Worldsheet aspects of heterotic compactifications with torsion

Fu-Yau compactifications from GLSMs

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★ *T-duality in gauged linear sigma-models with torsion*, D.I., arXiv:1306.6609, JHEP 1311, 093 (2013)

★ *One-loop corrections to heterotic torsional compactifications*, D.I. and M. Sarkis, work in progress

Introduction

Heterotic compactifications with torsion

- Compactifications with torsion, *i.e.* $H \neq 0$, difficult to deal with in supergravity
- Gauge bundle V, tangent bundle $T\mathcal{M}$ and H tied together by the Bianchi identity:

$$\mathrm{d}H = 2i\partial\bar{\partial}J = \frac{\alpha'}{4}\left(\mathrm{tr}\, R(\nabla_T)\wedge R(\nabla_T) - \mathrm{Tr}\,\mathcal{F}\wedge\mathcal{F}\right) + \mathcal{O}(\alpha'^2)$$

Choice of connection on $T\mathcal{M}$ in Bianchi

- In Bianchi: connection on $T\mathcal{M}$ with torsion
  - ★ $\mathrm{d}H$ should be a $(2,2)$ form ➥ choose Chern connection: $\nabla_T = \nabla_c$
  - ★ no $\mathcal{O}(\alpha')$ corrections to SUSY conditions ➥ $\nabla_T = \nabla_+ = \nabla(\omega + \frac{1}{2}H)$
  - ★ to satisfy e.o.m. at order $\mathcal{O}(\alpha')$: SUSY conditions, Bianchi and $R(\nabla_T)$ should be an $SU(3)$ instanton ➥ true for ∇_+ but at $\mathcal{O}(\alpha')$ only
- Bottom line: nearly impossible to get *exact* SUGRA solutions
- Moreover: Bianchi identity is non-linear in H ➥ no large-volume limit

➥ 2d worldsheet description more appropriate.

Important questions

- Are compactifications with torsion consistent beyond $\mathcal{O}(\alpha')$?
- What are their quantum symmetries?
- What are their moduli spaces?
- How to compute the four-dimensional effective action?

Prototypical example: Fu-Yau geometries ($\mathcal{N}=2$ in $d=4$)

Principal T^2 bundles over a $K3$ surface: $T^2 \hookrightarrow \mathcal{M}_6 \stackrel{\pi}{\to} \mathcal{S}$

History

- Duality from type IIB orientifolds with flux *(Dasgupta et al., 1999)*
- $SU(3)$-structure described by Goldstein and Prokushkin *(2002)*
- Fu & Yau: solution to Bianchi with Hermitian connection *(2006)*

- Choose $\rho_n \in H^2(\mathcal{S},\mathbb{Z}) \cap H^{1,1}(\mathcal{S})$, *i.e.* in $Pic(\mathcal{S})$, primitive ($J \wedge \rho_n = 0$)

★ Metric: $\boxed{ds^2 = e^{2A} ds^2(\mathcal{S}) + \frac{U_2}{T_2} \Theta\bar{\Theta}}$, $\Theta = dx + T dy + \mathfrak{w}^n \alpha_n$

with $d\Theta = 2\pi \mathfrak{w}^n \pi^\star \rho_n$, complex charges $\mathfrak{w}^n$ in lattice $\mathbb{Z} + T\mathbb{Z}$

- Torsion: $H = \star_{\mathcal{S}} de^{2A} - \frac{U_2}{T_2} \mathrm{Re}\left(\bar{\Theta} \wedge \star_{\mathcal{S}} d\Theta\right)$

★ Gauge bundle: pullback of HYM bundle on $\mathcal{S}$: $F^{0,2} = F^{2,0} = 0$, $J \lrcorner F = 0$

★ Bianchi identity: tadpole condition $\int_{\mathcal{S}} \left(\frac{U_2}{T_2} ||\mathfrak{w}^n \rho_n||^2 - \frac{1}{2} \mathrm{tr} F \wedge F \right) = 24$

Gauged linear sigma-model approach

- Worldsheet description? gauged linear sigma-models with torus bundles ➥ torsion GLSMs *(Adams, Ernedjberg, Lapan 2006)*
- Flow in the IR to non-linear sigma-models on Fu-Yau geometries

In this talk: two applications

- Exact T-duality transformations in torsional backgrounds, including topology-changing dualities
- Computation of the new-supersymmetric index through localization

Outline

1. (0,2) Gauged linear sigma-models with torsion
   - (0,2) superfields and Lagrangians
   - Torsion GLSM
2. T-duality
   - Perturbative dualities of Fu-Yau geometries
   - A GLSM proof of T-duality
   - Topology change
3. New supersymmetric index
   - Index for $K3 \times T^2$ compactifications
   - New index for Fu-Yau GLSMS: setting the stage
   - Computation through localization
   - Final result
   - Geometrical formula: a conjecture
4. Conclusions

(0,2) Gauged linear sigma-models with torsion

(0,2) superspace and superfields

Superspace coords $(x^{\pm}, \theta^+, \bar{\theta}^+)$: $D_+ = \partial_{\theta^+} - i\bar{\theta}^+\nabla_+$, $\bar{D}_+ = -\partial_{\bar{\theta}^+} + i\theta^+\nabla_+$

Superfields

- Chiral superfield (charged):

  $$\bar{D}_+\Phi = 0 \implies \Phi = \phi + \sqrt{2}\theta^+\psi_+ - i\theta^+\bar{\theta}^+\nabla_+\phi$$

- Fermi superfield: $\bar{D}_+\Gamma = 0 \implies \Gamma = \gamma_- + \sqrt{2}\theta^+ G - i\theta^+\bar{\theta}^+\nabla_+\gamma_-$
- Vector superfields (in WZ gauge):

  $$\mathcal{A} = \theta^+\bar{\theta}^+ A_+ \ , \quad \mathcal{V} = A_- - 2i\theta^+\bar{\mu}_- - 2i\bar{\theta}^+\mu_- + 2\theta^+\bar{\theta}^+ D$$

  ➥ Field-strength is chiral: $\Upsilon = \bar{D}_+(\partial_-\mathcal{A} + i\mathcal{V}_-)$

Lagrangian

$$L = -\frac{1}{2}\int \mathrm{d}^2\theta^+ \, \bar{\Phi}_I(\partial_- + iQ_I\mathcal{V})\Phi_I - \frac{1}{2}\int \mathrm{d}^2\theta^+ \, \bar{\Gamma}^a\Gamma^a - \frac{1}{8e^2}\int \mathrm{d}^2\theta^+ \, \bar{\Upsilon}\Upsilon$$
$$- \int \mathrm{d}\theta^+ \, \Gamma^a J_a(\Phi_I) + \frac{t}{4}\int \mathrm{d}\theta^+ \, \Upsilon + c.c.$$

Calabi-Yau non-linear sigma-model from the GLSM

Vacuum: CY hypersurface with holomorphic gauge bundle V

- K3 example: $\mathcal{L}_0 = \Gamma_0 J^0(\Phi_I) = \Gamma_0(\Phi_1^4 + \cdots + \Phi_4^4)$ with $Q_i = 1$
  - ★ D-term: $\sum_I Q_I |\phi_I|^2 - \mathrm{Im}\,(t) = 0$ ➥ $\mathbb{CP}^3$ of Kähler modulus t
  - ★ F-term: $J^0(\phi_I) = 0$ in $\mathbb{CP}^3$ ➥ Fermat quartic
- Gauge bundle: $\mathcal{L}_G = \Gamma^m \, P G_m(\Phi_I)$
  - ★ In CY regime $\langle p \rangle = 0$ ➥ Yukawa couplings $G_m(\phi_I)\gamma_-^m \psi_+^P$
  - ★ Massless left-moving fermions: sections of the monad bundle

$$0 \longrightarrow V \longrightarrow \oplus \mathcal{O}(q_m) \xrightarrow{\otimes G_m} \mathcal{O}(Q_P) \longrightarrow 0$$

Anomalies

- Gauge anomaly: $\boxed{\delta_\Xi L_{eff} = \frac{\mathfrak{A}}{8} \int \mathrm{d}\theta^+ \, \Xi \, \Upsilon \, + c.c.}$, with $\mathfrak{A} = Q_i Q^i - q_n q^n$
- A GLSM flowing to a $(0,2)$ heterotic NLSM should have $\mathfrak{A} = 0$ and
  - ★ Global right-moving $U(1)_R$ ➥ IR $\mathcal{N} = 2$ superconformal algebra
  - ★ Global left-moving $U(1)_L$ ➥ GSO projection with $\mathbb{Z}_2 \subset U(1)_L$

Torsion multiplet

Gauge anomaly $\mathfrak{A} = Q_i Q^i - q_n q^n$ is a measure of $ch_2(V) - ch_2(T)$
➥ solutions with $\mathrm{d}H \neq 0$, *i.e.* torsion?

The idea

- Hint: Fayet-Iliopoulos term $\frac{t}{4}\int \mathrm{d}\theta^+\, \Upsilon$
  ➥ $\mathrm{Re}(t)$ is a constant B-field in target space
- Field strength Υ is chiral (unlike $(2,2)$ susy)

  ➥ Field-dependent FI term (axial coupling):

$$-\frac{ih}{4}\int \Omega\Upsilon + c.c., \text{ with } h \in \mathbb{Z} \text{ and } \Omega \sim \Omega + 2i\pi$$

- Corresponds to a non-trivial B-field
- D-term ➥ $\mathbb{C}^\star$ fibration over the 'base GLSM'

★ If shift-symmetry of Ω gauged ➥ classical non-gauge invariance!

$$L_T = -\frac{i}{4}\int \mathrm{d}^2\theta \left(\Omega + \bar{\Omega} + 2\mathfrak{w}\mathcal{A}\right)\left(\partial_-(\Omega - \bar{\Omega}) + 2i\mathfrak{w}\mathcal{V}\right) - \frac{ih}{4}\int \mathrm{d}\theta^+\, \Omega\Upsilon + c.c.$$

- Gauge transformation: $\delta_\Xi \Omega = i\mathfrak{w}\Xi \implies \delta L_T = \frac{h\mathfrak{w}}{4}\int \mathrm{d}\theta^+\, \Xi\Upsilon$

Torsion GLSMs

Generic case ➥ T^2 of moduli (T, U) & charge $\mathfrak{w} = \mathfrak{w}_1 + T\mathfrak{w}_2$ *(D.I. 2013)*

$$L_T = -\frac{iU_2}{8T_2}\int d^2\theta\, (\Omega_1 + \bar{\Omega}_1 + T_1(\Omega_2 + \bar{\Omega}_2) + 2(\mathfrak{w}_1 + T_1\mathfrak{w}_2)\mathcal{A})\, (\partial_-(\Omega_1 - \bar{\Omega}_1 + T_1(\Omega_2 - \bar{\Omega}_2)) + 2i(\mathfrak{w}_1 + T_1\mathfrak{w}_2)\mathcal{V})$$
$$- \frac{i}{8}U_2 T_2 \int d^2\theta\, (\Omega_2 + \bar{\Omega}_2 + 2\mathfrak{w}_2\mathcal{A})\, (\partial_-(\Omega_2 - \bar{\Omega}_2) + 2i\mathfrak{w}_2\mathcal{V}) - \frac{ih^\ell}{4}\int d\theta^+\, \Upsilon\, \Omega_\ell + c.c.$$
$$+ \frac{i}{8}U_1 \int d^2\theta \Big[(\Omega_1 + \bar{\Omega}_1 + 2\mathfrak{w}_1\mathcal{A})\, (\partial_-(\Omega_2 - \bar{\Omega}_2) + 2i\mathfrak{w}_2\mathcal{V}) - (\Omega_2 + \bar{\Omega}_2 + 2\mathfrak{w}_2\mathcal{A})\, (\partial_-(\Omega_1 - \bar{\Omega}_1) + 2i\mathfrak{w}_1\mathcal{V}) \Big]$$

Compact models

- So far $(\mathbb{C}^\star)^2$ fibrations over a $K3$ base
- Decoupling of $\mathrm{Re}(\omega_\ell)$? ➥ both kinetic and FI Lagrangians contain 'dangerous' terms in $D\,\mathrm{Re}\,(\omega_\ell) + \frac{i}{\sqrt{2}}\mu_-\psi_{+,\ell} + cc$
- If the coefficient in front vanishes ➥ decoupled radial multiplet and <u>torsion multiplet</u>

Coupling to gauge superfields becomes chiral:

$$\frac{U_2}{T_2}\int d^2\theta^+ \left[|\mathfrak{m}|^2 \mathcal{A}\mathcal{V} - \frac{i}{2}\mathcal{A} \underbrace{\left(\mathrm{Re}(\mathfrak{m})\partial_-(\Omega_1 - \bar{\Omega}_1) + \mathrm{Re}(T^\star\mathfrak{m})\partial_-(\Omega_2 - \bar{\Omega}_2) \right)}_{\text{left current}} \right] + t.d.$$

Torus moduli stabilization and tadpole conditions

Moduli quantization

- Decoupling of radial multiplet if:

$$\begin{cases} \frac{U_2}{T_2}\mathrm{Re}(\mathfrak{w}) - U_1\mathfrak{w}_2 + h^1 = 0\,, \\ \frac{U_2}{T_2}\mathrm{Re}(T^\star\mathfrak{w}) + U_1\mathfrak{w}_1 + h^2 = 0\,. \end{cases}$$

- Quantization conditions from gauge instantons: $h^\ell \in \mathbb{Z}$.
  ➥ one-dimensional subspace of T^2 moduli space

Next section: T-dualities give further moduli quantization conditions.

Anomaly cancellation and tadpole condition

- Using quantization conds, the gauge anomaly is canceled by the torsion multiplet if $\boxed{Q_i Q^i - q_n q^n = \frac{2U_2}{T_2}|\mathfrak{w}|^2\,,}$
- With the relevant choice of charges for chiral and Fermi multiplets:
  ➥ integrated Bianchi identity of Fu-Yau compactifications

T-duality

Perturbative dualities of Fu-Yau geometries

- T-dualities: *exact* symmetries of toroidal compactifications
- Generalization to non-trivial backgrounds ?
  - ★ Exchanging metric and B-field components
  - ★ Mixing of gauge bundle and metric for heterotic solutions

T-dualities of Fu-Yau solutions in supergravity

- T-dualities of T^2 fibrations with H-flux *(Bouwknegt, Evslin, Mathai 2004)*
- Topology changing dualities (gauge ↔ torus bundles) *(Evslin & Minasian, 2009)*

- As Fu-Yau solutions have no large (torus) volume limit, are these statements reliable?
- Are there global obstructions?
- Corrections by worldsheet instantons?

A GLSM proof of T-duality

- T-dualities proven *exactly* starting from a torsion GLSM *(D.I. 2013)*
- Adapt Rocek & Verlinde quotient method *(1992)*

Extra ingredients

- An extra pair of vector superfields $(\hat{\mathcal{A}}, \hat{\mathcal{V}})$, with no kinetic terms
- A Lagrange multiplier chiral superfield Δ to enforce $\hat{\Upsilon} = 0$
- Minimal couplings of Ω_ℓ shift-symmetry and/or axial couplings

Example: $\mathcal{G}_s$ duality ($s \in \mathbb{Z}$)

$$\mathcal{L} = -\frac{iR^2}{8}\left(\Omega_1 + \bar{\Omega}_1 + 2\mathfrak{w}_1\mathcal{A} + 2\hat{\mathcal{A}}\right)\left(\partial_-(\Omega_1 - \bar{\Omega}_1) + 2i\mathfrak{w}_1\mathcal{V} + 2i\hat{\mathcal{V}}\right)$$
$$+ \cdots + \frac{h_1}{4}\left[\mathcal{V}(\Omega_1 + \bar{\Omega}_1 + 2\hat{\mathcal{A}}) + i\mathcal{A}\left(\partial_-(\Omega_1 - \bar{\Omega}_1) + 2i\hat{\mathcal{V}}\right)\right]$$
$$+\frac{1}{4}\left(\hat{\mathcal{V}}(\Delta + \bar{\Delta}) + i\hat{\mathcal{A}}\partial_-(\Delta - \bar{\Delta})\right) + \frac{s}{4}\left(\hat{\mathcal{V}}(\Omega_2 + \bar{\Omega}_2) + i\hat{\mathcal{A}}\partial_-(\Omega_2 - \bar{\Omega}_2)\right)$$

- Gauge variation w.r.t. 'unhatted' gauge sym.: $\frac{1}{4}(h^1 + s\mathfrak{w}_2)\int d\theta^+ \Xi \hat{\Upsilon}$
  ➥ Δ should have a non-zero shift-charge

Duality

- Integrating out Δ ➥ original theory as $\hat{\Upsilon} = 0$ is enforced
- Integrating out $(\hat{\mathcal{A}}, \hat{\mathcal{V}})$ ➥ T-dual model with Δ as dual coordinate

- Dual model is a torsion GLSM for a Fu-Yau geometry
- $\mathcal{G}_s$ duality: $\begin{cases} U & \mapsto \tilde{U} = T \\ T & \mapsto \tilde{T} = U + s \end{cases}, \begin{cases} \mathrm{Re}(\tilde{\mathfrak{w}}) = \frac{U_2}{T_2}\mathrm{Re}(\mathfrak{w}), \\ \mathrm{Re}\left((\overline{U}+s)\tilde{\mathfrak{w}}\right) = \frac{U_2}{T_2}\mathrm{Re}\left((\overline{U}+s)\mathfrak{w}\right) \end{cases}$
- Duality along Ω_2:

  $$\mathcal{I} \ : \quad \begin{cases} U & \mapsto \tilde{U} = -1/T \\ T & \mapsto \tilde{T} = U \end{cases} \quad \text{and} \quad \begin{cases} \mathrm{Re}(\tilde{\mathfrak{w}}) = \frac{U_2}{T_2}\mathrm{Re}(\tilde{T}\mathfrak{w}), \\ \mathrm{Re}\left(\overline{U}\tilde{\mathfrak{w}}\right) = -\frac{U_2}{T_2}\mathrm{Re}\left((\overline{TU}\mathfrak{w}\right) \end{cases}$$
- $\mathcal{G}_0$, $\mathcal{G}_1$ and $\mathcal{I}$ generate the full duality group (see next slide)
- Not corrected by worldsheet instantons unlike Hori-Vafa mirror symmetry as we gauge a *shift charge*: instead of susy QED with massless flavor, admitting vortex solutions, massive $U(1)$ gauge theory ➥ dualities are exact

Perturbative toroidal dualities

Duality group

$$O(2,2;\mathbb{Z}) = \underbrace{PSL(2;\mathbb{Z})_T}_{T\to\frac{aT+b}{cT+d}} \times \underbrace{PSL(2;\mathbb{Z})_U}_{U\to\frac{\alpha U+\beta}{\gamma U+\delta}} \times \underbrace{\mathbb{Z}_2}_{\text{T-duality along x}} \times \underbrace{\mathbb{Z}_2}_{\text{parity}}$$

- $(\mathfrak{w}_1, \mathfrak{w}_2)$ transforms as a doublet of $PSL(2;\mathbb{Z})_T$.
- Under a $U \to -1/U$ duality, $\mathfrak{w} \mapsto -\bar{U}\mathfrak{w}$
- Elliptic curve $E_T = \mathbb{C}/(\mathbb{Z} + T\mathbb{Z})$ should admit a non-trivial endomorphism *(complex multiplication)* $\phi : \begin{cases} E_T & \to & E_T \\ z & \mapsto & -\bar{U}z \end{cases}$

Moduli quantization

T and U valued in an *imaginary quadratic number field* $\mathbb{Q}(\sqrt{D})$:

$$T \in \mathbb{Q} + \sqrt{D}\,\mathbb{Q} \quad \text{with} \qquad D = b^2 - 4ac < 0 \quad , \qquad a,b,c \in \mathbb{Z}\,.$$

- Precisely the conditions for a $c = 2$ CFT in 2d with a T^2 target space to be rational *(Gukov & Vafa 2002)*

Heterotic dualities

- Heterotic compactifications on T^2 have a larger $O(2,2,\mathbb{Z})$ duality group ➥ in the present context, mixing of torus and gauge bundles
- Difficulity: mixing of quantum anomalies and gauge-variant terms

Recipe

- Free Fermi multiplet Γ of charge q ➥ section of $\mathcal{O}(q)$ over $\mathcal{S}$
- Bosonize the left-moving fermion γ_-
  ➥ left chiral boson
- Embbed into chiral superfield B with gauged shift symmetry, of shift-charge q (at fermionic radius $R = 1/\sqrt{2}$)
  ➥ anomaly becomes classical $\mathcal{L} = \frac{iq}{8} \int \mathrm{d}\theta^+ \, B\Upsilon + c.c.$
- Consider 'auxiliary' 2-torus with B and *e.g.* Ω_1
  ➥ auxiliary torsion multiplet Θ of moduli (u, t)

Topology change

Torus/gauge bundles duality

- $u \to -1/u$ duality exchanges shift charge $\mathfrak{w}_1$ and Fermi multiplet charge q
  ➥ Duality between compactifications with different topologies:
  $K3 \times S^1$ with an Abelian gauge bundle and $S^1 \hookrightarrow \mathcal{M}_5 \to K3$
- Non-orthogonal torii: duality if $U = 2T$ originally
  ➥ $T_1 \neq 0$ gives after duality a line bundle over the total space $\mathcal{M}_6$:
  $-\frac{i}{4} T_1(\Omega_2 + \bar{\Omega}_2 + 2\mathfrak{w}_2 \mathcal{A})\, \partial_-(B - \bar{B})$ (generalized Wilson line)

- Warning: did not take into account the spin structures (left GSO)
  ➥ communicated through duality as large gauge tranformations

★ Should study carefully the global charges of the model

New supersymmetric index

New supersymmetric index for $K3 \times T^2$ compactifications

- Defined as $Z_{new}(\tau) = \frac{1}{\eta^2}\mathrm{Tr}_{\mathrm{R}}\left[J_R^0(-1)^{F_R} q^{L_0 - c/24}\bar{q}^{\bar{L}_0-\bar{c}/24}\right]$

  ➥ trace over Hilbert space of $(c,\bar{c}) = (22,9)$ CFT for $K3 \times T^2$
- Counts the BPS states of $K3 \times T^2$ compactifications *(Harvey, Moore 1995)*

$$\frac{i}{2} Z_{new} = \sum_{\mathrm{BPS\ vectors}} q^{\Delta}\bar{q}^{\bar{\Delta}} - \sum_{\mathrm{BPS\ hypers}} q^{\Delta}\bar{q}^{\bar{\Delta}}$$

- Building block of threshold corrections to gauge/grav. couplings

New susy index and $K3$ elliptic genus

- $K3$ elliptic genus: $Z_{K3}(\tau, y) = \mathrm{Tr}_{\mathrm{RR}}\left[e^{2i\pi y J_L^0}(-1)^F q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24}\right]$
- Consider a bundle $V \subset SO(2n) \subset E_8$ and a T^2 lattice $\Gamma_{2,2}$:

$$Z_{new}(\tau) = \frac{1}{2i\eta^{12}}\Gamma_{2,2}\, E_4(\tau,0) \sum_{\gamma,\delta=0}^{1} q^{\gamma^2}\left(\frac{\theta_1(\tau, \frac{\gamma\tau+\delta}{2})}{\eta}\right)^{8-n} Z_{K3}\left(\tau, \frac{\gamma\tau+\delta}{2}\right)$$

★ Fixed by modular properties ➥ universality of thresholds *(Kiritis et al., 1997)*

★ In particular, *Matthieu moonshine* for *any* gauge bundle *(Cheng et al. 2013)*

New index for Fu-Yau GLSMS: setting the stage

- Right fermions ψ_+, $\bar{\psi}_+$ in the torus factor free
- We split $J^0_R = \bar{J}^{\mathrm{K3}}_{0,R} + \bar{J}^{tor}_{0,R}$ with $\bar{J}^{tor}_R \sim \bar{\psi}_+ \psi_+ + \cdots$
- We have $\mathrm{Tr}_{\mathrm{R}} \left[\bar{J}^{\mathrm{K3}}_{0,R} (-1)^{F_R} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24} \right] = 0$, since:
  - ★ Pair of fermionic zero modes of opposite F_R in T^2 factor
  - ★ The $K3$ SCFT has $\mathcal{N} = (0,4)$ superconformal symmetry

Left GSO projection

Assume the existence of a $\mathbb{Z}_2 \subset U(1)_L$ left symmetry. Then

$$Z_{new} = \frac{E_4(\tau,0)}{2\eta^{10}} \sum_{\gamma,\delta=0}^{1} q^{\gamma^2} \mathrm{Tr}_{\mathrm{RR}} \left[e^{i\pi J^L_0(\gamma\tau+\delta)} \bar{J}^{tor}_0 (-1)^{F_R} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24} \right]$$

where the trace is over the Hilbert space of $K3 \times T^2$ with the free fermions from the first E_8 factor

Supersymmetric localization I: the idea

- Choose a supercharge $\mathcal{Q}$ of your favorite SUSY QFT
- Aim : compute $\langle \mathcal{O} \rangle$ with $\mathcal{Q}\mathcal{O} = 0$
- Find V such that (i) $\mathcal{Q}^2 V = 0$ and (ii) $\mathcal{Q}V|_{bos}$ positive definite
- Deformed path integral

$$\langle \mathcal{O} \rangle_t = \int \mathcal{D}\Phi \, \mathcal{O}(\Phi) \, e^{-S[\phi] - t\mathcal{Q}V}$$

★ As $\partial_t \langle \mathcal{O} \rangle_t = 0$ ➥ compute in the $t \to \infty$ limit *(Pestun, 2012)*

- Exact result: one-loop fluctuations around the saddle points $\mathcal{Q}V(\Phi_0) = 0$

$$Z = \int_{\Phi_0, \mathcal{Q}V(\Phi_0)=0} \mathrm{d}\Phi_0 \, e^{-S[\Phi_0]} \, \mathcal{O}_{class}[\Phi_0] \, Z_{one-loop}$$

with $\Phi = \Phi_0 + \frac{1}{\sqrt{t}} \delta\Phi$ ➥ quadratic fluctuations only

GLSM elliptic genus through localization

Benini et al., 2013

- Supercharge $\mathcal{Q} = Q^+ + \bar{Q}^+$ ➥ *whole* GLSM action is $\mathcal{Q}$-exact
- For instance $\mathcal{S}_{c.m.} = \int \mathrm{d}^2x\, \mathcal{Q}(2 - \phi\nabla_w\psi_+ - i\bar{\phi}_+\mu\phi) + t.d. := \mathcal{Q}\, V_{c.m.}$

$$\mathcal{Z}(e,f,g) = \int \mathcal{D}[A,\mu,D] e^{-\frac{1}{e^2}\mathcal{Q}\, V_{v.m.}} \int \mathcal{D}[\phi^i,\psi^i] e^{-\frac{1}{g^2}\mathcal{Q}\, V_{c.m.}} \int \mathcal{D}[\gamma^m] e^{-\frac{1}{f^2}\mathcal{Q}\, V_{f.m.}}$$

- Path integral independent of e, f, g ➥ take $e, f, g \to 0$ limit
- Integral localizes on BPS configurations, *i.e.* annihilated by $\mathcal{Q}$: saddle points of the Euclidean (free) action

Localization locus

- Flat gauge connection on the worldsheet torus $A_G = \frac{\bar{u}}{2i\tau_2}\mathrm{d}w - \frac{u}{2i\tau_2}\mathrm{d}\bar{w}$
- Gaugino zero-modes μ_0, $\bar{\mu}_0$ and constant D-term D_0

★In addition, background gauge field for $U(1)_L$: $A_L = \frac{\bar{y}}{2i\tau_2}\mathrm{d}w - \frac{y}{2i\tau_2}\mathrm{d}\bar{w}$

$$Z = \lim_{e\to 0} \int \mathrm{d}^2u \int_{\mathbb{R}} \mathrm{d}D_0\, e^{-\frac{D_0^2}{2e^2}} \int \mathrm{d}\mu_0\, \mathrm{d}\bar{\mu}_0\; Z^{free}_{\text{one-loop}}(D_0,\mu_0,\bar{\mu}_0,u,y)$$

Localization for the torsion GLSM

- Fermions ψ_+, $\bar{\psi}_+$ in torsion multiplet have a pair of R zero modes
  ➥ as for $K3 \times T^2$, $Z_{new} \sim \text{Tr}\left[\bar{J}_0^{tor} \cdots\right]$
- Remark: NLSMs for Fu-Yau have $(0,2) \oplus (0,4)$ *(Melnikov, Minasian 2012)*
- Hence we consider a path integral:

$$\mathcal{Z} = \int \mathcal{D}[A_\pm, \mu_-, D] e^{-\frac{1}{e^2}\mathcal{Q}\, V_{v.m.}} \int \mathcal{D}[\phi^i, \psi^i] e^{-\frac{1}{g^2}\mathcal{Q}\, V_{c.m.}} \times \int \mathcal{D}[\gamma^m] e^{-\frac{1}{f^2}\mathcal{Q}\, V_{f.m.}} \int \mathcal{D}[\omega, \psi] \left(\int \mathrm{d}x\; \bar{\psi}_+ \psi_+\right) e^{-S[\omega,\psi,A_\pm]}$$

★ The operator $\int \mathrm{d}x\, \bar{\psi}_+ \psi_+$ is <u>not</u> invariant under $\mathcal{Q}$
➥ why should localization work?

- Moreover we DO expect that the index depends on the T^2 lattice sum
- Hint: At the fermionic radius, related by T-duality to a case where it <u>does</u> work (Abelian gauge bundle over $K3$)

SUSY localization and torsion multiplet

- Torsion multiplet:
  - ★ no couplings to D and μ_+, $\bar{\mu}_+$ (gauginos),
  - ★ right-moving fermions ψ_+, $\bar{\psi}_+$ gauge neutral
- Integral over torsion multiplet can be done exactly ➥ gaussian
- As ψ_+, $\bar{\psi}_+$ gauge-neutral, $e^{-W[A]} = \int \mathcal{D}[\omega, \psi] e^{-S[\omega,\psi,A]}$ does not depend on $\int \mathrm{d}x\, \bar{\psi}_+ \psi_+$ insertion, up to a constant
- Supersymmetric (but gauge-variant) functional in $\mathcal{A}$ and $\mathcal{V}$?
- Then the remaining path integral could be done using supersymmetric localization, *i.e.* in the free-field limit $e, f, g \to 0$

Slightly more complicated:

- Decoupling of fermions ψ_+, $\bar{\psi}_+$ of the torsion multiplet obtained in the Wess-Zumino gauge
- In this gauge, SUSY variation $\delta_\epsilon W[A] \sim A_+ (\epsilon \bar{\mu}_- + \bar{\epsilon} \mu_-)$
- Going back to the WZ gauge by a supergauge transformation:
  ➥ SUSY variation of W precisely cancelled by the anomalous gauge variation of the 'base' GLSM measure

Localization II: one-loop fluctuations

Chiral multiplet

- Determinant: $Z_c(u,\bar{u},y,D_0) = \prod_{m,n} \frac{m+n+\bar{\tau}+Q\bar{u}+L\bar{y}}{|m+n+\tau+Qu+Ly|^2+iQD_0}$
- In the limit $D_0 \to 0$, it gives
  $Z_c(u,\bar{u},y,0) = e^{\frac{\pi}{\tau_2}(\mathrm{Im}\,(Qu+Ly))^2} \frac{i\eta(\tau)}{\theta_1(\tau,Qu+Ly)}$
- To saturate the gaugino zero-modes: correlator

$$\left\langle \int \mathrm{d}^2x\ \mu Q\bar{\psi}\phi \int \mathrm{d}^2x\ \bar{\mu} Q\psi\bar{\phi} \right\rangle_{free}$$

Final result:

$$-\frac{i}{\pi} Z_c \sum_{m,n} \frac{Q^2}{(|m+n+\tau+Qu+Ly|^2+iQD_0)(m+n\bar{\tau}+Q\bar{u}+L\bar{y})} = -\frac{1}{\pi D_0}\frac{\partial}{\partial \bar{u}} Z_c(u,\bar{u},y,D_0)$$

➥ Poles in the $D_0 \to 0$ limit whenever

$$u \in \mathfrak{M}_{sing} = \{u,\, Q_i u + L_i y = 0 \mod \mathbb{Z}+\tau\mathbb{Z}\} := \underbrace{\mathfrak{M}^+_{sing}}_{\text{all } Q_i>0} \cup \underbrace{\mathfrak{M}^-_{sing}}_{\text{all } Q_i<0}$$

Fermi multiplets

Determinant of a Weyl fermion with twisted boundary conditions:

$$Z_{f,Q,L} = \det\left(\bar{\partial} + \frac{Qu+Ly}{2\tau_2}\right) = e^{-\frac{\pi}{\tau_2}(\mathrm{Im}\,(Qu+Ly))^2} \frac{i\theta_1(\tau,Qu+Ly)}{\eta(\tau)}$$

Abelian vector multiplet

Setting aside the zero modes μ_0, $\bar{\mu}_0$ and D_0 one gets simply:

$$Z'_v = \eta(\tau)^2$$

➥ Remark: taking into account gauge-fixing and the ghost sector does not bring modifications to the results

Torsion multiplet

- Crucial: no couplings to D_0 and gaugino zero-modes μ_0, $\bar{\mu}_0$
- Path integral over ψ_+, $\bar{\psi}_+$ gives $\frac{1}{2i\pi} \frac{\partial}{\partial\theta}\Big|_{\theta=0} \frac{\theta_1(\bar{q}, e^{2i\pi\theta})}{\eta(\bar{q})} = -i\eta(\bar{q})^2$
- For simplicity: orthogonal torus without B-field ($U_1 = T_1 = 0$)
- Bosonic fields: two terms like

$$\boxed{S = \frac{R^2}{4\pi}\int d^2w \left(\partial\theta\bar{\partial}\theta - 2A_w\bar{\partial}\theta + A_w A_{\bar{w}}\right)}$$

with flat connection $A = \mathfrak{w}A_G + \lambda A_L$.

- For $R = R_f$: bosonization of the chiral Schwinger model (Dirac fermion with chiral gauge coupling) *(Jackiw,Rajaraman 1984)*

➥ Coeff. of $A_w A_{\bar{w}}$ regularization-dependent (bosonization ambiguities)

- Appears also in the context of holomorphic factorization *(Witten 1991)*
- Determinant for a chiral boson: holomorphic square root of a non-chiral determinant (as with Dirac operator)

Contribution of S^1 fibers

★ At fermionic radius: sum over spin structures

$$Z = \frac{1}{\eta\bar{\eta}} e^{-\frac{\pi}{\tau_2}(\text{Im}\,(\mathfrak{w}u + \mathfrak{l}y))^2} \frac{1}{2} \sum_{\gamma,\delta=0}^{1} \vartheta\begin{bmatrix}\gamma\\ \delta\end{bmatrix}(\tau, \mathfrak{w}u + \mathfrak{l}y)\vartheta\begin{bmatrix}\gamma\\ \delta\end{bmatrix}(\bar{\tau}, 0)$$

- In terms of theta-functions at level two ➥ easier to generalize:

$$Z = \frac{1}{\eta\bar{\eta}} e^{-\frac{\pi}{\tau_2}(\text{Im}\,(\mathfrak{w}u + \mathfrak{l}y))^2} \sum_{m\in\mathbb{Z}_4} \Theta_{m,2}(\tau, \mathfrak{w}u + \mathfrak{l}y)\Theta_{m,2}(\bar{\tau}, 0)$$

★ At rational (radius)^2: $R = \sqrt{r/s}$ with r, s coprime

★Zero-modes path integral followed by Poisson resummation

★Holomorphic square root gives

$$Z = \frac{1}{\eta\bar{\eta}} e^{-\frac{2\pi R^2}{\tau_2}(\text{Im}\,(\mathfrak{w}u + \mathfrak{l}y))^2} \sum_{\substack{m+\bar{m}\equiv 0 \mod s\\ m-\bar{m}\equiv 0 \mod r}} \Theta_{m,rs}\left(\tau, \tfrac{2}{s}(\mathfrak{w}u + \mathfrak{l}y)\right)\Theta_{\bar{m},rs}(\bar{\tau}, 0)$$

➥ Remark: invariant under T-duality: $R \mapsto 1/R$, $\mathfrak{w} \mapsto R^2\mathfrak{w}$

Localization III: putting everything together

- Last steps as for a standard GLSM *(Benini et al., 2013)*

  ➥ assume no gauge/global anomalies

- Poles at $D_0 = \frac{i}{Q_i}|m + n\tau + Q_i u + L_i y|^2$ approach real axis as $u \to u_\star \in \mathfrak{M}_{sing}$
- Step 1: excise singularities ➥ cut disks Δ_ϵ around $u_\star \in \mathfrak{M}_{sing}$
- Step 2: move D_0 contour upwards (without hitting poles)
- Step 3: path integral gives $\frac{\partial}{\partial \bar{u}} Z_{c.m.}(u, \bar{u}, D_0) \times (\mathit{holomorphic})$

  ➥ $\int \mathrm{d}^2 u \;\longrightarrow$ contour integral over the boundaries $\oint_{\partial\Delta_\epsilon} \mathrm{d}u$

$$Z = \lim_{e,\epsilon \to 0} \int_{\Gamma_+} \frac{\mathrm{d}D_0}{2i\pi D_0} e^{-\frac{D_0^2}{2e^2} - itD_0} \oint_{\partial\Delta_\epsilon} \mathrm{d}u \; f(u, D_0)$$

- Step 4: No poles from the lower half-plane in $\epsilon \to 0$ limit
- Step 5: Each $u_\star \in \mathfrak{M}^+_{sing}$ gives a residue in the $e, \epsilon \to 0$ limit

$$Z = - \sum_{u_j \in \mathfrak{M}^+_{sing}} \oint_{u = u_j} \mathrm{d}u \; f(u, 0)$$

Final result

One gets finally for the path integral:

$$Z(\tau,\bar\tau,y) = \mathrm{Tr}_{\mathrm{RR}}\left[\bar J_0^{tor}(-1)^{F_R} e^{2i\pi y J_0^L} q^{L_0-c/24}\bar q^{\bar L_0-\bar c/24}\right]$$

$$= \sum_{u_j\in\mathfrak{M}^+_{sing}} i\oint_{u_j} \mathrm{d}u \prod_i \frac{i\eta(\tau)}{\theta_1(\tau, Q_i u + L_i y)} \prod_n \frac{i\theta_1(\tau, q_n u + L_n y)}{\eta(\tau)} \times$$

$$\times \prod_{\ell=1}^{2} \sum_{m_\ell,\bar m_\ell} \Theta_{m_\ell, r_\ell s_\ell}\left(\tau, \frac{2}{s_\ell}(\mathfrak{w}_\ell u + \mathfrak{l}_\ell y)\right) \Theta_{\bar m_\ell, r_\ell s_\ell}(\bar\tau, 0)$$

In terms of which the new supersymmetric index reads

$$Z_{new}(\tau,\bar\tau) = \frac{E_4(\tau,0)}{2\eta^{10}} \sum_{\gamma,\delta=0}^{1} q^{\gamma^2} Z\left(\tau,\bar\tau,\frac{\gamma\tau+\delta}{2}\right) .$$

Geometrical formula: a conjecture

- Geometrical formulation of the $(0,2)$ elliptic genus *(Kawai,Mohri 1994)*
- Generalization to Fu-Yau:

★ Using the splitting principle, factorize the total Chern classes of the gauge and tangent bundles

$$c(V) = \prod_{k=1}^{r} (1+v_k) \text{ and } c(T) = \prod_{j=1}^{D} (1+\xi_j)$$

★ Principal T^2 bundle ➥ set of a.s.d. $(1,1)$-forms ω_a on the base $\mathcal{S}$

$$Z_{new} = \frac{E_4(\tau,0)}{2\eta^{10}} \sum_{\gamma,\delta=0}^{1} q^{\gamma^2} \left(\frac{\theta_1(\tau, \frac{\gamma\tau+\delta}{2})}{\eta} \right)^{8-n} \times$$

$$\int_{\mathcal{S}} \Bigg(\prod_{k=1}^{n} \frac{\theta_1(\tau, v_k + \frac{\gamma\tau+\delta}{2})}{\eta} \prod_{j=1}^{2} \frac{\eta}{\theta_1(\tau,-\xi_j)} \, \xi_j$$

$$\prod_{\ell} \sum_{m_\ell,\bar{m}_\ell} \Theta_{m_\ell, r_\ell s_\ell} \left(\tau, \frac{2}{s_\ell} (\mathfrak{w}^a_\ell \omega_a + \frac{\gamma\tau+\delta}{2}) \right) \Theta_{\bar{m}_\ell, r_\ell s_\ell}(\bar{\tau}, 0) \Bigg)$$

➥ Remark: in Fu-Yau geometries $c(T\mathcal{M}_6) = \pi^\star c(T\mathcal{S})$

Conclusions

Summary

- GLSMs provide useful tools to probe compactifications with torsion
- Applications to a broad class of $\mathcal{N} = 2$ heterotic compactifications
  ➥ Fu-Yau geometries
- Exact statements about T-duality along torus bundles
  ➥ no instanton corrections
- Topology changing dualities between torsional and non-torsional models
- Computation of the new supersymmetric index using supersymmetric localization
  ➥ unlike $K3 \times T^2$, seemingly non-universal result

To-do list

- Generalization to non-orthogonal torii with B-field (rational $c = 2$ theory ➥ hol. factorization)
- Consider explicit examples of index computations
  ➥ SUSY vs. non-SUSY ? (self- vs. anti-self-dual two forms ω_n)
- Invariance of the new SUSY index under (topology changing) dualities
- Is the moonshine phenomenon visible?
  ➥ interesting case of study, as non-universal
- Generalization to higher rank worldsheet gauge groups
  ➥ Jeffrey-Kirwan residues *(Benini et al. II, 2013)*
- Explicit computation of threshold corrections
  ➥ Modular integrals using Niebur-Poincaré series (unfolding unappropriate) *(Angelantonj, Florakis, Pioline 2011)*