Entanglement entropies in critical 1+1d systems

1 Entanglement entropies

Consider a quantum system in a pure (normalized) state $|\psi\rangle$. Suppose that the Hilbert space decomposes as $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Then, in the state $|\psi\rangle$, the two subsystems A and B can be entangled. In other words, measuring an observable in A may affect the observables in B.

▶ The Schmidt decomposition theorem asserts that there exists *orthonormal* states $\{|u_j\rangle\}$ in $\mathcal{H}_A$, $\{|v_j\rangle\}$ in $\mathcal{H}_B$, and (strictly) positive numbers $\{p_j\}$ such that

$$|\Psi\rangle = \sum_{j=1}^{r} \sqrt{p_j}\, |u_j\rangle \otimes |v_j\rangle \,. \tag{1}$$

▶ Prove the above result. Hint : start from a generic state of the form $|\psi\rangle = \sum_{i,j} \psi_{ij}\, |e_i\rangle \otimes |f_j\rangle$, where $\{|e_i\rangle\}$ and $\{|f_j\rangle\}$ are orthonormal bases of $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. Then consider the singular value decomposition of the rectangular matrix (ψ_{ij}).

▶ Show that the Schmidt rank r and Schmidt coefficients $\{p_1, \cdots, p_r\}$ are well defined (*i.e.* independent of any possible choice made when performing the Schmidt decomposition).

▶ The Schmidt coefficients are positive numbers subject to $\sum_j p_j = 1$. One way to quantify the amount of entanglement between A and B in the state $|\Psi\rangle$ is to consider the von Neumann entropy

$$S = -\sum_{j=1}^{r} p_j \log p_j \,. \tag{2}$$

and more generally the Rényi entropies

$$S_n = \frac{1}{1-n} \log \sum_{j=1}^{r} p_j^n \,, \qquad \mathrm{Re}(n) > 1 \,. \tag{3}$$

▶ Show that $\lim_{n\to 1} S_n = S$.

▶ **Example 1.** For a state of the form $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$, show that ρ_A has the form of a density matrix for a pure state. Compute the Rényi and von Neumann entropies for A.

▶ **Example 2.** Consider a system of two $\frac{1}{2}$-spins, in the state

$$|\psi\rangle = \cos\lambda\, |\uparrow\downarrow\rangle + e^{i\alpha} \sin\lambda\, |\downarrow\uparrow\rangle \,, \tag{4}$$

with real λ and α. Compute the reduced density matrix for the first spin.

It will be useful in the following to rewrite these entropies in terms of the reduced density matrix ρ_A

$$\rho_A = \mathrm{Tr}_B\Big(|\psi\rangle\langle\psi|\Big) \,. \tag{5}$$

where the partial trace of $\mathcal{H}_B$ is defined via

$$\mathrm{Tr}_B\Big((|e_i\rangle \otimes |f_j\rangle)\,(\langle e_k| \otimes \langle f_\ell|)\Big) = \delta_{j\ell}\, |e_i\rangle\langle e_k|\,. \tag{6}$$

where $\{|e_i\rangle\}$ and $\{|f_j\rangle\}$ are bases of $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively.

▶ Show that

$$S = -\mathrm{Tr}(\rho_A \log \rho_A)\,, \tag{7}$$

and

$$S_n = \frac{1}{1-n} \log \mathrm{Tr}\left(\rho_A^n\right)\,, \qquad \mathrm{Re}(n) > 1\,. \tag{8}$$

Compute the corresponding Rényi and von Neumann entropies.

2 Rényi entropies in CFT

In the following, we consider a 1+1d quantum system at the critical point, described by a conformal field theory $\mathcal{M}$. The system has size L, with periodic boundary conditions. We denote by H_L the Hamiltonian of this periodic system, and we suppose it has a single ground state $|\psi_0\rangle$ (assumed to be normalized). We want to relate the density matrix $\rho = |\psi_0\rangle\langle\psi_0|$ to some partition functions on the cylinder of circumference L. We introduce some boundary states $|\mathrm{in}\rangle$ and $\langle\mathrm{out}|$, such that the overlaps $\langle\psi_0|\mathrm{in}\rangle$ and $\langle\mathrm{out}|\psi_0\rangle$ are nonzero.

▶ Argue that the density matrix can be obtained as the limit

$$\rho = \lim_{M\to\infty} \frac{e^{-MH_L/2}|\mathrm{in}\rangle\langle\mathrm{out}|e^{-MH_L/2}}{\langle\mathrm{out}|e^{-MH_L}|\mathrm{in}\rangle}\,. \tag{9}$$

Interpret graphically this expression.

▶ We suppose the degrees of freedom are local. Let $A = [u, v]$ be an interval of the periodic system, and B its complement. The corresponding decomposition of the Hilbert space reads $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, and we consider the reduced density matrix ρ_A, as defined above. For any pair of basis states $e, e' \in \mathcal{H}_A$, show that $(\rho_A)_{ee'}$ is given by $\lim_{M\to\infty}[Z_{ee'}(M)/Z(M)]$, where $Z(M)$ is the partition function of the cylinder of length M, with boundary conditions $|\mathrm{in}\rangle$ and $\langle\mathrm{out}|$ at the extremities of the cylinder, and $Z_{ee'}(M)$ is the partition function of the same cylinder, with a cut along the interval A, and boundary conditions $|e\rangle$ and $\langle e'|$ on the sides of the cut. Draw a picture for $Z_{ee'}(M)$.

▶ In the following, we restrict to the case when n is a positive integer. Argue that $\mathrm{Tr}(\rho_A^n)$ is given by

$$\mathrm{Tr}(\rho_A^n) = \lim_{M\to\infty} \frac{Z_n(M)}{[Z_1(M)]^n}\,, \tag{10}$$

where $Z_n(M)$ is the partition function on $\Sigma_n(M)$, the n-th cover of the cylinder of length M, with branch points at u and v. This is a n-sheeted surface, where each sheet is a replica of the cylinder, and the sheets are connected cyclically along the interval $[u, v]$.

▶ We write $\Sigma_n = \lim_{M\to\infty} \Sigma_n(M)$. Show that the function

$$w \mapsto t = \left(\frac{\sin\frac{\pi(w-u)}{L}}{\sin\frac{\pi(w-v)}{L}}\right)^{1/n} \tag{11}$$

provides a conformal mapping from Σ_n to the complex plane. Deduce the genus of Σ_n.

One way to obtain the partition function Z_n is by computing it directly, but this method involves addressing the conical singularities located at the branch points u and v. Instead, an alternative approach that bypasses these difficulties can be taken. The strategy involves interpreting the branch cut that connects u and v as a defect line, which resembles the one between two disorder operators in the Ising model, introduced in tutorial 5 on the bootstrap in the Ising model. Thus, instead of working on a replicated surface, we work on the standard cylinder, but each site has N copies of the original degrees of freedom. This means that we work with the replicated CFT $\mathcal{M}_n = \mathcal{M}^{\otimes n}$, where $\mathcal{M}$ is the CFT describing the original model. In the following we will refer to $\mathcal{M}$ as the mother CFT. These N copies are uncoupled, except for the interaction across the defect line, where the interaction couples copy j to copy $j+1$ (where j is defined modulo N). In this way we have reduced the problem of computing $\mathrm{Tr}(\rho_A^n)$ to a two-point correlation function of so-called *twist operators*

$$\mathrm{Tr}(\rho_A^n) = \langle\!\langle \tau^\dagger(u)\tau(v) \rangle\!\rangle_{\mathrm{cyl}} \tag{12}$$

where $\langle\!\langle \dots \rangle\!\rangle_{\mathrm{cyl}}$ denotes the average value in the replicated model $\mathcal{M}_n$, on the infinite cylinder.

▶ Prove that the above construction is indeed a two-point function, in the sense that it only depends on u and v and not on the position of the defect line (apart from the fact that it must join u and v).

▶ Argue that within the state-operator correspondence, the twist field gets mapped to the lowest energy state of a system made of N copies of the original quantum chain with specific twisted boundary conditions (which ones ?).

▶ Using the finite size scaling of the energy for a system of size L, compute the scaling dimension of the twist field.

▶ Show that the twist field is scalar. Deduce its conformal dimensions.

▶ Compute the Rényi entropy $S_n([u,v])$ in the ground state of the periodic system of size L. Same question for a finite interval $[u,v]$ of an infinite system, (i) in the ground state, and (ii) in the thermal state of inverse temperature β.

▶ Optional question. Compute the partition function $Z_{r,s}(\tau)$ of the CFT $\mathcal{M}_n$ on the torus $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ with a defect line of order r along the cycle $z \to z+1$ and a defect line of order s along the cycle $z \to z+\tau$. Here computing $Z_{r,s}(\tau)$ means expressing it in terms of the partition function of the mother theory $Z(\tau)$.

3 Orbifold Virasoro algebra

We consider the replicated CFT $\mathcal{M}_n = \mathcal{M}^{\otimes n}$. For any local operator $\mathcal{O}$ of the mother CFT, we denote by $\mathcal{O}_a$ the operator $\mathcal{O}$ acting on the replica a in $\mathcal{M}_n$. We introduce the defect operators τ and $\tau^\dagger$ associated to the $\mathbb{Z}_n$ symmetry of rotation of replicas. This means that

$$\mathcal{O}_a(e^{2i\pi}z, e^{-2i\pi}\bar{z}) \cdot \tau(0,0) = \mathcal{O}_{a-1}(z,\bar{z}) \cdot \tau(0,0)\,, \tag{13}$$

$$\mathcal{O}_a(e^{2i\pi}z, e^{-2i\pi}\bar{z}) \cdot \tau^\dagger(0,0) = \mathcal{O}_{a+1}(z,\bar{z}) \cdot \tau^\dagger(0,0)\,, \tag{14}$$

where the indices $a \pm 1$ are understood modulo n.

The replicated model $\mathcal{M}_n$ is often called the $\mathbb{Z}_n$ orbifold of $\mathcal{M}$. Each replica a carries a copy of the stress-energy components $T_a(z)$ and $\bar{T}_a(\bar{z})$. The holomorphic components obey the OPEs

$$T_a(z)T_b(w) = \delta_{ab}\left[\frac{c/2}{(z-w)^4} + \frac{2T_b(w)}{(z-w)^2} + \frac{\partial T_b(w)}{z-w}\right] + \mathrm{reg}_{z\to w}\,. \tag{15}$$

We introduce the discrete Fourier modes

$$T^{(r)}(z) = \sum_{a=0}^{n-1} e^{2i\pi ar/n}\, T_a(z)\,, \qquad r \in \mathbb{Z}_n\,. \tag{16}$$

▶ Compute the OPE $T^{(r)}(z)\cdot T^{(s)}(w)$. Examine the case $r = s = 0$, and show that the symmetric component $T^{(0)}(z)$ satisfies the OPE of the stress-energy tensor for $\mathcal{M}_n$. Deduce the value $\widehat{c}$ of the central charge of $\mathcal{M}_n$. Show that $T^{(s)}(z)$ is primary for $s \neq 0$, and determine its conformal dimension.

▶ Determine the monodromy of T_a around τ – in other words, the transformation of $T_a(z)\cdot\tau(0,0)$ as $z \mapsto e^{2i\pi}z$. Same question for the monodromy of T_a around $\tau^\dagger$. Deduce the monodromies of $T^{(r)}$ around τ and $\tau^\dagger$. We shall write the mode decompositions as

$$T^{(r)}(z)\cdot\mathcal{O}_a(0,0) = \sum_m z^{-m-2}(L_m^{(r)}\cdot\mathcal{O}_a)(0,0)\,, \tag{17}$$

$$T^{(r)}(z)\cdot\tau(0,0) = \sum_m z^{-m-2}(L_m^{(r)}\cdot\tau)(0,0)\,, \tag{18}$$

$$T^{(r)}(z)\cdot\tau^\dagger(0,0) = \sum_m z^{-m-2}(L_m^{(r)}\cdot\tau^\dagger)(0,0)\,. \tag{19}$$

Here, $\mathcal{O}_a$ denotes the insertion of the local operator $\mathcal{O}$ at a regular point on the replica a, while $\tau, \tau^\dagger$ are the defect operators associated to the branch points. Using the monodromies found in the previous question, determine the range of the index m in each of the three above decompositions.

▶ Find the commutator $[L_m^{(r)}, \mathcal{O}_a(z,\bar{z})]$.

▶ Find the commutation relation $[L_m^{(r)}, L_k^{(s)}]$. This defines the orbifold Virasoro algebra $\mathrm{Vir}_n(\widehat{c})$. Show that $\mathrm{Vir}_n(\widehat{c})$ possesses a subalgebra isomorphic to $\mathrm{Vir}(\widehat{c})$.

▶ Consider the linear map $\varphi : \mathrm{Vir}_n(\widehat{c}) \to \mathrm{Vir}(c)$ defined as

$$\varphi(L_{m/n}^{(r)}) = \begin{cases} \frac{1}{n}L_m + A(c,n)\,\delta_{m,0}\,\mathbf{1} & \text{if } r \equiv -m \mod n \\ 0 & \text{otherwise,}\end{cases} \tag{20}$$

where $A(c,n)$ is a constant. Find the expression of $A(c,n)$ for which φ satisfies, for all $L_{m/n}^{(r)}$ and $L_{p/n}^{(s)}$,

$$[\varphi(L_{m/n}^{(r)}), \varphi(L_{p/n}^{(s)})] = \varphi([L_{m/n}^{(r)}, L_{p/n}^{(s)}])\,, \tag{21}$$

and thus it defines a morphism of algebras. Consider a Virasoro module V, and the corresponding homomorphism $\rho : \mathrm{Vir}(c) \to \mathrm{End}\, V$. Let $\widehat{V}$ be a vector space isomorphic to V, and $\theta : V \to \widehat{V}$ the associated linear isomorphism. Show that the map

$$\widehat{\rho}: \quad \mu \mapsto \theta\cdot\rho(\varphi(\mu))\cdot\theta^{-1} \tag{22}$$

is a homomorphism from $\mathrm{Vir}_n(\widehat{c})$ to $\mathrm{End}\,\widehat{V}$. As a result, $\widehat{V}$ is a $\mathrm{Vir}_n(\widehat{c})$ module.

▶ The character of $\widehat{V}$ reads

$$\widehat{\chi}_{\widehat{V}}(q) = \mathrm{Tr}_{\widehat{V}} \left(q^{L_0^{(0)} - \widehat{c}/24} \right) . \tag{23}$$

Relate $\widehat{\chi}_{\widehat{V}}(q)$ to the character $\chi_V(q)$ from the mother CFT.

▶ Consider a lowest weight Virasoro module V_h, with lowest weight state $|h\rangle$. We denote by $\widehat{V}_{\widehat{h}}$ the corresponding $\mathrm{Vir}_n(\widehat{c})$ module, with lowest weight state $|\widehat{h}\rangle\rangle$. Compute the conformal dimension $\widehat{h}$. If h is a degenerate dimension for the Virasoro algebra, then show that $|\widehat{h}\rangle\rangle$ admits a null state. Compute this null state in the cases $h = h_{11}$ and $h = h_{12}$.

▶ Argue that $|\tau\rangle = |\widehat{0}\rangle\rangle$, and write the null vector condition for $|\tau\rangle$. Given a primary operator ϕ from the mother CFT, use the null vector condition on $|\tau\rangle$ to derive a differential equation for the correlation function $\langle\langle \tau(\infty)\phi_a(z,\bar{z})\phi_b(w,\bar{w})\tau(0)\rangle\rangle_{\mathbb{C}}$. Find the solution which is covariant under global conformal maps.

▶ Independently of the previous question, find the surface R_n such that

$$\langle\langle \tau(\infty)\phi_a(z,\bar{z})\phi_b(w,\bar{w})\tau(0)\rangle\rangle_{\mathbb{C}} = \langle \phi(x,\bar{x})\phi(y,\bar{y})\rangle_{R_n} ,$$

and relate the positions x and y to z and w. Compute the above correlation function in the mother CFT, using a conformal mapping $R_n \to \mathbb{C}$.

▶ Using the above results, compute the correlation function on the infinite cylinder

$$\langle\langle \tau(u,\bar{u})\phi_a(u',\bar{u}')\phi_b(v',\bar{v}')\tau(v,\bar{v})\rangle\rangle_{\mathrm{cyl}} .$$

Compute the form of this function as $u' \to u$ and $v' \to v$. What is the physical interpretation of this limit ?