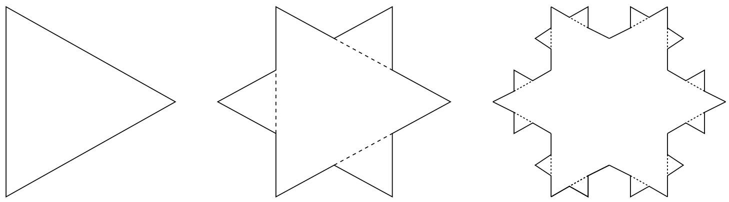
Fractal dimension of critical polygons

1 Preliminary example: the Koch snowflake

Let us start with a simple example of a *deterministic* fractal: the Koch snowflake, defined by the following sequence of polygons:



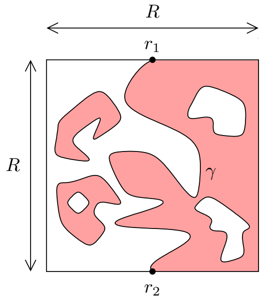
We consider the limit of this curve after an infinite number of iterations. The fractal dimension of a curve is defined as follows. For any $\epsilon > 0$, let $N(\epsilon)$ be the minimum number of disks of radius ϵ needed to cover the curve. If there exists a constant $A > 0$ and an exponent $d_f > 0$ such that $N(\epsilon) \sim A\epsilon^{-d_f}$, then we say that the curve is a fractal of dimension d_f.

► Show that the Koch snowflake is a fractal, and compute its fractal dimension.

2 Ising domain walls

We consider the critical Ising model on a square lattice of mesh size a, in a box of size $R \times R$. For any even integer k, let $\psi_k(r)$ be the operator which generates k "legs" of domain walls in the vicinity of r. More precisely, ψ_k inserts k marked paths which are not allowed to connect with each other, except in the vicinity of another operator ψ_ℓ. For instance, the two-point function $\langle\psi_2(r)\psi_2(r')\rangle$ gives the probability that r and r' sit on the same domain wall. We call x_k the scaling dimension of ψ_k.

Let r_1 and r_2 be two given boundary points, and fixed boundary conditions $\sigma = +1$ on the left boundary, and $\sigma = -1$ on the right boundary. These boundary conditions force the existence of an open domain wall γ joining the points r_1 and r_2.



► Show that the length of γ scales as $N \propto (R/a)^{d_f}$, and express d_f in terms of the scaling dimensions of the ψ_k's. The exponent d_f is the fractal dimension of critical Ising domain walls.

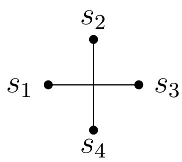
We admit that, for critical Ising domain walls, the scaling exponent of ψ_k is $\Delta_k = k^2/6 - 1/24$.

▶ Using this result, compute the fractal dimension of critical Ising domain walls.

3 The O(n) model and self-intersecting dense polygons

Consider the O(n) vector model on the square lattice. The variables are classical spins s_j living on the edges of the lattice, each spin $s = (s^1, \ldots, s^n)$ satisfies $s \cdot s = 1$, and we use an integration measure $d\mu(s)$ such that $\int d\mu(s)(s^\alpha)^k = 0$ for any integer $k \notin \{0, 2\}$, and any index α. The energy of four spins s_1, s_2, s_3, s_4 around a vertex v is

$$E_v(s_1, s_2, s_3, s_4) = -J_1(s_1 \cdot s_2 + s_3 \cdot s_4) - J_2(s_1 \cdot s_4 + s_2 \cdot s_3) - J_3(s_1 \cdot s_3 + s_2 \cdot s_4)\,.$$

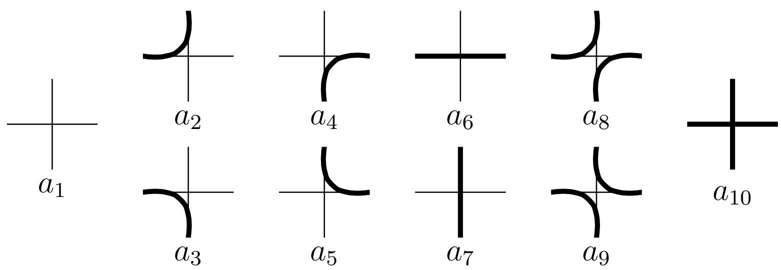


Here J_1, J_2, J_3 are positive coupling constants.

Consider a square lattice of size $M \times N$, with periodic boundary conditions. The partition function at invese temperature β reads

$$Z = \int \prod_j d\mu(s_j)\, \exp\left[-\beta \sum_v E_v(\{s_j\})\right]\,.$$

▶ Write an exact high-temperature expansion of the partition function, and show that each term in the expansion is represented by a configuration of polygons, with the possible configurations around a vertex:

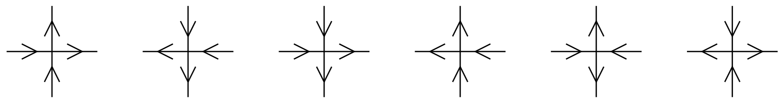


▶ Compute the local Boltzmann weights associated to the vertex configurations, and the weight associated to a closed polygon.

▶ Give the interpretation of the two-point function $\langle s_i \cdot s_j \rangle$ in terms of the polygon model.

In the following, we consider the dense version of the polygon model, with $a_1 = \cdots = a_7 = 0$, and we denote the nonzero weights as $a_8 = x, a_9 = y, a_{10} = z$. This model is known as the Brauer model. Moreover, we specialise to the case O($n = 2$).

▶ Show that the dense polygon model is equivalent to the six-vertex model, with vertex weights



and relate (x, y, z) to (a, b, c).

The six-vertex model is characterised by the anisotropy parameter

$$\Delta = \frac{a^2 + b^2 - c^2}{2ab} \,,$$

and it is critical for $-1 \leq \Delta \leq 1$. Along this critical line, the scaling limit is governed by the free compact boson action

$$A[\phi] = \frac{g}{4\pi} \int d^2r \, \partial_\mu \phi \, \partial^\mu \phi \,, \qquad \phi \equiv \phi + 2\pi \,,$$

where $\pi g = \mathrm{Arccos}(\Delta)$.

▶ Determine the critical line of the dense polygon model, and compute the coupling constant g in terms of the Boltzmann weights x, y, z.

▶ For any integer $m \in \mathbb{Z}$, let $Z_m(\epsilon, R)$ be the partition function of the free compact boson on the ring $\epsilon \leq |r| \leq R$, with Neumann boundary conditions $n_\mu \partial^\mu \phi = 0$ (here n_μ is the unit normal vector across the boundary), and with the condition that $\phi \to \phi + 2\pi m$ under a rotation of angle 2π. Compute the ratio $Z_m(\epsilon, R)/Z_0(\epsilon, R)$. Deduce the scaling exponent of the operator which inserts a line defect associated to the jump $\phi \to \phi + 2\pi m$. Finally, deduce the fractal dimension of a polygon.