \infty$ in momentum space, the inner loop, which is the same as the one-loop correction to the photon self-energy, will give a divergence of the same type as we have studied before, which manifests itself as a $1/\epsilon$ factor in dimensional regularization.

From the remaining k integral (the correction to the vertex), as in (3.330), we will get a $1/\epsilon$ pole plus a logarithmic correction in the external momentum, giving when multliplied by the previous factor a contribution to the vertex function the form

$$\Gamma^\mu \sim \frac{1}{\epsilon} \ln\left(-p_1^2/\mu^2\right)\gamma^\mu + \cdots$$

This type of divergences are called *overlapping divergences*.

This is potentially a catastrophe, as *no local counter-term could absorb this type of divergences*: local counterterms, built with a finite number of fields and their derivatives, are at most polynomial in the momentum and cannot reproduce logarithmic terms in p^2.

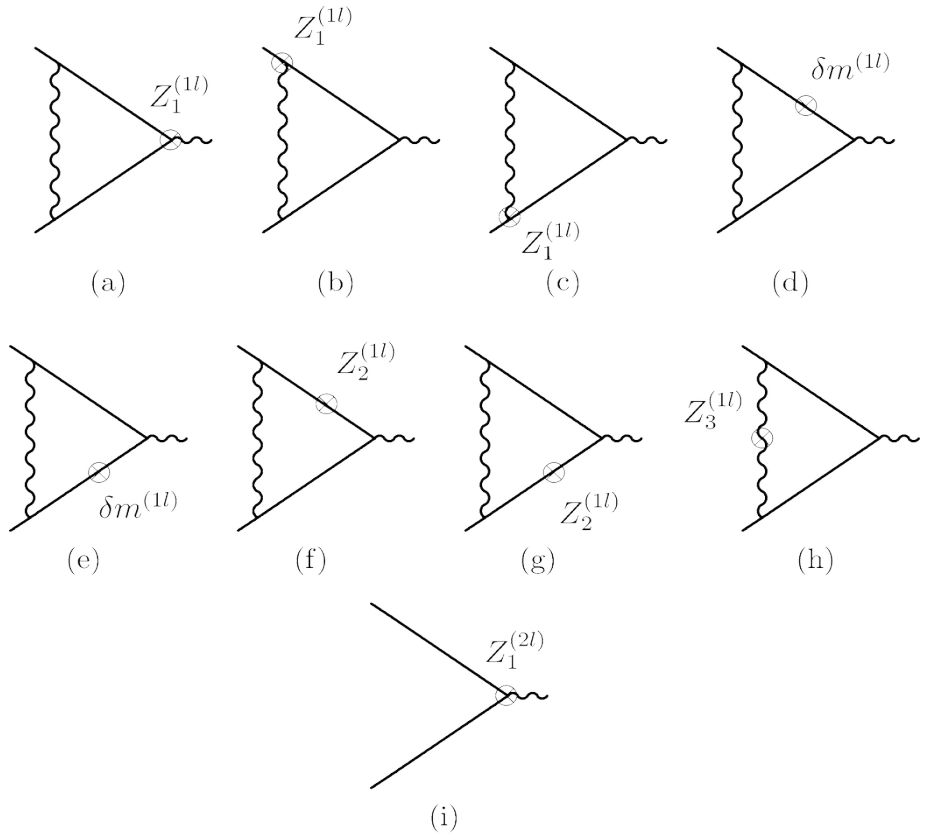


Figure 3.6: *Diagrams with counter-terms at order α^2, from Bonciani, Mastrolia and Remiddi.*

We should not forget however, that at the same order α^2 in perturbation theory we have two type of diagrams with counter-terms, see fig. 3.6.:

(i) diagrams with one loop and one insertion of an order α counter-term,

(ii) an order α^2 contribution from the δ_1 counter-term,

The diagram (h) can precisely take care of the problem highlighted above, *i.e.* substracting (h) from (g) eliminates the inner loop UV divergence. For all the other diagrams at order α^2 with divergences from one-loop diagrams displayed on figure 3.5 the same type of substraction occurs, see fig. 3.7.

Then, taking care of the remaining divergence in the form factor F_1 requires renormalization using the α^2 contribution of the δ_1 counter-term, given by the diagram (i).

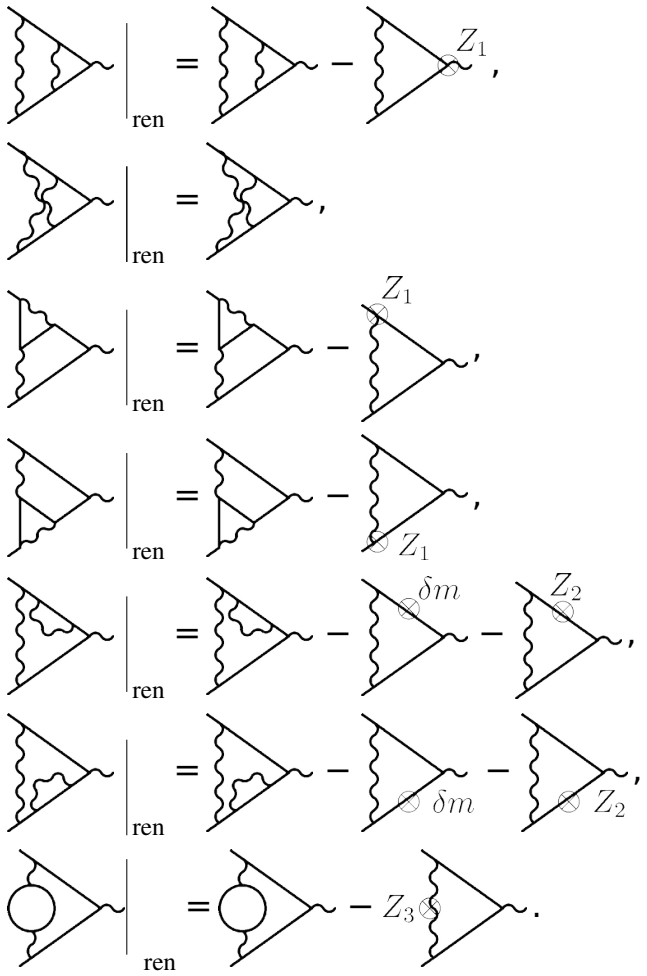


Figure 3.7: *Renormalization of sub-diagrams at order* α^2*, from Bonciani, Mastrolia and Remiddi.*

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Part 4

Introduction to path integral quantization

The path integral approach to quantum field theory is complementary to the canonical quantization method employed in the previous chapter. It allows in particular to obtain in an easier way properties related to the symmetries of the theory, such as Ward identities, and will be instrumental in the quantization of non-Abelian gauge theories that you will study during the second semester.

4.1 Path integral in quantum mechanics

We assume that the quantum mechanical system is governed by a time-independent Hamiltonian H. The main object of interest, to make the connection between canonical quantization and path integral approaches to quantization, is the *propagator*. Let us remind that the propagator is defined as the overlap between position states at different times:

$$\mathcal{U}\big(\vec{x}_I, \vec{x}_F \big| t_F - t_I\big) \stackrel{\text{def.}}{=} \langle \vec{x}_F, t_F | \vec{x}_I, t_I \rangle = \langle \vec{x}_F | \, e^{-iH(t_F - t_I)} \, | \vec{x}_I \rangle \ . \tag{4.1}$$

In this expression, $|\vec{x}, t\rangle = e^{iHt} |\vec{x}\rangle$ is an "eigenvector" of the Heisenberg position operator:

$$\vec{\mathcal{X}}_H \, |\vec{x}, t\rangle = e^{iHt} \vec{\mathcal{X}} e^{-iHt} \, e^{iHt} \, |\vec{x}\rangle = |\vec{x}, t\rangle \ \vec{x} \, . \tag{4.2}$$

Note that, since that the position operator has no eigenstates in $\mathcal{L}^2(\mathbb{R}^3)$, it is more appropriate to think of the propagator as the kernel of time evolution of wave-functions, in the sense that an initial wave-function $\psi_I(\vec{x})$ at $t = t_I$ will evolve in time following:

$$\psi(\vec{x}, t_F) = \int d^3y \, \mathcal{U}\big(\vec{y}, \vec{x} \big| t_F - t_I\big) \psi_I(\vec{y}) \, . \tag{4.3}$$

This said, let us split the time interval with N infinitesimal intermediate steps of size

$$\delta t \stackrel{\text{def.}}{=} \frac{T}{N+1} \ , \quad T = t_F - t_I \, . \tag{4.4}$$

One can slice the time evolution operator accordingly, using multiple times the resolution of the identity in position space:

$$\begin{aligned}\mathcal{U}\big(\vec{x}_{\rm I},\vec{x}_{\rm F}\big|t_{\rm F}-t_{\rm I}\big) &= \langle\vec{x}_{\rm F}|\,e^{-iH\delta t}\left(\int d^3x_N\,|\vec{x}_N\rangle\,\langle\vec{x}_N|\right)e^{-iH\delta t}\cdots e^{-iH\delta t}\left(\int d^3x_1\,|\vec{x}_1\rangle\,\langle\vec{x}_1|\right)e^{-iH\delta t}\,|\vec{x}_{\rm I}\rangle \\ &= \int d^3x_1\cdots d^3x_n\,\mathcal{U}\big(\vec{x}_N,\vec{x}_{\rm F},\big|\delta t\big)\,\cdots\,\mathcal{U}\big(\vec{x}_1,\vec{x}_2\big|\delta t\big)\,\mathcal{U}\big(\vec{x}_{\rm I},\vec{x}_1\big|\delta t\big)\,. \end{aligned} \tag{4.5}$$

Let us assume that the Hamiltonian of the system takes the usual form:

$$H = \frac{\vec{\mathcal{P}}^2}{2m} + V\big(\vec{\mathcal{X}}\big)\,, \tag{4.6}$$

in terms of the momentum operator $\mathcal{P}$ and the position operator $\mathcal{X}$. Using the Baker–Campbel–Hausdorff formula one can split:

$$e^{-i\delta t\,\frac{\vec{p}^2}{2m}}e^{-i\delta t\,V(\vec{\mathcal{X}})} = e^{-i\delta t\,H-(\delta t)^2\left[\frac{\vec{p}^2}{2m},V(\vec{\mathcal{X}})\right]+\cdots} \tag{4.7}$$

In the limit $N\to\infty$, the commutator is negligible (as well as the nested commutators that follow) so one can express the infinitesimal propagator as:

$$\begin{aligned}\mathcal{U}\big(\vec{x},\vec{y}\big|\delta t\big) &\overset{\delta t\to 0}{\simeq} \int\frac{d^3p}{(2\pi)^3}\,\langle\vec{y}|\overbrace{e^{-i\delta t\,\frac{\vec{p}^2}{2m}}\,|\vec{p}\rangle}^{|\vec{p}\rangle e^{-i\delta t\,\frac{\vec{p}^2}{2m}}}\langle\vec{p}|\overbrace{e^{-i\delta t\,V(\vec{\mathcal{X}})}\,|\vec{x}\rangle}^{|\vec{x}\rangle e^{-i\delta t\,V(\vec{x})}} \\ &\simeq \int\frac{d^3p}{(2\pi)^3}e^{i\vec{p}\cdot(\vec{y}-\vec{x})}e^{-i\delta t\left(\frac{\vec{p}^2}{2m}+V(\vec{x})\right)}\end{aligned} \tag{4.8}$$

Renaming $t_{\rm I}\to t_0$ and $t_{\rm F}\to t_{N+1}$ as well as $\vec{x}_{\rm I}\to\vec{x}_0$ and $\vec{x}_{\rm F}\to\vec{x}_{N+1}$, one can write formally the full propagator as:

$$\mathcal{U}\big(\vec{x}_{\rm I},\vec{x}_{\rm F}\big|T\big) \;=\; \lim_{N\to\infty}\int\prod_{\ell=1}^{N}d^3x_\ell\int\prod_{n=1}^{N+1}\left(\frac{d^3p_n}{(2\pi)^3}e^{i\delta t\left(\vec{p}_n\cdot\frac{\vec{x}_n-\vec{x}_{n-1}}{\delta t}-H\left(\vec{x}_n,\vec{p}_n\right)\right)}\right) \tag{4.9}$$

At leading order in δt, one can replace $\vec{x}_n\to(\vec{x}_n+\vec{x}_{n-1})/2$ in the Hamiltonian. We consider eqn. (4.9) as *defining* the functional integral in the continous limit:

$$\mathcal{U}\big(\vec{x}_{\rm I},\vec{x}_{\rm F}\big|T\big) \;=\; \int_{\vec{x}(t_{\rm I})=\vec{x}_{\rm I}}^{\vec{x}(t_{\rm F})=\vec{x}_{\rm F}}\mathcal{D}\vec{x}(t)\mathcal{D}\vec{p}(t)e^{i\int_{t_{\rm I}}^{t_{\rm F}}\left(\vec{p}\cdot\dot{\vec{x}}-H(\vec{x},\vec{p})\right)} \tag{4.10}$$

For a Hamiltonian of the standard form (4.6), *i.e.* quadratic in the momentum, the momentum integral can be performed explicitly in the discretized representation. The integral

$$\int\frac{d^3p}{(2\pi)^3}e^{-i\delta t\,\frac{\vec{p}^2}{2m}+i\vec{p}\cdot(\vec{y}-\vec{x})}\,, \tag{4.11}$$

becomes a gaussian integral after Wick rotation $\delta t = -i\delta\tau$. One can instead continue the time parameter slightly below the real axis, *i.e.* replace $\delta t \mapsto (1 - i\epsilon)\delta t$. We obtain then:

$$\lim_{\epsilon\to 0+} \int \frac{d^3p}{(2\pi)^3} e^{-i(1-i\epsilon)\delta t \frac{\vec{p}^2}{2m} + i\vec{p}\cdot(\vec{y}-\vec{x})} = \left(\frac{-im}{2\pi\delta t}\right)^{3/2} e^{\frac{im}{2\delta t}\left(\vec{y}-\vec{x}\right)^2} . \tag{4.12}$$

So putting everything together one has:

$$\mathcal{U}\big(\vec{x}_I, \vec{x}_F\big|T\big) \;=\; \lim_{N\to\infty} \left(\frac{-im}{2\pi\delta t}\right)^{3(N+2)/2} \int \prod_{\ell=1}^{N} d^3x_\ell \, e^{i\delta t \sum_{n=1}^{N} \left(\frac{m}{2}\left(\frac{\vec{x}_n - \vec{x}_{n-1}}{\delta t}\right)^2 - V\left(\frac{x_n + x_{n-1}}{2}\right)\right)} \tag{4.13}$$

which defines formally, after absorbing the (infinite) pre-factor in the measure $\mathcal{D}\vec{x}$, the path integral:

$$\mathcal{U}\big(\vec{x}_I, \vec{x}_F\big|T\big) \;=\; \int_{\vec{x}(t_I)=\vec{x}_I}^{\vec{x}(t_F)=\vec{x}_F} \mathcal{D}\vec{x}(t) \; e^{\,i\int_{t_I}^{t_F} L\left(\vec{x},\dot{\vec{x}}\right) dt} \tag{4.14}$$

This formula is interpreted as a sum of all possible continuous paths between $\vec{x}(t_I) = \vec{x}_I$ and $\vec{x}(t_F) = \vec{x}_F$, weighted by the factor $\exp iS/\hbar$. In the semi-classical limit this integral is dominated by the stationary points $\delta S/\delta\vec{x}(t) = 0$, *i.e.* the classical trajectories. In the Wick-rotated theory, it can be shown that the functional measure $\mathcal{D}\vec{x}(t)$ can be defined rigorously using the concept of *Wiener processes*.

Correlation functions. Besides the propagator, the path integral formalism can also be used to compute correlation functions. We work here in 1d to simplify the notations. Consider, for $t_I < t_1 < t_2 < \cdots < t_n < t_F$, the correlation function of Heisenberg position operators:

$$\begin{aligned} \mathcal{G}(t_1, t_2, \ldots, t_n) \;&=\; \langle x_F, t_F| \, \mathcal{X}_H(t_n) \cdots \mathcal{X}_H(t_1) \, |x_I, t_I\rangle \\ &= \langle x_F| \, e^{-i(t_F - t_n)H} \mathcal{X} e^{-i(t_n - t_{n-1})H} \cdots e^{-i(t_2 - t_1)H} \mathcal{X} e^{-i(t_1 - t_I)H} \, |x_I\rangle \; . \end{aligned} \tag{4.15}$$

One can express each of the evolution operators in terms of the propagator, starting from the right:

$$\mathcal{X} e^{-i(t_1 - t_I)H} \, |x_I\rangle = \mathcal{X} \int dx_1 \; |x_1\rangle \, \langle x_1| \, e^{-i(t_1 - t_I)H} \, |x_I\rangle = \int dx_1 \; |x_1\rangle \, x_1 \, \mathcal{U}\big(x_I, x_1\big|t_1 - t_I\big) \, . \tag{4.16}$$

giving overall:

$$\mathcal{G}(t_1, t_2, \ldots, t_n) = \int dx_1 \cdots dx_n \mathcal{U}\big(x_n, x_F\big|t_F - t_n\big) x_n \cdots x_2 \mathcal{U}\big(x_1, x_2\big|t_2 - t_1\big) x_1 \, \mathcal{U}\big(x_I, x_1\big|t_1 - t_I\big) \, . \tag{4.17}$$

For each of the propagators $\mathcal{U}(x_k, x_{k+1}|t_{k+1} - t_k)$ one can use the path integral representation (4.14).

Crucially, for two adjacent propagators with an intermediate point integrated over, one can write:

$$\begin{aligned} \int dx_2 \, \mathcal{U}\big(x_2, x_3\big|t_3 - t_2\big) \mathcal{U}\big(x_1, x_2\big|t_2 - t_1\big) \;&=\; \int dx_2 \int_{x(t_2)=x_2}^{x(t_3)=x_3} \mathcal{D}x \, e^{i\int_{t_2}^{t_3} L} \int_{x(t_1)=x_1}^{x(t_2)=x_2} \mathcal{D}x \, e^{i\int_{t_1}^{t_2} L} \\ &\overset{!}{=} \int_{x(t_1)=x_1}^{x(t_3)=x_3} \mathcal{D}x \, e^{i\int_{t_1}^{t_3} L} \end{aligned} \tag{4.18}$$

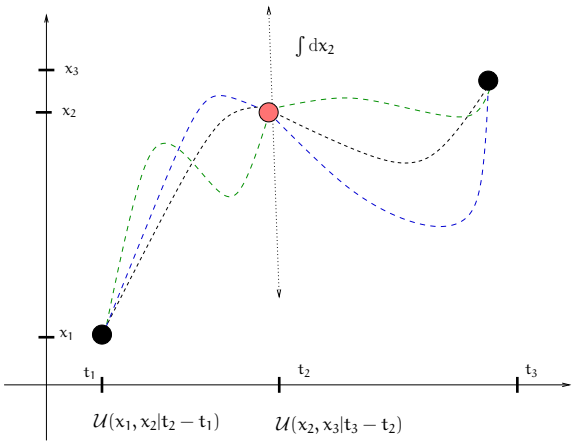


Figure 4.1: Insertion of an intermediate point in the propagator.

It means that joining the sum over paths from x_1 to x_2 to the sum over paths from x_2 to x_3 and integrating over the intermediate point x_2 amounts to sum over all paths from x_1 to x_3, see fig. 4.1. The same holds if there is an insertion of x_2 (or of any function of x_2) at the junction of the paths (which occurs for $t = t_2$):

$$\int dx_2\, \mathcal{U}(x_2, x_3|t_3 - t_2) x_2 \mathcal{U}(x_1, x_2|t_2 - t_1) = \int_{x(t_1)=x_1}^{x(t_3)=x_3} \mathcal{D}x\, x(t_2)\, e^{i\int_{t_1}^{t_3} L}\,, \tag{4.19}$$

because the integral over all paths involve in particular the integral over all positions at t_2. In this way, we arrive to the path-integral expression:

$$\mathcal{G}(t_1, t_2, \ldots, t_n) = \int_{x(t_I)=x_I}^{x(t_F)=x_F} \mathcal{D}x(t)\, e^{i\int_{t_I}^{t_F} L}\; x(t_1)\cdots x(t_n)\,. \tag{4.20}$$

Crucially, there is no notion of path ordering in eqn. (4.20), because the $x(t_k)$ are just functions and not operators, while at the starting point of the computation, eqn. (4.15), it was important that the insertion times where ordered a $t_I < t_1 < t_2 < \cdots < t_n < t_F$. It means that, for a generic choice of $t_1, \ldots t_n$ in the interval $]t_I, t_F[$, one should consider the time-ordred product and write:

$$\boxed{\int_{x(t_I)=x_I}^{x(t_F)=x_F} \mathcal{D}x(t)\, e^{i\int_{t_I}^{t_F} L}\; x(t_1)\cdots x(t_n) = \langle x_F, t_F|\, \mathsf{T}\, \mathcal{X}_H(t_n)\cdots \mathcal{X}_H(t_1)\, |x_I, t_I\rangle\,,} \tag{4.21}$$

Let us conclude this discussion with an important remark. In the path integral representation, the correlation function depends only on the time variables $t_1, \ldots, t_n$ through the functions $x(t_1), \ldots, x(t_n)$. Therefore one has for instance:

$$\int_{x(t_I)=x_I}^{x(t_F)=x_F} \mathcal{D}x(t)\, e^{i\int_{t_I}^{t_F} L}\; \frac{dx(t_1)}{dt_1} x(t_2)\cdots x(t_n) = \frac{d}{dt_1} \int_{x(t_I)=x_I}^{x(t_F)=x_F} \mathcal{D}x(t)\, e^{i\int_{t_I}^{t_F} L}\; x(t_1)\, x(t_2)\cdots x(t_n) \tag{4.22}$$

Leading to

$$\int_{x(t_I)=x_I}^{x(t_F)=x_F} \mathcal{D}x(t)\, e^{i\int_{t_I}^{t_F} L}\ \frac{d}{dt_1}x(t_1)\, x(t_2)\cdots x(t_n) = \frac{d}{dt_1}\,\langle x_F, t_F|\, T\, \mathcal{X}_H(t_n)\cdots \mathcal{X}_H(t_1)\,|x_I, t_I\rangle\ . \quad (4.23)$$

Importantly, the right-hand side contains both an explicit dependence in t_1 through the Heisenberg operator $\mathcal{X}_H(t_1)$ and implicitly through the time-ordering. This will be important when we will revisit the Ward–Takahashi identity (3.164) and the Schwinger–Dyson equation (3.166) later on.

Generating functional. The time-ordred correlation functions between Heisenberg position operators can be conveniently repackaged in terms of the *generating functional* $Z_{IF}[J]$, defined as:

$$Z_{IF}\big[J\big] \stackrel{\text{def.}}{=} \langle x_F, t_F|\, T\, e^{i\int_{t_I}^{t_F} dt\, J(t)x(t)}\,|x_I, t_I\rangle \quad (4.24)$$

Recalling that the functional derivative can be summarized as:

$$\frac{\delta J(t)}{\delta J(t')} = \delta(t-t')\,, \quad (4.25)$$

we have:

$$\mathcal{G}(t_1, t_2, \ldots, t_n) = \langle x_F, t_F|\, T\, \mathcal{X}_H(t_1)\cdots \mathcal{X}_H(t_n)\,|x_I, t_I\rangle = (-i)^n \frac{\delta^n}{\delta J(t_1)\cdots\delta J(t_n)} Z_{IF}\big[J\big]\Big|_{J=0} \quad (4.26)$$

In the path-integral formalism, one can write:

$$Z_{IF}\big[J\big] \ = \int_{x(t_I)=x_I}^{x(t_F)=x_F} \mathcal{D}x(t)\, e^{i\int_{t_I}^{t_F} dt\left(L+J(t)x(t)\right)} \quad (4.27)$$

which can be viewed as the coupling of the quantum system to an external source J. Naturally, a variant of this definition exists for the phase space path integral of eqn. (4.10).

Let us consider an infinitesimal change of variables in the path integral (4.27) which preserves the boundary conditions:

$$x(t) \ \mapsto\ x(t) + \Delta(t)\ , \quad \Delta(t_I) = \Delta(t_F) = 0\,. \quad (4.28)$$

Naturally, such transformation should not change the expression of $Z[j]$. At first order in Δ, it leads to the equation:

$$\int_{x(t_I)=x_I}^{x(t_F)=x_F} \mathcal{D}x(t)\, e^{i\int_{t_I}^{t_F} dt\left(L+j(t)x(t)\right)} \int_{t_I}^{t_F} dt' \left(\frac{\delta S}{\delta x(t')} + J(t')\right)\Delta(t') = 0\,. \quad (4.29)$$

This should hold for any Δ. Considering $\Delta(t') = \delta(t-t')$, one has:

$$\int_{x(t_I)=x_I}^{x(t_F)=x_F} \mathcal{D}x(t)\, e^{iS+i\int_{t_I}^{t_F} J(t)x(t)\,dt} \left(\frac{\delta S}{\delta x(t)} + J(t)\right) = 0\,. \quad (4.30)$$

This is the path-integral version of the quantum-mechanical Schwinger–Dyson equation. We will come back to it in more detail in the next section.

Correlation functions in the vacuum. Up to now we have considered only amplitudes between position eigenstates. Indstead we can consider as well amplitudes between energy eigenstates; we will be mostly interested in vacuum-to-vacuum amplitudes. We can proceed as in chapter 2, see eqn. (2.115) and below. We assume that the energy spectrum has a finite gap above the ground state $|\Omega\rangle$, *i.e.* all energy levels E_n are strictly larger than E_Ω. We set $E_\Omega = 0$ by shifting the Hamiltonian if needed, and use the resolution of the identity to write:

$$|x,t\rangle = e^{iHt}\Big(|\Omega\rangle\langle\Omega| + \sum_{n>0}|n\rangle\langle n|\Big)|x\rangle = \langle\Omega|x\rangle\,|\Omega\rangle + \sum_{n>0}\langle n|x\rangle\, e^{-iE_n t}\,|n\rangle\ . \tag{4.31}$$

Assuming that the overlaps of the position eigenstates with the vacuum states do not vanish, one can project onto the ground state by taking the limit of large t, slightly below the real axis:

$$|x,t\rangle \xrightarrow{t\to(1-i\epsilon)\infty} \langle\Omega|x\rangle\,|\Omega\rangle\ ,\quad \epsilon>0\,. \tag{4.32}$$

As we have done before, in order to get rid of the pre-factor one can normalize by the partition function and get the correlation functions in the vacuum as:

$$\langle\Omega|\,T\ \mathcal{X}_H(t_1)\cdots\mathcal{X}_H(t_n)\,|\Omega\rangle = \lim_{T\to(1-i\epsilon)\infty}\frac{\int\mathcal{D}p\,\mathcal{D}x\,e^{i\int_{-T}^{T}(p\dot{x}-H)dt}\ x(t_1)\cdots x(t_n)}{\int\mathcal{D}p\,\mathcal{D}x\,e^{i\int_{-T}^{T}(p\dot{x}-H)dt}} \tag{4.33}$$

This result does not depend on the boundary conditions, as long as their overlap with the vacuum state do not vanish. Using the same $i\epsilon$ prescription for the generating functional (4.27), defining

$$Z\big[J\big] \stackrel{\text{def.}}{=} \lim_{T\to(1-i\epsilon)\infty}\int\mathcal{D}x(t)\,e^{i\int_{-T}^{T}dt\left(L+J(t)x(t)\right)}\,, \tag{4.34}$$

one can write:

$$\langle\Omega|\,T\ \mathcal{X}_H(t_1)\cdots\mathcal{X}_H(t_n)\,|\Omega\rangle = \frac{(-i)^n}{Z[0]}\frac{\delta^n}{\delta J(t_1)\cdots\delta J(t_n)}Z\big[J\big]\Big|_{J=0}\,. \tag{4.35}$$

An equivalent way to obtain the same projection onto the vacuum state is to deform slightly the Hamiltonian, *i.e.* to replace:

$$\int_{-\infty(1-i\epsilon)}^{\infty(1-i\epsilon)}dt(p\dot{x}-H)\ \longrightarrow\ \int_{-\infty}^{\infty}dt\big(p\dot{x}-(1-i\epsilon)H\big)\,. \tag{4.36}$$

For a Hamiltonian of the usual form (4.6), this prescription also helps making the momentum functional integral convergent, giving:

$$Z[J] = \int\mathcal{D}x\,\exp\left(i\int_{-\infty}^{\infty}dt\,\Big((1+i\epsilon)\frac{m}{2}\dot{x}^2(t) - (1-i\epsilon)V\big(x(t)\big)\Big) + J(t)x(t)\right)\,. \tag{4.37}$$

4.2 Path integral in scalar QFT

The transition from quantum mechanics to quantum field theory can be made by considering the latter on a lattice. One replaces the continuous field $\Phi(t,\vec{x})$ by a discrete set of Heisenberg operators living on a lattice Λ:

$$\Big\{\Phi_\lambda(t)\,,\ \lambda\in\Lambda=\vec{a}_1\mathbb{Z}+\vec{a}_2\mathbb{Z}+\vec{a}_3\mathbb{Z}\Big\}\,. \tag{4.38}$$

In order to avoid large-volume (*i.e.* infrared) problems, one can add periodic boundary conditions:

$$\Phi_{\lambda+N\vec{a}_i}(t)=\Phi_\lambda(t)\ ,\quad i=1,2,3\,. \tag{4.39}$$

Let us consider in this context a discrete version of the scalar field Hamiltonian. Explicitely we replace:

$$\begin{aligned} H=\int d^3x\,\Big(\tfrac{1}{2}\Pi_\phi^2(\vec{x})+\tfrac{1}{2}\big(\nabla\Phi(\vec{x})\big)^2+V\big(\Phi(\vec{x})\big)\Big)\\ \longrightarrow\ H_\Lambda=\sum_{\lambda\in\Lambda/N\Lambda}\Big(\tfrac{1}{2}\Pi_\lambda^2+\tfrac{1}{2}\sum_{\mu\in n(\lambda)}\big(\Phi_\lambda-\Phi_{\lambda+\mu}\big)^2+V\big(\Phi_\lambda\big)\Big)\,, \end{aligned} \tag{4.40}$$

where the sum over μ in the second term is over the nearest neighbors of λ in the lattice. Thus we have a quantum mechanical system with a finite number of degrees of freedom, corresponding to the pairs of conjugated operators $(\Phi_\lambda,\,\Pi_\lambda)$ with

$$[\,\Phi_\lambda(t),\,\Pi_{\lambda'}(t)\,]=i\delta_{\lambda,\lambda'\ \mathrm{mod}\ N\Lambda}\,. \tag{4.41}$$

In this context, the analogue of position states are:

$$|\phi,t\rangle\ ,\quad \Phi_\lambda(t)\,|\phi,t\rangle=|\phi,t\rangle\,\phi_\lambda(t)\,. \tag{4.42}$$

Transition amplitudes can be computed using the path integral method, which can be made rigorous, as mentionned before, after Wick rotation, or a least by shifting the time variable slightly below the real axis:

$$\langle\phi_F,t_F|\phi_I,t_I\rangle=\prod_{\rho\in\Lambda/N\Lambda}\int_{\phi_\rho(t_I)=\phi_{I\,\rho}}^{\phi_\rho(t_F)=\phi_{F\,\rho}}\mathcal{D}\phi_\rho\,\mathcal{D}\pi_\rho\,\exp i\int_{t_I}^{t_F}\Big(\sum_\sigma\pi_\sigma\dot{\phi}_\sigma-H\Big)\,dt \tag{4.43}$$

and similarly for correlation functions. A discrete quantum field theory constructed in this way is well-defined, and an evaluation of the path integral can be made numerically. This is particularly useful in the context of *lattice QCD*, as quantum chromodynamics is strongly coupled at low energies.

Instead, one can try to take the *continuum limit*, which is valid at length scales much larger than the lattice spacing, and remove the large-volume cutoff, giving the field theory path integral. One would write then:

$$\langle\phi_F,t_F|\phi_I,t_I\rangle=\int_{\phi(t_I,\vec{x})=\phi_I(\vec{x})}^{\phi(t_F,\vec{x})=\phi_F(\vec{x})}\mathcal{D}\phi\,\mathcal{D}\pi_\phi\ \exp i\int_{t_I}^{t_F}dt\int d^3x\,\Big(\pi_\phi\dot{\phi}-\mathcal{H}\Big)\ , \tag{4.44}$$

in terms of the Klein–Gordon Hamiltonian density $\mathcal{H}$ in the continuum. Note that the parameters of lattice theory, for example the mass parameter, do not necessarily stay the same once we move to larger scales and take the continuum limit. In general the theory needs to be *renormalized* to take these scale-dependent effects into account, something that we already studied in length from a slightly different perspective.

A mathematically rigorous formulation of the quantum field theory path integral has been obtained only in a handful of lower-dimensional examples, such as Liouville theory in two dimensions. A proper mathematical formulation of the four-dimensional theories of interest like QCD remains an open problem.

Correlation functions. Time-ordered correlation functions of scalar fields are expressed exactly in the same way as in quantum mechanics. One has:

$$\langle \phi_F, t_F | \, \mathsf{T}\, \Phi(x_1)\Phi(x_2)\cdots\Phi(x_n) \, | \phi_I, t_I \rangle = \int_{\phi(t_I,\vec{x})=\phi_I(\vec{x})}^{\phi(t_F,\vec{x})=\phi_F(\vec{x})} \mathcal{D}\phi\, \mathcal{D}\pi_\phi \; e^{i\int_{t_I}^{t_F} dt \int d^3x \left(\pi_\phi \dot{\phi} - \mathcal{H}\right)} \, \phi(x_1)\phi(x_2)\cdots\phi(x_n) \,. \quad (4.45)$$

Correlation function in the vacuum are obtained likewise by projecting onto the vacuum state $|\Omega\rangle$ as:

$$\langle \Omega | \, \mathsf{T}\, \Phi(x_1)\cdots\Phi(x_n) \, | \Omega \rangle \; = \lim_{T\to(1-i\epsilon)\infty} \frac{\int \mathcal{D}\phi\, \mathcal{D}\pi_\phi \, e^{i\int_{-T}^{T} dt \int d^3x \left(\pi_\phi \dot{\phi} - \mathcal{H}\right)} \; \phi(x_1)\cdots\phi(x_n)}{\int \mathcal{D}\phi\, \mathcal{D}\pi_\phi \, e^{i\int_{-T}^{T} dt \int d^3x \left(\pi_\phi \dot{\phi} - \mathcal{H}\right)}} \quad (4.46)$$

In order to perform the functional integral over the conjugate momentum field $\pi_\phi(x)$, one can instead, as before, trade the change of time integration contour for a deformation of the Hamiltonian:

$$\int_{-\infty(1-i\epsilon)}^{\infty(1-i\epsilon)} dt \int d^3x \left(\pi_\phi \dot{\phi} - \mathcal{H}\right) \quad \longrightarrow \quad \int d^4x \left(\pi_\phi \dot{\phi} - (1-i\epsilon)\mathcal{H}\right) , \quad (4.47)$$

with

$$\mathcal{H} = \frac{1}{2}\Pi_\phi^2(\vec{x}) + \frac{1}{2}\big(\nabla\Phi(\vec{x})\big)^2 + V\big(\Phi(\vec{x})\big) \,. \quad (4.48)$$

And obtain:

$$\int \mathcal{D}\pi_\phi \, e^{\int d^4x \left(-\frac{\epsilon+i}{2}\pi_\phi^2 + i\pi_\phi\dot{\phi}\right)} = \mathcal{N}\, e^{\frac{i}{2(1-i\epsilon)}\int d^4x\, \dot{\phi}^2} \,. \quad (4.49)$$

with an (infinite) normalization factor than can be absorbed in the measure $\mathcal{D}\phi$ or, equivalently, will drop from the vacuum correlation functions. We obtain, for instance for the vacuum amplitude:

$$\langle \Omega | \Omega \rangle = \int \mathcal{D}\phi \, \exp i \int d^4x \left((1+i\epsilon)\frac{1}{2}\dot{\phi}^2 - (1-i\epsilon)\frac{1}{2}(\nabla\phi)^2 - (1-i\epsilon)V(\phi) \right) , \quad (4.50)$$

which has to be understood as:

$$\langle\Omega|\Omega\rangle = \int \mathcal{D}\phi \; e^{i\int d^4x\,\mathcal{L}}\,, \tag{4.51}$$

with some $i\epsilon$ prescription to ensure convergence of the functional integral.

4.2.1 Generating functional

The vacuum correlation functions (4.46), as in quantum mechanics, can all be obtained from functional derivatives of a *generating functional* obtained by coupling the theory to an external source $J(x)$:

$$Z[J] \overset{\text{def.}}{=} \int \mathcal{D}\phi \;\; e^{i\int d^4x\,(\mathcal{L}+J\phi)}\,, \tag{4.52}$$

$$\mathcal{G}(x_1,\dots,x_n) \overset{\text{def.}}{=} \langle\Omega|\,T\,\Phi(x_1)\cdots\Phi(x_n)\,|\Omega\rangle \;=\; \frac{(-i)^n}{Z[0]}\frac{\delta^n}{\delta J(x_1)\cdots\delta J(x_n)}Z[J]\Big|_{J=0}\,, \tag{4.53}$$

where the $i\epsilon$ prescription is understood for the evaluation of the path integral.

Free scalar field. Let us compute explicitly the generating functional $Z[J]$ in the case of a free real scalar field. We first rewrite the $i\epsilon$-deformed Klein–Gordon action as a quadratic form:

$$\begin{aligned} S_\epsilon &= \int d^4x\,\left((1+i\epsilon)\frac{1}{2}\dot\phi^2 - (1-i\epsilon)\frac{1}{2}(\nabla\phi)^2 - (1-i\epsilon)\frac{1}{2}m^2\phi^2\right) \\ &\overset{\text{IBP}}{=} \frac{1}{2}\int d^4x \int d^4y\,\phi(y)\,\underbrace{\delta^{(4)}(x-y)\left(-(1+i\epsilon)\frac{\partial^2}{\partial(x^0)^2} + (1-i\epsilon)\big(\nabla_x^2 - m^2\big)\right)}_{iK(x,y)}\,\phi(x)\,. \end{aligned} \tag{4.54}$$

where we have introduced the *integral kernel* $K(x,y)$ of the quadratic form.

Next, the generating functional of time-ordered correlation functions can be explicitly evaluated as follows. We start with:

$$Z[J] = \int \mathcal{D}\phi \exp\int d^4x d^4y\left(-\frac{1}{2}K(x,y)\phi(x)\phi(y) + i\phi(x)J(y)\delta^{(4)}(x-y)\right) \tag{4.55}$$

We introduce the inverse Kernel $K^{-1}(x,y)$ which is solution of the integral equation:

$$\int d^4y\; K(x,y)K^{-1}(y,z) = \delta^{(4)}(x-z)\,. \tag{4.56}$$

We complete then the square in the exponential as follows:

$$\frac{1}{2}\phi(x)K(x,y)\phi(y) - i\phi(x)J(y)\delta^{(4)}(x-y)$$
$$= \frac{1}{2}\left(\phi(x) - i\int d^4u\, J(u)K^{-1}(u,x)\right)K(x,y)\left(\phi(y) - i\int d^4v\, K^{-1}(y,v)J(v)\right)$$
$$+ \frac{1}{2}\int d^4u\, d^4v\ J(u)K^{-1}(u,x)K(x,y)K^{-1}(y,v)\, J(v) \quad (4.57)$$

Then, shifting the functional integration variable as

$$\phi(x) \longrightarrow \phi(x) + \int d^4u\, J(u)K^{-1}(u,x)\,, \quad (4.58)$$

One can write

$$Z[J] = Z[0]\exp\left(-\frac{1}{2}\int d^4u\, d^4v\ K^{-1}(u,v)J(u)J(v)\right)\,, \quad (4.59)$$

where $Z[0]$ contains the result of the gaussian functional integral.

What remains to be done is finding an explicit expression of the inverse kernel $K^{-1}(x,y)$. First, using

$$\delta^{(4)}(x-y) = \int \frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}\,, \quad (4.60)$$

one can write the action (4.54), after integration by parts, as follows:

$$S_\epsilon \overset{IB}{=} \frac{1}{2}\int d^4x \int d^4y\, \phi(y) \underbrace{\int \frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}\left((1+i\epsilon)(p^0)^2 - (1-i\epsilon)\left(\vec{p}^2+m^2\right)\right)}_{iK(x,y)}\phi(x)\,. \quad (4.61)$$

Therefore, since for any function $f(p^0,\vec{p})$,

$$\int d^4y \frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}f(p^0,\vec{p})\int\frac{d^4q}{(2\pi)^4}\frac{e^{-iq\cdot(y-z)}}{f(q^0,\vec{q})} = \int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-z)} = \delta^{(4)}(x-z)\,, \quad (4.62)$$

the inverse Kernel can be written, not suprisingly, as a propagator or Green function of the Klein–Gordon differential operator:

$$K^{-1}(x,y) = \int\frac{d^4p}{(2\pi)^4}\frac{i\,e^{-ip\cdot(x-y)}}{(1+i\epsilon)(p^0)^2 - (1-i\epsilon)\left(\vec{p}^2+m^2\right)} \quad (4.63)$$

This $i\epsilon$ prescription is analoguous to the Feynman $i\epsilon$ prescription. Indeed, the zeroes of the denominator occur for

$$\left(p^0\right)^2 = \vec{p}^2 + m^2 - i\epsilon \times \underbrace{2(\vec{p}^2+m^2)}_{>0} + \mathcal{O}(\epsilon^2)\,. \quad (4.64)$$

So, trading $K^{-1}(x,y)$ for the Feynman propagator $D(x-y)$, we arrive to the final expression of the generating functional for the free scalar QFT:

$$\boxed{Z[J] = Z[0]\,\exp\left(-\frac{1}{2}\int d^4x\,d^4y\;D(x-y)J(x)J(y)\right)} \tag{4.65}$$

It is easy to see that one reproduces the known results for the correlation functions starting from this generating functional. First,

$$\begin{aligned}\frac{-1}{Z[0]}\frac{\delta^2}{\delta J(x_1)\delta J(x_2)}Z[J] &= \frac{\delta^2}{\delta J(x_1)\delta J(x_2)}\Big(1-\frac{1}{2}\int d^4x\,d^4y\;D(x-y)J(x)J(y)\;+\cdots\Big)\\ &= \frac{\delta}{\delta J(x_1)}\Big(\int d^4x\,D(x-x_2)J(x)\;+\cdots\Big) = D(x_1-x_2)+\cdots\end{aligned} \tag{4.66}$$

The higher order terms in the expansion of the exponential don't survive when $J=0$ is taken at the end of the computation. It is easy to see that correlators with an odd number of field vanish after imposing $J=0$. The next non-trivial correlation function is the 4-point function:

$$\begin{aligned}\mathcal{G}(x_1,\ldots,x_4) &= \frac{1}{Z[0]}\frac{\delta^4}{\delta J(x_1)\cdots\delta J(x_4)}Z[J]\Big|_{J=0}\\ &= \frac{1}{Z[0]}\frac{\delta^4}{\delta J(x_1)\cdots\delta J(x_4)}\frac{1}{2!}\frac{1}{4}\left(\int d^4w\,d^4x\,D(w-x)J(w)J(x)\right)\left(\int d^4y\,d^4z\,D(y-z)J(y)J(z)\right)\\ &= D(x_1-x_2)D(x_3-x_4)+D(x_1-x_3)D(x_2-x_4)+D(x_1-x_4)D(x_2-x_4)\,,\end{aligned}$$

which is of course the same as one obtains from Wick theorem, see eqn. (2.147) and fig. 2.6. More generally, Wick theorem for any number of fields follows directly from eqn. (4.65).

Functional determinants. Let us describe briefly the complications hidden in the Gaussian functional integral:

$$Z[0] = \int \mathcal{D}\phi\,\exp\int d^4x d^4y\left(-\frac{1}{2}K(x,y)\phi(x)\phi(y)\right)\,. \tag{4.67}$$

In finite dimension, the analogue of this quantity is an ordinary multi-variable gaussian integral. Let A be an $n\times n$ positive-definite symmetric matrix *i.e.* whose eigenvalues are strictly positive. We have:

$$\int dv_1\cdots dv_n\;\exp\left(-\frac{1}{2}v^T M v\right) = \frac{(2\pi)^{n/2}}{\sqrt{\det A}}\,. \tag{4.68}$$

To prove it, one simply makes the change of variables $v=Ow$, such that O^TMO is diagonal, where O is an orthogonal matrix (thus giving a trivial jacobian). Conveniently, one can notice that:

$$\det A = \exp\big(\mathrm{Tr}\,\ln A\big)\,, \tag{4.69}$$

which is easy to prove using a basis where A is diagonal.

This result can be, at least formally, extended to the Gaussian functional integral (4.67). Ignoring some constant (but infinite!) factors, that can be otherwise absorbed in the integration measure, we have:

$$\int \mathcal{D}\phi \exp \int d^4x d^4y \left(-\frac{1}{2} K(x,y)\phi(x)\phi(y) \right) = \frac{1}{\sqrt{\det K}} . \tag{4.70}$$

The evaluation of the *functional determinant* $\det K$ is rather delicate and plagued with ambiguities because its spectrum does not consist in a finite set of eigenvalues some regularization method is needed.

Let us assume that the integral kernel $K(x,y)$ is a self-adjoint and positive-definite operator with a fully discrete spectrum. Let us denote $\{\lambda_1, \lambda_2, \dots, \}$ the set of eigenvalues, ordered such that $\lambda_1 \leqslant \lambda_2 \leqslant \cdots$. In this case one can proceed as follows. First we define the *spectral zeta function* associated with this operator as:

$$\zeta_K(z) \stackrel{\text{def.}}{=} \sum_{n=1}^{\infty} \lambda_n^{-z} . \tag{4.71}$$

The series is convergent when the real part of z is bigger than a certain value that depends on the eigenvalue distributions.

This function is closely related to the Riemann zeta function. As for the latter, it can be analytically continued almost everywhere in complex plane. Assuming $z = 0$ belongs to its domain of analyticity, one define the *zeta-regularized determinant* as:

$$\widetilde{\det} K = e^{-\zeta_K'(0)} , \tag{4.72}$$

since

$$\zeta_K'(0) = \sum_n \ln \lambda_n \; = \; \mathrm{Tr} \, \ln K .$$

In the case of path integrals over complex scalar fields, that we have not considered so far, the relevant finite-dimensional integral involves a Hermitian $n \times n$ matrix and an integral over complex vectors. We have:

$$\int dv_1 \cdots dv_n dv_1^* \cdots dv_n^* \exp\left(-v^\dagger H v\right) = \frac{(2\pi)^n}{\det H} , \tag{4.73}$$

which can be easily obtained by *(i)* diagonalization and *(ii)* splitting into real and imaginary parts. Its functional version is:

$$\int \mathcal{D}\phi \mathcal{D}\phi^* \; \exp\left(-\int d^4x d^4y K(x,y)\phi^*(x)\phi(y) \right) = \frac{1}{\det K} . \tag{4.74}$$

These functional determinants drop from correlation functions, but still have physical content, in particular in the context of effective field theory. One is usually interested in the ratio of functional determinants, in which case part of the complications mentioned here is no longer present.

4.2.2 Perturbation theory

Perturbation theory can be usefully rephrased in terms of path integrals. Let us split the action of the theory as usual:

$$S[\phi] = \underbrace{\int d^4x\, \mathcal{L}_0}_{S_0[\phi]} + \underbrace{\int d^4x\, \mathcal{L}_{\rm INT}}_{S_{\rm INT}[\phi]} . \tag{4.75}$$

In bare perturbation theory, S_0 is expressed in terms of the un-renormalized fields and of the bare parameters (m_0, etc.); using instead renormalized perturbation theory, S_0 is expressed in terms of the renormalized fields and parameters (m etc.), and $\mathcal{L}_{\rm INT}$ contains the counter-terms. In both cases, using that, for any functional $F[\phi]$, one has:

$$F[\phi] e^{i\int d^4x\, J(x)\phi(x)} = F\left[-i\frac{\delta}{\delta J}\right] e^{i\int d^4x\, J(x)\phi(x)} , \tag{4.76}$$

the generating functional of correlation functions can be expressed as:

$$\begin{aligned} Z[J] &= \int \mathcal{D}\phi\, e^{iS_0[\phi]+iS_{\rm INT}[\phi]+i\int d^4x\, J(x)\phi(x)} = \int \mathcal{D}\phi\, e^{iS_{\rm INT}\left[-i\frac{\delta}{\delta J}\right]} e^{iS_0[\phi]+i\int d^4x\, J(x)\phi(x)} \\ &= e^{iS_{\rm INT}\left[-i\frac{\delta}{\delta J}\right]} \int \mathcal{D}\phi\, e^{iS_0[\phi]+i\int d^4x\, J(x)\phi(x)} \end{aligned} \tag{4.77}$$

Using eqn. (4.65) we reach the result:

$$Z[J] \;=\; Z_0[0]\, e^{\,iS_{\rm INT}\left[-i\frac{\delta}{\delta J}\right]}\, e^{-\frac{1}{2}\int d^4x\, d^4y\; D(x-y)J(x)J(y)} . \tag{4.78}$$

Here $Z_0[J]$ is for the free theory and $Z[J]$ for the interacting theory. To reformulate this relation in a more useful form, let us start with the expression:

$$\begin{aligned} e^{\frac{1}{2}\int d^4x d^4y\, \frac{\delta}{\delta\phi(x)} D(x-y) \frac{\delta}{\delta\phi(y)}} e^{iS_{\rm INT}[\phi]+i\int d^4x\, J(x)\phi(x)} &= e^{\frac{1}{2}\int d^4x d^4y\, \frac{\delta}{\delta\phi(x)} D(x-y) \frac{\delta}{\delta\phi(y)}} e^{iS_{\rm INT}\left[-i\frac{\delta}{\delta J}\right]} e^{i\int d^4x\, J(x)\phi(x)} \\ &= e^{iS_{\rm INT}\left[-i\frac{\delta}{\delta J}\right]} e^{\frac{1}{2}\int d^4x d^4y\, \frac{\delta}{\delta\phi(x)} D(x-y) \frac{\delta}{\delta\phi(y)}} e^{i\int d^4x\, J(x)\phi(x)} \\ &= e^{iS_{\rm INT}\left[-i\frac{\delta}{\delta J}\right]} e^{-\frac{1}{2}\int d^4x d^4y\, J(x) D(x-y) J(y)} e^{i\int d^4x\, J(x)\phi(x)} \end{aligned} \tag{4.79}$$

Hence one can write:

$$\boxed{\frac{Z[J]}{Z[0]} = \frac{e^{\frac{1}{2}\int d^4x d^4y\, \frac{\delta}{\delta\phi(x)} D(x-y) \frac{\delta}{\delta\phi(y)}}\; e^{iS_{\rm INT}[\phi]+i\int d^4x\, J(x)\phi(x)}\Big|_{\phi=0}}{e^{\frac{1}{2}\int d^4x d^4y\, \frac{\delta}{\delta\phi(x)} D(x-y) \frac{\delta}{\delta\phi(y)}}\; e^{iS_{\rm INT}[\phi]}\Big|_{\phi=0}}} \tag{4.80}$$

This is the functional analogue of eqn. (2.126) for correlation functions in perturbation theory.

Let us see how to reproduce the first non-trivial contribution to $Z[0]$ in ϕ^4 theory, *i.e.* the one-loop vacuum bubble  , using this formalism. We have the terms:

$$Z[0] = \left(1 + \frac{1}{2}\int d^4x d^4y \, \frac{\delta}{\delta\phi(x)} D(x-y) \frac{\delta}{\delta\phi(y)} + \frac{1}{2!}\frac{1}{4}\left(\int d^4w d^4x \, \frac{\delta}{\delta\phi(w)} D(w-x) \frac{\delta}{\delta\phi(x)}\right)\left(\int d^4y d^4z \, \frac{\delta}{\delta\phi(y)} D(y-z) \frac{\delta}{\delta\phi(z)}\right) + \cdots\right) \left(1 - \frac{i\lambda}{4!}\int d^4v \, \phi^4(v) + \mathcal{O}(\lambda^2)\right)\Big|_{\phi=0} \quad (4.81)$$

At order λ the term with only two functional derivatives does not survive when we take $\phi = 0$. We get then from the term with four functional derivatives:

$$Z[0] = 1 - \frac{i\lambda}{8}\int d^4x \, D(0)^2 + \mathcal{O}(\lambda^2)\,, \quad (4.82)$$

the same result as eqn. (2.155) with the right combinatorial factor obtain by enumerating the equivalent actions of the four functional derivatives.

4.2.3 Schwinger–Dyson equation and Ward–Takahashi identities

As we have mentionned in section 3.3 of chapter 3, the Schwinger–Dyson equations are obtained more naturally from the path-integral viewpoint than from the canonical quantization view point. Let us consider an infinitesimal change of integration variable in the generating functional $Z[J]$:

$$\phi \;\mapsto \widetilde{\phi} = \phi + \epsilon\Delta\phi + \mathcal{O}(\epsilon^2)\,. \quad (4.83)$$

We assume that the functional measure is invariant under this transformation:

$$\mathcal{D}\phi \stackrel{!}{=} \mathcal{D}\widetilde{\phi}\,. \quad (4.84)$$

We have then:

$$Z[J] = \int \mathcal{D}\phi \, e^{iS[\phi]+i\int J\phi} = \int \mathcal{D}\widetilde{\phi} \, e^{iS[\widetilde{\phi}-\epsilon\Delta\phi]+i\int(\widetilde{\phi}-\epsilon\Delta\phi)J}$$
$$= Z[J] - i\epsilon \int \mathcal{D}\widetilde{\phi} \, e^{iS[\widetilde{\phi}]+i\int J\widetilde{\phi}} \int d^4x \left(\frac{\delta S}{\delta\widetilde{\phi}(x)} + J(x)\right)\Delta\phi(x) + \mathcal{O}(\epsilon^2) \quad (4.85)$$

Since the change of variables should keep invariant the functional integral for any $\Delta\phi$, in particular for $\Delta\phi = \delta^{(4)}(x-x')$, we arrive to the equation:

$$\boxed{\int \mathcal{D}\phi \, \left(\frac{\delta S}{\delta\phi(x)} + J(x)\right) \, e^{iS[\phi]+i\int J\phi} = 0\,.} \quad (4.86)$$

This is the path-integral version of the Schwinger–Dyson equation. It states that the classical equations of motion in the presence of a source term, $\delta S/\delta\phi + J = 0$, hold inside the path integral.

To make the connection with formula (3.166) that we derived using canonical quantization, let us extract relations between correlation functions from eqn. (4.86):

$$\begin{aligned} 0 &= \frac{(-i)^n \delta^n}{\delta J(x_1)\cdots\delta J(x_n)} \int \mathcal{D}\phi \ \left(\frac{\delta S}{\delta\phi(x)} + J(x)\right) \ e^{\,iS[\phi]+i\int J\phi}\Big|_{J=0} \\ &= \int \mathcal{D}\phi \, \frac{\delta S}{\delta\phi(x)} \, \phi(x_1)\cdots\phi(x_n)\, e^{\,iS[\phi]} \\ &\qquad - i\sum_{\ell=1}^{n} \delta^{(4)}(x-x_\ell) \int \mathcal{D}\phi \, \frac{\delta S}{\delta\phi(x)} \, \phi(x_1)\cdots\cancel{\phi(x_\ell)}\cdots\phi(x_n)\, e^{\,iS[\phi]} \end{aligned} \tag{4.87}$$

Dividing by $Z[0]$, one gets the following relation between correlation functions:

$$\langle\Omega|\, T\, \frac{\delta S}{\delta\Phi(x)} \, \Phi(x_1)\cdots\Phi(x_n)\,|\Omega\rangle - i\sum_{\ell=1}^{n} \delta^{(4)}(x-x_\ell)\, \langle\Omega|\, T\, \Phi(x_1)\cdots\cancel{\Phi(x_\ell)}\cdots\Phi(x_n)\,|\Omega\rangle = 0\,. \tag{4.88}$$

To make the comparison with eqn. (3.166) there is subtlety that was already mentionned in the context of quantum mechanics, see eqn. (4.23) and below. The times derivatives appearing in $\delta S/\delta\phi$ should be put *outside* of the time-ordered correlation function. As such, they can act on the step functions occuring form time-ordering.

*
* *

The Ward–Takahashi identity (3.164) can be obtained using similar arguments. Let us consider a theory with a continuous symmetry $\Phi \mapsto \Phi + \epsilon\delta_s\Phi$ associated with a conserved current $\mathcal{J}^\mu$. We have, from eqn. (3.83):

$$\frac{\delta S}{\delta\phi(x)}\delta_s\phi(x) = -\partial_\mu\mathcal{J}^\mu(x)\,. \tag{4.89}$$

Note that neither the left-hand side nor the right-hand side are automatically zero, since we are working off-shell.

Inserting this relation in eqn. (4.85) for $\delta_s\phi(y) = \delta^{(4)}(x-y)\delta_s\phi(x)$, and assuming again invariance of the measure under the field transformation, we arrive to the relation:

$$\boxed{\int \mathcal{D}\phi\, e^{\,iS[\phi]}\Big(-\partial_\mu\mathcal{J}^\mu(x) + J(x)\delta_s\phi(x)\Big) = 0\,.} \tag{4.90}$$

Exercise: show that this path integral identity implies eqn. (3.164) for correlation functions between the conserved current and field operators. As in the case of the Schwinger–Dyson equation, one needs to make the time derivative acting outside of the time-ordering.

4.3 Fermionic path integrals

In canonical quantization, we have argued that the quantum fields associated with particles following the Fermi–Dirac statistics satisfied anti-commutation rather than commutation relations. While a classical complex scalar field is a map $\phi : \mathbb{R}^{1,3} \to \mathbb{C}$ we are led to assume that classical fermionic fields are maps from $\mathbb{R}^{1,3}$ to a set of anti-commuting numbers.

4.3.1 Grassman algebra.

Let V be an n-dimensional complex vector space with a basis $\mathcal{B} = \{\theta^\ell,\, \ell = 1, \ldots, n\}$.

We consider then the exterior product $\Lambda^2 V$, *i.e.* the antisymmetrized tensor product $V \otimes V$. Any vector in $\Lambda^2 V$ can be written as

$$\frac{1}{2}\sum_{i,j} \alpha_{ij}\theta^i \wedge \theta^j \qquad \theta^i \wedge \theta^j \stackrel{\text{def.}}{=} \theta^i \otimes \theta^j - \theta^j \otimes \theta^i\,, \tag{4.91}$$

with α_{ij} an antisymmetric $n \times n$ complex matrix. In the same way one can construct iteratively the exterior products $\Lambda^k V$, for $1 \leqslant k \leqslant n$. The Grassman algebra $\mathfrak{Gr}(n)$, or exterior algebra, is defined formally as the direct sum of all these antisymmetrized tensor products of V:

$$\mathfrak{Gr}(n) \stackrel{\text{def.}}{=} \mathbb{C} \oplus V \oplus \Lambda^2 V \oplus \Lambda^3 V \oplus \cdots \oplus \Lambda^n V = \bigoplus_{k=0}^{n} \Lambda^k V\,, \tag{4.92}$$

with the conventions $\Lambda^0 V = \mathbb{C}$ and $\Lambda V = V$. A generic element of $\mathfrak{Gr}(n)$ can be written as:

$$a = \alpha + \alpha_i\theta^i + \frac{1}{2}\alpha_{ij}\theta^i \wedge \theta^j + \frac{1}{3!}\alpha_{ijk}\theta^i \wedge \theta^j \wedge \theta^k + \cdots + \frac{1}{n!}\alpha_{i_1\cdots i_n}\theta^{i_1} \wedge \cdots \wedge \theta^{i_n}\,, \tag{4.93}$$

where the coefficents $\alpha_{i_1\cdots i_k}$ are all totally antisymmetric in their indices. $\mathfrak{Gr}(n)$ defines an algebra since the exterior product $\theta^i \wedge \theta^j$ induces a bilinear product on $\mathfrak{Gr}(n)$. To simplify notation, the exterior product sign is usually omitted:

$$\theta^i \wedge \theta^j \;\to\; \theta^i\theta^j\,, \quad \theta^i\theta^j = -\theta^j\theta^i\,. \tag{4.94}$$

The θ^i used in the construction of the Grassmann algebra are often referred to as *Grassmann numbers*.

The Grassmann algebra is a graded algebra in the following sense. Every element of $\mathfrak{Gr}(n)$ can be decomposed as the sum of an *even* element, with only terms in Λ^{2k}, and *odd* elements, with only terms in Λ^{2k+1}. To the former we assign a grade $s = 0$, and to the latter a grade $s = 1$. If a and b have respectively grade s_a and s_b, we have

$$ab = (-1)^{s_a s_b} ba\,. \tag{4.95}$$

Grassmann derivative. A Grassmann algebra can be endowed with notions of derivation and integration. Let f be an element of $\mathfrak{Gr}(n)$, *i.e.* a generic function of the Grassmann variables θ^i:

$$f(\theta) = \phi + \phi_i\theta^i + \frac{1}{2!}\phi_{ij}\theta^i\theta^j + \frac{1}{3!}\phi_{ijk}\theta^i\theta^j\theta^k \cdots + \frac{1}{n!}\phi_{i_1\cdots i_n}\theta^{i_1}\cdots\theta^{i_n}\,. \tag{4.96}$$

The derivative of f with respect to the variable θ^i is defined as:

$$\frac{\partial}{\partial\theta^i} f(\theta) = \phi_i + \phi_{ij}\theta^j + \frac{1}{2!}\phi_{ijk}\theta^j\theta^k + \cdots \tag{4.97}$$

To phrase it differently, the Grassmann derivative is a linear operator acting on the Grassmann variables as:

$$\frac{\partial\theta^i}{\partial\theta^j} = \delta^i{}_j\,, \tag{4.98}$$

and obeying a *graded Leibniz rule*: if f is of grade s, we have:

$$\frac{\partial}{\partial\theta^i}\big(f(\theta)g(\theta)\big) = \left(\frac{\partial f(\theta)}{\partial\theta^i}\right) g(\theta) + (-1)^s\, f(\theta)\left(\frac{\partial g(\theta)}{\partial\theta^i}\right)\,. \tag{4.99}$$

We observe in particular that:

$$\frac{\partial}{\partial\theta^i}\left(\theta^j\theta^k\right) = \delta^j{}_i\theta^k - \delta^k{}_i\theta^j\,. \tag{4.100}$$

It is often useful to call this derivative *left derivative* and to introduce also a *right derivative*. They are defined as:

$$\frac{\overrightarrow{\partial}}{\partial\theta^i}\left(\theta^j\theta^k\right) = \delta^j{}_i\theta^k - \delta^k{}_i\theta^j\,, \quad \left(\theta^j\theta^k\right)\frac{\overleftarrow{\partial}}{\partial\theta^i} = \delta^k{}_i\theta^j - \delta^j{}_i\theta^k\,. \tag{4.101}$$

In the case of a single Grassmann variable, *i.e.* the algebra $\mathfrak{Gr}(1)$, any function is characterized by only two numbers:

$$f(\theta) = a + b\theta\,, \quad \frac{\partial}{\partial\theta} f(\theta) = b\,. \tag{4.102}$$

Berezin integration. Integration over Grassmann variable is known as *Berezin integration*. As the ordinary integral over $\mathbb{R}$, it is designed to be linear and invariant under translation of the integration variable. Let us start with the case of a single variable. We impose:

$$\int \mathrm{d}\theta\,\big(\alpha f(\theta) + \beta g(\theta)\big) \;=\; \alpha\int \mathrm{d}\theta\, f(\theta) + \beta\int \mathrm{d}\theta\, g(\theta)\,, \tag{4.103a}$$

$$\int \mathrm{d}\theta\, f(\theta + c) \;=\; \int \mathrm{d}\theta\, f(\theta)\,. \tag{4.103b}$$

Since f is of the form (4.102), the condition (4.103b) implies that

$$\int d\theta = 0 . \tag{4.104}$$

The Berezin integral is then only defined by $\int d\theta\, \theta$, that we can normalize as:

$$\int d\theta\, \theta = 1 . \tag{4.105}$$

We can summarize these properties as:

$$f(\theta) = a + b\theta , \quad \int d\theta\, f(\theta) = b = \frac{\partial}{\partial\theta} f(\theta) . \tag{4.106}$$

Thus, quite surprisingly, integration is the same as derivation. We have the property:

$$\int d\theta\, \frac{\partial}{\partial\theta} f(\theta) = 0 , \tag{4.107}$$

meaning that there are no boundary terms in this integration. Let us define $\tilde{\theta} = M\theta$, $M \in \mathbb{C}$. We have:

$$\int d\theta\, f(\tilde{\theta}) = M \int d\theta\, f(\theta) = M \int d\tilde{\theta}\, f(\tilde{\theta}) , \tag{4.108}$$

which can be interpreted as a rule of change of variables:

$$d\tilde{\theta} = \tfrac{1}{M}\, d\theta . \tag{4.109}$$

Let us consider now a set of n Grassmann variables $\theta^1, \ldots, \theta^n$, is this order (the order defines an orientation), and a generic function $f(\theta)$ of the form (4.96). The n-dimensional Berezin integral of f is defined as:

$$\int d^n\theta\, f(\theta) \;=\; \int d\theta^n \cdots d\theta^1\, f(\theta) \;=\; \int d\theta^n \cdots d\theta^1\, \frac{1}{n!} a_{i_1 \cdots i_n} \theta^{i_1} \cdots \theta^{i_n} = a_{12\cdots n} \tag{4.110}$$

which is equivalent to:

$$\int d\theta^n \cdots d\theta^1\, \theta^{i_1} \cdots \theta^{i_n} = \epsilon_{i_1 \cdots i_n} . \tag{4.111}$$

Integrands with less than n Grassmann variables do not contribute.

Consider as before the linear change of variables $\tilde{\theta}^i = M^i{}_j \theta^j$, parametrized by an $n \times n$ matrix M, and consider the integral:

$$\begin{aligned} \int d^n\theta\, f(\tilde{\theta}) &= \int d\theta^n \cdots d\theta^2 d\theta^1\, \frac{1}{n!} \phi_{i_1 i_2 \cdots i_n} M^{i_1}_{j_1} M^{i_2}_{j_2} \cdots M^{i_n}_{j_n} \theta^{j_1} \theta^{j_2} \cdots \theta^{j_n} \\ &= \phi_{i_1 i_2 \cdots i_n} M^{i_1}_{\;1} M^{i_2}_{\;2} \cdots M^{i_n}_{\;n} \\ &= \det M\, \phi_{12\cdots n} \\ &= \det M \int d^n\tilde{\theta}\, f(\tilde{\theta}) , \end{aligned} \tag{4.112}$$

Hence we have the formula

$$d^n\tilde{\theta} = \frac{1}{\det M} d^n\theta , \tag{4.113}$$

which is the inverse of the corresponding formula for ordinary variables.

Gaussian Berezin integral. In preparation for the fermionic path integral let us consider a Gaussian integral over $\mathfrak{Gr}(n)$, with $n = 2m$ even:

$$I = \int d^n\theta \, \exp\Big(\frac{1}{2}\theta^i K_{ij}\theta^j\Big)\,, \tag{4.114}$$

where K is an antisymmetric $n \times n$ matrix. The only non-zero contribution from the m-th terms in the series expansion of the exponential:

$$\exp\Big(\frac{1}{2}\theta^i K_{ij}\theta^j\Big) = 1 + \frac{1}{2}\theta^i K_{ij}\theta^j + \cdots + \frac{1}{m!}\frac{1}{2^n}\theta^{i_1} K_{i_1 j_1}\theta^{j_1}\cdots\theta^{i_m} K_{i_m j_m}\theta^{j_m}\,. \tag{4.115}$$

It leads to:

$$I = \mathrm{Pf}\,(K) \quad , \qquad \mathrm{Pf}\,(K) \stackrel{\text{def.}}{=} \frac{1}{m!}\frac{1}{2^n}\epsilon_{i_1 j_1\cdots i_m j_m} K_{i_1 j_1}\cdots K_{i_m j_m}\,, \tag{4.116}$$

where we have defined the Pfaffian $\mathrm{Pf}\,(K)$. One can show that, for any antisymmetric matrix,

$$(\mathrm{Pf}\,(K))^2 = \det K\,. \tag{4.117}$$

Therefore, we have:

$$\boxed{\int d^n\theta \, \exp\Big(\frac{1}{2}\theta^i K_{ij}\theta^j\Big) \;=\; \sqrt{\det K}\,.} \tag{4.118}$$

Notice that this is the inverse of the ordinary gaussian integral over real numbers. A natural generalization is an integral with a linear term:

$$\begin{aligned}\int d^n\theta \, \exp\Big(\frac{1}{2}\theta^i K_{ij}\theta^j + \delta_{ij}\theta^i\eta^j\Big) \\ = \int d^n\theta \, \exp\Big(\frac{1}{2}(\theta^i - \eta_k (K^{-1})^{ki}) K_{ij}(\theta^j + (K^{-1})^{j\ell}\eta_\ell) + \frac{1}{2}\eta^j (K^{-1})_{j\ell}\eta^\ell\Big) \\ = \mathrm{Pf}\,(K)\, \exp\Big(\frac{1}{2}\eta^j (K^{-1})_{j\ell}\eta^\ell\Big)\,.\end{aligned} \tag{4.119}$$

Complexified Grassmann algebra. Let us consider an even-dimensonal Grassmann algebra $\mathfrak{Gr}(2n)$ with variables $\kappa^1, \cdots \kappa^{2n}$. We define the sets of complex Grassmann variables $\theta^1, \ldots, \theta^n$ and $\bar\theta^1, \ldots, \bar\theta^n$ by the linear transformations:[1]

$$\theta^\ell = \frac{\kappa^{2\ell-1} + i\kappa^{2\ell}}{\sqrt{2}} \quad , \qquad \bar\theta^\ell = \frac{\kappa^{2\ell-1} - i\kappa^{2\ell}}{\sqrt{2}} \quad , \qquad \ell = 1, \ldots, n\,. \tag{4.120}$$

We define the corresponding Berezin integral through:

$$\int d^n\theta d^n\bar\theta \; \bar\theta^1\theta^1\cdots\bar\theta^n\theta^n \stackrel{\text{def.}}{=} \int d\theta^n d\bar\theta^n \cdots d\theta^1 d\bar\theta^1 \; \bar\theta^1\theta^1\cdots\bar\theta^n\theta^n = 1\,. \tag{4.121}$$

[1] Equivalently, one can complexify the Grassmann algebra with variables $\theta^1, \ldots, \theta^n$ by declaring that θ^i and $\bar\theta^i$ are independent variables.

Let K be an antisymmetric $n \times n$ matrix. The Gaussian integral is given now by:

$$\int d^n\theta d^n\bar{\theta}\, \exp\left(\bar{\theta}^i K_{ij}\theta^j\right) = \frac{1}{n!}\int d^n\theta d^n\bar{\theta}\, \left(\bar{\theta}^{i_1} K_{i_1 j_1}\theta^{j_1}\right)\cdots\left(\bar{\theta}^{i_n} K_{i_n j_n}\theta^{j_n}\right)$$
$$= \frac{1}{n!}\epsilon_{i_1\cdots i_n}\epsilon_{j_1\cdots j_n} K_{i_1 j_1}\cdots K_{i_n j_n} = \det K\,. \quad (4.122)$$

Adding a source term one obtains:

$$\int d^n\theta d^n\bar{\theta}\, \exp\left(\bar{\theta}^i K_{ij}\theta^j + \bar{\theta}^j\eta_j + \bar{\eta}_j\theta^j\right) \;=\; \det K\, \exp\left(-\bar{\eta}^i (K^{-1})_{ij}\eta^j\right)\,. \quad (4.123)$$

4.3.2 Fermionic path integral in quantum mechanics.

Let us start the discussion with a very simple system with a single fermionic degree of freedom. We have a creation operator and an annihilation operator obeying the algebra:

$$\left\{\,\Psi,\,\Psi\,\right\} = \left\{\,\Psi^\dagger,\,\Psi^\dagger\,\right\} = 0\,, \quad \left\{\,\Psi,\,\Psi^\dagger\,\right\} = 1\,. \quad (4.124)$$

We define then the ground state $|0\rangle$ and the excited state $|1\rangle$ through:

$$\Psi\,|0\rangle = 0 \quad , \qquad |1\rangle = \Psi^\dagger\,|0\rangle\,. \quad (4.125)$$

Consider a state $|\psi\rangle$ given by the linear combination:

$$|\psi\rangle \stackrel{\text{def.}}{=} |0\rangle - \psi\,|1\rangle\,, \quad (4.126)$$

where ψ is a complex Grassmann variable. It is an eigenvector of the annihilation operator:

$$\Psi\,|\psi\rangle = -\Psi\,\psi\,|1\rangle = +\Psi\Psi^\dagger\,|0\rangle\,\psi = \psi\,|0\rangle = \psi\,|\psi\rangle\,, \quad (4.127)$$

where we have used that $|1\rangle = \Psi^\dagger\,|0\rangle$ is Grassmann-odd. It can be written as

$$|\psi\rangle = e^{-\psi\Psi^\dagger}\,|0\rangle\,, \quad (4.128)$$

thus is the analogue of a coherent state of the harmonic oscillator. The present system can be viewed as a *fermionic harmonic oscillator*. The inner product between such states is:

$$\langle\psi'|\psi\rangle = \langle 0|0\rangle + \bar{\psi}'\psi\,\langle 1|1\rangle = e^{\bar{\psi}'\psi}\,. \quad (4.129)$$

In terms of these states, the resolution of the identity can be written as:

$$\mathbb{I} = \int d\bar{\psi}d\psi\,|\psi\rangle\, e^{-\bar{\psi}\psi}\,\langle\psi|\,. \quad (4.130)$$

Indeed we have:

$$\int d\bar\psi d\psi\, |\psi\rangle\, e^{-\bar\psi\psi}\, \langle\psi| = \int d\bar\psi d\psi \big(\, |0\rangle - \psi\, |1\rangle\,\big)\big(1-\bar\psi\psi\big)\big(\,\langle 0| - \langle 1|\,\bar\psi\big)$$
$$= \underbrace{\Big(-\int d\bar\psi d\psi\, \bar\psi\psi\Big)}_{=1} |0\rangle\,\langle 0| + \underbrace{\Big(\int d\bar\psi d\psi\, \psi\bar\psi\Big)}_{=1} |1\rangle\,\langle 1|\,. \quad (4.131)$$

Let us consider now the transition amplitude between two spin states:

$$\langle \psi_F, t_F|\psi_I, t_I\rangle = \langle\psi_F|\, e^{-iH(t_F-t_I)}\, |\psi_I\rangle\,. \quad (4.132)$$

As in the case of the bosonic system, we split the time interval with N infinitesimal intermediate steps of size

$$\delta t \overset{\text{def.}}{=} \frac{T}{N+1}\,, \quad T = t_F - t_I\,. \quad (4.133)$$

Inserting the resolution of the identity between each time step, we have:

$$\langle \psi_F, t_F|\psi_I, t_I\rangle =$$
$$\lim_{N\to\infty}\int\prod_{\ell=1}^{N} d\bar\psi_\ell d\psi_\ell\, \langle\psi_F|\, e^{-iH\delta t}\, |\psi_N\rangle\, e^{-\bar\psi_N\psi_N}\, \langle\psi_N|\cdots|\psi_1\rangle\, e^{-\bar\psi_1\psi_1}\, \langle\psi_1|\, e^{-iH\delta t}\, |\psi_I\rangle\,. \quad (4.134)$$

Assume now that the Hamiltonian is of the form $H = \Psi^\dagger K\Psi$. We have then, using eqn. (4.127) and then eqn. (4.129):

$$\langle\psi_{\ell+1}|\, e^{-iH\delta t}\, |\psi_\ell\rangle = \langle\psi_{\ell+1}|\,(1 - iH\delta t)\,|\psi_\ell\rangle = \langle\psi_{\ell+1}|\,(1 - i\bar\psi_{\ell+1}K\psi_\ell\delta t)\,|\psi_\ell\rangle$$
$$= e^{\bar\psi_{\ell+1}\psi_\ell} e^{-i\bar\psi_{\ell+1}K\psi_\ell\delta t}\,. \quad (4.135)$$

It is convenient to factorize the result as:

$$\langle\psi_{\ell+1}|\, e^{-iH\delta t}\, |\psi_\ell\rangle = e^{\bar\psi_{\ell+1}\psi_{\ell+1}} e^{\bar\psi_{\ell+1}(\psi_\ell - \psi_{\ell+1})} e^{-i\bar\psi_{\ell+1}K\psi_\ell\delta t}\,, \quad (4.136)$$

in order to remove the exponential factors coming from the resolution of the identity (4.130) (except the first one). Renaming $\psi_F \to \psi_{N+1}$ and $\psi_I \to \psi_0$, we have

$$\langle \psi_F, t_F|\psi_I, t_I\rangle =$$
$$\lim_{N\to\infty}\int\prod_{\ell=1}^{N} d\bar\psi_\ell d\psi_\ell e^{\bar\psi_{N+1}\psi_{N+1}} \exp i\delta t \sum_{\ell=0}^{N}\left(\bar\psi_{\ell+1} i\frac{\psi_{\ell+1}-\psi_\ell}{\delta t} - \bar\psi_{\ell+1}K\psi_\ell\right) \quad (4.137)$$

which can be rewritten in the continuum as the fermionic path integral (absorbing the first exponential factor into the measure):

$$\boxed{\langle \psi_F, t_F|\psi_I, t_I\rangle = \int \mathcal{D}\bar\psi\mathcal{D}\psi\; e^{i\int_{t_I}^{t_F} dt\,\left(\bar\psi\, i\frac{d}{dt}\psi - H\right)}\,.} \quad (4.138)$$

4.3.3 Fermionic path integral in quantum field theory.

The steps from quantum mechanics to quantum field theory are exactly the same as those followed for the scalar QFT:

1. Consider a quantum mechanical system on a lattice Λ, with a fermionic degree of freedom attached to each node:
$$\left\{\Psi_\lambda(t)\,,\ \lambda\in\Lambda=\vec{a}_1\mathbb{Z}+\vec{a}_2\mathbb{Z}+\vec{a}_3\mathbb{Z}\right\}, \tag{4.139}$$
2. Couple those degrees of freedom by a discrete version of the Dirac Hamiltonian,
3. Take the continuum limit.

It leads to the path integral formula:
$$\langle\psi_F(\vec{x}_F),t_F|\psi_I(\vec{x}_I),t_I\rangle=\int\mathcal{D}\bar{\psi}(x)\mathcal{D}\psi(x)\ e^{i\int_{t_I}^{t_F}dt\int d^3x\,\mathcal{L}[\psi,\bar{\psi}]}\,, \tag{4.140}$$
where $\mathcal{L}[\psi,\bar{\psi}]$ is the Dirac Lagrangian density (here $\bar{\psi}$ means the Dirac conjugate $\psi^\dagger\gamma^0$).

In order to replace the initial and final states by the vacuum Ω, one can perform the same type of $i\epsilon$ manipulations as above: either replace take the limit $t_{I/F}\to\mp(1-i\epsilon)\infty$, or deform the Lagrangian density to get a damping factor in the limit $t_{I/F}\to\mp\infty$, for instance:
$$\bar{\psi}\big(i\partial\!\!\!/-m\big)\psi\ \longrightarrow\ \bar{\psi}\big(i\partial\!\!\!/-(1-i\epsilon)m\big)\psi \tag{4.141}$$
With this prescription implicit, the correlation functions in the vacuum for the free Dirac theory follow from the following generating function:

$$\boxed{Z_0\big[\eta,\bar{\eta}\big]=\int\mathcal{D}\bar{\psi}(x)\mathcal{D}\psi(x)\ e^{i\int d^4x\,\left(\mathcal{L}[\psi,\bar{\psi}]+\bar{\eta}(x)\psi(x)+\bar{\psi}(x)\eta(x)\right)}\,.} \tag{4.142}$$

It is understood that the functional derivatives w.r.t. η acts on the right and those w.r.t. $\bar{\eta}$ act on the left. One has for instance:
$$\langle\Omega|\,T\,\Psi(x)\bar{\Psi}(y)\,|\Omega\rangle=\frac{(-i)^2}{Z_0[0]}\frac{\overrightarrow{\delta}}{\delta\bar{\eta}(x)}Z_0[J]\frac{\overleftarrow{\delta}}{\delta\eta(y)}\ \Big|_{\eta=0,\,\bar{\eta}=0}\,. \tag{4.143}$$
To show this, one follows similar steps as below eqn. (4.55), using the functional analogue of the Grassmann Gaussian integral (4.123) and obtain:
$$Z_0\big[\eta,\bar{\eta}\big]=Z_0[0]e^{\int d^4xd^4y\,\bar{\eta}(x)D_F(x-y)\eta(y)}\,, \tag{4.144}$$
in terms of the Feynman propagator $D_F(x-y)$ of the free Dirac field.

4.4 Quantum effective action

In this last part of the notes we will present some tools, derived from the generating functionals, which have a lot in common with statistical physics. We will use the language of scalar QFT, although the same ideas can be applied to other types of quantum field theories.

Generating functional of connected diagrams. The starting point is the relation (4.52) defining the generating functional. This relation can be rephrased as:

$$\frac{Z[J]}{Z[0]} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 \cdots d^4x_n J(x_1) \cdots J(x_n)\, \mathcal{G}(x_1, \cdots, x_n)\,, \tag{4.145}$$

where $\mathcal{G}(x_1, \ldots x_n)$ is the time-ordrered correlation function of n fields in the vacuum.

Every correlation function $\mathcal{G}(x_1, \ldots x_n)$ receives contributions from *fully connected* diagrams but also from *partially connected* diagrams made of several pieces.[2] As we have seen, the fully connected correlation functions, whose diagrammatic expansion contains only fully connected Feynman diagrams, play a prominent role.

Let $\mathcal{G}_C(x_1, \ldots x_n)$ denote the sum of all fully connected contributions to the correlation function $\langle \Omega | \, T\, \phi(x_1) \cdots \phi(x_n) \, | \Omega \rangle$. We define the generating functional of fully connected correlation functions in a way similar to eqn. (4.145):

$$iW[J] \stackrel{\text{def.}}{=} \sum_{n=1}^{\infty} \frac{i^n}{n!} \int d^4x_1 \cdots d^4x_n J(x_1) \cdots J(x_n) \mathcal{G}_C(x_1, \cdots, x_n)\,. \tag{4.146}$$

Naturally, partially connected correlation functions can be obtained by product of fully connected ones, so a relation should exist between the generating functionals $Z[J]$ and $W[J]$. We claim that:

$$\boxed{\frac{Z[J]}{Z[0]} = e^{iW[J]} \iff iW[J] = \ln \frac{Z[J]}{Z[0]}\,.} \tag{4.147}$$

This can be seen by looking at the series expansion of the first equation:

$$\frac{Z[J]}{Z[0]} = 1 + iW[J] + \frac{1}{2!}\big(iW[J]\big)^2 + \frac{1}{3!}\big(iW[J]\big)^2 + \cdots \tag{4.148}$$

By differentiation w.r.t. J, the linear term will naturally provide the connected contributions to the correlation functions, while the quadratic terms will provide the contributions to the the correlation functions made of two pieces, and so on. The $1/n!$ factors are precisely those needed to avoid overcounting (permutation of the different pieces of a partially connected diagram).

[2] There are no contributions from vacuum bubbles since we divide by $Z[0]$.

1PI generating functional. Among the fully connected Feynman diagrams, the one-particle irreducible (1PI) diagrams play an important role, as they allow to resum all the quantum contributions to propagators and vertices. The generating functionnal of those diagrams is called the *1PI effective action* and denoted by $\Gamma[\phi]$.

Let us first consider the connected one-point function of the field in the presence of a source J. By definition, it is given by:

$$\varphi_J(x) = \frac{\delta W[J]}{\delta J(x)} = \langle\Omega|\,\Phi(x)\,|\Omega\rangle_J\,\big|_{\text{CON.}}\,. \tag{4.149}$$

In the absence of a source term, it can be chosen to vanish by shifting the field operator Φ by a constant:

$$\varphi_{J=0}(x) = \frac{\delta W[J]}{\delta J(x)}\Big|_{J=0} = \langle\Omega|\,\Phi(x)\,|\Omega\rangle_{J=0} = 0\,. \tag{4.150}$$

Let us assume that the relation (4.149) is invertible; namely, for a given scalar function $\phi(x)$, there exists a unique source term $J_\phi(x)$ such that $\phi(x) = \delta W[J]/\delta J_\phi(x)$. We define then $\Gamma[\phi$ as a Legendre transform of $W[J]$:

$$\Gamma[\phi] = W[J_\phi] - \int d^4x\,\phi(x)J_\phi(x)\,. \tag{4.151}$$

The functionnal $\Gamma[\phi]$ depends on ϕ only (and not on J) as the notation suggests. We have:

$$\frac{\delta\Gamma[\phi]}{\delta\phi(x)} = \int d^4y\,\underbrace{\frac{\delta W}{\delta J_\phi(y)}}_{\phi(y)}\frac{\delta J_\phi(y)}{\delta\phi(x)} - J_\phi(x) - \int d^4y\,\phi(y)\frac{\delta J_\phi(y)}{\delta\phi(x)} = -J_\phi(x)\,. \tag{4.152}$$

We would like to show that $i\Gamma[\phi]$, as the notation suggests, is in fact the generating functionnal of 1PI fully connected correlation functions. If we denote by $\Gamma^n(x_1,\ldots,x_n)$ the n-point 1PI fully connected correlation function, it would mean that:

$$i\Gamma[\phi] = \sum_{n=1}^{\infty}\frac{1}{n!}\int d^4x_1\cdots d^4x_n\,\phi(x_1)\cdots\phi(x_n)\Gamma^n(x_1,\ldots,x_n)\,, \tag{4.153}$$

or in other words:

$$\Gamma^n(x_1,\ldots,x_n) = \frac{\delta^n i\Gamma[\phi]}{\delta\phi(x_1)\cdots\phi(x_n)}\Big|_{\phi=0}\,. \tag{4.154}$$

Let us see first what we get for the two point function. On the one hand we obtain from (4.152) that:

$$\frac{\delta^2\Gamma[\phi]}{\delta\phi(x_1)\delta\phi(x_2)} = -\frac{\delta J_\phi(x_1)}{\delta\phi(x_2)}\,, \tag{4.155}$$

On the other hand we have:

$$\frac{\delta^2 W[J]}{\delta J(x_1)\delta J(x_2)} = \frac{\delta \phi_J(x_2)}{\delta J(x_1)} \tag{4.156}$$

Thus by the chain rule we have:

$$\int d^4x_2 \, \frac{\delta^2 W[J]}{\delta J(x_1)\delta J(x_2)} \frac{\delta^2 \Gamma[\phi]}{\delta\phi(x_2)\delta\phi(x_3)} = -\int d^4x_2 \frac{\delta\phi(x_2)}{\delta J(x_1)} \frac{\delta J(x_3)}{\delta\phi(x_2)} = -\delta^{(4)}(x_3 - x_1)\,. \tag{4.157}$$

In other words,

$$\frac{\delta^2\Gamma[\phi]}{\delta\phi\delta\phi} = -\left(\frac{\delta^2 W[J]}{\delta J\delta J}\right)^{-1}, \tag{4.158}$$

in the sense of the functional inverse. Since we have:

$$\Gamma^2(x_1,x_2) = \frac{\delta^2 i\Gamma[\phi]}{\delta\phi(x_1)\delta\phi(x_2)}\Big|_{\phi=0} \quad , \qquad \mathcal{G}_c(x_1,x_2) = -\frac{\delta^2 iW[J]}{\delta J(x_1)\delta J(x_2)}\Big|_{J=0}, \tag{4.159}$$

evaluating (4.158) at $J = 0$ (which implies using (4.150) that $\phi = 0$) we find that:

$$\Gamma^2 = -\mathcal{D}^{-1}, \tag{4.160}$$

i.e. $\Gamma^2(x_1,x_2)$ is equal to minus the functionnal inverse of the connected two-point function $\mathcal{G}_c(x_1,x_2)$, which is nothing other than the fully corrected propagator $\mathcal{D}(x_1-x_2)$. One can express $\mathcal{D}(x_1-x_2)$ in terms of the resummed propagator (2.207) as:

$$\mathcal{D}(x_1-x_2) = \int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ip\cdot(x_1-x_2)}}{p^2 - m_0^2 - \Sigma(p^2) + i\epsilon} \tag{4.161}$$

From which we deduce that:

$$\begin{aligned}\Gamma^2(x_1,x_2) &= \int \frac{d^4p}{(2\pi)^4}\, ie^{-ip\cdot(x_1-x_2)}\big(p^2 - m_0^2 - \Sigma(p^2) + i\epsilon\big) \\ &= \int \frac{d^4p}{(2\pi)^4}\, ie^{-ip\cdot(x_1-x_2)}\big(-i\Sigma(p^2)\big) - \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x_1-x_2)} \frac{p^2 - m_0^2 + \epsilon}{i}\end{aligned} \tag{4.162}$$

In other words, Γ^2 is the self-energy (the sum of all amputated 1PI diagrams with two external legs) minus the functional inverse of the tree-level propagator.

The two-point function is a bit special because of this substraction. Let us examine the three-point 1PI function which is representative of all n-point 1PI functions, using similar methods. We start with the third derivative of $W[J]$ and apply the chain rule:

$$\frac{\delta^3 W[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)} = -\frac{\delta}{\delta J(x_1)}\left(\frac{\delta^2\Gamma[\phi]}{\delta\phi\delta\phi}\right)^{-1}(x_2,x_3) = -\int d^4y \, \frac{\delta\phi(y)}{\delta J(x_1)} \frac{\delta}{\delta\phi(y)}\left(\frac{\delta^2\Gamma[\phi]}{\delta\phi\delta\phi}\right)^{-1}(x_2,x_3) \tag{4.163}$$

We now use the functionnal analogue of the matrix relation:

$$\frac{d}{dt}M^{-1}(t) = -M^{-1}(t)\Big(\frac{d}{dt}M(t)\Big)M^{-1}(t)\,: \tag{4.164}$$

It gives:

$$\frac{\delta^3 W[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)} = \int d^4y \, \frac{\delta\phi(y)}{\delta J(x_1)} \int d^4u d^4v \left(\frac{\delta^2\Gamma[\phi]}{\delta\phi\delta\phi}\right)^{-1}(x_2,u) \frac{\delta^3\Gamma[\phi]}{\delta\phi(y)\delta\phi(u)\delta\phi(v)} \left(\frac{\delta^2\Gamma[\phi]}{\delta\phi\delta\phi}\right)^{-1}(v,x_3) \quad (4.165)$$

Let us now evaluate this expression at $J = 0$ (therefore $\phi = 0$ as noticed before). Using the previous results, one can replace then:

$$\left(\frac{\delta^2\Gamma[\phi]}{\delta\phi\delta\phi}\right)^{-1}\Bigg|_{\phi=0}(x,y) = -\left.\frac{\delta^2 W[J]}{\delta J\delta J}\right|_{J=0}(x,y) = -i\mathcal{D}(x-y)\,, \quad (4.166)$$

as well as, using eqn. (4.149):

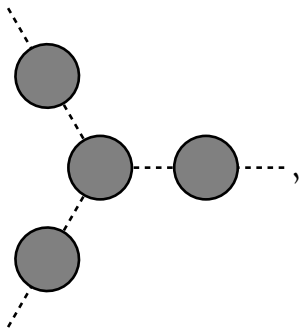
$$\left.\frac{\delta\phi}{\delta J}\right|_{J=0}(x,y) = \frac{\delta^2 W}{\delta J\delta J}(x,y) = i\mathcal{D}(x-y) \quad (4.167)$$

We obtain then the final relation:

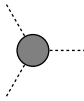
$$\frac{(-i)^3\delta^3 iW[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)} = i\int d^4y\, \mathcal{D}(x_1-y)\int d^4u\, \mathcal{D}(x_2-u)\int d^4v\, \frac{\delta^3\Gamma[\phi]}{\delta\phi(y)\delta\phi(u)\delta\phi(v)}\, \mathcal{D}(v-x_3) \quad (4.168)$$

This can be interpreted as follows. The left-hand side is the fully connected three-point function, which is diagramatically of the form:

$\mathcal{G}_c(x_1,x_2,x_3)$ =

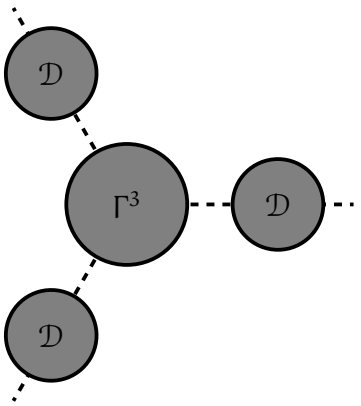


, (4.169)

i.e. the fully corrected 1PI vertex function  attached to the three copies of the fully corrected propagator .

The relation (4.168) can then be interpreted as:

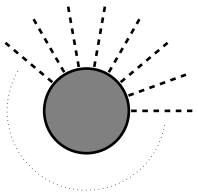
$\mathcal{G}_c(x_1, x_2, x_3)$ =



It shows than $\Gamma^3 = \delta^3\Gamma/(\delta J)^3$ is indeed, as claimed, the fully corrected 1PI 3-particle vertex function, which is the sum of the tree-level contribution (if any, for instance in ϕ^3 theory) and of all loop corrections.

The same holds for all n-point functions with $n \geqslant 3$:

$$\Gamma^n(x_1, \ldots, x_n) = \left.\frac{\delta^n\Gamma[\phi]}{\phi(x_1)\cdots\phi(x_n)}\right|_{\phi=0} = \qquad (4.170)$$



Quantum effective action. The generating functional $\Gamma[\phi]$ has a very nice interpretation. Let a consider a given connected n-point function in the full quantum theory. It can be computed using the *tree-level* Feynman rules by:

1. Replacing each n-point vertex (with $n \geqslant 3$) by the 1PI fully corrected vertex, *i.e.* the n-th functional derivative of $\Gamma[\phi]$.

2. Replacing each propagator by the fully corrected propagator $\mathcal{D}$.

We are lead to the following important conclusion.

Property 1 *Tree-level amplitudes computed with the functional* $\Gamma[\phi]$ *(rather than with the classical functional* $S[\phi]$*) give the fully quantum-corrected amplitudes.*

For this reason, it is appropriate to view $\Gamma[\phi]$ as the *quantum effective action* associated with the classical action $S[\phi]$. It can be computed order by order in perturbation theory.

*
* *

Unfortunately due to lack of time I have to stop here, but there would be much more to say!