One-loop corrections to heterotic flux compactifications

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★ *New supersymmetric index of heterotic compactifications with torsion*, D.I. and Matthieu Sarkis, JHEP 1512, 069 (2015)

★ *One-loop corrections to heterotic flux compactifications*, C. Angelantonj, D.I., M. Sarkis, to appear

Introduction

Heterotic compactifications with torsion

Supersymmetric heterotic compactifications

- Killing spinor eqn. : $\nabla(\omega - \frac{1}{2}\mathcal{H})\,\epsilon = 0 + \mathcal{O}(\alpha'^2)$
- Gauge, tangent bundles and 3-form $\mathcal{H}$ tied together by the Bianchi identity:

$$\mathrm{d}\mathcal{H} = \frac{\alpha'}{4}\left(\mathrm{tr}\, R(\nabla_T) \wedge R(\nabla_T) - \mathrm{Tr}\, \mathcal{F} \wedge \mathcal{F}\right) + \mathcal{O}(\alpha'^2)$$

Non-Kähler geometry

★ *General* SUSY compactification: $\begin{cases} \text{Complex manifold } \mathcal{M} \\ + \text{ Holomorphic vector bundle } \mathcal{V} \end{cases}$

★ 3-form flux $\mathcal{H} \leftrightarrow$ Non-Kähler manifold $\mathcal{H} = i(\bar{\partial} - \partial)J \neq 0$

Supergravity limit of flux compactifications?

- In Bianchi: connection with torsion, *e.g.* $\nabla_T = \nabla^+ = \nabla(\omega + \frac{1}{2}\mathcal{H})$

★ If $\mathcal{H} \neq 0$ at leading order ➥ no large-volume limit

➥ 2d worldsheet description more appropriate. Fortunately, no RR fluxes!

➥ general framework: $(0,2)$ superconformal theories in 2d

Important questions

- Are compactifications with torsion consistent beyond $\mathcal{O}(\alpha')$?
- What are their quantum symmetries?
- What are their moduli spaces?
- How to compute the four-dimensional effective action?

Gauged linear sigma-model (GLSM) approach

★Basic idea: *simple* 2d QFT in the same universality class as NLSM *(Witten, 96)*

UV-free 2d Abelian gauge theory $\xrightarrow{\text{low energy}}$ NLSM on (non)Kähler manifold

Toy example with (2,2) supersymmetry

- $U(1)$ theory with N chiral fields Φ_i of charge $Q = 1$

$\Downarrow$ Classical vacua

- Supersymmetric non-linear sigma-model on $\mathbb{CP}^{N-1}$ (ϕ^i: homogeneous coords)

➥ ⚠ non-conformal !

★Strategy: find "supersymmetry protected" quantities invariant under RG flow

In this talk

★How to extend this formalism to non-Kähler geometries?

★RG-invariant partition function for these theories ➥ SUSY index

★One-loop corrections to 4d effective action

Outline

Gauged linear sigma models basics

(0,2) superspace and superfields in two dimensions

Superspace coordinates $(x^{\pm}, \theta^+, \bar{\theta}^+)$: $D_+ = \partial_{\theta^+} - i\bar{\theta}^+\nabla_+$, $\bar{D}_+ = -\partial_{\bar{\theta}^+} + i\theta^+\nabla_+$

Superfields

- Chiral superfield: $\bar{D}_+\Phi = 0 \implies \Phi = \phi + \theta^+\psi_+ + \cdots$
- Fermi superfield: $\bar{D}_+\Gamma = 0 \implies \Gamma = \gamma_- + \theta^+ G + \cdots$ ← auxiliary fields ↓
- Vector superfields ➥ Chiral field-strength: $\Upsilon = \mu_- - i\theta^+(D - iF_{01}) + \cdots$

Lagrangian

$$L = L_{\text{KIN}} - \frac{1}{8e^2}\int d^2\theta^+ \, \bar{\Upsilon}\Upsilon \quad - \quad \underbrace{\int d\theta^+ \, \Gamma^a J_a(\Phi_I)}_{\text{Superpotential}} + \underbrace{\frac{t}{4}\int d\theta^+ \, \Upsilon}_{\text{Fayet-Iliopoulos term}} + h.c.$$

$$= L_{\text{KIN}} + \frac{1}{2e^2}\left(2i\bar{\mu}_-\partial_+\mu_- + D^2 + F_{01}^2\right) + \quad G^a J_a(\phi^i) - \gamma_-^a \psi_+^i \partial_i J_a \quad + \quad r\,D - \frac{\theta}{2\pi}F_{01} + h.c.$$

with $t = ir + \frac{\theta}{2\pi}$

Calabi-Yau NLSM from GLSM: K3 surface example

Fermat quartic

- $\mathcal{L}_0 = \underbrace{\Gamma_0}_{q_0=-4}(\Phi_1^4 + \underbrace{\cdots}_{Q_i=1} + \Phi_4^4)$

  ★ F-term: $\phi_1^4 + \cdots + \phi_4^4 = 0$ in $\mathbb{CP}^3$ ➥ Fermat quartic

- Gauge bundle: $\mathcal{L}_G = \underbrace{\Phi_0}_{Q_0<0}\underbrace{\Gamma^m}_{q_m} J_m(\Phi_I)$

  ★ D-term: $\sum_i |\phi_i|^2 = r + 4|\phi_0|^2$

  ★ In "Calabi-Yau regime" $r \gg 1$: $\langle \phi_0 \rangle = 0$
  ➥ $\mathbb{CP}^3$ of size r
  ➥ Yukawa couplings $J_m(\phi_I)\gamma_-^m \psi_+^0$

  ★ Massless left-moving fermions: sections of the holomorphic vector bundle $\mathcal{V}$

$$0 \longrightarrow \mathcal{V} \longrightarrow \oplus \mathcal{O}(q_m) \xrightarrow{\otimes J_m} \mathcal{O}(-Q_0) \longrightarrow 0$$

Anomalies

- Gauge anomaly: $\boxed{\delta_{\Xi} L_{eff} = \frac{\mathfrak{A}}{8} \int d\theta^{+}\, \Xi\, \Upsilon + c.c.}$, with $\mathfrak{A} = Q_i Q^i - q_n q^n$
- Consistent models need also 2 non-anomalous global symmetries:
  - ★ Right-moving $U(1)_R$ ➥ IR $\mathcal{N}=2$ superconformal algebra
  - ★ Left-moving $U(1)_L$ ➥ GSO projection

Towards GLSM with torsion

- Interpretation of gauge anomaly: $\mathfrak{A} \sim \mathrm{ch}_2\,(T_{\mathcal{M}}) - \mathrm{ch}_2\,(\mathcal{V})$
- Geometry of Fayet-Iliopoulos terms $\mathfrak{L}_{\mathrm{FI}} = \sum_a t_a \underbrace{\Upsilon^a}_{[h^a] \in H^{(1,1)}(\mathcal{M})}$

  ➥ Complexified Kähler class $[J] = \sum_a t_a [h^a]$ (Re $J \leftrightarrow B$-field)
- Idea: $t_a \to t_a(\Phi_i)$ gives $\mathrm{d}J \neq 0$

  ➥ non-zero torsion $\mathcal{H}$

GLSMs with torsion

Introducing the torsion multiplet

★Strategy: cancel gauge anomaly of $K3$ GLSM with axion fields *(Adams et al. 06)*

★Torsion chiral multiplet $\Theta = (\vartheta, \chi)$

★ ϑ : coordinate on T^2 with moduli $(T,\ U)$ ➥ metric $\mathrm{d}s^2 = \frac{U_2}{T_2}|\mathrm{d}x^1 + T\mathrm{d}x^2|^2$

★Gauge transformation: $\Theta \xrightarrow{\Xi} \Theta - M\,\Xi$ with $M \in \mathbb{Z} + T\mathbb{Z}$

Axial coupling

★Linear field-dependent Fayet Iliopoulos term (gauge-variant)

$$L = -\bar{M}\frac{U_2}{2T_2}\int \mathrm{d}\theta^+\,\Theta\Upsilon + h.c. \;\supset\; \bar{M}\frac{U_2}{2T_2}\,\vartheta\,F_{01}$$

★Cancels the gauge anomaly if $\mathfrak{A} + \frac{2U_2}{T_2}|M|^2 = 0$ ★

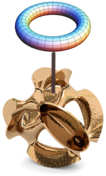
Important consistency condition (single-valuedness of action/duality covariance):
Moduli of T^2 such that the free boson ϑ defines a rational conformal field theory

(DI, 13)

Non-Kähler Geometry: Fu-Yau manifolds

★ At low energies: flows to a NLSM on a class of non-Kähler manifolds

★ Principal T^2 bundles over a $K3$ surface $\mathcal{S}$: *(Dasgupta et al., 99, Fu and Yau 2006)*



$$T^2 \hookrightarrow \mathcal{M}_6 \xrightarrow{\pi} \mathcal{S}$$

★ Metric: $\boxed{ds^2 = e^{2A(y)}\, ds^2(\mathcal{S}) + \frac{U_2}{T_2}\, |dx_1 + T dx_2 + \pi^\star \alpha|^2}$

- $d\alpha = \omega$ anti-self dual $(1,1)$ form, $\omega = \omega_1 + T\omega_2$ with $\frac{1}{2\pi}\omega_i \in H^2(\mathcal{S}, \mathbb{Z})$

★ Gauge bundle $\mathcal{V}$: Hermitian-Yang-Mills connection over $\mathcal{S}$:

$$F^{0,2} = F^{2,0} = 0\ , \quad J \wedge F = 0$$

★ Bianchi identity: topological condition

$$\int_{\mathcal{S}} \left(\frac{U_2}{T_2} \omega \wedge \star\bar{\omega} - \mathrm{ch}_2(\mathcal{V}) + \mathrm{ch}_2(T_{\mathcal{M}}) \right) = 0 \ \leftrightarrow \ \text{anomaly condition in GLSM} \ \star$$

Lessons

- Class of GLSMs with torsion $\overset{\text{low energy}}{\Longrightarrow}$ Conformal NLSM on Fu-Yau manifolds
- Most generic compactifications of heterotic strings with $\mathcal{N} = 2$ SUSY in 4d
  ➥ A subset of those are the well-studied $K3 \times T^2$ compactifications
- Many far-reaching insights made for $K3 \times T^2$ compactifications including:
  - ★ *Duality*: Heterotic on $K3 \times T^2 \leftrightarrow$ IIA on elliptically fibered Calabi-Yau 3-fold
  - ★ *Mathieu moonshine*: hidden symmetry (Mathieu group M_{24}) of NLSMs on $K3$

  ➥ due to $\mathcal{N} = 2$ SUSY constraints, many *exact* statements possible.
- General understanding should include Fu-Yau manifolds (generic case)

Supersymmetric index

Supersymmetric partition function

★Partition function of $(2,2)$ superconformal NLSM on Calabi-Yau: *elliptic genus*

$$Z_{\mathrm{E}}(q,y) = \mathrm{Tr}_{\mathrm{P}} \left(q^{\Delta} \bar{q}^{\bar{\Delta}} (-1)^F e^{2i\pi y J_0^R} \right) ,$$

with J_0^R zero-mode of the left-moving R-current in superconformal algebra
➥vanishes for $K3 \times T^2$ because of T^2 fermionic zero-modes

★$(0,2)$ NLSM on Fu-Yau ➥zero-modes $(\chi_0, \bar{\chi}_0)$ from torsion multiplet

★Assuming there exists a left current J^L (and right-moving $\bar{J}^R$), define
(DI & Sarkis, 15)

$$Z_{\mathrm{FY}}(q,\bar{q},y) = \mathrm{Tr}_{\mathrm{P}} \left(q^{\Delta} \bar{q}^{\bar{\Delta}} (-1)^F e^{2i\pi y J_0^L} \bar{J}_0^R \right) ,$$

★We have $\bar{J}^R = \bar{\chi}\chi + \underbrace{\cdots}_{\text{terms with no } \chi \text{ zero-modes}}$

➥ As we will argue, can be computed exactly in the GLSM thanks to SUSY !

Path integral formulation

★On a two-torus of complex structure τ ($q = \exp 2i\pi\tau$) we have:

$$Z_{\rm FY}(\tau,\bar\tau,y) = \int \mathscr{D}a_z \mathscr{D}a_{\bar z} \mathscr{D}\mu \mathscr{D}\bar\mu \mathscr{D}D \, e^{-\frac{1}{e^2} S_{\rm vector}[a,\mu,D] - t\, S_{\rm FI}(a,D)} \times$$

$$\times \int \mathscr{D}\phi_i \mathscr{D}\bar\phi_i \mathscr{D}\psi_i \mathscr{D}\bar\psi_i \, e^{-\frac{1}{g^2} S_{\rm chiral}[\phi_i,\psi_i,a,D,a_{\rm L}]} \times$$

$$\times \int \mathscr{D}\gamma_a \mathscr{D}\bar\gamma_a \mathscr{D}G_a \mathscr{D}\bar G_a \, e^{-\frac{1}{f^2} S_{\rm fermi}[\gamma_a,G_a,a,a_{\rm L}] - S_{\rm J}[\gamma_a,G_a,\phi_i,\psi_i]} \times$$

$$\times \int \mathscr{D}\vartheta \mathscr{D}\bar\vartheta \mathscr{D}\chi \mathscr{D}\bar\chi \, e^{-S_{\rm torsion}[\vartheta,\chi,a,a_{\rm L}]} \int \frac{{\rm d}^2 z}{2\tau_2} \bar\chi\chi \,,$$

with background gauge field for the $U(1)_{\rm L}$ global symmetry

$$a_{\rm L} = \frac{\pi y}{2i\tau_2}({\rm d}z - {\rm d}\bar z)$$

★Interacting theory with coupling constants $1/e^2$, t, $1/f^2$, $1/g^2$

Supersymmetric localization

Supersymmetric localization I: the idea

- Choose a supercharge $\mathcal{Q}$ of a supersymmetric QFT
  ➥ $\mathcal{Q}\, S[\Phi] = 0$ and $\mathcal{Q}$-invariant measure
- Aim : compute $\langle \mathcal{O} \rangle$ with $\mathcal{Q}\mathcal{O} = 0$
- Find functional $P[\Phi]$ such that $\mathcal{Q}\, P|_{bos}$ positive definite
- Deformed path integral

$$\langle \mathcal{O} \rangle_\ell = \int \mathcal{D}\Phi\, \mathcal{O}(\Phi)\, e^{-S[\phi] - \ell \mathcal{Q} P}$$

★ As $\frac{\partial}{\partial_\ell} \langle \mathcal{O} \rangle_\ell = -\int \mathcal{Q} \left(\mathcal{D}\Phi\, \mathcal{O}(\Phi)\, e^{-S[\phi] - \ell \mathcal{Q} P} P \right) = 0$

➥ compute in the $\ell \to \infty$ limit

- Exact result: one-loop fluctuations around the saddle points $\mathcal{Q}P(\Phi_0) = 0$

$$Z = \int_{\Phi_0, \mathcal{Q}P(\Phi_0)=0} \mathrm{d}\Phi_0\, e^{-S[\Phi_0]}\, \mathcal{O}_{class}[\Phi_0]\, Z_{one-loop}$$

Supersymmetric localization II: why we can

★In the $(0,2)$ GLSM ➥ natural choice of supercharge $\mathcal{Q} = \mathcal{Q}_+ + \bar{\mathcal{Q}}_+$

Good news

- Whole vector, chiral and fermi multiplets actions are $\mathcal{Q}$-exact
- $1/e^2 \to \infty$ limit for the gauge fields ➥ keep only gauge holonomies ($a_{z\bar{z}} = 0$)
- $1/f^2$, $1/g^2 \to \infty$ limit ➥ superpotential vanishes

Bad news ?

1. $\mathcal{O} = \int \frac{\mathrm{d}^2 z}{2\tau_2} \bar{\chi}\chi$ not $\mathcal{Q}$-invariant
2. Torsion multiplet action not gauge-invariant ➥ not $\mathcal{Q}$-invariant
3. Gauge anomaly ➥ measure $\mathscr{D}\Phi_i \mathscr{D}\Gamma_a$ not $\mathcal{Q}$-invariant

1. $\mathcal{Q}\mathcal{O}$ doesn't contain enough zero-modes $(\chi_0, \bar{\chi}_0)$ to contribute $\surd$
2.
3. $\Big\}$ In anomaly-free models $\mathcal{Q}\left(\mathscr{D}\Phi_i \mathscr{D}\Gamma_a \, e^{-S_{\text{torsion}}}\right) = 0$ $\surd$

Supersymmetric localization III: one-loop determinants

Localisation locus

- $e \to 0$ limit ➥ flat connections on worldsheet torus : $a = \frac{\pi \bar{u}}{2i\tau_2}\mathrm{d}z - \frac{\pi u}{2i\tau_2}\mathrm{d}\bar{z}$
- $f, g \to 0$ limit ➥ free chiral and Fermi multiplets
- Zero modes (generically) : aux. field D_0, gaugino zero-modes λ_0, $\bar{\lambda}_0$

One-loop determinants

- ambiguities in chiral determinant ➥ choice consistent with torsion multiplet
  $\mathrm{Det}\, \nabla(u) = e^{\frac{\pi}{\tau_2}(u^2 - u\bar{u})}\vartheta_1(\tau|u)$
  ➥ holomorphic section of holomorphic line bundle over space of gauge connections *(Witten, 96)*
- Chiral : $Z_\Phi = e^{-\frac{\pi}{\tau_2}(\upsilon^2 - \upsilon\bar{\upsilon})}\frac{i\,\eta(\tau)}{\vartheta_1(\tau|\upsilon)}$, $\upsilon = Q_i u + q_i^{\mathrm{L}} y$
- Fermi : $Z_\Gamma = e^{\frac{\pi}{\tau_2}(\upsilon^2 - \upsilon\bar{\upsilon})}\frac{\vartheta_1(\tau|\upsilon)}{i\,\eta(\tau)}$
- $U(1)$ vector : $Z_A(\tau, y) = -2i\pi\eta(\tau)^2 \mathrm{d}u$

Supersymmetric localization IV: the short story

- Torsion multiplet action not $\mathcal{Q}$-exact ➥ Z_{FY} depends on torus moduli T, U
- Torsion multiplet maps to a "rational" two-torus ➥ defined by two isomorphic rank two even lattices

$$\Gamma_{\text{L}} = \Gamma^{2,2}(T,U) \cap \mathbb{R}^{2,0} \ , \quad \Gamma_{\text{R}} = \Gamma^{2,2}(T,U) \cap \mathbb{R}^{0,2}$$

- Integral over gauge holonomies $u_\ell \in \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ ➥ poles in $Z_{one-loop}$ due to "accidental" bosonic zero modes
- Turn $\int d^2 u_\ell$ into contour integral around the poles
  ➥ Jeffrey-Kirwan residues *(Benini et al. 2013)*

★ $Z_{\text{FY}}(\tau, \bar{\tau}, y)$ has an intrinsic mathematical definition, as an non-holomorphic elliptic genus that I will use to present the result

Mathematical formulation

Geometrical data

★(0, 2) Non-linear sigma-model on Fu-Yau manifold characterized by

- Holomorphic tangent bundle $\mathcal{T}_{\mathcal{S}}$ over the base, with $c_1(\mathcal{T}_{\mathcal{S}}) = 0$
  ➥ splitting principle $c(\mathcal{T}_{\mathcal{S}}) = \prod_{i=1}^{2}(1 + \nu_i)$
- A rank r holomorphic vector bundle $\mathcal{V}$ over $\mathcal{S}$, with $c_1(\mathcal{V}) = 0$
  ➥ splitting principle $c(V) = \prod_{a=1}^{r}(1 + \xi_a)$
- A "rational" Narain lattice in $\mathbb{R}^{2,2}$ defined by a pair of isomorphic rank two lattices $\Gamma_{\rm L}$, $\Gamma_{\rm R}$ and a gluing map $\varphi \; : \Gamma_{\rm L}^*/\Gamma_{\rm L} \to \Gamma_{\rm R}^*/\Gamma_{\rm R}$
- A pair of anti-self-dual (1,1)-forms ω_1 and ω_2 in $H^2(\mathcal{S}, \mathbb{Z})$ characterizing the torus bundle

⚠ Constraint $\frac{U_2}{T_2}\, \omega \wedge \star\bar{\omega} - \text{ch}_2(\mathcal{V}) + \text{ch}_2\left(T_{\mathcal{M}}\right) = 0$ in cohomology

Non-holomorphic elliptic genus

$$Z_{\rm FY}(\mathcal{M},\mathcal{V},\omega|\tau,\bar\tau,y) = \frac{1}{\eta(\tau)^2}\int_{\mathcal{S}}\prod_{a=1}^{r}\frac{i\theta_1(\tau\,\big|\,\frac{\xi_a}{2i\pi}-y\,)}{\eta(\tau)}\prod_{i=1}^{2}\frac{\eta(\tau)\nu_i}{i\theta_1(\tau\,\big|\,\frac{\nu_i}{2i\pi}\,)}\times$$
$$\times\sum_{\mu\in\Gamma_{\rm L}^*/\Gamma_{\rm L}}\Theta_{\mu}^{\Gamma_{\rm L}}\left(\tau\,\Big|\,\frac{p_\omega}{2i\pi}\right)\Theta^{\Gamma_{\rm R}}_{\varphi(\mu)}\left(-\bar\tau\,\Big|\,0\right)$$

Notations

- $\eta(\tau)$: Dedekind eta-function
- $\theta_1(\tau|y)$: Jacobi theta-function
- Θ^{Γ}_{μ} : theta-function on lattice Γ ➥ $\Theta^{\Gamma}_{\mu}(\tau|\lambda) = \sum_{\gamma\in\Gamma+\mu} q^{\frac12\langle\gamma,\gamma\rangle_\Gamma}e^{2i\pi\langle\gamma,\lambda\rangle_\Gamma}$
- p_ω : two-dimensional vector of two-forms with $\langle p_\omega,p_\omega\rangle_{\Gamma_{\rm L}} = -\frac{2U_2}{T_2}\omega\wedge\star\bar\omega$
  ➥ belongs to formal extension of Γ_L over $H^2(\mathcal{S},\mathbb{Z})\oplus H^2(\mathcal{S},\mathbb{Z})$

★ Coincides with path integral computation (checked in simple examples)

Properties

In physics

- Exact path-integral computation ➥ valid for the superconformal non-linear sigma-model emerging in the IR
- Seed for computing the one-loop corrections to the 4d effective action
- Allows for quantitative tests of non-perturbative Heterotic/type II duality

In mathematics: generalized elliptic genus when $ch_2(V) \neq ch_2(T_{\mathcal{S}})$

- $\bar{\eta}^{-2} Z_{\text{FY}}(\tau, \bar{\tau}, y)$ transforms as weak Jacobi form of weight 0 and index $\frac{r}{2}$

$$\boxed{\bar{\eta}\left(\tfrac{a\bar{\tau}+b}{c\bar{\tau}+d}\right)^{-2} Z_{\text{FY}}\left(\tfrac{a\tau+b}{c\tau+d}, \tfrac{a\bar{\tau}+b}{c\bar{\tau}+d}, \tfrac{y}{c\tau+d}\right) = e^{\frac{i\pi\, r\, cy^2}{c\tau+d}} \bar{\eta}(\bar{\tau})^{-2} Z_{\text{FY}}(\tau, \bar{\tau}, y)} \quad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{Z})$$

- Terms with $p_L = 0$ ➥ generating function for indices of Dirac operators in representations given by the formal power series $\mathbb{V} = 1 - e^{-2i\pi y} V^\star + q(\mathcal{T}_{\mathcal{S}} + \mathcal{T}_{\mathcal{S}}^\star - e^{2i\pi y} V - e^{-2i\pi y} V^\star) + \cdots$
- In general: Dirac operator with non-zero 'KK momentum' along T^2 fiber?

Adding line bundles

General SUSY gauge bundles in FY compactifications

- Pullback of an HYM gauge bundle over the base $\mathcal{S}$:
  $F^{(2,0)} = F^{(0,2)} = 0$ and $J \wedge F = 0$
- Abelian bundles over the total space (would be Wilson lines for $K3 \times T^2$)
  $A = \mathfrak{T}^a \,\mathrm{Re}\, \left(\bar{V}_a \iota\right)$ with $\iota = \mathrm{d}x_1 + T\mathrm{d}x_2 + \pi^\star \alpha$, globally defined on $\mathcal{M}_6$

Abelian bundles in the GLSM

- Neutral Fermi multiplets $(\gamma_{-,a}, G_a)$ ➥ bosonize into chiral multiplets $B_a = (b_a, \psi_{-,a}, \psi_{+,a})$ with $\mathrm{Im}\,(b_a)$ at fermionic radius
- Consider a larger torus T^{2+n} with $V_a \leftrightarrow$ off-diagonal terms in the metric
- No single-valuedness constr. ➥ vector moduli space (S, V_a) in $\mathcal{N} = 2$ SUGRA

Contribution to the index

- Kill superfluous fermionic zero-modes ➥ $(J_0)^{16}$ insertion in the path integral
- SUSY localization argument still holds
- Factor out non-compact & right-moving compact bosons in B_a

Result for the index with most general gauge bundle

$$Z_{\text{FY}}(X,\mathcal{V},\omega|\tau,\bar\tau,z) = \int_{\mathcal{S}} \prod_{a=1}^{r} \frac{i\theta_1\left(\tau \,\middle|\, \frac{\xi_a}{2i\pi} - z\right)}{\eta(\tau)} \prod_{i=1}^{2} \frac{\eta(\tau)\nu_i}{i\theta_1(\tau \,|\, \frac{\nu_i}{2i\pi}\,)}$$
$$\sum_{(p_{\text{L}},p_{\text{R}})\in\Gamma_{10,2}} \exp\left(-\langle p_\omega, p_{\text{L}}\rangle\big|_{V_a=0}\right) \times \frac{q^{\frac{1}{4}|p_{\text{L}}|^2}}{\eta(\tau)^{18}} \frac{\bar q^{\frac{1}{4}|p_{\text{R}}|^2}}{\bar\eta(\bar\tau)^2}$$

With standard expressions for $(10,2)$ lattice (however T,U quantized here):

$$\frac{1}{4}|p_{\text{R}}|^2 = \frac{1}{4\left(T_2U_2 - \sum_a (V_2^a)^2\right)} \left|-n_1T + n_2 + w_1U + w_2\left(TU - \sum_a (V^a)^2\right) + N_aV^a\right|^2 ,$$
$$\frac{1}{4}|p_{\text{L}}|^2 = \frac{1}{4}|p_{\text{R}}|^2 + n_1w_1 + n_2w_2 + \frac{1}{4}N_aN^a$$

Threshold corrections

Threshold corrections: generalities

One-loop corrections to the gauge couplings : $G = \prod_a G_a$

$$\frac{4\pi^2}{g_a^2(\mu^2)^2}\Big|_{\text{one-loop}} = k_a \mathrm{Im}(S) + \underbrace{\frac{b_a}{4}\log M_s^2/\mu^2}_{\text{one-loop }\beta\text{-fct}} + \underbrace{\frac{1}{4}\Delta_a(M,\bar{M})}_{\text{moduli dependence}} + \textit{universal term}$$

★ $\mathcal{N}=2$, $d=4$ heterotic compactification:

$$\frac{b_a}{4}\log\frac{M_s^2}{\mu^2} + \frac{1}{4}\Delta_a(M,\bar{M}) = \frac{i}{4}\int_{\mathscr{F}}\frac{\mathrm{d}^2\tau}{\tau_2}\frac{1}{\eta^2}\mathrm{Tr}_{\bar{R}}\left[\left(Q_a^2 - \frac{k_a}{2\pi\tau_2}\right)q^{L_0-\frac{c}{24}}\bar{q}^{L_0-\frac{\bar{c}}{24}}e^{i\pi\bar{J}_0^R}\bar{J}_0^R\right]$$

➥ modified SUSY index with insertion of Casimir Q_a^2 for G_a factor

Heterotic/type II duality

- Heterotic threshold corrections ➥ one-loop prepotential for vector multiplets

$$F_{\mathrm{H}} = -\gamma_{ij}SV^iV^j + F_{\mathrm{H}}^{\text{1-loop}}(V^i) + \cdots$$

- CY_3 Type IIA dual: K3 fibration with a compatible T^2 fibration
  ➥ perturbative het. limit $\leftrightarrow$ large $\mathbb{P}^1$ limit in IIA
- Expansion of CY_3 vector multiplet prepotential ➥ qualitative duality check

Threshold corrections for FY compactifications

★Simplify geometric formulation of the index using Gritsenko's formula

$$\theta_1(\tau|z+\xi) = \exp\left\{-\frac{\pi^2}{6}E_2(\tau)\xi^2 + \frac{\theta_1'(\tau|z)}{\theta_1(\tau|z)}\xi - \sum_{\ell\geqslant 2}\wp^{(\ell-2)}(\tau,z)\frac{\xi^\ell}{\ell!}\right\}\theta_1(\tau|z)\,,\quad \wp^{(\ell)} := \frac{\partial^\ell}{\partial z^\ell}\wp$$

★Assume rank two bundle V over K3 in E_8 with $\int_{\mathcal{S}} \mathrm{ch}_2(V) = -n$

Correction to "hidden" E_8 gauge coupling

$$\Delta_{E_8} = \int_{\mathscr{F}} \frac{d^2\tau}{\tau_2}\frac{\hat{E}_2 E_4 - E_6}{24\Delta}\sum_{p_{\rm L},p_{\rm R}} q^{\frac{1}{2}p_{\rm L}^2}\bar{q}^{\frac{1}{2}p_{\rm R}^2}\left\{-\frac{n}{12}E_6 + \frac{n-24}{12}\hat{E}_2 E_6 - f(p_l|\omega)E_4\right\}$$

with

$$f(p_{\rm L}|\omega) := \int_{\mathcal{S}} \langle p_{\rm L}, p_\omega\rangle^2_{\Gamma_{\rm L}} - \frac{1}{4\pi\tau_2}\langle p_\omega, p_\omega\rangle_{\Gamma_{\rm L}}$$

➥ Each term separately modular invariant

★Likewise, one-loop corrections for other gauge factors and gravitational thresholds

➥ Computation of the modular integral over $\mathscr{F}$?

Modular integral and final result

Modular integral

- Expand (weakly almost holomorphic) modular forms in Δ_{E_8} into Niebur-Poincaré series *(Angelantonj, Florakis, Pioline 12)*

$$\mathcal{F}(s,w,z) = \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathcal{M}_s(-\Im z) e^{-2i\pi \Re z} \Big|_w \gamma$$

- Unfold the modular integral against the $\mathcal{F}(s,w,z)$ (rather than unfolding against the Narain lattice)

➥ appropriate here because momentum-dependent insertion $f(p_l|\omega)$

Final result for E_8 threshold

$$\Delta_{\mathrm{E}_8} = \sum_{\mathrm{BPS}} \Big\{ 1 + \frac{n-24}{6p_{\mathrm{L}}^2} + (p_{\mathrm{L}}^2/2 - 1)\log(1 - 2/p_{\mathrm{L}}^2) + \frac{1}{4}\left(1 - \frac{1}{3p_{\mathrm{L}}^2} - \frac{1}{p_{\mathrm{L}}^2} + (p_{\mathrm{L}}^2/2-1)\log(1-2/p_{\mathrm{L}}^2)\right) \int_{\mathcal{S}} \langle p_{\mathrm{L}}, p_\omega \rangle^2 + 2(n+12)\log 4\pi e^{-\gamma} T_2 U_2 |\eta(U)\eta(T)|^4 \Big\}$$

Conclusions

Summary

What has been understood

- Non-Kähler manifolds $\leftrightarrow$ gauge-variant couplings
- GLSMs for Fu-Yau manifolds $\leftrightarrow$ generic $\mathcal{N}=2$ heterotic vacua
- Supersymmetric localization ➥ exact supersymmetric partition function
- Defines a modified non-holomorphic elliptic genus, interesting on its own right
- Computation of threshold corrections for these flux compactifications

Ongoing work and directions for future studies

- Heterotic/type II duality beyond $K3 \times T^2$ *(Melnikov, Minasian, Theisen 14)*
- Thresholds: instanton corrections from Fourier expansion (T^2 fiber is not a 2-cycle in Fu-Yau geometries!)
- Generalization of $K3 \times T^2$ Mathieu moonshine to FY?
- More general non-Kähler compactifications