Entanglement entropies in minimal models

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Entanglement entropies in minimal models from null vectors
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1. Introduction

2. The null-vector approach

3. Entanglement entropies in the Yang-Lee model

4. Further studies of the cyclic orbifold
1. Introduction
Entanglement entropies in quantum systems

- Density matrix of the whole system $A + B$
  - Pure state $|\psi\rangle \Rightarrow \rho = |\psi\rangle \langle \psi|$
  - Mixed state at temperature $\beta \Rightarrow \rho = \frac{1}{Z} \exp(-\beta H)$
Entanglement entropies in quantum systems

Density matrix of the whole system $A + B$

- Pure state $|\psi\rangle$ $\Rightarrow$ $\rho = \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}$
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Reduced density matrix of subsystem $A$: $\rho_A = \text{Tr}_B(\rho)$
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  - Pure state $|\psi\rangle$ \implies $\rho = \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}$
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- Rényi entropy: $S_N(A) = \frac{1}{1-N} \log \text{Tr}_A(\rho_A^N)$
Entanglement entropies in quantum systems

- Example: two $1/2$-spins $A, B$

\[ |\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle \]

\[ \rho_A = |\psi_A\rangle \langle \psi_A | \Rightarrow S(A) = 0 \]

Region $A$ is effectively in a pure state.

\[ \psi = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle + e^{i\phi} |\downarrow\uparrow\rangle) \]

\[ \rho_A = \frac{1}{2} (|\uparrow\rangle \langle \uparrow| + |\downarrow\rangle \langle \downarrow|) \Rightarrow S(A) = \log 2 \]

Region $A$ is effectively in a thermal state.

Area law:
For $d \geq 1 + 1$, the entropy in the groundstate of a gapped, short-range Hamiltonian scales generally as $S(A) \propto \text{Area}(\partial A)$.

̸= Gapless systems in $d = 1 + 1$: Conformal Field Theory [Holzhey-Larsen-Wilczek '94, Calabrese-Cardy '04]

\[ S(A) \sim c \frac{\log \ell_A}{\ell_A} \]
Entanglement entropies in quantum systems

- Example: two 1/2-spins $A, B$
  - Product state: $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$
  - $\rho_A = |\psi_A\rangle\langle\psi_A| \implies S(A) = 0$
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Entanglement entropies in quantum systems

- Example: two 1/2-spins $A, B$
  - Product state: $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$
    
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  - Entangled state $\psi = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + e^{i\phi}|\downarrow\uparrow\rangle)$
    
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    $ρ_A = |ψ_A⟩⟨ψ_A|$ $⇒$ $S(A) = 0$
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  - Entangled state $ψ = \frac{1}{√2} (|↑↓⟩ + e^{iφ}|↓↑⟩)$
    $ρ_A = \frac{1}{2} (|↑⟩⟨↑| + |↓⟩⟨↓|)$ $⇒$ $S(A) = \log 2$
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- Area law:
  For $d \geq 1 + 1$, the entropy in the groundstate of a gapped, short-range Hamiltonian scales generally as $S(A) ∝ \text{Area}(∂A)$. 

[Holzhey-Larsen-Wilczek '94, Calabrese-Cardy '04]
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  $S(A) \sim \frac{c}{3} \log \ell_A$
The path-integral formalism for Rényi entropies

[Calabrese-Cardy '04]

\[ \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B : \]

\[ H = H_A \otimes H_B : \]

\[ \text{Reduced density matrix} \]

\[ \alpha(\rho_A)_{\alpha\beta} = \beta \]

\[ \text{Rényi entropy} \]

\[ \text{Tr}\left[ (\rho_A)^N \right] = Z(R^N) \]
The path-integral formalism for Rényi entropies

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(\rho_A)_{\alpha\beta} = \frac{\text{Tr}[\rho_A^N]}{Z_N} = \frac{\text{Tr}[\rho_B^N]}{Z_N}
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The path-integral formalism for Rényi entropies

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- Rényi entropy

$$ \text{Tr}[(\rho_A)^N] = = Z(\mathcal{R}_N) $$
Partition functions on Riemann surfaces

- Rényi entropy: $S_N = \frac{1}{1-N} \log \frac{Z(\mathcal{R}_N)}{Z^N}$
Partition functions on Riemann surfaces

- Rényi entropy: \( S_N = \frac{1}{1 - N} \log \frac{Z(\mathcal{R}_N)}{Z^N} \)

- One interval \([u, v]\):
  \[
  \begin{cases}
  \mathcal{R}_N & \to \mathbb{C} \cup \{\infty\} \\
  z & \mapsto w = \left(\frac{z-u}{z-v}\right)^{1/N}
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- Generic case:
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  \begin{align*}
  &A = \text{union of} \ p \text{ intervals} \\
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  \[\Rightarrow \text{genus}(\mathcal{R}_N) = (N - 1)(p - 1)\]
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- Example: \( p = 2, N = 4 \)

\( \mathcal{R}_N \cong \) [Diagram of a Riemann surface with four intervals]
Partition functions on Riemann surfaces

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- For \( c = 1 \): results available from [Zamolodchikov ’87], [Dixon, Friedan, Martinec, Shenker ’87], [Alvarez-Gaumé, Gost, Moore, Nelson, Vafa ’87], [Dijkgraaf, Verlinde, Verlinde ’88]
Overview of Entanglement Entropies in CFT

- Scaling argument $S \propto \frac{c}{3} \log \ell$ [Holzhey, Larsen, Wilczek '94]:
- Path-integral approach, EE by conformal mapping for $A = [u, v]$ [Calabrese, Cardy '04]
- Compute EE at $g > 0$ for $c = 1$ and/or Ising CFTs [Calabrese, Cardy, Tonni, Tagliacozzo, Alba, Datta, David, Misguich, Pasquier, Stéphan, Furukawa, Shiraishi, Essler, Campostrini, Nienhuis '06–'12]
- EE for excited states [Sierra, Alcaraz, Berganza, Palmai '12–'16]
- EE for integrable QFTs [Castro-Alvaredo, Doyon, Cardy, Blondeau-Fournier '07–'15]:
- EE, entanglement spectrum, fidelity using CTM [Franchini, Its, Korepin, Takhtajan, Evangelisti, Weston '11–'12]
- Entanglement after a quench [Cardy '11]
- EE for non-unitary CFTs [Castro-Alvaredo, Doyon, Ravanini, Bianchini, Levi, Couvreur, Jacobsen, Saleur '14–'17]
- Entanglement spectra in FQHE [Li, Haldane, Read, Rezayi, Dubail, Eisler, Peschel, Cardy, Tonni '08–'17]:
- . . .
2. The null-vector approach
The $\mathbb{Z}_N$ cyclic orbifold CFT

[Crnković-Sotkov-Stanishkov '89, Klemm-Schmidt '90, Borisov-Halpern-Schweigert '98]

“Replicate the degrees of freedom instead of the surface”
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- Local configuration of fields: $(\phi_1, \ldots, \phi_N)$

$\langle \tau(u_1) \tilde{\tau}(v_1) \ldots \tau(u_p) \tilde{\tau}(v_p) \rangle$

$\langle \Phi(\infty) \tau(u) \tilde{\tau}(v) \Phi(0) \rangle$, $\Phi := \phi_{12} \otimes \cdots \otimes \phi_{12}$
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- Twist operator inserting a branch point:

$$
\tau(0). (\phi_1, \ldots, \phi_{N-1}, \phi_N) (e^{2i\pi z}) = \tau(0). (\phi_2, \ldots, \phi_N, \phi_1)(z)
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- Examples:
  - $\langle \tau(u_1) \tilde{\tau}(v_1) \ldots \tau(u_p) \tilde{\tau}(v_p) \rangle$
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Generating algebra [Crnković-Sotkov-Stanishkov '89, Klemm-Schmidt '90, Borisov-Halpern-Schweigert '98]

- Family of currents $\hat{T}(r)(z) = \sum_{j=1}^{N} e^{2i\pi rj/N} T_j(z)$
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- Fourier modes: $\hat{L}_m^{(r)} = \frac{1}{2i\pi} \int dz \ z^{m+1} \hat{T}^{(r)}(z) \quad m \in \mathbb{Z}/N$
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- Orbifold Virasoro algebra:

$$\left[ \hat{L}^{(r)}_m, \hat{L}^{(s)}_n \right] = (m - n)\hat{L}^{(r+s)}_{m+n} + \frac{Nc}{12} m(m^2 - 1) \delta_{m+n,0} \delta_{r+s,0}$$
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- Operator content
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- Operator content
  - Untwisted primary fields: $\phi_1 \otimes \cdots \otimes \phi_N$, $h = h_1 + \cdots + h_N$
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- Operator content
  - Untwisted primary fields: $\phi_1 \otimes \cdots \otimes \phi_N$, $h = h_1 + \cdots + h_N$
  - Twisted primary fields: $\tau_\phi = :\tau\phi:$
Induction procedure

[Crnković-Sotkov-Stanishkov '89, Borisov-Halpern-Schweigert '98]

- Consider (radial) quantisation around a branch point
Induction procedure

[Crnković-Sotkov-Stanishkov '89, Borisov-Halpern-Schweigert '98]

- Consider (radial) quantisation around a branch point
- Conformal mapping $z \mapsto w = z^{1/N}$ in the vicinity of $z = 0$: 

\[
\hat{T}(r)(z) \mapsto w^2 - 2N \sum_{j=1}^{N} e^{2i\pi(j+2)N} T(e^{2i\pi jN}w) + c24 z^2(N-1)N \delta r, 0
\]

\[
\hat{L}(r)m \mapsto \frac{1}{N} L Nm + c24 (N-1)N \delta r, 0 \delta m, 0
\]

- Applications:
  1. $\tau_{\phi}(z) \mapsto w(1-N)h_{\phi}(w)$
  2. If $\phi$ degenerate at level $\ell$ then $\tau_{\phi}$ degenerate at level $\ell/N$

- Basic example: $L - 1 = 0 \implies \hat{L}(-1)^{-1/N} \tau_{\phi} = 0$
Induction procedure

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- Consider (radial) quantisation around a branch point
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  - $\hat{T}^{(r)}(z) \mapsto w^{2-2N} \sum_{j=1}^{N} e^{\frac{2i\pi(j+2)}{N}} T(e^{\frac{2i\pi j}{N}} w) + \frac{c}{24z^2} \left(N - \frac{1}{N}\right) \delta_{r,0}$
Induction procedure

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  - $\hat{L}_{m} \mapsto \frac{1}{N} L_{Nm} + \frac{c}{24} \left( N - \frac{1}{N} \right) \delta_{r,0} \delta_{m,0}$

- Applications:
  1. $\tau \phi$ has dimension $\hat{h} \phi = c_{24} \left( N - \frac{1}{N} \right)$
  2. If $\phi$ degenerate at level $\ell$ then $\tau \phi$ degenerate at level $\ell/N$
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  - $\widehat{T}^{(r)}(z) \mapsto w^{2-2N} \sum_{j=1}^{N} e^{\frac{2i\pi(j+2)}{N}} T(e^{\frac{2i\pi j}{N}} w) + \frac{c}{24z^2} \left( N - \frac{1}{N} \right) \delta_{r,0}$
  - $\widehat{L}_m \mapsto \frac{1}{N} L_{Nm} + \frac{c}{24} \left( N - \frac{1}{N} \right) \delta_{r,0} \delta_{m,0}$
  - $\tau_\phi(z) \mapsto w^{(1-N)h_\phi} \phi(w)$
Induction procedure

[Crnković-Sotkov-Stanishkov '89, Borisov-Halpern-Schweigert '98]

▪ Consider (radial) quantisation around a branch point

▪ Conformal mapping \( z \mapsto w = z^{1/N} \) in the vicinity of \( z = 0 \):

  ▪ \( \hat{T}^{(r)}(z) \mapsto w^{2-2N} \sum_{j=1}^{N} e^{\frac{2i\pi(j+2)}{N}} T(e^{\frac{2i\pi j}{N}} w) + \frac{c}{24z^2} \left( N - \frac{1}{N} \right) \delta_{r,0} \)

  ▪ \( \hat{L}_{m}^{(r)} \mapsto \frac{1}{N} L_{Nm} + \frac{c}{24} \left( N - \frac{1}{N} \right) \delta_{r,0}\delta_{m,0} \)

  ▪ \( \tau_{\phi}(z) \mapsto w^{(1-N)h_{\phi}} \phi(w) \)

▪ Applications:
Consider (radial) quantisation around a branch point

Conformal mapping $z \mapsto w = z^{1/N}$ in the vicinity of $z = 0$:

- $\widehat{T}^{(r)}(z) \mapsto w^{2-2N} \sum_{j=1}^{N} e^{\frac{2i\pi(j+2)}{N}} T(e^{\frac{2i\pi j}{N}} w) + \frac{c}{24z^2} \left(N - \frac{1}{N}\right) \delta_{r,0}$

- $\widehat{L}^{(r)} \mapsto \frac{1}{N} L_{Nm} + \frac{c}{24} \left(N - \frac{1}{N}\right) \delta_{r,0}\delta_{m,0}$

- $\tau_\phi(z) \mapsto w^{(1-N)h_\phi} \phi(w)$

Applications:

1. $\tau_\phi$ has dimension $\widehat{h}_\phi = \frac{c}{24} (N - \frac{1}{N}) + \frac{h}{N}$
Induction procedure

[Crnković-Sotkov-Stanishkov '89, Borisov-Halpern-Schweigert '98]

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- Applications:
  1. \( \tau_\phi \) has dimension \( \hat{h}_\phi = \frac{c}{24} \left( N - \frac{1}{N} \right) + \frac{h}{N} \)
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▶ Basic example: $L_{-1} \mathbf{1} = 0 \Rightarrow \hat{L}^{(-1)}_{-1/N} \tau = 0$
The $\mathbb{Z}_N$ orbifold of a minimal model

- EE in minimal model $\mathcal{M}(p, q)$ with $c = 1 - \frac{6(p-q)^2}{pq}$
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\[
\begin{cases}
L_{-1}1 = 0 \\
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The $\mathbb{Z}_N$ orbifold of a minimal model

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  - \[
  \begin{array}{ccc}
  \epsilon & \sigma & 1 \\
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- $\Rightarrow$ \[
  \begin{cases}
  \hat{L}_{-1/N} \tau = 0 \\
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From null vectors to differential equations

- Four-point functions in terms of \( x = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)} \):

\[
\langle \mathcal{O}_1(z_1, \bar{z}_1) \ldots \mathcal{O}_4(z_4, \bar{z}_4) \rangle = (\ldots) \langle \mathcal{O}_1(\infty)\mathcal{O}_2(1)\mathcal{O}_3(x, \bar{x})\mathcal{O}_4(0) \rangle
\]

In standard CFT, action of \( L^{-m} \) for \( m \in \mathbb{N} \): [BPZ '84]

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\langle \mathcal{O}_1(\infty)\mathcal{O}_2(1)\mathcal{O}_3(x, \bar{x})\mathcal{O}_4(0) \rangle = L^{-m} \langle \mathcal{O}_1(\infty)\mathcal{O}_2(1)\mathcal{O}_3(x, \bar{x})\mathcal{O}_4(0) \rangle
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In orbifold Virasoro algebra: action of \( \hat{L}^{-m} \) for \( m \in \mathbb{Z}^+ \)?

Strategy: obtain linear relation between \( \langle \mathcal{O}_1(\hat{L}^{-r_1}m_1)\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4 \rangle \), \( \langle \mathcal{O}_1(\hat{L}^{-r_2}m_2)\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4 \rangle \), \( \langle \mathcal{O}_1\mathcal{O}_2(\hat{L}^{-r_3}m_3)\mathcal{O}_3\mathcal{O}_4 \rangle \), \( \langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3(\hat{L}^{-r_4}m_4)\mathcal{O}_4 \rangle \) using orbifold Ward identities.

Closed-contour relations:

\[
\oint dz \left( z - 1 \right)^{m_2+1} \left( z - x \right)^{m_3+1} z^{m_4+1} \langle \mathcal{O}_1(\infty)\mathcal{O}_2(1)\mathcal{O}_3(x, \bar{x})\mathcal{O}_4(0) \rangle = 0
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  \]

- In standard CFT, action of \( L_{-m} \) for \( m \in \mathbb{N} \): \[\text{[BPZ '84]}\]
  \[
  \langle \mathcal{O}_1(\infty)\mathcal{O}_2(1)\mathcal{O}_3(x, \bar{x})(L_{-m}\mathcal{O}_4)(0) \rangle = L_{-m}\langle \mathcal{O}_1(\infty)\mathcal{O}_2(1)\mathcal{O}_3(x, \bar{x})\mathcal{O}_4(0) \rangle
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- In standard CFT, action of $L_{-m}$ for $m \in \mathbb{N}$: [BPZ '84]

$$\langle O_1(\infty)O_2(1)O_3(x, \bar{x})(L_{-m}O_4)(0) \rangle = L_{-m}\langle O_1(\infty)O_2(1)O_3(x, \bar{x})O_4(0) \rangle$$

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\[
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\langle O_1 O_2 (\hat{L}^{(r_3)}_{m_3} O_3) O_4 \rangle , \langle O_1 O_2 O_3 (\hat{L}^{(r_4)}_{m_4} O_4) \rangle
\]

using orbifold Ward identities = Closed-contour relations:

\[
\int_C dz \ (z - 1)^{m_2+1} (z - x)^{m_3+1} z^{m_4+1} \\
\times \langle O_1(\infty) O_2(1) O_3(x, \bar{x}) \hat{T}^{(r)}(z) O_4(0) \rangle = 0
\]

where \( O_j \in [k_j] \Rightarrow m_j \in \mathbb{Z} + r k_j / N \)
Summary of the null-vector approach

1. Consider eigenstate $|\psi\rangle$ of $H_{M(p,q)}$
   - Write Rényi EE $S_N(A) = \frac{1}{1-N} \log G(x, \bar{x})$
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   \[
   G(x, \bar{x}) = \sum_{i=1}^{M} X_i |I_i(x)|^2 = \sum_{j=1}^{M} Y_j |J_j(x)|^2
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6. Solve consistency for $\{X_i\} \leftrightarrow \{Y_j\}$ (“conformal bootstrap”)
3. Entanglement entropies in the Yang-Lee model
The Yang-Lee edge singularity

[ Yang-Lee '52 ]

- Classical Ising model in magnetic field:

\[
Z(J, H) = \sum_{\{\sigma\}} \exp \left( J \sum_{\langle i, j \rangle} \sigma_i \sigma_j + H \sum_j \sigma_j \right)
\]
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[Yang-Lee '52]

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- Distribution of zeros of \( Z(J, H) \) for \( T > T_c \):

\[ \text{Im } H \uparrow \]

\[ \text{Re } H \]

\[ ih_c \]

\[ -ih_c \]
The Yang-Lee model as a minimal model of CFT
[Cardy ’85]

- **Scaling limit:** fixed \( T > T_c \) and \( H = i h \rightarrow i h_c \) [Fisher ’78]
  - Density of zeros \( \rho(h, T) \sim (h - h_c)^\sigma \)
  - Free energy \( F(h, T) = \int dx \rho(x, T) \log(h - ix) \)
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- Scaling features
  - No internal symm, only one non-trivial primary field $\phi$
  - OPE: $\phi \times \phi \rightarrow 1 + \phi$ Scaling law: $\sigma = \frac{h_{\phi}}{1 - h_{\phi}}$
  - high-T exp. $\rightarrow \sigma \simeq -0.163 \rightarrow h_{\phi} \simeq -0.195$

Corresponds to minimal model $\mathcal{M}(5, 2) = 1\phi\phi^1$

Central charge $c = -\frac{22}{5}$, dimensions $h_1 = 0, h_{\phi} = -\frac{1}{5}$

Exactly solved RSOS model in same universality class [Andrews-Baxter-Forrester ’84]
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The RSOS quantum chain

- Basis states: \( |a_1, a_2, \ldots a_L \rangle \) with \( \left\{ a_i \in \{1, \ldots, p - 1\} \right\} \) and \( |a_i - a_{i+1}| = 1 \)
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- Hamiltonian: \( H = - \sum_{i=1}^{L} e_i \) with PBC

\( \lambda = \pi \left( \frac{p-q}{p} \right) \) with \( p > q \) coprime → scaling limit = \( M(p, q) \)

\( \text{The YL case: } p = 5, \lambda = \frac{3}{5} \pi \)
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- **Action on states:**

\[e_i|\ldots a_{i-1}, a_i, a_{i+1} \ldots\rangle = \delta_{a_{i-1}, a_{i+1}} \sum_{a_i'} \frac{\sin \lambda a_i'}{\sin \lambda a_i} |\ldots a_{i-1}, a'_i, a_{i+1} \ldots\rangle\]
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EEs of a non-unitary model

see also [Bianchini, Castro-Alvaredo, Doyon, Levi, Ravanini '14]

- $h_1 = 0$: the conformally invariant state is $|1\rangle$. 

EEs of a non-unitary model

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- $h_\phi = -1/5 < 0$: the ground state is $|\phi\rangle$. 
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EE in ground state $|\phi\rangle$:

$$\langle \Phi(\infty)\tau_\phi(1)\tilde{\tau}_\phi(x, \bar{x})\Phi(0) \rangle = G(x, \bar{x}) \quad \text{with } \Phi = \phi \otimes^N$$
$N = 3$ Rényi entropy in state $|1\rangle$
$N = 3$ Rényi entropy in the ground state $|\phi\rangle$
4. Further studies of the cyclic orbifold
Coulomb Gas approach

- Dictionary between minimal CFTs and “imaginary Liouville” action [Dotsenko-Fateev '84]:

$$A(\phi) = \int d^2x \left[ (\nabla \phi)^2 + 2iQR\phi + \# e^{ib\phi} + \# e^{-i\phi/b} \right]$$

with $2Q = b - 1/b$
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- Central charge \(c = 1 - 24Q^2\)

Vertex operators \(V_\alpha = e^{i\alpha\phi}\), \(h_\alpha = h_{2Q-\alpha} = \alpha^2 - 2Q\alpha\)

Kac table \(\Phi_{rs} \leftrightarrow \alpha_{rs} = \frac{(1-r)b}{2} - \frac{(1-s)}{2b}\)
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  \[ \sum_j \alpha_j + 2(g - 1)Q = 0 \]
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- Correlation functions: \(\langle \Phi_1 \ldots \Phi_n \rangle \leftrightarrow \langle V_{\alpha_1} \ldots V_{\alpha_n} Q^k Q^\ell \rangle\)
  - Neutrality: \(\sum_j \alpha_j + 2(g - 1)Q + kb - \ell/b = 0\)
Some working examples with the CG

- One-interval, generic $N$ entropy in the state $\phi_{21}$
  - Recall $\alpha_{21} = -b/2$
  - $\langle \Phi(0)\tau_h(z, \bar{z})\tilde{\tau}_h(1)\Phi(\infty) \rangle$ with $\Phi = \phi_{21}^N$
  - Insert $Q_+^N = (\oint dz V_b)^N$
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  - For $N = 2$: we obtain full action of monodromy group on conformal blocks
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- Two-interval $N = 2$ entropy in the vacuum state
  - Minimal model $\mathcal{M}(p, q)$: CG parameter $b = \sqrt{q/p}$
  - Four-point function $\langle \tau(u_1, \bar{u}_1)\tilde{\tau}(v_1, \bar{v}_1)\tau(u_2, \bar{u}_2)\tilde{\tau}(v_2, \bar{v}_2) \rangle$
  - Associate vertex charges $(0, 2Q)$ to $(\tau, \tilde{\tau})$
  - Neutrality: $4Q + (p - 2)b - (q - 2)/b = 0$
  - $(p - 1)(q - 1)$ choices of contours?
Modular invariance for $\mathbb{Z}_N$ orbifolds

- In any rational CFT:
  - Torus partition function: $Z(\tau, \bar{\tau}) = \sum_j |\chi_j(\tau)|^2$
  - Modular S-matrix: $\chi_j(-1/\tau) = \sum_k S_{jk} \chi_k(\tau)$
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- Apply to $\mathbb{Z}_N$ orbifold of $\mathcal{M}(p, q)$ (for prime $N$):
  - Describe set of characters
  - Explicit $S$-matrix: (non-trivial even for $N = 2$)
  - Obtain fusion rules for twisted and untwisted operators
Conclusions and perspectives

- Current results
  - “Standard” computations of EE: only $g = 0$ or $c \in \{\frac{1}{2}, 1\}$
  - New approach based on $\mathbb{Z}_N$-orbifold of Virasoro algebra
  - Works for minimal models (twist has two null vectors)
  - Applied at $g = 0$ for YL: gives non-trivial $\langle \ldots \tau \tilde{\tau} \tau \ldots \rangle$
  - Tested at $g = 1$ for YL+Ising: recover $\{\chi_j\}$

- Future work
  - Find systematic derivation of differential equations?
  - Use mod. invariance to get fusion rules in $\mathbb{Z}_N$-orbifold?
  - Construct Coulomb-Gas formalism for conformal blocks in $\mathbb{Z}_N$-orbifold? [joint with O. Blondeau-Fournier (Laval)]
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Thank you!