Heterotic Torsional Backgrounds, Threshold Corrections and Mock Modular Forms

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Montreal, June 2012

L .Carlevaro, D.I. and M. Petropoulos, arXiv:0812.3391
L .Carlevaro, E. Dudas and D.I., arXiv:1207.xxxx

Introduction

Heterotic compactifications

- $\mathcal{N} = 1$ heterotic compactifications correspond to solvable perturbative worldsheet CFT only at very special points: toroidal orbifolds, Gepner models, free fermions,...
- Usually the geometrical interpretation is lost – although the topological data matches geometrical Calabi-Yau compactifications in the relevant cases

Smooth compactifications

- Instead one can consider in supergravity a CY compactification
- One specifies a gauge bundle V on the compactification manifold. Needs to satisfy the modified Bianchi identity for the NSNS two-form
- Satified automatically for the standard embedding of the $SU(3)$ spin connection in the gauge connection ➥ however not interesting
- For more general gauge bundles, torsion is usually needed ➥ fluxes appear naturally even without trying to stabilize moduli

Heterotic Flux compactifications (torsional)

- Less understood that type II models, as *not* conformally CY, not even Kähler manifolds appear
- One needs therefore to deal with explicit examples (few are known)
- However, perturbative heterotic compatifications are potentially accessible to worldsheet CFT methods because *(i)* no R-R flux and *(ii)* the dilaton is not stabilized perturbatively
- Stepstone : heterotic gauged linear sigma-models *(Adams, Lapan,...)*

Local models of flux compactifications

- Simpler descriptions can be found near smoothed singularities (as Klebanov-Strassler throats in type IIB)➥ local models
- In heterotic, examples with a solvable worldsheet CFT. Simplest is (a T^2 fibration over) warped Eguchi-Hanson space with Abelian bundle.
- Other example: resolved orbifoldized conifold
- Allows to study some stringy aspects of flux compactifications ➥ in this talk, hypermultiplets contributions to gauge thresholds

Outline

$\mathcal{N}=1$ heterotic vacua in four dimensions

- An $\mathcal{N}=1$ heterotic compactification with torsion corresponds to a 6d manifold admitting a covariantly constant spinor w.r.t. the torsionfull spin connection:

$$\delta\psi_n = (\nabla_n - \tfrac{1}{4}\not{\mathcal{H}}_n)\eta = 0$$

- Defines an $SU(3)$ structure, characterized by the $SU(3)$-invariant real 2-form $J_{mn} = -i\eta^\dagger\Gamma_{mn}\eta$ and complex 3-form $\Omega_{mnp} = \eta^t\Gamma_{mnp}\eta$
- Allows to define a complex structure from Ω (such that J is $(1,1)$ and Ω is $(3,0)$) and a metric
- The susy conditions are written as generalized calibrations for wrapped fivebranes:

$$\begin{cases} \mathrm{d}(e^{-2\Phi}\Omega) & = 0 \\ \mathrm{d}(e^{-2\Phi}J\wedge J) & = 0 \\ \mathrm{d}(e^{-2\Phi}J) & = e^{-2\Phi}\star_6\mathcal{H} \end{cases}$$

- The solutions are complex but non Kähler manifolds

Gauge bundle and Bianchi identity

- The gauge bundle ($Spin(32)/\mathbb{Z}_2$ or $E_8 \times E_8$) has to satisfy then Hermitean Yang-Mills equations: $\mathcal{F}^{a\bar{b}} J_{a\bar{b}} = \mathcal{F}^{ab} = \mathcal{F}^{\bar{a}\bar{b}} = 0$
- Existence of massive spinorial reps of the gauge group gives the constraint $c_1(V) \in H^2(2\mathbb{Z})$

Bianchi identity

- Strongest form: $\mathrm{d}\mathcal{H} = \alpha' \left[\mathrm{tr} R(\omega_-) \wedge R(\omega_-) - \mathrm{Tr}_v F \wedge F \right]$ in forms
- Usually: $ch_2(V) = p_1(T\mathcal{S})$ in cohomology

  - Non-linear constraint (as $R(\omega_-)$ constructed out of the torsionfull spin connection $\omega - \frac{1}{2}\mathcal{H}$)
  - Use of different connections on $T\mathcal{M}$ ➥ higher order corrections
  - A simple solution (strongest form): *standard embedding* of the spin connection in the gauge connection ➥ CY
  - These conditions can be generalized to include fivebranes sources ➥ non-perturbative vacua (may have perturbative type I dual).

Eguchi-Hanson space

- EH gravitational instanton in 4d ➥ hyper-Kähler (J_i self-dual)
- Resolution of an A_1 singularity (non compact $\mathbb{C}^2/\mathbb{Z}_2$ orbifold)

Explicit metric

- Introduce $SU(2)$ left-invariant one-forms σ_i^{L}, $i = 1, \ldots, 3$
- $\mathrm{d}s^2_{\mathrm{EH}} = (1 - \frac{a^4}{r^4})^{-1}\mathrm{d}r^2 + \frac{r^4}{4}\left[\sigma_1^2 + \sigma_2^2 + (1 - \frac{a^4}{r^4})\sigma_3^2\right]$
- Hopf fibration $S^1 \to S^3 \to S^2$ degenerates at the bolt $r = a$
  ➥ non-trivial two-cycle Σ
- No conical singularity ➥ asympt. $S^3/\mathbb{Z}_2$ ➥ $\pi_1(\mathrm{EH}) = \mathbb{Z}_2$

- 2^{d} cohomology class generated by a normalizable, anti-self-dual $(1,1)$-form $\omega_{[2]}$, locally written

$$\omega_{[2]} = -\frac{a^2}{4\pi}\mathrm{d}\left(\frac{\sigma_3}{r^2}\right)$$

Gauge bundle over Eguchi-Hanson

- Consider $\mathcal{N}=1$ SYM on Eguchi-Hanson space
- Hermitean YM equations $\mathcal{F} \lrcorner J_i = 0$ ➥ $\mathcal{F}$ anti-self dual

Abelian instanton in EH space

- $\mathcal{F} = -2\pi \, \omega_{[2]} \, \vec{\ell} \cdot \vec{T}$

➥ $\int ch_2(V) = -\frac{1}{4} \sum_\alpha \ell_\alpha^2 \mathrm{tr}(iT^\alpha)^2$

- Abelian bundles with vector structure : $\ell_\alpha \in \mathbb{Z}$
- Abelian bundles without vector structure: $\ell_\alpha \in \mathbb{Z} + \frac{1}{2}$ *(Berkooz et al., 96)*

Conformally Eguchi-Hanson ansatz

- Heterotic on $\mathrm{EH} \times T^2$ with Abelian instanton background
- Susy conditions lead to five-brane like ansatz *(Cvetic, Lu, Pope)*

Heterotic supergravity solution

- $ds^2 = dx_\mu dx^\mu + + \frac{U_2}{T_2} \left| dx^1 + T dx^2 \right|^2 + H\, ds^2_{\mathrm{EH}}$
- $e^{2\Phi} = g_s^2 H$
- $\mathcal{H} = \star^{\mathrm{EH}} dH$

- Bianchi identity $d\mathcal{H} = \alpha' \left(\mathrm{tr}\mathcal{R}(\omega_-)^2 - \mathrm{Tr}_v \mathcal{F}^2 \right)$
  ➥ for an Abelian bundle :
  $$\Delta_{\mathrm{EH}} H = \alpha' \star^{\mathrm{EH}} \mathrm{tr}\mathcal{R}(\omega_-)^2 + \frac{8\alpha' a^4}{r^8} \sum_\alpha \ell_\alpha^2 \mathrm{Tr}_v (T^\alpha)^2$$
- Generalization➥ T^2 fibration $\frac{U_2}{T_2} \left| dx^1 + T dx^2 + \alpha \right|^2$, $d\alpha \in H^{1,1}(EH)$
- Bianchi solved using the *Chern connection* for $Q_5 = 0$ *(Fu, Tseng, Yau, 2008)*
- These are local models of Dasgupta-Rajesh-Sethi compactifications $(T^2 \to \mathcal{M}_6 \stackrel{\pi}{\to} K3)$ *(Dasgupta, Rajesh, Sethi 99)*

Large charge approximation

- In the large charge limit $\vec{\ell}^2 \gg 1$ the $\mathcal{R}^2$ contribution to Bianchi is negligible
  ➥ solution $\boxed{H(r) = 1 + \frac{2\alpha' Q_5}{r^2}}$ with $Q_5 = -\frac{\alpha'}{2}\ell_\alpha^2 \mathrm{Tr}_v(T^\alpha)^2$
- Varying dilaton. capped at the bolt : $e^{2\Phi_{\text{MAX}}} = g_s^2 H(a)$

Blow-down limit

- blow-down limit $a \to 0$ (g_s fixed) ➥ EH degenerates to $\mathbb{C}^2/\mathbb{Z}_2$
- Abelian instanton becomes point-like
- Small instantons equivalent to *stack of* $\sim Q_5$ *heterotic fivebranes*, transverse to $\mathbb{C}^2/\mathbb{Z}_2$
  ➥ linear dilaton blows up for $r \to 0$: *strong coupling*

Integrated Bianchi identity : tadpole condition

- Bianchi identity not satisfied by this ansatz beyond the large charges limit ($\mathcal{R}^2$ term) ➥ α' corrections
- Barring this, integrated Bianchi gives $\int_{\frac{S^3}{\mathbb{Z}_2}} \mathcal{H} = 8\pi\alpha' \left[3 + \int \mathrm{ch}_2(V)\right]$
- Leads to tadpole-like condition $Q_5 = \vec{\ell}^2 - 6$

Tadpole-free models ?

- Case $\vec{\ell} = (1,1,2,10^{13})$ ➥ peculiar blow-up of standard $\mathbb{C}^2/\mathbb{Z}_2$ orb.
- Case $\vec{\ell} = (1^6, 0^{10})$ ➥ claimed to be a blown-up perturbative orbifold with shift vector $(1^6, 0^{10})$ mod internal lattice $\mathcal{L}$ *(Honecker, Trapletti)*
- w/o vector structure $\vec{\ell} = (\frac{1}{2}^{15}, -\frac{3}{2})$ ➥ dual of Gimon-Polchinski *(Berkooz et al.)*

- Outside the scope of sugra as strong α' corrections ➥ more latter...

Note that the blow-down limit of our models is *not* an orbifold point.

Abelian bundle on EH in the double scaling limit

- String coupling grows in the blow-down limit of some 2-cycle
  ➥ defines a double-scaling limit of this cycle as $g_s \to 0$ with $g_s \frac{\sqrt{\alpha'}}{a}$ fixed
- Stays at fixed distance of the cycle in a units: $\cosh \rho = (r/a)^2$ fixed
  ➥ space conformal to Eguchi-Hanson, weakly curved and coupled

Heterotic sugra solution in the double scaling limit

$$\begin{aligned}
ds^2 &= dx_\mu dx^\mu + \frac{\alpha' Q_5}{2}\left[d\rho^2 + \sigma_1^2 + \sigma_2^2 + \tanh^2 \rho\, \sigma_3^2\right] \\
e^{2\Phi} &= \frac{2g_s^2 \alpha' Q_5}{a^2} \frac{1}{\cosh \rho} \\
\mathcal{H} &= -\frac{\alpha' Q_5}{2} \tanh^2 \rho\, \Omega(S^3) \\
\mathcal{A} &= \frac{\sigma_3}{2 \cosh \rho} \vec{\ell} \cdot \vec{T}
\end{aligned}$$

- The blow-up parameter a disappears from the metric (eaten by dilaton zero-mode)

'Near-horizon' CFT of the blow-down limit

- In the blow-down limit, $\mathbb{Z}_2$ orbifold of the fivebrane near-horizon solution (Callan-Harvey-Strominger)
- Worldsheet conformal field theory : $SU(2)$ $\mathcal{N} = (0,1)$ WZW model at level $2(Q_5 - 1)$ and linear dilaton of background charge $\mathcal{Q} = 1/\sqrt{Q_5}$
  ➥ singular background
- Action of $\mathbb{Z}_2$ orbifold $g(z,\bar{z}) \to -g(z,\bar{z})$: lens space $S^3/\mathbb{Z}_2$
- Orbifold action in gauge group : in cases with vector structure, $Spin(32)/\mathbb{Z}_2$ lattice unaffected

CFT for the warped Eguchi-Hanson space

- The heterotic Eguchi-Hanson solution with line bundle has a worldsheet CFT *also* in the double scaling limit
- Chiral gauging of an $SL(2,\mathbb{R})_{2Q_5} \times SU(2)_{2Q_5}$ $\mathcal{N}=(0,1)$ WZW model : $(g,h) \in SL(2,\mathbb{R}) \times SU(2) \to (e^{i\sigma_3 \tilde{\beta}} g e^{i\sigma_3 \beta}, h e^{i\sigma_3 \beta})$
- Couple the left worldsheet gauge field $(\tilde{\beta})$ to the 32 het. fermions
- Embedding in $\mathfrak{so}(32)$ Cartan: shift vector $\vec{\ell}$
  ➥ worldsheet anomalies cancelled for $Q_5 = \vec{\ell}^2 - 1$

Lowest ordrer background fields

- Asymmetric coset ➥ bosonization useful *(Johnson)*
- Integrate out classically worldsheet gauge fields $(A, \bar{A})$ ➥ Σ-model on conformally EH after re-fermionization
- 4-fermion term $\frac{2}{\cosh \rho}(\bar{\psi}^1\bar{\psi}^2 + \bar{\psi}^3\bar{\psi}^\rho)\ell_n \psi^{2n-1}\psi^{2n}$ ➥ Abelian bundle

➥ exact solution to Bianchi identity for any Q_5 *(E. Herrera-Cordero and D. I., work in progress)*

Worldsheet instantons and Liouville potentials

- So far, the $\mathbb{Z}_2$ orbifold, put "by hand" in the CFT ➥ is there an analogue of no-conical singularity condition of sugra ?
- 'K-theory' condition on the bundle $c_1(V) \in H^2(2\mathbb{Z})$, i.e $\sum \ell_i \in 2\mathbb{Z}$
  ➥ role in worldsheet CFT ?
- $\frac{SL(2)}{U(1)}$ corrected by worldsheet non-perturbative effects *(Fateev, Zamolodchikov)*

Liouville-like potentials for Abelian bundles

- In our heterotic coset, the dynamically generated $\mathcal{N} = (2,0)$ Liouville potential reads $\mu_L(\bar{\psi}^\rho + \bar{\psi}^3)e^{-\sqrt{\ell^2-1}(\rho+iY_R^3)-i\vec{\ell}\cdot\vec{X}_L}$
  $(\partial x_L^n = \psi^{2n-1}\psi^{2n})$
- Belongs to the twisted sector of the $\mathbb{Z}_2$ orbifold
- Orbifold and (right) GSO invariant only if $c_1(V) \in H^2(2\mathbb{Z})$

Tadpole-free model with vector structure : $\vec{\ell} = (1^6, 0^{10})$

- Massive $U(1) \in U(6)$: $m = 2/\sqrt{5\alpha'}$
- Unbroken $SU(6) \times SO(20)$: non-normalizable operators $A^{ij}_{\mu} \partial X^{\mu} \bar{\psi}^i \bar{\psi}^j$

Massless hypermultiplets (localized chiral operators)

- First type : $e^{-\frac{J\rho}{\sqrt{\ell^2-1}} - i\vec{n}\cdot\vec{X}_L} V_{J-1}$ ➥ $2(\mathbf{1},\mathbf{15})_{\mathrm{U}} + (\mathbf{1},\mathbf{1})_{\mathrm{T}} + (\mathbf{1},\mathbf{15})_{\mathrm{T}}$
- Second type : $e^{-\frac{J\rho}{\sqrt{\ell^2-1}} - i\vec{n}\cdot\vec{X}_L} V_{J-1} \bar{\psi}^{\alpha}$ ➥ $(\mathbf{20},\mathbf{6})_{\mathrm{U}} + (\mathbf{20},\mathbf{6})_{\mathrm{T}}$
- Third type : $\vec{\ell} \cdot \bar{\partial} \vec{X}_L e^{-\frac{J\rho}{\sqrt{\ell^2-1}} - i\vec{n}\cdot\vec{X}_L} V_{J-1}$ ➥ $(\mathbf{1},\mathbf{1})_{\mathrm{U}} + (\mathbf{1},\mathbf{15})_{\mathrm{T}}$

Blow-down limit ➥ not at orbifold point

- Blow-down limit: linear dilaton $\mathcal{Q} = 1/\sqrt{5}$ ➥ as 5 fivebranes
- *same* bundle from blown-up perturbative $\mathbb{T}^4/\mathbb{Z}_2$ orbifold *(Honecker et al.)*
- Spectrum found there ➥ different multiplicities

T^2 fibrations over *EH* and moduli stabilization

- Add Kaluza-Klein gauge fields ➥ T^2 fibered over Eguchi-Hanson

Fibered EH solution in the double scaling limit

$$ds^2 = dx_\mu^2 + \frac{U_2}{T_2}\left|dx^1 + q_1\mathcal{A}_1 + T(dx^2 + q_2\mathcal{A}_2)\right|^2 + 2\alpha' Q_5\left[d\rho^2 + \sigma_1^2 + \sigma_2^2 + \text{th}^2\rho\, \sigma_3^2\right]$$

- $\mathcal{A}_{1,2} \sim \sqrt{Q_5}\,\frac{\sigma_3}{2\cosh\rho}$
- Bianchi identity ➥ $Q_5 = \frac{2U_2}{T_2}\left|q_1 + Tq_2\right|^2$
- 'Self-dual' background under T-duality along T^2: $B_{x^{1,2}\sigma^3} = G_{x^{1,2}\sigma^3}$ (square torus)
- Generically, invariance under T^2 duality group constrains the T^2 moduli U and T

- The worldsheet CFT is constructed along the same lines as for internal gauge fields
- However the T^2 CFT has to be at a rational point in the moduli space of T^2 CFTs for consistency

Rational torii

- S^1 CFT (free boson) is rational for $R^2/\alpha' \in \mathbb{Q}$
- T^2 CFT is rational if T, U valued in the same imaginary quadratic number field K:
  $K = \mathbb{Q}(\sqrt{D}) \sim \mathbb{Q} + \mathbb{Q}\sqrt{D}$ w. $D = b^2 - 4ac < 0$ and $gcd(a, b, c) = 1$

- One can then construct a Liouville potential using a generator of the anti-holomorphic W algebra of the T^2 CFT
- Gives a neat worldsheet understanding of moduli stabilization
  ➥ generalization to fibrations of a K3 Gepner model over EH

Gauge threshold corrections in heterotic compactifications

- Consider a $K3 \times T^2$ heterotic compactification with $\mathcal{N}=2$ susy in four dimensions (without torsion)
- For an unbroken gauge factor $(G_a)_{k_a} \subset G_{4d}$, the one-string-loop correction is given by *(Lerche et al., 88)*

$$\boxed{\frac{4\pi^2}{g_a(\mu^2)} = \frac{k_a}{L} + \frac{b_a}{4} \log\left(\frac{M_s^2}{\mu^2}\right) + \frac{\Delta_a(M,\bar{M})}{4}} \text{ with } L^{-1} = \Re S - \tfrac{1}{4}\Delta_{\text{UNIV}}$$

- Given $n_{\mathbf{R}}$ hypermultiplets in the representation $\mathbf{R}$ of G_a, $b_a = \sum_{\mathbf{R}} n_{\mathbf{R}} T_a(\mathbf{R}) - 2T_a(Adj_a)$
- Moduli dependence through $\Delta_a(M,\bar{M})$ (contributions from massive modes)
- Can be computed explicitely at the orbifold points $T^2 \times T^4/\Gamma$.

Gauge threshold from $K3$ elliptic genus

- $b_a \log\left(\frac{M_s^2}{\mu^2}\right) + \Delta_a = -i\int_{\mathcal{F}} \frac{\Gamma_{2,2}(T,U)}{\bar{\eta}^2}\mathcal{I}_a$ ($\mathcal{F}$: fund. domain of T^2)
- With $\mathcal{I}_a = \text{Tr}^{\text{K3}}_{*,R}\left(\left[Q_a^2 - \frac{k_a}{4\pi\tau_2}\right] e^{i\pi\bar{J}_0^R} q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}} e^{2i\pi\nu\cdot J_0}\right)\Big|_{\nu=0}$
- Q_a^2 : quadratic casimir of G_a and $\bar{J}^R(\bar{z})$: R-current of $\mathcal{N}=2$ SCA
- Elliptic genus independent of $\bar{\tau}$ by worldsheet supersymmetry

➥ Universal form, obtained from demanding the absence of charged tachyons

$$\Lambda_a = \frac{k_a}{8}\int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\tau_2}\Gamma_{2,2}(T,U)\,\frac{D_{10}E_{10} - 528\eta^{24}}{20\eta^{24}} + \frac{b_a}{4}\int_{\mathcal{F}}\frac{\mathrm{d}^2\tau}{\tau_2}\Gamma_{2,2}(T,U)$$

with the covariant derivative on modular forms $D_r\Phi_r = \left(\frac{i}{\pi}\frac{\partial}{\partial\tau} + \frac{r}{2\pi\tau_2}\right)\Phi_r$

✔ The last term is IR-divergent, signaling the presence of massless states (hypers and vectors)

Gauge thresholds in local models - what does it compute?

Physical considerations

- Threshold corrections in our local model of (warped $K3$) $\times T^2$
- Massless gauge fields : *non-normalizable* operators ➥ frozen, do not contribute to the index
- In a genuine compactification, the local threshold expected to compute the contribution of hypers localized in a throat to the gauge couplings (gauge fields in the bulk).

Technical issues

- String spectrum : *discrete representations* (localized states) and *continuous representations* (plane waves)
- Under S-transformation : $(\textit{discrete}) \to (\textit{discrete}) + (\textit{continuous})$
- Infinite volume canceling the fermionic zero-modes ➥ holomorphicity of the elliptic genus no longer holds
- More involved non-holomorphic contributions to the threshold: non-holomorphic contribution from non-localized states

Elliptic genus : contribution from localized states

- Case $\vec{\ell} = (1, \ell, 0^{14})$ ➥ (twisted) $\mathcal{N} = (2,2)$ Liouville theory appear
- Localized states $\leftrightarrow$ BPS representations of the $c > 3$, $\mathcal{N} = 2$ superconformal algebra (characters $Ch_d(j, m; \tau, \nu)\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$)
- Contribution to the elliptic genus $E(\tau, \nu)$ can be computed algebraically or by free field techniques

$$\boxed{E_{loc}(\tau,\nu) = \frac{1}{2}\sum_{j=1}^{k/2}(\chi_{k-2}^{j-1} + \chi_{k-2}^{k/2-j}) \sum_{n\in\mathbb{Z}_{2\ell}} iCh_d(j, \ell(n+\tfrac{1}{2}); \tau, \nu)\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]}$$

- Red term rephrased in terms of level 1 Appell-Lerch sums :

$$q^{-\frac{(j-\frac{\ell+1}{2})^2}{k}} \frac{1}{\ell}\sum_{N=0}^{\ell-1} e^{-\frac{i\pi N}{\ell}} A_1\left(\frac{1}{\ell}(\nu + (\frac{\ell+1}{2} - j)\tau + N, \frac{\nu}{\tau}|\tau\right) \frac{i\vartheta_1(\tau,\nu)}{\eta^3}$$

Appell-Lerch sums

- At level K, $A_K(u, v|\tau) = e^{i\pi K u}\sum_n (-)^{Kn} e^{2i\pi n v} \frac{q^{Kn(n+1)/2}}{1-e^{2i\pi u}q^n}$
- 'Almost' modular form, *e.g.* :
  $A_1\left(\frac{u}{\tau}, \frac{v}{\tau}\right| - \frac{1}{\tau}\left.\right) = \tau e^{\pi i \frac{(2v-u)u}{\tau}}\left[A_1(u, v|\tau) + \frac{1}{2}M(u-v|\tau)\, i\vartheta_1(v|\tau)\right]$
- Mordell integral $M(\nu|\tau) = \int dx\, q^{\frac{x^2}{2}} e^{-2i\pi n u x}/\cosh(\pi x)$

Maths intermede : Mock modular forms

- *Mock* modular forms $\mathbb{M}_K \; : \{\tau|\tau_2 \geqslant 0\} \to \mathbb{C}$ of weight K *almost* transform as a modular forms of the same weight (Zwegers, 2002)
- To $h \in \mathbb{M}_k$ one associates a *shadow* $g = \mathcal{S}(h)$, holomorphic modular form of weight $2 - K$
- Shadow map $\mathcal{S}$ ($\mathbb{R}$-linear) : such that $\hat{h} = h + g^*$ is a modular form of weight K, with $g^* = (\frac{i}{2})^{K-1} \int_{-\bar{\tau}}^{i\infty} \mathrm{d}z (z+\tau)^{-K} \overline{g(-\bar{\tau})}$.
- If h holomorphic, non-hol. encoded in the shadow: $\frac{\partial \hat{h}}{\partial \bar{\tau}} = \frac{1}{\tau_2^K} \overline{g(\tau)}$
- Trivial example: $\hat{E}_2 = E_2 - \frac{3}{\pi \tau_2}$ (constant g) ➡ mod. form of weight 2

Appell-Lerch sums

One gets non-holomorphic Jacobi forms of weight one
$\hat{A}_1(u,v|\tau) = A_1(u,v|\tau) + \frac{i}{2}\vartheta_1(v|\tau) R(u-v|\tau)$
with $R(\nu|\tau) = \sum_{n \in \mathbb{Z}+\frac{1}{2}} (-)^{n-1/2} \left[\mathrm{sign}(n) - E((n+\nu_2/\tau_2)\sqrt{2\tau_2})\right] e^{-i\pi\nu n} q^{-\frac{n^2}{2}}$

Back to elliptic genus

✔ Adding the shadow to the Appell-Lerche sum, the modular-covariant elliptic genus of warped Eguchi-Hanson reads

$$\widehat{E}(\nu|\tau) = \frac{1}{2}\sum_{j=1}^{k/2} \left(\chi_{k-2}^{j-1} + \chi_{k-2}^{k/2-j}\right) q^{-\frac{1}{k}\left(j-\frac{\ell+1}{2}\right)^2} \times$$

$$\times \frac{i}{\ell}\sum_{m=0}^{\ell-1} e^{-\pi i \frac{m}{\ell}} \widehat{A}_1\left(\frac{1}{\ell}\left(\nu + \left(\frac{\ell+1}{2} - j\right)\tau + m\right), \frac{\nu}{\ell}\middle|\tau\right) \frac{\vartheta_1(\nu|\tau)}{\eta(\tau)^3}\,.$$

✔ Elliptic genus of the $\mathcal{N}=(2,2)$ Liouville piece independently computed by modular-invariant regularization of the path integral *(Troost 2010)*

✔ The shadow is interpreted as the contribution from continuous representations (*i.e.* bulk modes) ➥ Non-BPS representations, but infinite volume of 'internal' target space cancels the fermionic zero-modes

SO(28) threshold from elliptic genus

- In this example, unbroken gauge group $SO(28) \times U(1)$.
- $SO(28)$ threshold ➥ spectral flow of elliptic genus + 'gauge' factors

$$\Lambda^{\ell}_{SO(28)} = -\frac{1}{384}\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2}\,\Gamma_{2,2}(T,U)\,\frac{1}{2}\sum_{j=1}^{k/2}\left(\chi^{J-1}_{k-2}+\chi^{k/2-j}_{k-2}\right) q^{-\frac{1}{k}\left(j-\frac{\ell+1}{2}\right)^2}\times$$
$$\times \sum_{(a,b)\neq(1,1)} \frac{i}{\ell}\sum_{m=0}^{\ell-1}\frac{1}{\eta}e^{\pi i\frac{m}{\ell}}\,\widehat{A}_1\left(\left(\frac{\ell a+1}{2}-J\right)\frac{\tau}{\ell}+\frac{b-1}{2}+\frac{m}{\ell}\right),\frac{(a-1)\tau+(b-1)}{2}\Big|\tau\right)\times$$
$$\times\frac{1}{\eta^{20}}\left(\hat{E}_2+(-)^b\vartheta\begin{bmatrix} a+b+1 \\ a\end{bmatrix}^4-(-)^a\vartheta\begin{bmatrix} b \\ a+b+1\end{bmatrix}^4\right)\vartheta\begin{bmatrix} a \\ b\end{bmatrix}^{16}.$$

$\vec{\ell}=(1^2,0^{14})$ case : enhanced $\mathcal{N}=(4,4)$

$$\Lambda^{1}_{SO(28)} = \frac{1}{24}\Lambda(K3)(T,U)+\frac{1}{8}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2}\Gamma_{2,2}(T,U)\frac{1}{288}\frac{\hat{F}(\hat{E}_2E_4-E_6)E_4}{\eta^{21}}$$

- First term ➥ $\frac{1}{24}$ of K3 threshold with instanton numbers $(16,8)$
- Second term ➥ $\hat{F}=F+g^*$ with F encoding the effect of the flux and warping, and the shadow g^* the contribution of bulk modes

Modular integral: unfolding technique

- Integration over the fundamental domain $\mathcal{F}$ of a non-holomorphic modular form ?
- Fundamental domain ➥ to avoid overcounting due to $PSL(2,\mathbb{Z})$ symmetry
- Idea: reduce the full sum over winding modes on T^2 to a sum over *orbits* of $PSL(2,\mathbb{Z})$ *(Dixon, Kaplunovsky, Louis 91)*

$$\Gamma_{2,2}(T,U) = \frac{T_2}{\tau_2} \sum_{n_1,n_2,m_1,m_2} \exp\left[2\pi i T \det A - \frac{\pi\, T_2}{\tau_2\, U_2} \left|(1,\ U)\ A \begin{pmatrix} \tau \\ -1 \end{pmatrix}\right|\right] ,$$

where the matrix A encodes mappings of the worldsheet onto T^2:

$$A = \begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix} , \qquad n_i,\ m_i \in \mathbb{Z}, \quad i = 1, 2 \,.$$

➥ the fundamental domain can be unfolded as we get the set of modular transformations mapping the fundamental domain inside the half-strip $\mathcal{S}$ or the upper half-plane $\mathbb{C}^+$

'Perturbative' corrections

- zero orbit ($A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$) and degenerate orbits ($A = \begin{pmatrix} 0 & j \\ 0 & p \end{pmatrix}$)

$$-\frac{T_1}{8}\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\hat{\mathcal{I}} - \frac{T_1}{8}\int_{\mathcal{S}} \frac{d^2\tau}{\tau_2^2} \sum_{(j,p)\neq(0,0)} e^{-\frac{\pi T_1}{\tau_2 U_1}|j+ipU|^2}\hat{\mathcal{I}}$$

- Localized modes contribution (with $E(z,k) = \frac{1}{\zeta(2k)} \sum_{(j_1,j_2)\neq(0,0)} \frac{(\mathrm{Im}\, z)^k}{|j_1+j_2 z|^{2k}}$) :

$$\Lambda^{pert}_{loc} = \frac{19\pi}{(24)^2}\, T_1 - \frac{1}{4}\left(\log|\eta(iU,1)|^4 + \log\left(T_1 U_1 \mu^2\right)\right) - \frac{29\pi}{1440}\frac{E(iU,2)}{T_1}$$

- Involved term from bulk modes (d_i's Fourier coeffs of Eisenstein series)

$$\Lambda^{pert}_{bulk} = \frac{1}{96}\left(-\frac{d_1}{2}\left(\log|\eta(iU,1)|^4 + \log\left(T_1 U_1 \mu^2\right)\right) - \frac{\pi d_2}{30}\frac{E(iU,2)}{T_1} - \right.$$

$$\left.\sum_{n=0}^{\infty}(-)^n\left[\sum_{m=0}^{\infty}(-)^m \frac{\pi^m}{m!}\frac{\left(\sqrt{2}(n+\frac{1}{2})\right)^{2m+1}}{m+\frac{1}{2}} T_1^{m+\frac{1}{2}}\left(d_1\left(\frac{n(n+1)}{2}\right)E^*\left(iU, m+\frac{1}{2}\right) - \frac{3d_2\left(\frac{n(n+1)}{2}\right)}{\pi T_1}E^*\left(iU, m-\frac{1}{2}\right)\right)\right]\right)$$

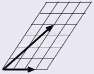
✔ These contributions are mapped to tree-level and perturbative contributions in the type I dual theory

Non-perturbative corrections (worldsheet instantons)

- non-degenerate orbits (the rest) ➥ worldsheet instantons on T^2

  $\int_{\mathbb{C}_+} \frac{d^2\tau}{\tau_2^2} \sum_{k>j\geq 0} \sum_{p\neq 0} e^{-2\pi k p T_1 - \frac{\pi T_1}{\tau_2 U_1}|k\tau - j - ipU|^2} \hat{\mathcal{I}}$
- Localized states contribution (with $\Box = (\mathcal{U}_2)^2\, \partial_{\mathcal{U}} \bar{\partial}_{\mathcal{U}}$)

  $\Lambda^{non-pert}_{loc} = -\frac{1}{4} \sum_{k>j\geq 0} \sum_{p\neq 0} \frac{1}{kp}\, e^{-2\pi k p T} \left[1 + \frac{1}{kpT_1} \Box \right] \hat{\mathcal{I}}_{loc.}(\mathcal{U})$



- $\mathcal{U} = (j + iU)/k$ is the 'induced' Kähler modulus of the torus wrapped by the instanton
- Similar type as in compact models ($i.e.$ for $T^4/\mathbb{Z}_2$) *(Bachas et al., 97)*
- More unusual contribution from bulk modes ➥ all inverse powers of T_1 despite $\mathcal{N} = 2$ supersymmetry

- In the type I dual these are mapped to D-instantons
- More interesting in $T^2 \to \mathcal{M}_6 \xrightarrow{\pi} EH$: worldsheet instantons can only wrap a finite number of times due to topology change

Conclusions

- We studied local models of flux heterotic compactifications to four dimensions, with line bundles
- Remarkably these throats admit weakly coupled solvable worldsheet CFT descriptions, valid *even* for large curvatures
- Computation of threshold corrections ➥ uses the machinery of mock modular forms, also very usefull in supersymmetric black holes microstates counting

Future directions

- Description of non-Abelian bundles
- More about conifold solutions: new branches, one-loop corrections,..
- Rational points of compact models ➥ gauged linear sigma-model techniques *(Adams, Lapan 2008)*
- Holographic understanding ➥ $\mathcal{N} = 2$ LSTs with compact Coulomb branches, confining $\mathcal{N} = 1$ theories, ...

Warped Conifold I : supergravity ansatz

- Local model of heterotic $\mathcal{N}=1$ compactification in 4d
  ➥ conifold singularity
  hypersurface $z_1^2+z_2^2+z_3^2+z_4^2=0$ in $\mathbb{C}^4$
- Singular cone over a a $T^{1,1}\sim(SU(2)\times SU(2))/U(1)$ base :
  S^1 fibration over $S^2\times S^2$ ➥ $S^3\times S^2$ topology
- Singularity regularized with a finite S^2 (blow-up) or a finite S^3 (deformation)
  ➥ the latter case is used in type IIB (Klebanov-Strassler) as the compact 3-cycle can support RR 3-form flux corresponding to fractional D3-branes
- In heterotic needs instead a 2 or 4-cycle (by Hodge duality) to support magnetic flux.
- We will consider the latter, which is topologically possible for the orbifold *conifold*$/\mathbb{Z}_2$

Heterotic supergravity ansatz

$$\mathrm{d}s^2 = \mathrm{d}x^\mu \mathrm{d}x_\mu + \frac{3}{2} H(r) \Big[\frac{\mathrm{d}r^2}{f^2(r)} + \frac{r^2}{6} \big(\mathrm{d}\theta_1^2 + \sin^2\theta_1 \, \mathrm{d}\phi_1^2 + \mathrm{d}\theta_2^2 + \sin^2\theta_2 \, \mathrm{d}\phi_2^2 \big) + \frac{r^2}{9} f(r)^2 \big(\mathrm{d}\psi + \cos\theta_1 \, \mathrm{d}\phi_1 + \cos\theta_2 \, \mathrm{d}\phi_2 \big)^2 \Big]$$

$$\mathcal{H}_{[3]} = \frac{\alpha' k}{6} g_1(r)^2 \left(\Omega_1 + \Omega_2 \right) \wedge \left(\mathrm{d}\psi + \cos\theta_1 \, \mathrm{d}\phi_1 + \cos\theta_2 \, \mathrm{d}\phi_2 \right)$$

- The S^1 fiber is deformed by the squashing factor $f(r) \leqslant 1$
  ➥ non-Kähler warped conifold with torsion
- Unlike in type II the $\mathbb{R}^{3,1}$ space-time is unwarped (in string frame)
- We will find below a 'bolt' for some $r = a$ ($f(a) = 0$), as in Eguchi-Hanson space
  ➥ conical singularity removed by the orbifold $\psi \sim \psi + 2\pi$

Susy, Bianchi and HYM

- The SUSY calibration conditions give the system

$$\begin{cases} f^2 H' = -2\alpha' k g_1^2 / r^3 \\ r^3 H f f' + 3r^2 H(f^2-1) + \alpha' k g_1^2 = 0 \end{cases}$$

- The HYM equations are satisfied with a line bundle ($\vec{p}$, $\vec{q}$ gives embedding in the Cartan $\vec{T}$):

Gauge bundle

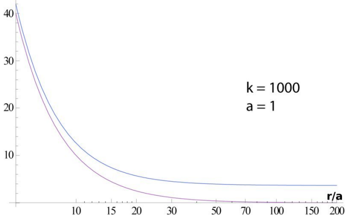
$$\mathcal{A} = \tfrac{1}{2}\vec{p}\cdot\vec{T}(\cos\theta_1\,\mathrm{d}\phi_1 - \cos\theta_2\,\mathrm{d}\phi_2) + \vec{q}\cdot\vec{T}\left(\tfrac{a}{r}\right)^4(\mathrm{d}\psi + \cos\theta_1\,\mathrm{d}\phi_1 + \cos\theta_2\,\mathrm{d}\phi_2)$$

- Second magnetic field ($\vec{q}$) responsible for the resolution of the conifold singularity ➥ free parameter a (blow-up parameter)

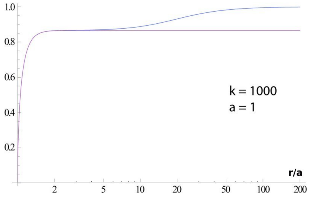
- In the large charges limit ($\vec{p}^2$,$\vec{q}^2 >> 1$) the $\mathcal{R}^2$ term in Bianchi identity is negligible (checked 'on-shell')
- $\mathrm{d}\mathcal{H} = \alpha' \mathrm{Tr}\mathcal{F}^2$ ➥ $g_1^2(r) = \frac{3}{4}(1-(\frac{a}{r})^4)$ and $k = \vec{p}^2 = 4\vec{q}^2$

Numerical solution

- One can solve numerically the susy equations for $f(r)$ and $H(r)$
- Smooth and weakly coupled everywhere
- For $r \to \infty$ one finds the usual Ricci-flat conifold and constant dilaton, however with non-zero NSNS and magnetic charges
- For $r \to a$, the S^1 fiber degenerates (bolt)
  ➥ one finds a non-Ricci flat conifold resolved by a 4-cycle



Conformal factor $H(r)$



Resolution function $f(r)$

both numerical (blue) and near-horizon (purple) solutions are plotted

Analytical solution in the double-scaling limit

- String coupling diverges in the blow-down limit (point-like instanton)
  ➥ One can define a double-scaling limit $g_s \to 0$ with $g_s \alpha' / a^2$ fixed
- It isolates the near-bolt region $a^2 \leqslant r^2 \ll \alpha' k$ from the asymptotically Ricci-flat region
- In this regime, one can solve analytically the susy equations

Double-scaling limit of the warped orbifoldized deformed conifold

$$ds^2 = \frac{\alpha' k}{2r^2} \left[\frac{dr^2}{1 - \frac{a^8}{r^8}} + \frac{r^2}{8} \left(d\theta_2^2 + \sin^2\theta_1 \, d\phi_1^2 + d\theta_2^2 + \sin^2\theta_2 \, d\phi_2^2 \right) + \frac{r^2}{16} \left(1 - \frac{a^8}{r^8}\right) \left(d\psi + \cos\theta_1 \, d\phi_1 + \cos\theta_2 \, d\phi_2 \right)^2 \right]$$

$$e^{2\Phi} = e^{2\Phi_0} \frac{(\alpha' k)^2}{r^4}$$

$$\mathcal{A} = \left[\frac{1}{2} \left(\cos\theta_1 \, d\phi_1 - \cos\theta_2 \, d\phi_2 \right) \vec{p} + \left(\frac{a}{r} \right)^4 \left(d\psi + \cos\theta_1 \, d\phi_1 + \cos\theta_2 \, d\tilde{\phi}_2 \right) \vec{q} \right] \cdot \vec{H}$$

$$\mathcal{B}_{[2]} = \frac{\alpha' k}{8} \left(\cos\theta_1 \, d\phi_1 + \cos\theta_2 \, d\phi_2 \right) \wedge d\psi$$

➥ Smooth weakly coupled background without sources (as $r \geqslant a$)

Worldsheet CFT in the blow-down limit

- In the blow-down limit one gets a linear dilaton $\times$ $T^{1,1}$
  ➥ Non-Einstein $[SU(2)^2]/U(1)$ coset w. torsion and & line bundle
- Obtained in the worldsheet CFT as a $\boxed{U(1)_{\mathrm{L}} \backslash^{SU(2)_k \times SU(2)_k}}$ asymmetrically gauged $\mathcal{N} = (1,0)$ WZW model (left action)
- Classical 'anomaly' of the gauging cancelled by an action in the $\widehat{SO(32)}_1$ or $(\widehat{E_8} \times \widehat{E_8})_1$ anti-holomorphic CFT
- Specified by a 16-dim vector of charges $\vec{p}$, with $k = \vec{p}^2$
  ➥ in space-time, Abelian magnetic field
- Possible to add an orbifold acting as
  $(g_1(z,\bar{z}), g_2(z,\bar{z})) \to (-g_1(z,\bar{z}), -g_2(z,\bar{z}))$

Worldsheet CFT in the double-scaling limit

- The heterotic orbifoldized warped conifold resolved by a 4-cycle has a worldsheet CFT *also* in the double scaling limit
- Complicated asymmetrically gauged WZW model

$$\frac{SL(2)_{k/2} \times U(1) \backslash SU(2)_k \times SU(2)_k}{U(1)_{\mathrm{L}} \times U(1)_{\mathrm{R}}}$$

➥ Needs gauging in $\widehat{G}_1$ specified by $\vec{q}$, with $k = \vec{p}^2 = 4\vec{q}^2 - \underbrace{4}_{\alpha' \text{ corr.}}$

Lowest ordrer background fields

- Asymmetric coset ➥ careful bosonization is needed *(Johnson)*
- Integrate out classically worldsheet gauge fields $(A, \bar{A})$ ➥ Σ-model corresponding to SUGRA solution
- 4-fermion term $\frac{2}{\cosh \rho}(\psi^1\psi^2 + \psi^3\psi^\rho)\ell_n \bar{\psi}^{2n-1}\bar{\psi}^{2n}$ ➥ Abelian bundle

One can extract from the gauged WZW model the background fields to all orders in α' ➥ exact solution to Bianchi !

Algebraic solution of the CFT

- Worldsheet CFT solved by standard coset construction ➥ full one-loop partition function known
- Left $\mathcal{N} = 2$ SCA ➥ $\mathcal{N} = 1$ 4d SUSY in spacetime as expected

Spectrum

- Discrete $SL(2)$ representations (localized bound states) ➥ massive $U(1)$ gauge field along $\vec{q} \cdot \vec{T}$ and set of massless 4d chiral multiplets localized at the bolt
- Unbroken gauge bosons and graviton ➥ non-normalizable massless modes and continuum of massive modes

Localized states: interacting d.o.f. of the blow-down theory (kind of Little String Theory) ?

Worldsheet instantons and Liouville potentials

- So far, the $\mathbb{Z}_2$ orbifold of $T^{1,1}$ put 'by hand' in the CFT ➥ is there an analogue of no-conical singularity condition of sugra ?
- Condition on the bundle $c_1(V) \in H^2(2\mathbb{Z})$ ➥ role in worldsheet CFT ?
- $\frac{SL(2)}{U(1)}$ corrected by worldsheet non-perturbative effects *(Fateev, Zamolodchikov)*

Liouville-like potentials for Abelian bundles

- In our heterotic coset, the dynamically generated $\mathcal{N} = (2,0)$ Liouville potential reads $\mu_L \int (\psi^\rho + \psi^3) e^{-\frac{\sqrt{\vec{q}^2-4}}{2}(\rho + iY_L^3) - \frac{i}{2}\vec{q}\cdot\vec{X}_R} + c.c.$
  ($\bar{\partial} X_R^n = \bar{\psi}^{2n-1}\bar{\psi}^{2n}$ is the bosonized Cartan)
- Belongs to the twisted sector of the $\mathbb{Z}_2$ orbifold
- Orbifold and (right) GSO invariant only if $c_1(V) \in H^2(2\mathbb{Z})$: Explicitly $\sum_\ell (p_\ell \pm q_\ell) \equiv 0 \mod 4$

Remark : for $Spin(32)/\mathbb{Z}_2$ both bundles with or without vector structure are possible