Massless scalar field in Hamiltonian formalism

The point of this tutorial is to illustrate the operator formalism of CFT, in particular the notions of radial ordering, Hilbert space, (Euclidean) time evolution. Recall the Euclidean action of the massless real valued scalar field ϕ

$$S[\phi] = \frac{\beta}{4\pi}\int_M g(d\phi, d\phi)dV_g \tag{1}$$

1 Canonical quantization on the cylinder

We consider the scalar field ϕ on the (flat) cylinder $\mathbb{R}\times S^1$. For clarity we go back to real-time, *i.e.* Minkowski space, with the compact direction being space, while time runs in the non-compact direction. Physically we are working on a one-dimensional quantum system, with space coordinate x on a circle. We denote by L the perimeter, and we identify x and $x+L$. The corresponding metric is

$$\eta = dt^2 - dx^2 \tag{2}$$

In these coordinates the action reads

$$S[\phi] = \int_{\mathbb{R}} \mathbf{L}dt, \qquad \mathbf{L} = \frac{\beta}{4\pi}\int_0^L \left\{(\partial_t\phi)^2 - (\partial_x\phi)^2\right\} dx \tag{3}$$

Note that we could have included β in the Lagrangian if we prefer to think of β as $1/\hbar$. The scalar field is real and periodic, thus

$$\phi(x,t) = \sum_n e^{\frac{2\pi i n}{L}x}\phi_n(t), \qquad \phi_{-n}(t) = \overline{\phi_n(t)} \tag{4}$$

and the Fourier modes decouple. Accordingly the Lagrangian splits into

$$\mathbf{L} = \frac{\beta L}{4\pi}\sum_{n\in\mathbb{Z}}\left\{\left|\dot{\phi}_n\right|^2 - \left(\frac{2\pi n}{L}\right)^2 |\phi_n|^2\right\} = \frac{\beta L}{4\pi}\dot{\phi}_0^2 + \frac{\beta L}{2\pi}\sum_{n>0}\left\{\left|\dot{\phi}_n\right|^2 - \left(\frac{2\pi n}{L}\right)^2 |\phi_n|^2\right\} \tag{5}$$

The zero mode term $\mathbf{L}_0 = \frac{\beta L}{4\pi}\dot{\phi}_0^2$ can be interpreted as the Lagrangian of a one-dimensional (non-relativistic) particle of mass $m = \beta L/2\pi$ on the line with position $\phi_0 \in \mathbb{R}$. The conjugate momentum is

$$\pi_0 = \frac{\beta L}{2\pi}\dot{\phi}_0 \tag{6}$$

and within canonical quantization we have $[\phi_0, \pi_0] = i$. Thus in the Heisenberg picture

$$\phi_0(t) = \phi_0 + \frac{2\pi t}{\beta L}\pi_0, \qquad [\phi_0, \pi_0] = i \tag{7}$$

Likewise the nth term

$$\mathbf{L}_n = \frac{\beta L}{2\pi}\left\{\left|\dot{\phi}_n\right|^2 - \left(\frac{2\pi n}{L}\right)^2 |\phi_n|^2\right\} \tag{8}$$

describes a two-dimensional harmonic oscillator (see Appendix) with mass $m = \beta L/\pi$ and frequency $\omega_n = \frac{2\pi n}{L}$, with ϕ_n begin the position of the particle in the complex plane. The conjugate momenta are

$$\pi_n = \frac{\beta L}{2\pi}\dot{\bar{\phi}}_n, \qquad \bar{\pi}_n = \frac{\beta L}{2\pi}\dot{\phi}_n \tag{9}$$

and canonical quantization means

$$[\phi_n, \pi_n] = i, \qquad [\bar{\phi}_n, \bar{\pi}_n] = i \tag{10}$$

Note that $\phi_n^\dagger = \overline{\phi_n} = \phi_{-n}$, and likewise $\pi_n^\dagger = \overline{\pi_n} = \pi_{-n}$. Thus we introduce for $n \neq 0$

$$a_n = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{\beta}}\pi_n - in\sqrt{\beta}\bar{\phi}_n\right), \qquad \bar{a}_n = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{\beta}}\bar{\pi}_n - in\sqrt{\beta}\phi_n\right), \qquad n \in \mathbb{Z}^* \tag{11}$$

1. Check that

$$[a_n, a_m] = n\delta_{n+m}, \qquad [\bar{a}_n, \bar{a}_m] = n\delta_{n+m}, \qquad [a_n, \bar{a}_m] = 0 \tag{12}$$

This is (two copies of) the U(1) Kac-Moody algebra. Note that $a_n^\dagger = a_{-n}$: a_n is an annihilation operator for $n > 0$, and a creation operator for $n < 0$. In terms of these operators the Hamiltonian is

$$H = \frac{2\pi}{L}\left(a_0^2 + \sum_{n>0}(a_{-n}a_n + \bar{a}_{-n}\bar{a}_n + n)\right) \tag{13}$$

where we introduced $a_0 = \pi_0/\sqrt{2\beta}$

2. What is the energy spectrum ? Check that the energy gap is in $1/L$.

3. What can you say about the vacuum energy ?

4. Show that in the Heisenberg picture

$$a_n(t) = e^{-i\frac{2\pi n}{L}t}a_n, \qquad \bar{a}_n(t) = e^{-i\frac{2\pi n}{L}t}\bar{a}_n \tag{14}$$

5. Deduce that in the Hamiltonian formalism we have

$$\phi(x,t) = \phi_0 + \frac{2\pi t}{\beta L}\pi_0 + \frac{i}{\sqrt{2\beta}}\sum_{n\neq 0}\frac{1}{n}\left(e^{\frac{2\pi i n}{L}(x-t)}\bar{a}_n + e^{-\frac{2\pi i n}{L}(x+t)}a_n\right) \tag{15}$$

Observe that the operator $\phi(x,t)$ obeys the classical equation of motion

$$\left(\partial_t^2 - \partial_x^2\right)\phi(x,t) = 0\,.$$

What would be the analogue in the path-integral formalism ?

6. We now go back to Euclidean time by replacing $t \to -it$ [$t = -i\tau$; $w = \tau + ix$]. This means

$$\phi(x,y) = \phi_0 - i\frac{\pi(w+\bar{w})}{\beta L}\pi_0 + \frac{i}{\sqrt{2\beta}}\sum_{n\neq 0}\frac{1}{n}\left(e^{-\frac{2\pi n}{L}w}a_n + e^{-\frac{2\pi n}{L}\bar{w}}\bar{a}_n\right) \tag{16}$$

in terms of $w = t + ix$.

2 Radial quantization on the plane

Let (r, θ) denote polar coordinates on the Euclidean plane (with the origin removed).

7. What is the metric in these coordinates ? We now introduce $t = \log r$. Check that (t, θ) are isothermal coordinates, and deduce that

$$\phi(z, \bar{z}) = \phi_0 - \frac{i}{2\beta} \log(z\bar{z})\, \pi_0 + \frac{i}{\sqrt{2\beta}} \sum_{n \neq 0} \frac{1}{n} \left(z^{-n} a_n + \bar{z}^{-n} \bar{a}_n\right) \tag{17}$$

8. Recall that the two-components of the current associated to translations of the field $\phi(x, t) \to \phi(x, t) + a$ are given by

$$J = i\partial_z \phi, \qquad \bar{J} = -i\partial_{\bar{z}} \phi \tag{19}$$

They are primary fields of conformal dimension $(1, 0)$ and $(0, 1)$, respectively. Check that

$$J(z) = \frac{1}{\sqrt{2\beta}} \sum_{n \in \mathbb{Z}} z^{-n-1} a_n \tag{20}$$

where $a_0 = \bar{a}_0 = \pi_0 / \sqrt{2\beta}$ enjoys $[\phi_0, a_0] = \frac{1}{\sqrt{2\beta}}$.

9. Recall that on the plane the scalar field Green's function is given by

$$\langle \phi(z, \bar{z}) \phi(w, \bar{w}) \rangle = -\frac{1}{2\beta} \log |z - w|^2 .$$

Argue (using Wick theorem, see Appendix) that the OPE of J with itself reads

$$J(z) J(w) = \frac{1}{2\beta} \frac{1}{(z - w)^2} + \text{reg} \tag{21}$$

10. Recover the commutation relation $[a_n, a_m] = n \delta_{n+m,0}$ using the above OPE and writing the commutator as a contour integral.

3 Stress-energy tensor on the plane

At the classical level we have already seen that

$$T_{\mu\nu} = -\beta \left(\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial^\rho \phi \partial_\rho \phi \right) \tag{24}$$

which in complex coordinates reads

$$T = -\beta \left(\partial_z \phi\right)^2, \qquad \bar{T} = -\beta \left(\partial_{\bar{z}} \phi\right)^2 . \tag{25}$$

At the quantum level such products are ill-defined, and in flat space a common cure is normal ordering. Thus on the flat plane we declare

$$T = -\beta \, :\! (\partial_z \phi)^2 \!: \quad , \qquad \bar{T} = -\beta \, :\! (\partial_{\bar{z}} \phi)^2 \!: \quad . \tag{26}$$

The attentive reader will object that this construction of the stress-energy tensor is rather dishonest. Indeed the stress-energy tensor as defined in the lecture is obtained via functional derivation of the partition function with respect to the metric. In a previous tutorial the partition function was defined for an arbitrary background metric via zeta regularization, so in principle this functional derivation can be computed. Normal ordering is a different type of regularization, to which we already resorted to define vertex operators. However in the presence of curvature normal ordering yields a stress-energy tensor which is not covariantly conserved. This deficiency can be corrected by adding a term proportional to $Rg_{\mu\nu}$ (which is nothing but the Weyl anomaly !). We will simply admit that on the plane normal order gives the correct stress-energy tensor.

11. Check the following OPE using Wick theorem

$$T(z)T(w) = \frac{1}{2}\frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg} \tag{27}$$

12. Compute L_n in terms of a_n, namely show that

$$L_n = \frac{1}{2}\sum_p :a_p a_{n-p}: \tag{28}$$

In particular

$$L_0 = \frac{1}{2}a_0^2 + \sum_{p>0} a_{-p}a_p \tag{29}$$

13. As a sanity check, recover the Virasoro algebra with $c = 1$.

14. Compute the stress-energy tensor on the cylinder using the usual conformal mapping. Recall that the Hamiltonian is given in term of the stress-energy tensor as

$$H = \frac{1}{2\pi}\int_0^L \left(T(0,x) + \overline{T}(0,x)\right) dx$$

Compare with the Hamiltonian obtained in the first section using canonical quantization (in particular the ground-state energy). Notice how on the cylinder normal ordering does not yield the correct SET (the term responsible for the Casimir effect, *i.e.* the one coming from the Schwarzian derivative when mapping the plane to the cylinder, is missing).

4 Vertex operators

Vertex operators are defined as

$$V_q(z,\bar{z}) = :e^{iq\phi(z,\bar{z})}: = \sum_n \frac{(iq)^n}{n!} :\phi(z,\bar{z})^n:$$

15. Compare with the definition used in the tutorial about the scalar field in curved space.

16. Compute the OPE $T(z)V_q(w)$, and recover the fact that V_q is a primary field with conformal dimension $h_q = \frac{q^2}{4\beta}$.

17. Compute the OPE $J(z)V_q(w)$, and show that V_q is annihilated by all positive modes a_n, $n > 0$. V_q is said to be primary *w.r.t.* the Kac-Moody algebra generated by the a_n's. Argue that any Kac-Moody primary is automatically a Virasoro primary. Are there fields that are Virasoro primary but not Kac-Moody primary ?

18. Let $|q\rangle$ denote the state corresponding to the operator V_q. How is $|q\rangle$ characterized ?

19. Using Wick theorem, argue that for A and B linear combinations of creation and annihilation operators

$$:e^A::e^B: = e^{\overline{A\,B}} :e^{A+B}: \tag{30}$$

From this conclude that

$$\left\langle :e^{iq_1\phi(x_1)}: \cdots :e^{iq_p\phi(x_p)}:\right\rangle = \delta_{\sum_j q_j,0} \prod_{i<j} |z_i - z_j|^{\frac{q_i q_j}{\beta}} . \tag{31}$$

A Two-dimensional harmonic oscillator in complex coordinates

$$\mathbf{L} = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - \frac{1}{2}m\omega^2\left(x^2 + y^2\right) \tag{32}$$

Instead of working with x and y, we can use $z = x + iy$ and $\bar{z}$ as variables.

$$\mathbf{L} = \frac{1}{2}m\dot{z}\dot{\bar{z}} - \frac{1}{2}m\omega^2 z\bar{z} \tag{33}$$

The canonical momenta are

$$p = \frac{\partial \mathbf{L}}{\partial \dot{z}} = \frac{m}{2}\dot{z} = \frac{1}{2}\left(p_1 - ip_2\right) , \qquad \bar{p} = \frac{m}{2}\dot{\bar{z}}, \tag{34}$$

and the Hamiltonian is

$$H = p\dot{z} + \bar{p}\dot{\bar{z}} - \mathbf{L} = \frac{2}{m}p\bar{p} + \frac{m\omega^2}{2}z\bar{z} \tag{35}$$

Canonical quantization is obtained through $[z, p] = i\hbar$ and $[\bar{z}, \bar{p}] = i\hbar$. In position representation

$$p = -i\hbar\frac{\partial}{\partial z}, \qquad \bar{p} = -i\hbar\frac{\partial}{\partial \bar{z}} \tag{36}$$

Note that $\bar{z} = z^\dagger$ and $\bar{p} = p^\dagger$. The problem is essentially solved upon introducing creation and annihilation operators

$$a = \sqrt{\frac{m\omega}{\hbar}}\left(\frac{z}{2} + \frac{i}{m\omega}\bar{p}\right), \qquad a^\dagger = \sqrt{\frac{m\omega}{\hbar}}\left(\frac{\bar{z}}{2} - \frac{i}{m\omega}p\right) \tag{37}$$

$$\bar{a} = \sqrt{\frac{m\omega}{\hbar}}\left(\frac{\bar{z}}{2} + \frac{i}{m\omega}p\right), \qquad \bar{a}^\dagger = \sqrt{\frac{m\omega}{\hbar}}\left(\frac{z}{2} - \frac{i}{m\omega}\bar{p}\right) \tag{38}$$

These are subject to the commutation relation $[a, a^\dagger] = 1$ and $[\bar{a}, \bar{a}^\dagger] = 1$ (with mixed commutators vanishing) and the Hamiltonian is

$$H = \hbar\omega\left(a^\dagger a + \bar{a}^\dagger \bar{a} + 1\right) \tag{39}$$

Moreover in Heisenberg picture $X(t) = e^{i\frac{t}{\hbar}H} X e^{-i\frac{t}{\hbar}H}$ twe have

$$a(t) = e^{-i\omega t}a, \qquad a^\dagger(t) = e^{i\omega t}a^\dagger \tag{40}$$

$$\bar{a}(t) = e^{-i\omega t}\bar{a}, \qquad \bar{a}^\dagger(t) = e^{i\omega t}\bar{a}^\dagger \tag{41}$$

B Wick theorem (for boson)

Let A_i be arbitrary linear combinations of (bosonic) creation and annihilation operators (in particular all commutators $[A_i, A_j]$ are numbers). Given a reference state $|0\rangle$, we can decompose A_i as $A_i = A_i^+ + A_i^-$, such that $A_i^-|0\rangle = 0$ and $\langle 0|A_i^+ = 0$. Normal ordering of a product $A_1 \cdots A_n$ is then defined as first expanding the product $\prod_i (A_i^+ + A_i^-)$, and then in each term moving all creation operators A_i^+ to the left (in whichever order since they all commute). Then Wick theorem asserts that

$$\begin{aligned} A_1 \cdots A_n = {:}A_1 \cdots A_n{:} &+ \sum_{(ij)} {:}A_1 \cdots \overbrace{A_i \cdots A_j} \cdots A_n{:} \\ &+ \sum_{(ij)(rs)} {:}A_1 \cdots \overbrace{A_i \cdots A_j} \cdots \overbrace{A_r \cdots A_s} \cdots A_n{:} + \cdots \end{aligned} \tag{42}$$

where the first sum runs on single pair contractions, the second sum runs on double contractions *etc.* A contraction is simply

$$\overbrace{A_i A_j} = A_i A_j - {:}A_i A_j{:} = \langle 0|A_i A_j|0\rangle \tag{43}$$

We will also use the following slight generalization:

$$\begin{aligned} {:}A_1 \cdots A_p{:}\ A_{p+1} \cdots A_n = {:}A_1 \cdots A_n{:} &+ \sum_{(ij)} {:}A_1 \cdots \overbrace{A_i \cdots A_j} \cdots A_n{:} \\ &+ \sum_{(ij)(rs)} {:}A_1 \cdots \overbrace{A_i \cdots A_j} \cdots \overbrace{A_r \cdots A_s} \cdots A_n{:} + \cdots \end{aligned} \tag{44}$$

in which in the left hand side pair contractions between the first p operators are not allowed.