Correlation functions of primary operators

In these exercises, we shall find the simple form of one-, two- and three-point functions of primary operators in a conformally invariant QFT, in various geometries. We shall denote by $(e_1, \dots, e_d)$ an orthonormal basis of $\mathbb{R}^d$.

1 Generic space dimension d

1. Let r be a point in $\mathbb{R}^d$. Find two conformal transformations of $\mathbb{R}^d \to \mathbb{R}^d$ with different scaling factors at r, which both map r to e_1. Deduce that, for any primary operator ϕ, one has $\langle \phi(r) \rangle_{\mathbb{R}^d} = 0$.

2. We consider the half-space $\mathbb{H}_d = \{r \in \mathbb{R}^d\,, \quad r.e_1 > 0\}$. Using a properly chosen conformal transformation of $\mathbb{H}_d \to \mathbb{H}_d$, show that, for any primary operator ϕ, the one-point function $\langle \phi(r) \rangle_{\mathbb{H}_d}$ is a function of the distance to the boundary, namely $r_\perp = r.e_1$. Determine this function, up to a multiplicative constant, in terms of $r_\perp$ and x, the scaling dimension of ϕ.

3. Consider the inversion transformation $r \mapsto r' = r/|r|^2$. Show that it is an involution. Using the associated transformation of the Euclidean metrics, show that the inversion is conformal, with a scaling factor $\lambda(r) = |r|^2$. If we have two points x, y and their images x', y', show that
$$|x' - y'| = \frac{|x - y|}{|x|\,|y|}\,.$$

4. Let r_1 and r_2 be two points in $\mathbb{R}^d$. Using translations, rotations and scaling transformations, find a conformal map of $\mathbb{R}^d \to \mathbb{R}^d$ which maps (r_1, r_2) onto $(0, e_1)$. Deduce that for any two primary operators ϕ_1, ϕ_2, the function $\langle \phi_1(r_1)\phi_2(r_2) \rangle_{\mathbb{R}^d}$ must be of the form $\langle \phi_1(r_1)\phi_2(r_2) \rangle_{\mathbb{R}^d} = \text{const} \times |r_1 - r_2|^{-x_1 - x_2}$, where x_j is the scaling dimension of ϕ_j. Relate $\langle \phi_1(r_1)\phi_2(r_2) \rangle_{\mathbb{R}^d}$ to $\langle \phi_1(r'_1)\phi_2(r'_2) \rangle_{\mathbb{R}^d}$, where $r'_j = r_j/|r_j|^2$. By considering the behaviour of this function as $r_1 \to \infty$, show that the multiplicative constant is zero, unless $x_1 = x_2$.

5. Let $r_1, \dots, r_{n-1}, r_n = R$ be n points in $\mathbb{R}^d$, all distinct from the origin. Let $\phi_1, \dots, \phi_n$ be some primary operators. Using the inversion, show that the correlation function $\langle \phi_1(r_1) \dots \phi_{n-1}(r_{n-1})\phi_n(R) \rangle_{\mathbb{R}^d}$ is of the form:
$$\langle \phi_1(r_1) \dots \phi_{n-1}(r_{n-1})\phi_n(R) \rangle_{\mathbb{R}^d} \underset{R \to \infty}{\sim} |R|^{-2x_n} f(r_1, \dots, r_{n-1})\,.$$
Thus, we shall denote
$$\langle \phi_1(r_1) \dots \phi_{n-1}(r_{n-1})\phi_n(\infty) \rangle_{\mathbb{R}^d} = \lim_{R \to \infty} \left[|R|^{2x_n} \langle \phi_1(r_1) \dots \phi_{n-1}(r_{n-1})\phi_n(R) \rangle_{\mathbb{R}^d} \right]\,.$$
Show how to adapt the above argument to the case when one of the r_j's is at the origin.

6. We consider the three-point function of primary operators $\langle\phi_1(r_1)\phi_2(r_2)\phi_3(r_3)\rangle_{\mathbb{R}^d}$. We denote $r_{ij} = r_i - r_j$. Let ε be a small vector. Find a conformal map of $\mathbb{R}^d \to \mathbb{R}^d$ which maps (r_1, r_2, r_3) onto $(\widetilde{r}_{13}/|\widetilde{r}_{13}|^2, \widetilde{r}_{23}/|\widetilde{r}_{23}|^2, R)$, where $\widetilde{r}_{ij} = r_{ij} + \varepsilon$ and $R = \varepsilon/|\varepsilon|^2$. Perform the corresponding transformation on the correlation function. Using only translations, dilatations and rotations, find a conformal map of $\mathbb{R}^d \to \mathbb{R}^d$ which maps $(\widetilde{r}_{13}/|\widetilde{r}_{13}|^2, \widetilde{r}_{23}/|\widetilde{r}_{23}|^2, R)$ onto $(0, e_1, \widetilde{R})$, where $\widetilde{R} \to \infty$ when $\varepsilon \to 0$. Perform the corresponding transformation on the correlation function, and show that

$$\langle\phi_1(r_1)\phi_2(r_2)\phi_3(r_3)\rangle_{\mathbb{R}^d} = \frac{\langle\phi_1(0)\phi_2(e_1)\phi_3(\infty)\rangle_{\mathbb{R}^d}}{|r_{12}|^{x_1+x_2-x_3}\,|r_{13}|^{x_1+x_3-x_2}\,|r_{23}|^{x_2+x_3-x_1}}\,.$$

2 The case of two dimensions

1. Let z_1, z_2, z_3, z_4 be four complex numbers. Find a Moebius transformation [*i.e.* a map of the form $z \mapsto w(z) = \frac{az+b}{cz+d}$, where a, b, c, d are complex constants], such that the images of z_1, z_3, z_4 are $0, 1, \infty$, respectively. Compute $\eta = w(z_2)$, the image of the fourth point under the Moebius transformation. If $\phi_1, \phi_2, \phi_3, \phi_4$ are primary operators, find a relation between

$$\langle\phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\phi_3(z_3,\bar{z}_3)\phi_4(z_4+\varepsilon,\bar{z}_4+\bar{\varepsilon})\rangle_{\mathbb{C}}$$

and

$$\langle\phi_1(0)\phi_2(\eta,\bar{\eta})\phi_3(1)\phi_4(R,\bar{R})\rangle_{\mathbb{C}}\,,$$

where ε is a small complex number, and $R = w(z_4+\varepsilon)$. Let $\varepsilon \to 0$ in the end of the calculation, to obtain an expression of $\langle\phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\phi_3(z_3,\bar{z}_3)\phi_4(z_4,\bar{z}_4)\rangle_{\mathbb{C}}$ in terms of the positions $z_j, \bar{z}_j$, the conformal dimensions the ϕ_j's, and the function $\langle\phi_1(0)\phi_2(\eta,\bar{\eta})\phi_3(1)\phi_4(\infty)\rangle_{\mathbb{C}}$. To simplify the calculations, you can restrict to the case when $\phi_1 = \phi_2 = \phi_3 = \phi_4 = \phi$, with conformal dimensions $h, \bar{h}$.

2. Let $\mathcal{M}_L$ be the infinite cylinder of circumference L, defined as $\mathcal{M}_L = \mathbb{C}/iL\mathbb{Z}$. This means that points in $\mathcal{M}_L$ are identified modulo iL. Show that the map

$$\begin{cases} \mathcal{M}_L & \to \mathbb{C} \\ z & \mapsto w(z) = \exp(2\pi z/L) \end{cases}$$

is conformal. Using the knowledge of the two-point function on the plane for a primary operator ϕ, find the expression of the two-point function on the cylinder, namely $\langle\phi(z_1,\bar{z}_1)\phi(z_2,\bar{z}_2)\rangle_{\mathcal{M}_L}$.

3. Using the same ideas as above, compute the one-point function of a primary operator on the infinite strip:

$$\mathcal{S}_L := \{z \in \mathbb{C}\,, 0 < \mathrm{Im}(z) < L\}\,.$$

4. Show that the Moebius transformation $z \mapsto i(1+z)/(1-z)$ maps the unit disc to the upper half-plane. Deduce the one-point function of a primary operator on the unit disc. Discuss the behaviour of this function close to the boundary of the disc.