Massless scalar field in curved space

1 The classical problem

We consider a massless real valued scalar field ϕ on an oriented two-dimensional Riemannian manifold (M, g) with action

$$S[\phi] = \frac{\beta}{4\pi}\int_M g(d\phi, d\phi)\omega_g \tag{1}$$

where ω is the canonical Riemannian volume form. In local coordinate x^μ this means

$$S[\phi] = \frac{\beta}{4\pi}\int g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\, dV(x), \qquad dV(x) = \sqrt{|g(x)|}d^2x \tag{2}$$

where $|g(x)| = g_{11}(x)g_{22}(x) - g_{12}^2(x)$ is the determinant of the matrix $g_{\mu\nu}(x)$, and $g^{\mu\nu}(x)$ its inverse matrix, namely $g^{\mu\nu} = \frac{1}{|g|}\begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11}\end{pmatrix}$.

1. Check that the action is Weyl invariant. What happens for a massive scalar field with action

$$S[\phi] = \frac{\beta}{4\pi}\int \left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2\right) dV(x)\,? \tag{3}$$

2. Using Stokes theorem (see Appendix), recast the action as

$$S[\phi] = \frac{\beta}{4\pi}\int \phi\Delta_g\phi\, dV \tag{4}$$

   where Δ_g is the laplacian as given in local coordinates by

$$\Delta_g\phi = -\frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}g^{\mu\nu}\partial_\nu\phi\right) = -\nabla^\mu\nabla_\mu\phi \tag{5}$$

3. Show that the equation of motion for the massless scalar field is: $\Delta_g\phi = 0$.
4. Check $\Delta_{e^{2\sigma}g} = e^{-2\sigma}\Delta_g$, and conclude that the classical solutions are indeed Weyl invariant.
5. Compute the classical stress-energy tensor (*a la* Hilbert). Check that it is traceless and that $\nabla_\mu T^{\mu\nu}$ vanishes on-shell.
6. This theory is manifestly invariant under $\phi \to \phi + a$ for any constant a. What is the (Noether) current associated with this global U(1) symmetry ?

2 Gaussian QFT and zeta regularization

The path-integral formulation of the free scalar field is Gaussian : $Z = \int [D\phi]e^{-\frac{\beta}{4\pi}\int\phi\Delta_g\phi\, dV}$. By analogy with the finite-dimensional case (see Appendix A), it is natural do postulate that the partition function of the free boson is

$$Z_g = \frac{1}{\sqrt{\det\frac{\beta}{(2\pi)^2}\Delta_g}} \tag{6}$$

But it turns out that defining the above determinant is a rather subtle affair. Before doing so, let us recall some facts about the laplacian on a Riemann manifold. The laplacian or Laplace-Beltrami operator Δ_g is ubiquitous in physics, and it controls for instance: (i) heat diffusion, (ii) wave propagation, and (iii) the Schrödinger equation for a free (non-relativistic) particle of mass m evolving on M, whose Hamiltonian is $H = \frac{\hbar^2}{2m}\Delta_g$. On a compact Riemannian manifold M without boundary, Δ_g has a discrete non-negative spectrum. Let $\lambda_0 \le \lambda_1 \le \cdots$ be its eigenvalues, with L^2 normalized eigenstates[1]

$$\Delta\phi_n = \lambda_n\phi_n, \qquad \lambda_n \ge 0$$

and the kernel is made of all locally constant functions. Thus the dimension of the kernel is the number of connected components of M. This is not a surprise, these are nothing but the classical solutions. From now on we will assume M to be connected, in which case the kernel is one-dimensional and is simply the subspace of all constant functions. In quantum mechanics notations[2]

$$\Delta|\phi_0\rangle = 0, \qquad \phi_0(x) = \langle x|\phi_0\rangle = \frac{1}{\sqrt{\mathrm{Vol}(M)}}$$

where $\mathrm{Vol}(M)$ is the total volume of the (compact) manifold M. If we decompose

$$\phi(x) = \sum_{n=0}^{\infty} \alpha_n\phi_n(x) \tag{7}$$

in the path integral formulation (6), it appears that the modes α_n are independent gaussian random variables with a variance $\left(\frac{\beta}{4\pi}\lambda_n\right)^{-1}$.

Upon trying to define the determinant of the laplacian, we face several issues. A first issue is the presence of a zero eigenvalue. Since we want to define a non-trivial determinant, we exclude this vanishing eigenvalue, and we denote by $\det'(\Delta)$ the corresponding determinant with the zero eigenvalue omitted. A careful treatment of this zero mode requires to define the partition function as

$$Z = \sqrt{\frac{\mathrm{Vol}(M)}{\det'(\Delta)}} \tag{8}$$

A second, more dramatic issue is that the product of all non zero eigenvalues $\prod_n' \lambda_n$ is divergent (see eq. (11)). So we must propose a different, regularized definition for $\det'(\Delta)$.

A particularly elegant way to regularize functional determinants is the so-called **zeta regularization**. First defined the meromorphic function

$$\zeta_\Delta(s) = \sum_{n;\lambda_n>0} \frac{1}{\lambda_n^s} = \sum_n{}' \frac{1}{\lambda_n^s} \tag{9}$$

where $\sum'$ stands for the sum over all non-zero eigenvalues. A straightforward (but formal) calculation yields

$$\zeta_\Delta'(s) = \sum_n{}' \frac{\ln\lambda_n}{\lambda_n^s} \qquad \text{and therefore} \qquad \zeta_\Delta'(0) = \sum_n{}' \ln\lambda_n = \ln\prod_n{}'\lambda_n \tag{10}$$

This formal computation motivates the following definition : $\boxed{\det'(\Delta) = e^{-\zeta_\Delta'(0)}\,.}$

The actual meaning of the above equation is as follows. The series defining $\zeta_\Delta(s)$ is convergent for $\mathrm{Re}(s) > d/2$. This comes from the asymptotic behavior of the eigenvalues of second order elliptic operators on d-dimensional manifolds (as shown by Weyl):

$$\lambda_n^{d/2} \sim \frac{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}+1\right)}{\mathrm{vol}_g(M)} n \tag{11}$$

[1] Here we work with the Hilbert space of maps from M to $\mathbb{R}$ (with L^2 scalar product, using the volume $\omega = \sqrt{|g|}dx^1 \wedge \cdots \wedge dx^d$).

[2] We choose position eigenstates normalized as scalars $\langle x|y\rangle = \delta_y(x)$, where the Dirac delta $\delta_y(x)$ is normalized such that $\int_M f(x)\delta_y(x)\sqrt{|g(x)|}d^2x = f(y)$. This means that $1 = \int_M |x\rangle\langle x|\,dV(x)$.

Furthermore $\zeta_\Delta(s)$ is holomorphic in the variable s, and it turns out that it can be analytically continued to a holomorphic function in the vicinity of $s = 0$. Thus $\zeta'_\Delta(0)$ makes sense. We will now prove this.

7. Using the identity

$$\frac{1}{\lambda^s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-t\lambda} dt \tag{12}$$

show that the zeta-function ζ_Δ can be expressed in terms of the Heat-kernel

$$K(t,x,y) = \langle x| e^{-\Delta t} |y\rangle \tag{13}$$

i.e. the solution of the Heat equation $\partial_t K(t,x,y) = -\Delta_x K(t,x,y)$ with initial condition: $\lim_{t\to 0^+} K(t,x,y) = \delta_y(x)$.

8. Using the small time asymptotic of the Heat kernel on a Riemann surface (without boundary) :

$$K(t,x,x) = \frac{1}{4\pi t} + \frac{R(x)}{24\pi} + O(t) \tag{15}$$

show that $\zeta_\Delta(s)$ can be analytically continued to $\mathbb{C} \setminus \{1\}$, with a single pole at $s = 1$. What is $\zeta_\Delta(0)$? And the residue at $s = -1$?

9. We want to compute the effect of a Weyl transformation $\tilde{g}_{\mu\nu}(x) = e^{2\sigma(x)} g_{\mu\nu}(x)$ on the functional determinant of the laplacian Δ. Let us consider the effect of an infinitesimal Weyl transformation

$$g_{\mu\nu}(x) \to e^{2\sigma(x)} g_{\mu\nu}(x) \sim (1 + 2\sigma(x)) g_{\mu\nu}(x) \tag{18}$$

At first order the Laplacian transforms as $\Delta \to (1 - 2\sigma(x))\Delta$. Argue that the variation of the nth eigenvalue is (at first order in perturbation theory)

$$\delta\lambda_n = -2\lambda_n \int_M \phi_n^*(x)\phi_n(x)\delta\sigma(x)\, dV(x) \tag{19}$$

and deduce the variation of the zeta function

$$\delta\zeta_\Delta(s) = 2\frac{s}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \int_M \left(K(t,x,x) - \frac{1}{\mathrm{Vol}(M)} \right) \delta\sigma(x)\, dV(x) \tag{20}$$

10. Show that

$$\delta\zeta'_\Delta(0) = 2 \int_M \left(\frac{R(x)}{24\pi} - \frac{1}{\mathrm{Vol}(M)} \right) \delta\sigma(x)\, dV(x) \tag{21}$$

11. Deduce that the Weyl anomaly is $c = 1$, through

$$\left.\frac{\delta}{\delta\sigma(x)}\right|_{\sigma=0} Z_{e^{2\sigma}g} = \frac{1}{24\pi} R(x) Z_g \tag{22}$$

i.e.

$$Z_{e^{2\sigma}g} = Z_g \left(1 + \frac{1}{24\pi} \int_M R(x)\sigma(x)\sqrt{|g(x)|} + O(\sigma^2) \right) \tag{23}$$

3 Green's function

Recall that we can decompose the field $\phi(x)$ into eigenmodes $\phi(x) = \sum_{n=0}^{\infty} \alpha_n \phi_n(x)$, and that in the path integral formulation (6), the coefficients α_n are independent gaussian random variables with a variance $\left(\frac{\beta}{2\pi}\lambda_n\right)^{-1}$. There is a problem with the zero mode α_0, which would seemingly be a uniform random variable over $\mathbb{R}$. Since this is not quite well-defined, so far we ignore this issue by removing the zero mode

$$\overline{\phi}(x) = \phi(x) - \frac{1}{\mathrm{vol}(M)} \int_M \phi(y) dV(y) = \sum_{n>0} \alpha_n \phi_n(x) \tag{24}$$

One can readily compute the two-point function

$$\langle \overline{\phi}(x) \overline{\phi}(y) \rangle = \frac{2\pi}{\beta} \sum_{n>0} \frac{1}{\lambda_n} \phi_n(x) \phi_n(y) = \frac{2\pi}{\beta} G(x,y) \tag{25}$$

where G is the *Green's function*. It could also have been characterized as the being the unique solution of[3]

$$\Delta_x G(x,y) = \delta_y(x) - \frac{1}{\mathrm{Vol}(M)}, \qquad \int_M G(x,y) dV(x) = 0 \tag{26}$$

After a Weyl transformation $g \to \tilde{g} = e^{2\sigma} g$ the Green's function is of the form

$$\tilde{G}(x,y) = G(x,y) + F_\sigma(x) + F_\sigma(y) + C \tag{27}$$

where G is the Green's function with metric g, C a constant, and F_σ is the following smooth function

$$F_\sigma(x) = -\frac{1}{\mathrm{Vol}_{\tilde{g}}(M)} \int_M G(x,u)\, d\tilde{V}(u) \tag{28}$$

Proving this is not difficult and is left to the reader.

12. Show that on a generic surface (M,g), the short-distance behavior of the Green's function is

$$G(x,y) = -\frac{1}{2\pi} \log \mathrm{dist}_g(x,y) + O(1), \qquad (x \to y)$$

where $\mathrm{dist}_g(x,y)$ is the geodesic distance between x and y (hint : use convenient local coordinates).

4 Vertex operators

Recall that for a real symmetric matrix A with strictly positive eigenvalues and any vector B (possibly complex)

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}x^t A x} e^{Q^t x}\, d^n x = \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det A}} e^{\frac{1}{2} Q^t A^{-1} Q}$$

and one would be tempted to define

$$\left\langle e^{i \sum_j q_j \phi(x_j)} \right\rangle = \left\langle e^{\int Q(x)\phi(x)\, dV(x)} \right\rangle, \qquad Q(x) = i \sum_j q_j \delta(x - x_j)$$

[3] In the absence of an inverse for Δ, this is the next best thing. Indeed equation in (26) is nothing but

$$\Delta G = 1 - |\varphi_0\rangle\langle\varphi_0|, \qquad \Delta|\varphi_0\rangle = 0$$

and this means that G is the inverse of Δ as long as one restricts to $(\ker\Delta)^\perp$.

and therefore

$$\left\langle e^{i\sum_j q_j\phi(x_j)}\right\rangle = \exp\left[\frac{1}{2}\int Q(x)G(x,y)Q(y)\,\omega(x)\omega(y)\right] = \delta_{\sum_j q_j,0}\prod_{j,k} e^{-\frac{\pi}{\beta}q_jq_kG_g(x_j,x_k)}$$

where the neutrality condition $\delta_{\sum_j q_j,0}$ comes from integration over the zero-mode, formally

$$\frac{\int_{\mathbb{R}} e^{i\phi_0\sum_j q_j}\,d\phi_0}{\int_{\mathbb{R}} d\phi_0} = \delta_{\sum_j q_j,0}$$

13. Check that with this naive definition, the correlation function would formally be invariant under $g \to e^{2\sigma}g$.

    This is consistent with the expectation that vertex operators, at the classical level, are simply functions on the manifold, and therefore they have a vanishing scaling dimension. However in the double product $\prod_{j,k} e^{-\frac{\pi}{\beta}q_jq_kG_g(x_j,x_k)}$ the terms $j=k$ are divergent since

    $$G_g(x,y) \sim -\frac{1}{2\pi}\log \mathrm{dist}_g(x,y)$$

    In terms of Wick's theorem, these divergences come from the self-contractions. So one has to regularize the above expression. One possibility is to modify the Green's function at coincident points as follows

    $$\tilde{G}_g(x,x) = \lim_{y\to x}\left(G_g(x,y) + \frac{1}{4\pi}\log \mathrm{dist}_g^2(x,y)\right)$$

    Note that this prescription is local and covariant : it depends only on the spacetime geometry in an arbitrarily small neighborhood of x. On the Euclidean plane $\tilde{G}_g(x,x) = 0$, and this regularization amounts to completely remove self-contractions, so it is equivalent to normal ordering. In curved space self-contractions are not removed but merely made finite.

    Within this regularization scheme one thus defines the correlation function of Vertex operators as

    $$\boxed{\left\langle : e^{iq_1\phi(x_1)} : \cdots : e^{iq_p\phi(x_p)} :\right\rangle_g = \delta_{\sum_j q_j,0}\prod_{j<k} e^{-\frac{2\pi}{\beta}q_jq_kG_g(x_j,x_k)}\prod_j e^{-\frac{\pi}{\beta}q_j^2\tilde{G}_g(x_j,x_j)}}$$

    in which the colons $: e^{iq\phi} :$ are a reminder of the above regularization. Morally

    $$: e^{iq\phi(x)} := \text{``}\ \frac{e^{iq\phi(x)}}{(\mathrm{dist}_g(x,x))^{\frac{q^2}{2\beta}}}\ \text{''}$$

    In particular on the Euclidean plane we recover

    $$\left\langle : e^{iq_1\phi(x_1)} : \cdots : e^{iq_p\phi(x_p)} :\right\rangle = \delta_{\sum_j q_j,0}\prod_{i<j}|z_i - z_j|^{\frac{q_iq_j}{\beta}}\ .$$

14. Check that (up to an irrelevant additive constant ; why is it irrelevant ?)

    $$\tilde{G}_{e^{2\sigma}g}(x,x) = \tilde{G}_g(x,x) + 2F_\sigma(x) + \frac{1}{2\pi}\sigma(x)$$

    and deduce that

    $$\boxed{\left\langle e^{i\sum_j q_j\phi(x_j)}\right\rangle_{e^{2\sigma}g} = \prod_j e^{-\Delta_j\sigma(x_j)}\left\langle e^{i\sum_j q_j\phi(x_j)}\right\rangle_g, \qquad \Delta_j = \frac{q_j^2}{2\beta}}$$

This means that the vertex operator $:e^{iq\phi}:$ is a primary field, and that its scaling dimension is $\Delta(q) = \frac{q^2}{2\beta}$. Note that these anomalous dimensions are proportional to $\hbar = 1/\beta$, as befits quantum corrections.

The above regularization is a good illustration of the fact that the regularization of a field induces an anomalous behavior under a particular symmetry, here rescaling.

5 U(1) current

The charge neutrality $\sum_j q_j = 0$ in the correlation function of vertex operators can also be understood as the invariance of the theory under $\phi(x) \to \phi(x) + a$ for any constant $a \in \mathbb{R}$. At the level of the classical field theory the corresponding Noether current is

$$\nabla_\mu J^\mu = 0 \text{ on-shell}, \qquad J^\mu = \beta g^{\mu\nu}\partial_\nu \phi \tag{42}$$

(it is a good exercice to check this).

15. Consider now an infinitesimal transformation $\phi(x) \to \tilde{\phi}(x) = \phi(x) + \epsilon(x)$. Assuming that the path-integral measure $D[\Phi]$ is invariant under such rigid translations ($D[\tilde{\Phi}] = D[\Phi]$) argue that there are no quantum corrections to the classical expression of the current J^μ. Show that $\nabla_\mu J^\mu$ vanishes in correlation functions away from field insertions.

16. In isothermal coordinates $d^2s = e^{2\sigma}dzd\bar{z}$, let $J = J_z = \beta\partial_z\phi$ and $\bar{J} = J_{\bar{z}} = \beta\partial_{\bar{z}}\phi$. Argue that the field $\phi(x)$ can be decomposed (locally[4]) as $\phi(x) = \varphi(z) + \overline{\varphi}(\bar{z})$ for some holomorphic field $\varphi(z)$ and deduce that $\partial_{\bar{z}}J \simeq 0, \quad \partial_z\bar{J} \simeq 0$.

17. What is the classical behavior of $J_\mu = \beta\partial_\mu\phi$ under an infinitesimal diffeomorphism $x^\mu \to x^\mu + \epsilon^\mu(x)$? In particular, what is its scaling dimension ?

18. We are going to argue that J_μ has no anomalous dimension. To see this, consider for instance the two-point function of J_μ is given by

$$\langle J_\mu(x)J_\nu(x')\rangle_g = -\beta^2\partial_\mu\partial'_\nu G_g(x,x') \tag{46}$$

where ∂_μ stands for $\partial/\partial x^\mu$ and ∂'_ν stands for $\partial/\partial x'^\nu$. Check that

$$\langle J_\mu(x)J_\nu(x')\rangle_{e^{2\sigma}g} = \langle J_\mu(x)J_\nu(x')\rangle_g \tag{47}$$

It follows from Wick's theorem that the above Weyl invariance also holds for k-point functions $\langle J_{\mu_1}(y_1)\cdots J_{\mu_k}(y_k)\rangle$ as well. Homework : check that

$$\begin{aligned}&\langle J_{\mu_1}(y_1)\cdots J_{\mu_k}(y_k) : e^{iq_1\phi(x_1)} : \cdots : e^{iq_p\phi(x_p)} :\rangle_{e^{2\sigma}g} \\ &= \prod_{j=1}^{p} e^{-\frac{q_j^2}{2\beta}\sigma(x_j)}\langle J_{\mu_1}(y_1)\cdots J_{\mu_k}(y_k) : e^{iq_1\phi(x_1)} : \cdots : e^{iq_p\phi(x_p)} :\rangle_g\end{aligned} \tag{48}$$

The above behavior tells us two things. First J_μ transforms as a primary field under Weyl rescaling, second it has vanishing anomalous dimension. Therefore its scaling dimension is $\Delta = 1$ (and what is its spin ?). Note that the absence of anomalous dimension is a generic feature of conserved currents. The stress-energy tensor is another example.

[4]On any simply connected domain a harmonic function can be described as the real part of a holomorphic function. But this may not be true globally, as the harmonic function $\log|z|^2 = \log z + \log\bar{z}$ on $\mathbb{C}^*$ shows.

19. Derive the Ward identity associated to this U(1) symmetry. Namely show that under $\tilde{\phi}(x) = \phi(x) + \epsilon(x)$ we have

$$\delta_\epsilon O(x) = \frac{1}{2\pi} \int_B \epsilon(y) \left(\nabla_\mu J^\mu\right)(y) O(x)\, dV(y) \tag{49}$$

where B is an arbitrary neighborhood of x. What is the contact term of $\nabla_\mu J^\mu$ with a vertex operator $: e^{iq\phi} :$? With the current J_μ ?

20. Upon integrating by parts the above Ward identity can be rewritten as

$$\begin{aligned}\delta_\epsilon O(x) = &-\frac{1}{2\pi} \int_B \partial_\mu \epsilon(y) J^\mu(y) O(x)\, dV(y) \\ &+ \frac{1}{2\pi} \oint_{\partial B} \epsilon_{\mu\rho}\, \epsilon(y) J^\mu(y) O(x) \sqrt{|g(y)|} dy^\rho\end{aligned} \tag{50}$$

where B is an arbitrary neighborhood of x and the boundary ∂B is oriented clockwise. Show that for $\epsilon(x) = \epsilon$ this can means

$$\delta_\epsilon O(x) = \frac{\epsilon}{2\pi i} \oint_{\circlearrowleft} J(z) O(x) dz + \frac{\epsilon}{2\pi i} \oint_{\circlearrowright} \bar{J}(\bar{z}) O(x) d\bar{z} \tag{51}$$

in which the integration contour circles around x. Deduce the OPE of $J(z)$ with a vertex operator.

A Gaussian integration

In this section A denote a real $n \times n$ symmetric matrix A with strictly positive eigenvalues, and $\phi = (\phi_1, \cdots, \phi_n)$ a vector in $\mathbb{R}^n$.

- Show that

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}\phi^t A \phi}\, d^n\phi = \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det A}} \tag{52}$$

  Let's denote by Z the above quantity.

- Check that the two-point function $\langle \phi_i \phi_j \rangle$ is given by

$$\langle \phi_i \phi_j \rangle = \frac{1}{Z} \int_{\mathbb{R}^n} \phi_i \phi_j e^{-\frac{1}{2}\phi^t A \phi}\, d^n\phi = \left(A^{-1}\right)_{ij} \tag{53}$$

  One can then compute higher-point correlation functions using Wick's theorem

$$\langle \phi_{i_1} \cdots \phi_{i_{2p-1}} \rangle = 0, \qquad \langle \phi_{i_1} \cdots \phi_{i_{2p}} \rangle = \sum_{\text{pairings } P} \langle \phi_{i_{P(1)}} \phi_{i_{P(2)}} \rangle \cdots \langle \phi_{i_{P(2p-1)}} \phi_{i_{P(2p)}} \rangle \tag{54}$$

  A proof of Wick's theorem can be obtained by taking derivatives *w.r.t.* the b_j's of the next identity.

- Show that for any vector $b = (b_1, \cdots, b_n)$, possibly complex, one has

$$\left\langle e^{\sum_j b_j \phi_j} \right\rangle = \frac{1}{Z} \int_{\mathbb{R}^n} e^{-\frac{1}{2}\phi^t A \phi} e^{b^t \phi}\, d^n\phi = e^{\frac{1}{2} b^t A^{-1} b} \tag{55}$$

B Stokes theorem in Riemannian geometry

On a n-dimensional compact manifold M (without boundary) Stokes theorem ensures that

$$\int_M d\alpha = 0$$

for any $n-1$ form α. In particular this implies that for any vector field X

$$\int_M \mathcal{L}_X \omega_g = 0$$

where $\mathcal{L}_X$ is the Lie derivative along X and ω_g is the Riemnannian volume form. This follows form the Cartan formula $\mathcal{L}_X = d\iota_X + \iota_X d$ where ι_X is the interior product, and the fact that $d\omega_g = 0$. In fact $\mathcal{L}_X \omega_g$ is nothing but $\mathrm{div}_g(X)\omega_g$, so we have

$$\int_M \mathrm{div}_g(X)\omega_g = 0$$

In local coordinates this reads

$$\int_M \frac{1}{\sqrt{|g|}} \partial_\rho \left(\sqrt{|g|} X^\rho \right) dV = 0$$

Since $\Gamma^\mu{}_{\mu\rho} = \frac{1}{\sqrt{|g|}} \partial_\rho \sqrt{|g|}$, we can rewrite this as

$$\int_M \nabla_\rho X^\rho \, dV = 0$$

In the presence of a boundary ∂M, Stokes theorem becomes

$$\int_M d\alpha = \int_{\partial M} \alpha$$

Note that the boundary ∂M of an oriented Riemannian manifold (M, g) is itself naturally an oriented Riemannian manifold, as the boundary inherits the metric and orientation of M. The natural integration form on ∂M is simply $\iota_N \omega_g$, where N is the unit outward normal field along ∂M. Within this framework one can rewrite Stokes theorem as

$$\int_M \mathrm{div}_g(X)\omega_g = \int_{\partial M} \iota_X \omega_g = \int_{\partial M} g(N, X) \iota_N \omega_g$$

Indeed the two forms $\iota_X \omega_g$ and $g(N, X)\iota_N \omega_g$ are equal once restricted to ∂M. To see this, notice the decomposition $X = g(X, N)N + Y$, where $g(Y, N) = 0$, yielding $\iota_X \omega_g = g(N, X)\iota_N \omega_g + \iota_Y \omega_g$. Finally $\iota_Y \omega_g$ vanishes once restricted to ∂M. Often in physics textbook, the above equality is written in coordinates :

$$\int_M \nabla_\rho X^\rho \, dV_M = \int_{\partial M} N_\rho X^\rho dV_{\partial M}$$

In two dimensions we will prefer the following alternative version

$$\int_M \mathrm{div}_g(X)\omega_g = \int_{\partial M} \iota_X \omega_g$$

or in coordinates, writing $\omega_g = \sqrt{|g|} dx^1 \wedge dx^2$

$$\int_M \nabla_\rho X^\rho \sqrt{|g|} d^2x = \int_{\partial M} \epsilon_{\rho\mu} X^\rho \sqrt{|g|} dx^\mu$$