Conformal Field theory for 2d Statistical Mechanics

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Lecture 1

Critical Phenomena and Scale Invariance

1.1 Phase transitions

1.1.1 First- and second-order transitions

Let us start by considering a sample of material in a definite physical phase, with homogeneous measurable properties: density, elasticity, magnetisation, *etc.* Any measure of these quantities, performed anywhere in the sample, should give the same result. Hence, if the sample has the shape of an $L \times L \times L$ cube, a portion of size $\frac{L}{2} \times \frac{L}{2} \times \frac{L}{2}$ should have the same properties. If we iterate this procedure, we finally reach a scale where all the elementary constituents are correlated, and the above argument is not valid anymore. This scale is called the *correlation length*, and we will denote it by ξ. For example, in a crystal, ξ is usually of the order of a few interatomic spacings. The correlation length depends on the external parameters (pressure, temperature, magnetic field, *etc.*). An *order parameter* is defined as a local quantity which characterises the phases of the system, *i.e.* whose values are different from one phase to another. Examples : the local density of a fluid, the (components of) local magnetisation in a magnet, *etc.*

When one or several external parameters are varied, the properties of the material may change drastically: this is called a *phase transition*. We distinguish two types of

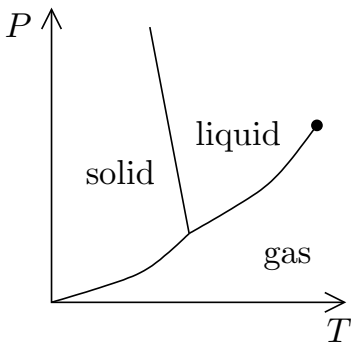


Figure 1.1: The phase diagram of water. The critical point is indicated by a black dot.

phase transitions:

- A phase transition is of first order if the two phases remain different at the transition. An example is the liquid-gas transition of water at 100°C. The order parameter (in the case of water, the local density ρ) is discontinuous at the phase transition. In this case, the two phases coexist at the transition, and an extensive amount of energy (latent heat) must be exchanged with the environment during the transition. This reflects the fact that, when the two phases coexist, the respective spatial domains of each phase have a definite size: *the correlation length ξ remains finite at a first-order transition.* In the example of water, ξ is the average size of liquid droplets which coexist with the gas phase during the transition.

- A phase transition is of second order if the two phases become identical at the transition, and hence all the order parameters are continuous. This is the case for the critical endpoint of the liquid-gas transition of water, at temperature 374°C and pressure 218 atm. When approaching this point along the transition line, the latent heat vanishes. In the vicinity of the critical point, liquid droplets of various scales appear, up to the size of the sample. *The correlation length diverges at a second-order transition.* As a consequence, for a large range of spatial scales, the spatial distribution of physical properties (*e.g.* the local density) is *self-similar* : it remains unchanged as one "zooms" into a subregion. *The system displays scale invariance at a second-order transition.*

Let us make the notion of scale invariance more precise, by discussing order parameters and their correlation functions. Let us consider a system in dimension d, close to a second-order phase transition, and denote the order parameter generically by $S(r)$. From mean-field arguments [neglecting local fluctuations of $S(r)$] one may show that, at short distances, the two-point correlation function $G(r)$ obeys a Laplace equation, and therefore it scales as $G(r) \propto 1/r^{d-2}$. To take spatial fluctuations into account, we should assume that $G(r)$ depends non-trivially on the ratios r/ξ and a/ξ, where a is the microscopic distance:

$$G(r) := \langle S(0)S(r)\rangle = \frac{1}{r^{d-2}} f(r/\xi, a/\xi)\,. \tag{1.1.1}$$

Now suppose that f vanishes as a power law in its second argument: $f(u,v) \propto_{v\to 0} v^\eta$, and compute the susceptibility χ, in the regime $a \ll \xi$.

$$\chi = \int \mathrm{d}^d r\, G(r) = \int \frac{\mathrm{d}^d r}{r^{d-2}} f(r/\xi, a/\xi) = \xi^2 \int \frac{\mathrm{d}^d u}{u^{d-2}} f(u, a/\xi) \propto \xi^{2-\eta}\, a^\eta\,. \tag{1.1.2}$$

The exponent η is called the anomalous dimension.

In this course, we shall concentrate on second-order phase transitions. The central task of the course will be to characterise the various classes of second-order phase transitions by determining scaling exponents such as η, and multipoint correlation functions like $G(r)$.

1.1.2 Spontaneous symmetry breaking

Consider a material whose internal interactions enjoy some (discrete or continuous) symmetry for any value of the external parameters. We shall illustrate this situation on the example of a magnetic system, described by the canonical ensemble, with temperature T

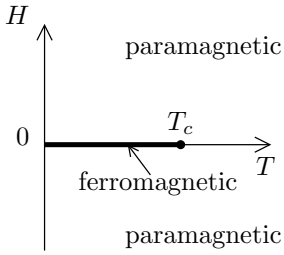
and external magnetic field H. In the canonical ensemble (also called Gibbs measure), the probability of a spin configuration $[S] = \{S(r)\}$ is given by:

$$\mathbb{P}_{T,H}[S] = \frac{1}{Z(T,H)} \times \exp\left(-\frac{\mathcal{H}_{\text{int}}[S] + H\int \mathrm{d}^d r\, S(r)}{k_B T}\right), \tag{1.1.3}$$

where $\mathcal{H}_{\text{int}}[S]$ is the classical Hamiltonian encoding the interactions between the spins. The external magnetic field couples to the total magnetisation $M = \int \mathrm{d}^d r S(r)$. Let us consider the case when this interaction is symmetric under spin reversal:

$$\mathcal{H}_{\text{int}}[S] = \mathcal{H}_{\text{int}}[-S] \tag{1.1.4}$$

We assume moreover that the interaction is ferromagnetic, *i.e.* it favours the configurations where neighbouring spins are aligned. For any spatial dimension $d > 1$, one observes experimentally or numerically the following phase diagram:



In the $H > 0$ (resp. $H < 0$) half-plane, the system is in a paramagnetic phase, and the total magnetisation M is positive (resp. negative). For $H = 0$ and $T > T_c$, the system is in a disordered phase, with $M = 0$. Along the line $0 < T < T_c$ at $H = 0$, the system is in a ferromagnetic phase, *i.e.* it has a non-zero total magnetisation $\pm M_0$. The latter is a symmetry-broken phase: although the Hamiltonian $\mathcal{H}_{\text{int}}$ is symmetric under spin reversal, the state of the system is not symmetric. This ferromagnetic line represents a first-order phase transition between the two paramagnetic phases, because the order parameter M is discontinuous across the transition: it goes from $+M_0$ to $-M_0$ as H changes sign. Note that $M_0 \to 0$ as $T \to T_c$ along this line. In contrast, at the critical point $(T = T_c, H = 0)$ the system goes through a second-order phase transition as T is varied.

We have described above the simplest example of symmetry breaking in a spin system, where the symmetry group is $\mathbb{Z}_2$. Note that other discrete or continuous symmetry groups may give rise to symmetry-broken phases.

1.1.3 Critical exponents

The above example of magnetic systems is convenient to introduce scaling exponents. We first introduce the reduced temperature and magnetic field:

$$t := \frac{T - T_c}{T_c}, \qquad h := \frac{H}{k_B T_c}. \tag{1.1.5}$$

Let us list the most common critical exponents, defined by the behaviour of the system in the vicinity of the critical point:

- The specific heat $C = \frac{dF}{dT} \propto |t|^{-\alpha}$.
- The spontaneous magnetisation $M_0 = \lim_{H\to 0^+} M \propto |t|^{\beta}$ for $t < 0$.
- The zero-field susceptibility $\chi \propto |t|^{-\gamma}$.
- The magnetisation at critical temperature $M \propto |h|^{1/\delta}$.
- The divergence of the correlation length $\xi \propto |t|^{-\nu}$.
- The anomalous dimension at the critical point $G(r) = \langle S(0)S(r)\rangle \propto 1/|r|^{d-2+\eta}$.

1.1.4 Simple lattice models

To describe a phase transition, one considers a discrete model as simple as possible to allow some computations (sometimes up to a complete solution), but with enough ingredients to capture the important features of the transition. Moreover, even when the degrees of freedom are of quantum nature (*e.g.* atomic spins in a crystal), in many cases one may still capture the essential physical features by considering a classical discrete model, *i.e.* by neglecting the quantum fluctuations at ordinary temperatures. Furthermore, in the case of spin systems, one may simplify the model even more by only including the (classical) fluctuations of one component of the spin. This gives rise to the Ising model, with Hamiltonian:

$$\mathcal{H}[S] = -\sum_{i,j} J_{ij} S_i S_j - H \sum_i S_i \,, \tag{1.1.6}$$

where $[S]$ denotes a configuration of spins with values $S_i = \pm 1$, living on the sites of some regular lattice, representing the position of nuclei of the crystal. The set of parameters J_{ij} are the coupling constants, which determine the interaction between spins. A standard choice (which is the only one we shall discuss in this course) is to set

$$J_{ij} = \begin{cases} J & \text{if } i \text{ and } j \text{ are adjacent sites}\,, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1.7}$$

This defines the nearest-neighbour Ising model:

$$\mathcal{H}_{\text{Ising}}[S] = -J \sum_{\langle i,j\rangle} S_i S_j - H \sum_i S_i \,, \tag{1.1.8}$$

where $\langle i, j\rangle$ denotes adjacent sites on the lattice. This model has an internal $\mathbb{Z}_2$ symmetry (global spin reversal $S_i \mapsto -S_i$). When the lattice is regular, the model also enjoys translation invariance. In the scaling limit, it exhibits the behaviour described in Sec. 1.1.2, with a $\mathbb{Z}_2$ broken symmetry phase ending at a second-order critical point.

Various generalisations of the Ising model may be considered. *Clock models* are classical spin models where the degrees of freedom can take N values equally spaced on the unit circle, with an interaction invariant under global spin rotation and spin reversal. The internal symmetry group is then $\mathbb{D}_N$, the dihedral group. It is convenient to introduce $\omega = e^{2i\pi/N}$, and to use complex notations for the spins $S_i \in \{1, \omega, \dots, \omega^{N-1}\}$. A generic nearest-neighbour interaction can be written as:

$$\mathcal{H}_{\text{clock}} = -\sum_{\langle i,j\rangle} \sum_{k=1}^{N-1} \frac{J_k}{2} (S_i^k S_j^{-k} + S_i^{-k} S_j^k) \,. \tag{1.1.9}$$

These models have a richer phase diagram than Ising: in particular, there can be more than two ordered phases. For $N = 2$ one recovers the Ising model.

Another way of generalising the Ising model is to consider n-dimensional vector spin variables $\boldsymbol{S}_i$ living on the sphere $\boldsymbol{S}_i^2 = 1$, with an interaction invariant under global spin rotation. The internal symmetry group is then $\mathrm{O}(n)$, the group of isometric linear transformations. A generic nearest-neighbour Hamiltonian will be of the form:

$$\mathcal{H}_{\mathrm{O}(n)} = -\sum_{\langle i,j\rangle} \sum_{k=1}^{\infty} J_k \, (\boldsymbol{S}_i \cdot \boldsymbol{S}_j)^k \,. \tag{1.1.10}$$

An important difference with the Ising model is that the internal symmetry group is continuous (and unitary). As a consequence, from the Mermin-Wagner theorem, in two spatial dimensions there cannot be a broken-symmetry phase. The case $n = 2$ is of specific importance: it is usually called the XY model. It exhibits a peculiar type of phase transition, called the Berezinskii–Kosterlitz–Thouless transition. The XY model may be obtained as the $N \to \infty$ limit of a $\mathbb{Z}_N$ clock model.

1.2 The Renormalisation Group

1.2.1 Block-spin variables

The Renormalisation Group (RG), as applied to problems of Statistical Mechanics, is considered in this course as a conceptual framework for the physical theory of critical phase transitions. The main ideas may be exposed by considering the example of block-spin transformations on the Ising model. We start with the Ising model with spin variables $S_i = \pm 1$ on a regular lattice of mesh size a, with an interaction defined by the Hamiltonian $\mathcal{H}[S]$. We set some rescaling factor $\lambda > 1$ (practically, λ is a positive integer), and we form blocks of $n_b = \lambda^d$ neighbouring variables, which we replace by block-spin variables $S'_\alpha = \pm 1$, using some definite rule. This will result in an Ising model on a lattice of mesh λa, with a new Hamiltonian $\mathcal{H}'[S']$. Let us explain how this new Hamiltonian is defined. Let α denote a block of n_b spins, and $(S_1^{(\alpha)}, \dots S_{n_b}^{(\alpha)})$ be the set of original variables included in this block. The block-spin rule can be encoded by some function μ :

$$\mu : \begin{cases} \{+1,-1\}^{n_b} & \to \{+1,-1\} \\ (S_1^{(\alpha)}, \dots S_{n_b}^{(\alpha)}) & \mapsto S'_\alpha \,. \end{cases} \tag{1.2.1}$$

We want the Boltzmann weights on block-spin variables to be formed by the sum over internal degrees of freedom in each block:

$$e^{-\mathcal{H}'[S']} := \sum_{[S]} \prod_{\alpha} \delta[S'_\alpha - \mu(S_1^{(\alpha)}, \dots S_{n_b}^{(\alpha)})] \, e^{-\mathcal{H}[S]} \,. \tag{1.2.2}$$

Note that we have absorbed the inverse temperature $\beta = 1/k_B T$ into the definition of $\mathcal{H}$. By construction, the partition function is invariant under this process:

$$Z = \sum_{[S]} e^{-\mathcal{H}[S]} = \sum_{[S']} e^{-\mathcal{H}'[S']} \,. \tag{1.2.3}$$

Any correlation function which depends only on block-spin variables is also left invariant:

$$\frac{1}{Z}\sum_{[S]}\mathcal{A}[\mu(S_1^{(\alpha)},\dots S_{n_b}^{(\alpha)}),\mu(S_1^{(\beta)},\dots S_{n_b}^{(\beta)}),\dots]e^{-\mathcal{H}[S]} = \frac{1}{Z}\sum_{[S']}\mathcal{A}(S'_\alpha,S'_\beta,\dots)e^{-\mathcal{H}'[S']}\,. \tag{1.2.4}$$

However, some information is lost during the process, since the internal degrees of freedom of each block are summed over, and replaced by a single spin variable S'_α : the new Hamiltonian is a *coarse-grained* version of the original one, sharing the same internal symmetry group – $\mathbb{Z}_2$ in the Ising case. Even if the original Hamiltonian has only nearest-neighbour interactions, the new Hamiltonian may include more general interactions. Generically, if we take the original (reduced) Hamiltonian to be of the form:

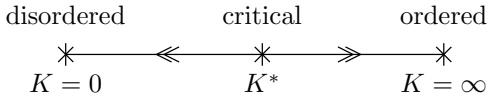
$$\mathcal{H} = -\sum_{\langle i,j\rangle} K_1 S_i S_j - \sum_{\langle\langle i,j\rangle\rangle} K_2 S_i S_j - \dots \tag{1.2.5}$$

where $\langle i,j\rangle$ denotes a pair of nearest-neighbour sites, $\langle\langle i,j\rangle\rangle$ a pair of next-nearest neighbours, *etc.* and we denote by $K = (K_1, K_2, \dots)$ the set of coupling constants, then we expect $\mathcal{H}'$ to be of the same form as $\mathcal{H}$, with different values of the coupling constants $K' = (K'_1, K'_2, \dots) = \mathcal{R}(K)$. The map $\mathcal{R}$ is called the renormalisation group (RG) transformation. When the RG procedure is iterated, the vector K converges to some attractive fixed point of $\mathcal{R}$, which corresponds to a physical phase of the system. The regions of attraction of different phases are typically separated by transition lines.

1.2.2 Qualitative RG flow for Ising with $d > 1$

The above 1d example is very special, because, up to a constant term in the Hamiltonian, the interaction remains of the same form after an RG transformation: it only has nearest-neighbour interaction, with a single parameter K. In higher dimensions, RG transformations typically give rise to more complicated interactions, and in most cases they cannot be treated by an exact computation. The basic reason is that a spin S_i at the boundary of a block α may interact with several neighbouring blocks $\beta, \gamma, \dots$ When summing over this S_i, one typically produces interactions between α and its neighbours, but also between, say, β and γ, which may not be adjacent to one another.

In the case of the zero-field Ising model, we assume that the scaling behaviour can still be analysed by looking only at the RG flow for the nearest-neighbour coupling K. The RG flow consistent with the phase diagram described above for the Ising model with dimension $d > 1$ is:



The regions $0 \le K < K^*$ and $K^* < K \le \infty$ correspond respectively to the disordered and ferromagnetic phases. The repulsive fixed point is the critical point where a second-order transition occurs. Although we do not have an exact expression for the RG transformation, we can still analyse the divergence of the correlation length in the vicinity of the critical point.

Let us define the RG exponent y as:

$$y := \frac{\log\left|\left[\frac{d\mathcal{R}}{dK}\right]_{K^*}\right|}{\log\lambda}\,, \qquad \text{so that} \qquad \left|\left[\frac{d\mathcal{R}}{dK}\right]_{K^*}\right| = \lambda^y\,. \tag{1.2.6}$$

Since K^* is a repulsive fixed point, we have $|d\mathcal{R}/dK| > 1$. By construction, the scaling factor always satisfies $\lambda > 1$. Hence, $y > 0$ for a repulsive RG fixed point. When we linearise the RG transformation for $K \simeq K^*$, we get

$$K' = \mathcal{R}(K) = K^* + \frac{d\mathcal{R}}{dK}(K - K^*) \qquad \Rightarrow \qquad |K' - K^*| = \lambda^y |K - K^*| \,. \tag{1.2.7}$$

The correlation length is always rescaled as $\xi(K') = \xi(K)/\lambda$. After p iterations, starting from some value $K^{(0)}$, we get

$$|K^{(p)} - K^*| = \lambda^{py} |K^{(0)} - K^*| \,, \qquad \xi(K^{(p)}) = \frac{K^{(0)}}{\lambda^p} \,. \tag{1.2.8}$$

Eliminating p between these equations, for any $K = K^{(p)}$ we get the relation

$$\xi(K) \propto |K - K^*|^{-1/y} \,. \tag{1.2.9}$$

Hence, the correlation exponent ν is simply given by the inverse of the RG exponent y:

$$\nu = \frac{1}{y} \,. \tag{1.2.10}$$

1.3 Scale invariance

1.3.1 Scaling variables and scaling operators

Let us now consider the general situation of an RG fixed point $K^* = (K_1^*, K_2^*, \dots)$ in the multidimensional space of parameters. In the vicinity of K^*, when we apply the RG tranformation $K' = \mathcal{R}(K)$, we get the linear approximation:

$$K'_a - K^*_a = \sum_b \left. \frac{\partial K'_a}{\partial K_b} \right|_{K^*} (K_b - K^*_b) \,. \tag{1.3.1}$$

Let $J_{ab} = \partial K'_a / \partial K_b$ be the Jacobian matrix at $K = K^*$, and $\{e_j\}$ the left eigenvectors of J, with eigenvalues $\{\mu_j\}$:

$$\forall j \,, \qquad e_j^t \, J = \mu_j \, e_j^t \,. \tag{1.3.2}$$

We assume that all the eigenvalues μ_j are positive[1]. From these objects, we define the *scaling variables* $\{u_j\}$ and the associated RG exponents $\{y_j\}$:

$$u_j := e_j^t \cdot (K - K^*) \,, \qquad y_j := \frac{\log \mu_j}{\log \lambda} \,. \tag{1.3.3}$$

In the vicinity of the critical point K^*, the scaling variables transform under an RG iteration of factor λ as:

$$\boxed{u'_j = \lambda^{y_j} \, u_j \,.} \tag{1.3.4}$$

The physical properties of the critical point depend on the signs of the exponents y_j:

- If $y_j > 0$, the scaling variable u_j is *relevant*. A small deviation of u_j from the critical value $u_j^* = 0$ drives the system away from the fixed point.

[1] If some negative eigenvalues μ_j appear, we can apply the RG tranform twice, and redefine $J \to J^2, \lambda \to \lambda^2$, so that all eigenvalues become positive.

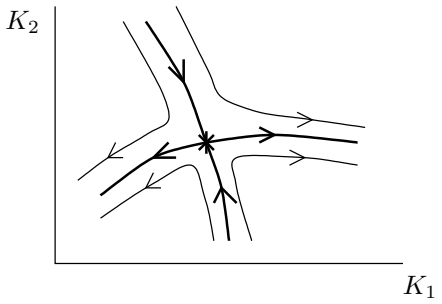


Figure 1.2: An example of RG flow in two-parameter space, in the case of a fixed point with one relevant eigenvalue and one irrelevant eigenvalue of the RG operator $\mathcal{R}$.

- If $y_j < 0$, the scaling variable u_j is *irrelevant*. A small deviation of u_j from zero is destroyed under the RG, and leaves the system at its critical point.
- If $y_j = 0$, the scaling variable u_j is *marginal*. More detailed analysis is required to predict the phase diagram. In typical situations, a critical line of fixed points parameterised by u_j passes through K^*.

The general structure of the RG flow in the vicinity of K^* can be inferred simply from the signs of the y_j's: see Fig. 1.2.

Consider the set of parameters $\{K_a\}$ of the Hamiltonian, and denote $\{S_a\}$ the associated operators. Let $\{f_j\}$ be the right eigenvectors of the matrix (J_{ab}) defined in Sec. 1.3.1, so that the bases $\{e_j\}$ and $\{f_j\}$ are dual to one another. We can write the coupling terms as

$$\sum_a (K_a - K_a^*) S_a = \sum_j u_j \widehat{S}_j \,, \qquad \text{where} \qquad \widehat{S}_j := \sum_a (f_j)_a \, S_a \,. \tag{1.3.5}$$

The operators $\widehat{S}_j$ are called *scaling operators*.

1.3.2 Scaling of correlation functions

Let us now study how correlation functions transform under an RG iteration. We first look at the correlation functions of spin operators in a magnetic system, on the domain $\mathcal{D} \cap a\mathbb{Z}^d$. It is convenient to introduce a non-uniform magnetic field $h(r)$, and to assume that the spatial variations of this field are small enough, so that it transforms locally as a uniform field under RG:

$$h(r) \to h'(r') = \lambda^{y_h} \, h(r) \,. \tag{1.3.6}$$

The n-point correlation function reads

$$G_{K,a}(r_1, \dots, r_n) = \langle S(r_1) \dots S(r_n) \rangle_{K, \mathcal{D} \cap a\mathbb{Z}^d} = \frac{\partial^n \log Z_{K,a}[h]}{\partial h(r_1) \dots \partial h(r_n)} \,. \tag{1.3.7}$$

The partition function

$$Z_{K,a}[h] = \sum_{[S]} e^{-\mathcal{H}_{K,a}[S] + \sum_r h(r) S(r)} \tag{1.3.8}$$

is preserved by the RG transformation, and thus we have

$$G_{K',\lambda a}(r'_1,\dots r'_n) = \frac{\partial^n \log Z_{K',\lambda a}[h']}{\partial h'(r'_1)\dots\partial h'(r'_n)} = \frac{\partial^n \log Z_{K,a}[h]}{\partial h'(r'_1)\dots\partial h'(r'_n)}\,. \tag{1.3.9}$$

In each block labelled by the position $r'_j = r_j/\lambda$, there are λ^d original spins, and we assume that they all couple equally to the local magnetic field $h'(r'_j)$. Hence, each derivative $\partial/\partial h'(r'_j)$ acts as $\lambda^d \times \lambda^{-y_h} \times \partial/\partial h(r_j)$ on $Z_{K,a}[h]$. As a result, we obtain the identity:

$$G_{K',\lambda a}(r_1/\lambda,\dots r_n/\lambda) = \lambda^{n\Delta_h} G_{K,a}(r_1,\dots,r_n)\,, \qquad \Delta_h := d - y_h\,. \tag{1.3.10}$$

In particular, at the critical point $K = K^*$, we get $G_{K,a} \propto a^{n\Delta_h}$. To get a finite correlation function as $a \to 0$, we introduce the renormalised scaling operator for physical coordinates $r \in \mathbb{R}^d$:

$$s(r) := \text{const} \times \lim_{a\to 0} \big[a^{-\Delta_h} S(r/a)\big]\,, \tag{1.3.11}$$

where the constant is chosen so that $\langle s(r)s(0)\rangle_{\mathbb{R}^d} \sim 1/r^{2\Delta_h}$ as $r \to \infty$. From the RG relation (1.3.10), and using the fact that the domain $\mathcal{D} \cap \lambda a\mathbb{Z}^d$ is trivially equivalent to $\mathcal{D}/\lambda \cap a\mathbb{Z}^d$, we get the scale covariance relation:

$$\langle s(r_1)\dots s(r_n)\rangle_{K^*,\mathcal{D}} = \lambda^{-n\Delta_h}\langle s(r_1/\lambda)\dots s(r_n/\lambda)\rangle_{K^*,\mathcal{D}/\lambda}\,. \tag{1.3.12}$$

From a straightforward generalisation of the above argument, the critical correlation functions have the *scale covariance* property at the critical point:

$$\boxed{\langle \mathcal{O}_1(r_1)\dots\mathcal{O}_n(r_n)\rangle_{K^*,\mathcal{D}} = \left(\prod_{j=1}^n \lambda^{-\Delta_j}\right)\langle \mathcal{O}_1(r_1/\lambda)\dots\mathcal{O}_n(r_n/\lambda)\rangle_{K^*,\mathcal{D}/\lambda}\,,} \tag{1.3.13}$$

where the scaling dimensions are given by

$$\boxed{\Delta_j = d - y_j\,,} \tag{1.3.14}$$

and the renormalised scaling operators $\mathcal{O}_j(r)$ are given by

$$\boxed{\mathcal{O}_j(r) := \text{const} \times \lim_{a\to 0}\big[a^{-\Delta_j} S_j(r/a)\big]\,.} \tag{1.3.15}$$

The relation (1.3.13) can be expressed compactly by saying that the scaling operator $\mathcal{O}_j$ transforms as

$$\mathcal{O}_j(r) \to \lambda^{-\Delta_j}\,\mathcal{O}_j(r/\lambda) \tag{1.3.16}$$

under a scale transformation $r \mapsto r/\lambda$.

As an immediate consequence, the two-point function of a rotationally invariant scaling operator $\mathcal{O}_j$ on the full Euclidean space is of the form $\langle \mathcal{O}_j(r_1)\mathcal{O}_j(r_2)\rangle_{\mathbb{R}^d} = \text{const}/|r_1 - r_2|^{2\Delta_j}$. By convention, the normalisation of scaling operators is chosen so that

$$\boxed{\langle \mathcal{O}_j(r_1)\mathcal{O}_j(r_2)\rangle = \frac{1}{|r_1 - r_2|^{2\Delta_j}}\,.} \tag{1.3.17}$$

We thus see that $\mathcal{O}_j$ has an anomalous dimension $\eta_j = d + 2 - y_j$.

Remark. In the vicinity of the critical point, if all relevant parameters are set to their critical value except t, the identity (1.3.10) imposes a scaling form for G:

$$G(r_1, \dots, r_n|t) = \xi^{-n\Delta_h}\, \Psi_S(r_1/\xi, \dots, r_n/\xi)\,, \qquad \xi \propto t^{-\nu}\,, \tag{1.3.18}$$

where we have used $\nu = 1/y_t$. Similarly, if a single relevant parameter t is slightly different from its critical value $t^* = 0$, we have the scaling form:

$$\langle \mathcal{O}_1(r_1) \dots \mathcal{O}_n(r_n) \rangle_t = \left(\prod_{j=1}^{n} \xi^{-\Delta_j} \right) \Psi(r_1/\xi, \dots, r_n/\xi)\,, \qquad \xi \propto t^{-\nu}\,. \tag{1.3.19}$$

1.4 Conformal invariance

1.4.1 Conformal mappings

In the previous section, we have seen from RG arguments that, at the critical point, correlation functions of scaling operators are expected to be covariant under scale transformations: see (1.3.13). The basic assumption of CFT is that this property extends to more general space transformations: the conformal maps, *i.e.* the diffeomorphisms which locally preserve angles. Let us give here a heuristic description of conformal mappings – their formal definitions and properties will be treated in detail in Chapter 2.

We start with conformal maps from the full Euclidean space $\mathbb{R}^d$ to itself: they are called *global* conformal maps. In fact, we shall work with the compactified space $\mathbb{R}^d \cup \{\infty\}$. For instance, in two dimensions, the compactified plane $\mathbb{R}^2 \cup \{\infty\}$ is diffeomorphic to the sphere, through the stereographic projection. In general dimension d, one can show that any global conformal map is a composition of:

- Translations
- Dilatations
- Rotations
- The inversion map $I : r \mapsto r' = r/|r|^2$.

Exercise. The goal is to show that the inversion map locally preserves angles. Consider three points $a, b = a + \epsilon, c = a + \delta$, with ϵ and δ two independent infinitesimal vectors. We call a', b', c' the images of a, b, c, and we define $\epsilon' = b' - a'$ and $\delta' = c' - a'$. Compute ϵ' and δ' at order one in ϵ and δ. Compare the quantities $\epsilon \cdot \delta/(|\epsilon|\,|\delta|)$ and $\epsilon' \cdot \delta'/(|\epsilon'|\,|\delta'|)$.

A generic conformal transformation is defined as a diffeomorphism from one domain $\mathcal{D} \subset \mathbb{R}^d$ to another domain $\mathcal{D}'$, which locally preserves angles. Consider an infinitesimal volume element $dr_1 \dots dr_d$ located at position r. Under a conformal transformation $r \mapsto r'$, the volume element is subject to a translation by $(r' - r)$, followed by a linear transformation which preserves angles, *i.e.* an orthogonal transformation, composed of a dilatation and a rotation. As a result, the infinitesimal volume element transforms as

$$dr_1 \dots dr_d \to \lambda(r)^d\, dr'_1 \dots dr'_d\,, \tag{1.4.1}$$

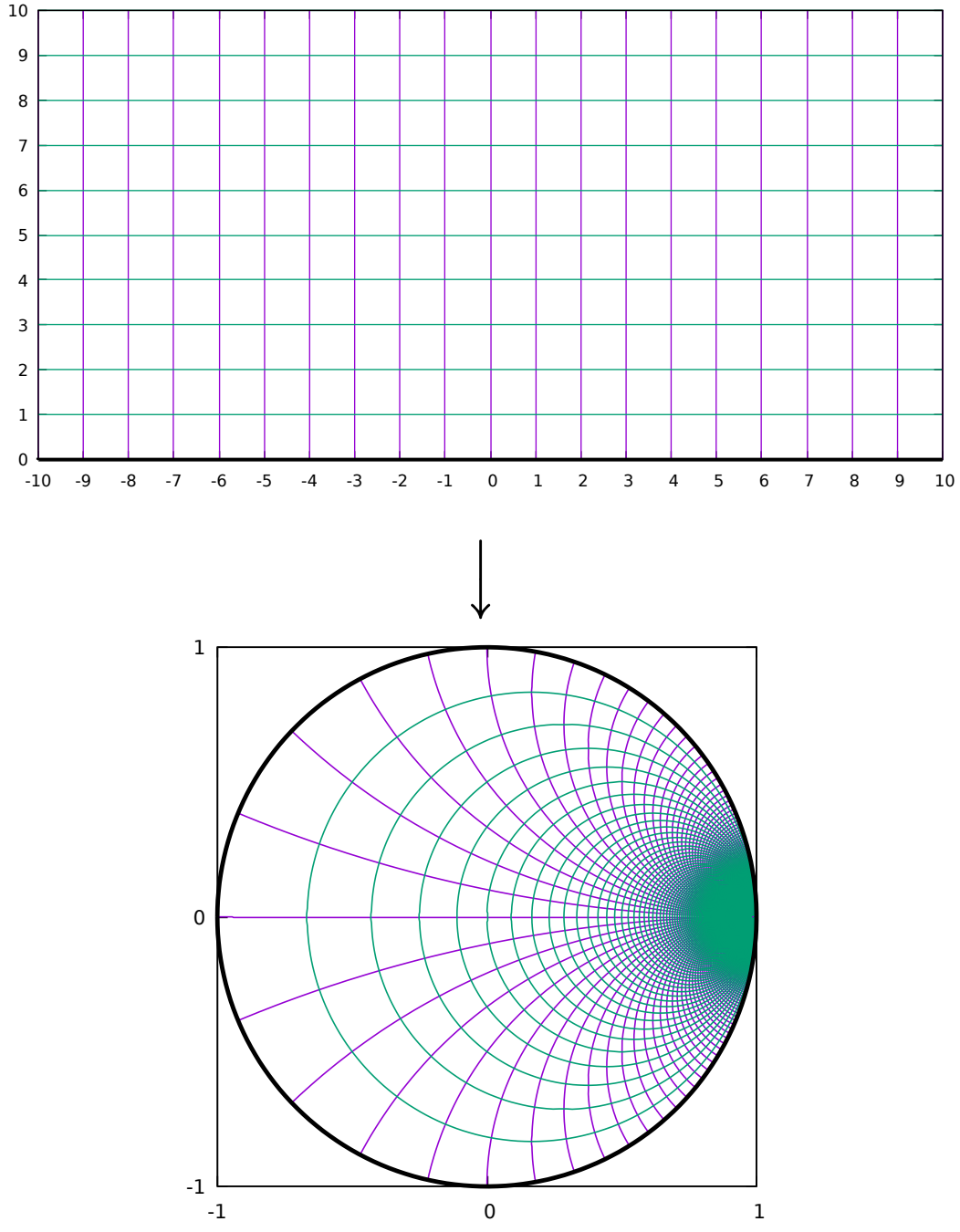
where the dilatation factor $\lambda(r)$ is related to the Jacobian as

$$\lambda(r) = \left[\det\left(\frac{\partial r'_\mu}{\partial r_\nu}\right)\right]^{-1/d} . \tag{1.4.2}$$

This situation can be visualised by choosing a regular lattice in $\mathcal{D}$, and examining its image under the conformal transformation. An example is given below in two dimensions, with the conformal mapping from the upper half plane $\mathbb{H}$ to the unit disc $\mathbb{D}$:

$$\begin{cases} \mathbb{H} & \to \mathbb{D} \\ z & \mapsto w(z) = \frac{z-i}{z+i} . \end{cases} \tag{1.4.3}$$

The square lattice on $\mathbb{H}$ is mapped to the deformed grid in $\mathbb{D}$ illustrated here:



When the lattice step a tends to zero, the image of an elementary square cell located at position z becomes a square located at position $w(z)$, of side $a \times |dw/dz|$.

1.4.2 Primary operators

We are now in the position to present the main assumption of conformal field theory: we expect that, at the critical point, there exists a collection of scaling operators, called the *primary operators*, whose correlation functions obey a covariance equation generalising (1.3.13) to any conformal mapping. If the domains $\mathcal{D}$ and $\mathcal{D}'$ are related by a conformal mapping:

$$\begin{cases} \mathcal{D} & \to \mathcal{D}' \\ r & \mapsto r' \,, \end{cases} \tag{1.4.4}$$

and $\phi_1, \dots \phi_n$ are scalar primary operators with scaling dimensions $\Delta_1, \dots, \Delta_n$, then we have the identity:

$$\boxed{\langle \phi_1(r_1) \dots \phi_n(r_n) \rangle_{\mathcal{D}} = \left(\prod_{j=1}^{n} \left| \frac{\partial r'_j}{\partial r_j} \right|^{\Delta_j/d} \right) \langle \phi_1(r'_1) \dots \phi_n(r'_n) \rangle_{\mathcal{D}'} \,,} \tag{1.4.5}$$

where $|\partial r'/\partial r|$ denotes the determinant of the Jacobian matrix of the map (1.4.4), and the local scaling factor is $\lambda(r) = |\partial r'/\partial r|^{-1/d}$. It is important to stress that, on both sides of (1.4.5), the average values correspond to the continuum limit of the domains $\mathcal{D}$ and $\mathcal{D}'$ *with the same lattice discretisation*: if the RHS is the continuum limit of, say, the square lattice Ising model, then the lattice model corresponding to the LHS also lives on the square lattice, *not* its image by the conformal mapping. The relation (1.4.5) is summarised by saying that each primary operator transforms as

$$\phi_j(r) \to \left| \frac{\partial r'}{\partial r} \right|^{\Delta_j/d} \phi_j(r') \tag{1.4.6}$$

inside a correlation function. This assumes that the ϕ_j's are scalar under rotations. Non-scalar operators will be described in the two-dimensional case.

Remark 1. Note that not all operators are primary. For instance, if ϕ_j is a primary operator, then $\partial_\mu \phi_j$ is not a primary in general, since

$$\partial_\mu \phi_j(r) \to \lambda(r)^{-\Delta_j} \partial_\mu r'^\nu \, \partial'_\nu \phi_j(r') - \Delta_j \, \partial_\mu \lambda(r) \, \lambda(r)^{-\Delta_j - 1} \, \phi_j(r') \,, \tag{1.4.7}$$

and the second term is not a derivative of ϕ_j.

Remark 2. A larger class of operators is given by the *quasi-primary operators*, for which one imposes the covariance relation (1.4.5) only for global conformal mappings, *i.e.* in the case $\mathcal{D} = \mathcal{D}' = \mathbb{R}^d$, where the conformal map is a composition of translations, dilatations, rotations and special conformal transformations. Of course, any primary operator is also quasi-primary.

Remark 3. Using the inversion map, one can easily show that, for fixed $r_2, \dots, r_n$, the correlation function $\langle \phi_1(r_1) \dots \phi_n(r_n) \rangle$ scales as $|r_1|^{-2\Delta_1}$ when $r_1 \to \infty$. Hence, to define the correlation functions on the compactified space $\mathbb{R}^d \cup \{\infty\}$, we introduce the convention

$$\langle \phi_1(\infty)\phi_2(r_2) \dots \phi_n(r_n) \rangle_{\mathbb{R}^d} \coloneqq \lim_{R\to\infty} \left[|R|^{2\Delta_1} \times \langle \phi_1(R)\phi_2(r_2) \dots \phi_n(r_n) \rangle_{\mathbb{R}^d} \right], \tag{1.4.8}$$

for any primary or quasi-primary operators $\phi_1, \dots, \phi_n$.

Two-point and three-point functions of primary operators. Let ϕ_1, ϕ_2 be two primary operators. Using a translation, followed by a dilatation and a rotation, one can write

$$\langle \phi_1(r_1)\phi_2(r_2) \rangle_{\mathbb{R}^d} = \langle \phi_1(r_1 - r_2)\phi_2(0) \rangle_{\mathbb{R}^d} = |r_1 - r_2|^{-\Delta_1-\Delta_2} \langle \phi_1(u)\phi_2(0) \rangle_{\mathbb{R}^d}, \tag{1.4.9}$$

where u is an arbitrary unit vector. If, instead, we first apply an inversion, we get

$$\begin{aligned} \langle \phi_1(r_1)\phi_2(r_2) \rangle_{\mathbb{R}^d} &= |r_1|^{-2\Delta_1} |r_2|^{-2\Delta_2} \langle \phi_1(r_1')\phi_2(r_2') \rangle_{\mathbb{R}^d} \\ &= |r_1|^{-2\Delta_1} |r_2|^{-2\Delta_2} |r_1' - r_2'|^{-\Delta_1-\Delta_2} \langle \phi_1(u)\phi_2(0) \rangle_{\mathbb{R}^d} \\ &= (|r_1|/|r_2|)^{\Delta_1-\Delta_2} |r_1 - r_2|^{-\Delta_1-\Delta_2} \langle \phi_1(u)\phi_2(0) \rangle_{\mathbb{R}^d}, \end{aligned} \tag{1.4.10}$$

where we have denoted $r_j' = r_j/|r_j|^2$, and we have used the identity

$$|r_1' - r_2'| = \frac{|r_1 - r_2|}{|r_1|\,|r_2|}. \tag{1.4.11}$$

As a result, we get

$$\boxed{\langle \phi_1(r_1)\phi_2(r_2) \rangle_{\mathbb{R}^d} = \begin{cases} \text{const} \times |r_1 - r_2|^{-2\Delta_1} & \text{if } \Delta_1 = \Delta_2, \\ 0 & \text{otherwise.} \end{cases}} \tag{1.4.12}$$

By convention we choose the normalisation of primary operators so that

$$\langle \phi_j(r_1)\phi_j(r_2) \rangle_{\mathbb{R}^d} = \frac{1}{|r_1 - r_2|^{2\Delta_j}}. \tag{1.4.13}$$

Using a similar argument, we get the three-point function of scalar primary operators:

$$\langle \phi_1(r_1)\phi_2(r_2)\phi_3(r_3) \rangle_{\mathbb{R}^d} = \frac{C_{123}}{|r_1 - r_2|^{\Delta_{12}^3} |r_1 - r_3|^{\Delta_{13}^2} |r_2 - r_3|^{\Delta_{23}^1}}, \tag{1.4.14}$$

with the notation $\Delta_{ij}^k = \Delta_i + \Delta_j - \Delta_k$. The constant C_{123} cannot be set to an arbitrary value, because the primary operators are already normalised by their two-point function. This constant is given by

$$C_{123} = \langle \phi_1(0)\phi_2(u)\phi_3(\infty) \rangle_{\mathbb{R}^d}, \tag{1.4.15}$$

for any unit vector u. Higher correlation functions (four-point, five-point ...) of primary operators are more complicated objects, and their spatial dependence is not immediately determined by conformal covariance. A much deeper analysis is needed to compute them.

1.4.3 The two-dimensional case

Throughout this course, we shall concentrate on the case $d = 2$. We introduce the complex coordinates:

$$z = r^0 + ir^1 \,, \qquad \bar{z} = r^0 - ir^1 \,. \tag{1.4.16}$$

The associated derivatives read:

$$\partial := \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial r^0} - i\frac{\partial}{\partial r^1}\right), \qquad \bar{\partial} := \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial r^0} + i\frac{\partial}{\partial r^1}\right). \tag{1.4.17}$$

In two dimensions, the global conformal maps take the form of *Möbius transformations*, which read in complex coordinates

$$z \mapsto \frac{az+b}{cz+d}\,, \tag{1.4.18}$$

where a, b, c, d are complex constants conventionnally normalised so that $ad - bc = 1$. These transformations form a group, isomorphic to the special linear group $\mathrm{SL}(2,\mathbb{C})$ of 2×2 complex matrices with determinant one.

To emphasize the fact that (correlation functions of) a scaling operator $\mathcal{O}$ typically have nonzero derivatives $\langle \partial\mathcal{O} \dots \rangle \neq 0$ and $\langle \bar{\partial}\mathcal{O} \dots \rangle \neq 0$, we shall denote it by $\mathcal{O}(z,\bar{z})$. A scaling operator is called scalar if it transforms trivially under a rotation θ, *i.e.* $\mathcal{O}(z,\bar{z}) \to \mathcal{O}(e^{i\theta}z, e^{-i\theta}\bar{z})$ in a correlation function. Non-scalar operators are those which transform as

$$\mathcal{O}(z,\bar{z}) \to e^{-i\theta s}\,\mathcal{O}(e^{i\theta}z, e^{-i\theta}\bar{z})\,, \tag{1.4.19}$$

where the real number s is called the *conformal spin*. Hence, a general scaling operator $\mathcal{O}_j$ is characterised by its scaling dimension Δ_j and its conformal spin s_j. Alternatively, we shall use the *conformal dimensions*, defined as

$$\Delta_j = h_j + \bar{h}_j \,, \qquad s_j = h_j - \bar{h}_j \,. \tag{1.4.20}$$

The conformal covariance relation for general (possibly non-scalar) primary operators now reads

$$\boxed{\langle \phi_1(z_1,\bar{z}_1)\dots\phi_n(z_n,\bar{z}_n)\rangle_{\mathcal{D}} = \left[\prod_{j=1}^{n}\left(\frac{\partial w_j}{\partial z_j}\right)^{h_j}\left(\frac{\partial \bar{w}_j}{\partial \bar{z}_j}\right)^{\bar{h}_j}\right]\langle \phi_1(w_1,\bar{w}_1)\dots\phi_n(w_n,\bar{w}_n)\rangle_{\mathcal{D}'}\,,} \tag{1.4.21}$$

where we have considered the conformal map:

$$\begin{cases} \mathcal{D} & \to \mathcal{D}' \\ z & \mapsto w = w(z)\,. \end{cases}$$

In short, we denote

$$\phi_j(z,\bar{z}) \to \left(\frac{\partial w}{\partial z}\right)^{h_j}\left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{\bar{h}_j}\phi_j(w,\bar{w})\,. \tag{1.4.22}$$

Equations (1.4.5) and (1.4.21) are the fundamental relations for primary operators. As we shall see, they encode a rich symmetry, and it will be the basis for the construction of CFT.

Finally, let us give the two- and three-point functions of primary operators in 2d:

$$\langle \phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\rangle_{\mathbb{C}} = \begin{cases} (z_1-z_2)^{-2h_1}\,(\bar{z}_1-\bar{z}_2)^{-2\bar{h}_1} & \text{if } (h_1,\bar{h}_1)=(h_2,\bar{h}_2)\,, \\ 0 & \text{otherwise.} \end{cases} \tag{1.4.23}$$

and

$$\langle \phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\phi_3(z_3,\bar{z}_3)\rangle_{\mathbb{C}} = \frac{C_{123}}{z_{12}^{h_{12}^3}\, z_{13}^{h_{13}^2}\, z_{23}^{h_{23}^1} \times \bar{z}_{12}^{\bar{h}_{12}^3}\, \bar{z}_{13}^{\bar{h}_{13}^2}\, \bar{z}_{23}^{\bar{h}_{23}^1}}\,, \tag{1.4.24}$$

where we have used the notations $z_{ij} = z_i - z_j$, and $h_{ij}^k = h_i + h_j - h_k$, and similarly for $\bar{z}_{ij}$ and $\bar{h}_{ij}^k$.

1.5 Exercises

1.5.1 Decimation procedure in the 1d Ising model

Before we make the RG picture more precise, let us perfom a particular block-spin procedure, namely the spin decimation, on the one-dimensional case, and compute explicitly its RG transformation. We start with the 1d Ising Hamiltonian:

$$\mathcal{H}[S] = -\sum_j K S_j S_{j+1}\,. \tag{1.5.1}$$

As a first step, we write the exact expansion $e^{KS_1S_2} = \cosh K\,(1+xS_1S_2)$, where $x = \tanh K$, so that the Boltzmann weight reads:

$$e^{-\mathcal{H}[S]} = \cosh^N K \times \prod_j (1 + xS_jS_{j+1})\,, \tag{1.5.2}$$

where N is the number of sites of the system. We form blocks of three spins, and we choose the *decimation* rule:

$$\mu(S_1,S_2,S_3) := S_2\,. \tag{1.5.3}$$

The rescaling factor is thus $\lambda = 3$. Let α,β be two adjacent spin blocks. The factors of $e^{-\mathcal{H}[S]}$ involving α and β are

$$\cosh^3 K \times (1 + xS'_\alpha S_3^{(\alpha)})(1 + xS_3^{(\alpha)}S_1^{(\beta)})(1 + xS_1^{(\beta)}S'_\beta)\,. \tag{1.5.4}$$

Expanding this product and summing over $S_3^{(\alpha)}$ and $S_1^{(\beta)}$, we get

$$4\cosh^3 K(1 + x^3\, S'_\alpha S'_\beta) = \frac{4\cosh^3 K}{\cosh K'}\, e^{-K'S'_\alpha S'_\beta}\,, \tag{1.5.5}$$

where the new coupling constant is given by:

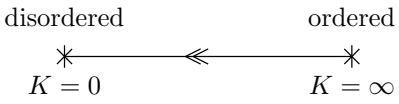
$$K' = \tanh^{-1}(\tanh^3 K)\,. \tag{1.5.6}$$

Hence the coarse-grained Hamiltonian is given by

$$\mathcal{H}'[S'] = N\varphi(K) - \sum_\alpha K' S'_\alpha S'_{\alpha+1}\,, \tag{1.5.7}$$

where $\varphi(K) = \frac{1}{3}\log(2\sqrt{1+3\cosh^2 2K})$. This constant term only plays a role in the normalisation of the partition function. In terms of the variable $x = \tanh K$, the RG transformation (1.5.6) corresponds to the map $x \mapsto x' = x^3$. The physical range for this variable

is $0 \leq x \leq 1$. Let us analyse the behaviour of the system when the RG is iterated many times. The fixed point $x = 0$ is attractive, and corresponds to $K = 0$, *i.e.* to infinite temperature. The fixed point $x = 1$ is repulsive, and corresponds to $K = +\infty$, *i.e.* zero temperature. This may be summarised in a diagram of RG flow:



The main idea of the scaling hypothesis is that the large-scale behaviour of the system is predicted by this RG flow. In this case, the situation is very simple, and does not involve any phase transition. For any finite value of K, the system remains in the disordered phase governed by the fixed point $K = 0$. The other fixed point, $K = \infty$, is instable under RG, and represents the totally ordered (ferromagnetic) phase.

The correlation length transforms under an RG iteration as a geometric sequence:

$$\xi(K') = \frac{\xi(K)}{\lambda}\,, \qquad \tanh K' = \tanh^3 K\,. \tag{1.5.8}$$

Hence, we may write the correlation length as a function of K along the RG flow:

$$\xi(K) = \frac{\text{const}}{|\log\tanh K|^{\lambda/3}} = \frac{\text{const}}{|\log\tanh K|}\,. \tag{1.5.9}$$

The correlation length decreases from $\xi = \infty$ to $\xi = 0$ along the RG flow. Close to the repulsive fixed point $K = \infty$, the correlation length diverges as $\xi \propto e^{2K}$.

1.5.2 Scaling functions and relations between critical exponents

Let us first discuss the transformation of the free energy density under an RG iteration. Under the RG transformation $K \to K'$, the configuration energy may vary by an extensive amount $N\varphi(K)$, where N is the number of sites of the system:

$$\mathcal{H}_K[S] \to \widetilde{\mathcal{H}}_{K'}[S'] = \mathcal{H}_{K'}[S'] + N\varphi(K)\,. \tag{1.5.10}$$

From the conservation of the partition function

$$Z(K) = \sum_{[S]} e^{-\mathcal{H}_K[S]} = \sum_{[S']} e^{-\widetilde{\mathcal{H}}_{K'}[S']}\,, \tag{1.5.11}$$

we get the inhomogeneous relation for the free energy density $f(K) = -[\log Z(K)]/N$:

$$f(K) = \lambda^{-d} f(K') + \varphi(K)\,. \tag{1.5.12}$$

Since $\varphi(K)$ must be analytic around K^*, one can construct a regular solution f_{reg} of (1.5.12), say by writing the power series expansion around K^*. Thus the physical free energy has the form

$$f(K) = f_{\text{reg}}(K) + f_{\text{sing}}(K)\,, \qquad \text{with} \qquad f_{\text{sing}}(K) = \lambda^{-d} f_{\text{sing}}(K')\,. \tag{1.5.13}$$

Consider the situation where only two scaling variables are relevant, and all other variables are irrelevant. In the case of a magnetic system, suppose we may identify, at linear order, these variables as the reduced temperature and magnetic field:

$$u_t \propto \frac{T - T_c}{T_c} := t\,, \qquad u_h \propto \frac{H}{k_b T_c} := h\,. \tag{1.5.14}$$

The singular free energy $f_{\text{sing}}(u_t, u_h)$ then satisfies:

$$f_{\text{sing}}(t,h) = \lambda^{-d} f_{\text{sing}}(\lambda^{y_t} t, \lambda^{y_h} h)\,, \tag{1.5.15}$$

where λ is the scaling factor of an RG iteration. After p iterations, we get

$$f_{\text{sing}}(t,h) = \lambda^{-dp} f_{\text{sing}}(\lambda^{py_t} t, \lambda^{py_h} h)\,. \tag{1.5.16}$$

We suppose that, close enough to the fixed point, this sequence is interpolated by an RG flow line $(\mu^{y_t} t, \mu^{y_h} h)$ parameterised by the real variable μ:

$$\forall \mu > 0\,, \qquad f_{\text{sing}}(t,h) = \mu^{-d} f_{\text{sing}}(\mu^{y_t} t, \mu^{y_h} h)\,. \tag{1.5.17}$$

We want to use this relation, to infer a scaling form for f_{sing}. First, let us fix some arbitrary value t_0. Let (t,h) be some point close to the fixed point. This point sits on the RG flow line $(\mu^{y_t} t, \mu^{y_h} h)$. For $\mu = (t/t_0)^{-1/y_t}$, we get the point $(t_0, h(t/t_0)^{-y_h/y_t})$ on the same flow line. We then define the variable:

$$u(t,h) := h/t^{y_h/y_t}\,.$$

Note that this variable is invariant along the RG flow line. We introduce the *scaling function*:

$$\Phi(u) := t_0^{-d/y_t} f_{\text{sing}}(t_0, t_0^{y_h/y_t}\, u)\,.$$

From (1.5.17), we get:

$$f_{\text{sing}}(t,h) = t^{d/y_t} \times \Phi\left(\frac{h}{t^{y_h/y_t}}\right). \tag{1.5.18}$$

As a consequence of the scaling form (1.5.18), we can compute the specific heat exponent α as follows:

$$C = \frac{\partial^2 f_{\text{sing}}}{\partial t^2}(t,0) \propto t^{d/y_t - 2} \qquad \Rightarrow \quad \alpha = 2 - d/y_t\,. \tag{1.5.19}$$

Similar arguments yield the spontaneous magnetisation and susceptibility exponents in terms of y_t, y_h:

$$\beta = \frac{d - y_h}{y_t}\,, \tag{1.5.20}$$

$$\gamma = \frac{2y_h - d}{y_t}\,. \tag{1.5.21}$$

1.6 References

- John Cardy, *Scaling and renormalization in statistical physics*, Cambridge ; New York : Cambridge University Press, 1996.
- Michel Le Bellac, *Des phénomènes critiques aux champs de jauge*, Intereditions, 1988.

Lecture 2

Conformal transformations

In the previous chapter scaling operators were introduced, and it was argued using RG arguments that (at criticality) correlation functions transform nicely under rescalings (1.3.13). This means that scaling operators transform covariantly under scale transformations

$$\mathcal{O}(x) \to \lambda^{-\Delta}\mathcal{O}(x/\lambda), \qquad \text{for } x \to x' = x/\lambda \text{ a scale transformation} \tag{2.0.1}$$

where Δ is the scaling dimension of the operator $\mathcal{O}$. We're going to take a leap and assume that at criticality the emergent scale invariance is promoted to *conformal invariance*. What might seem at first glance like a very strong assumption is in practice "always" statisfied, at least for any "reasonable" system. In particular in two dimensions, Zamolodchikov proved that scale invariance implies conformal invariance under broad conditions, namely unitarity plus discrete spectrum of scaling dimension.

But what is a conformal transformation? In some sense it is a *local* scale transformation. In particular, for the class of *primary operators*, the above transformation law for scaling operators will generalise to

$$\phi(x) \to \lambda(x)^{-\Delta}\phi(x'), \qquad \text{for } x \to x' = f(x) \text{ a conformal transformation} \tag{2.0.2}$$

in which the scalar function $\lambda(x) = \left[\det(\partial x'_\mu/\partial x^\nu)\right]^{-1/d}$ can be interpreted as a local dilation factor.

Before moving on to the study of conformally invariant field theories, it is necessary to discuss conformal transformations. This is the point of this chapter.

2.1 Space transformations and isometries

Unless otherwise specified we are working in the Euclidean setting : all directions are spacelike, and there is no time. This is mostly a matter of convention, as the following could also be formulated in the Minkoswski setting. We will use indiscriminately the terms *space* and *spacetime*.

We will be mostly interested in the d-dimensinal Eucidean space, that is $\mathbb{R}^d$ equipped with the usual notion of distance. More generally we can (and we will) consider arbitrary Riemannian manifolds (M, g), that is smooth manifolds M endowed with a (positive-definite) metric g.

While we will use some terminology of differential and Riemannian geometry, we will keep it to a minimum and try to give expressions in local coordinates as much as possible in order not to block the reader unfamiliar with these notions.

An extremely brief appendix on Riemannian geometry containing a few definitions and formulae is given at the end of these lecture notes, mostly to fix notations/conventions.

By a space transformation we mean a diffeomorphism $f : M \to M$, that is to say a smooth bijection whose inverse f^{-1} is also smooth. Note that space transformations as we just defined are *not* change of coordinates, but actual transformations of the manifold which may be visualized as smooth deformations of a continuous medium.

2.1.1 Flow generated by a vector field, exponential map

The set of diffeomorphisms $\mathrm{Diff}(M)$ of a given manifold M is a huge group, and describing all possible diffeomorphisms is a rather daunting task. There is however a very convenient way to construct many diffeomorphisms using vector fields.

Consider the stationary flow of a fluid : any point of the fluid has a particular velocity, and in this way there is a vector field associated to any flow. The converse is also true: it is possible to associate a flow to a vector field having that vector field as its velocity.

The crucial point is that the time evolution of such a flow is a diffeomorphism. Let's see how this works in more detail.

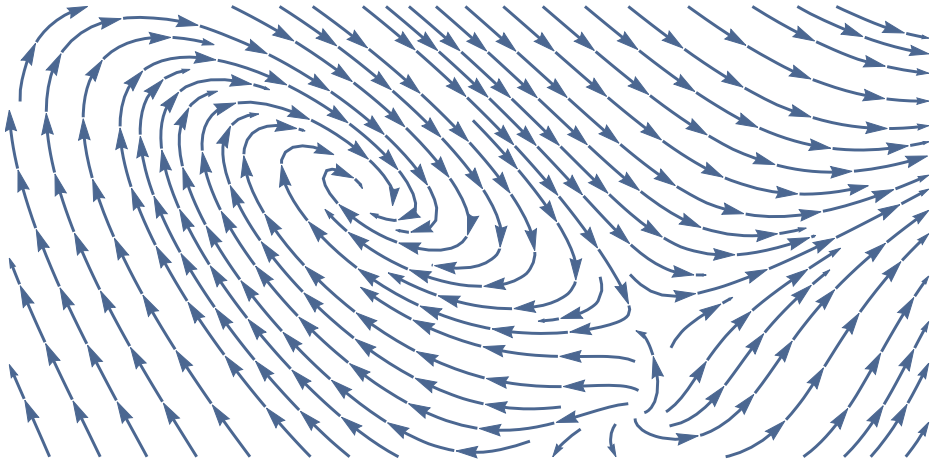


Figure 2.1: A two-dimensional flow

Let's start from an arbitrary vector field ξ. An integral curve $t \to c(t)$ is a solution of

$$c'(t) = \xi(c(t))\,. \tag{2.1.1}$$

In local coordinates $\xi = \xi^\mu(x)\partial_\mu$ the curve $c(t) = (c^1(t), \cdots, c^d(t))$ obeys the following system of differential equations

$$\frac{dc^\mu}{dt} = \xi^\mu(c^1, \cdots, c^d)\,. \tag{2.1.2}$$

These equations simply say that the vector tangent to the curve at any point $c(t)$ along the curve is precisely the vector $\xi(c(t))$, and so the curve $t \to c(t)$ is tangent at each point to the vector field ξ.

For a given point x, there is a unique integral curve[1] $t \to c_x(t)$ with initial condition $c_x(0) = x$, and furthermore $c_x(t)$ depends smoothly on x. This means that the map

$$f_t : x \to c_x(t) \tag{2.1.3}$$

is a smooth map. Moreover it is clear that if $t \to c(t)$ is an integral curve, then so is $t \to c(t+s)$. From the abovementioned unicity this means that $c_x(t+s) = c_{c_x(s)}(t)$, or equivalently

$$f_{t+s} = f_t \circ f_s = f_s \circ f_t \,. \tag{2.1.4}$$

Because of the above relation, the functions f_t are sometimes called the one-parameter group associated to the vector field ξ. In particular f_t is bijective, with smooth inverse f_{-t}. All this means that $x \to f_t(x)$ is a diffeomorphism. It is the time evolution at time t of the flow induced by the vector field ξ. One should think of the vector field ξ as the generator of the infinitesimal transformation

$$f_t(x) = x + t\xi + O(t^2) \,. \tag{2.1.5}$$

The flow f_t of the vector field ξ is also called the exponential map and is denoted $\exp t\xi$ instead of f_t. The exponential notation is motivated by the above mentioned properties, namely

- $\exp t\xi|_{t=0}$ is the identity map
- $\frac{d}{dt} \exp t\xi(x) = \xi \cdot (\exp t\xi(x))$
- $\exp t\xi \circ \exp s\xi = \exp(t+s)\xi$

▸ **Example 1 :** The flow induced on $\mathbb{R}$ by the constant vector field $\xi = a\partial_x$ is the translation by ta

$$f_t(x) = \exp(ta\partial_x) = x + ta \tag{2.1.6}$$

Clearly $f_t \circ f_s = f_{t+s}$.

▸ **Example 2 :** The flow induced on $\mathbb{R}$ by the vector field $\xi = x\partial_x$ is the dilation by a factor e^t:

$$f_t(x) = e^t x \tag{2.1.7}$$

It is easy to check that f_{-t} is indeed the inverse of f_t.

[1] This curve may or may not be defined at all times : it can happen that the flow blows up to infinity in finite time (see example 2 and special conformal transformations for instance). For now we simply ignore this fact and suppose that the flow is well defined at all times. Vector fields with this property are called *complete*. For instance this is the case of any vector field with compact support, so *a fortiori* any vector field on a compact manifold.

▸ **Example 3 :** The flow induced on $\mathbb{R}^2$ by the vector field $\xi = -x_2\partial_1 + x_1\partial_2$ is the rotation

$$f_t\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{2.1.8}$$

▸ **Example 4 :** The flow induced on $\mathbb{R}$ by the vector field $\xi = x^2\partial_x$ is

$$f_t(x) = \frac{x}{1-tx}. \tag{2.1.9}$$

It is only defined for $xt < 1$, since the flow blows up to infinity as $t \to 1/x$. An elegant way to cure this is to add a point at infinity. This amounts to work on the one-point compactification of the real line $\mathbb{R} \cup \{\infty\}$, *i.e.* the circle S^1, as given by the inverse stereographic projection. The stereographic projection is the following diffeomorphism from the unit circle minus the north pole $N = (0,1)$ to the real line (see Fig (2.2)) :

$$(x_1, x_2) \to \frac{x_1}{1-x_2} \tag{2.1.10}$$

If we parametrize the circle by θ as in (see Fig (2.2)), this means

$$x = \tan\frac{\theta}{2} \tag{2.1.11}$$

The vector field ξ now reads on the circle $\xi = 2\sin^2\frac{\theta}{2}\,\partial_\theta$, and it is perfectly well behaved at the North pole $\theta = \pi$ (see Fig. (2.3)). Since it has compact support, the corresponding flow on the circle is defined at all times.

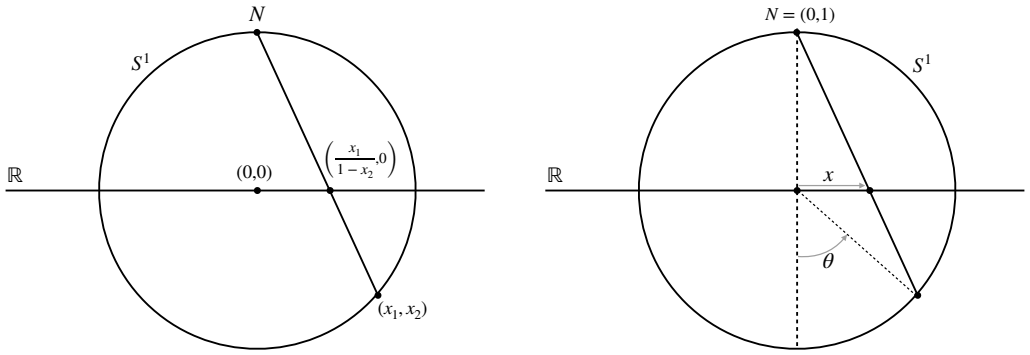


Figure 2.2: The one-point compactification of the real line is diffeomorphic to the unit circle $S^1 = \{(x_1, x_2) \in \mathbb{R}^2,\ x_1^2 + x_2^2 = 1\}$. The North pole $N = (0,1)$ corresponds to the point added at infinity.

▸ **Exercise** : compute this flow, and check that $\cot f_t(\theta)/2 = -t + \cot\theta/2$. What is this flow in the variable $1/x$?

▸ **Exercise :** compute the flows of the following vector fields in $\mathbb{R}^d$

- $\xi = a^\mu\partial_\mu$ for some constant vector a^μ.
- $\xi = x^\mu\partial_\mu$

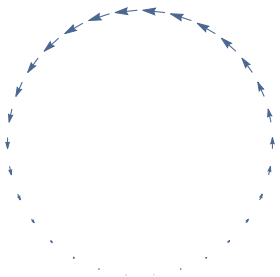


Figure 2.3: The vector field $\xi = 2\sin^2\frac{\theta}{2}\,\partial_\theta$ on the circle S^1.

- $\xi = A^\mu{}_\nu x^\nu \partial_\mu$
- $\xi = \left(2a\cdot x\, x^\mu - x^2 a^\mu\right)\partial_\mu$.

For the last one, use the one-point compactification of $\mathbb{R}^d$ (as given by the inverse stereographic projection) to construct a flow on the d-dimensional sphere S^d.

2.1.2 Isometries in Euclidean space

Perhaps the most familar transformations of Euclidean space $\mathbb{R}^d$ are isometries *i.e.* distance-preserving diffeomorphisms. In Euclidean space the distance between two points x and y is of course $d(x,y) = \|x-y\|$, and a map f is distance-preserving when

$$d(f(x), f(y)) = d(x,y) \tag{2.1.12}$$

Isometries of the Euclidean space form a $\frac{d(d+1)}{2}$-dimensional Lie group called the Euclidean group[2] $\mathrm{ISO}(d)$. Isometries are compositions of a translation and an orthogonal transformation

$$f(x) = a + R\cdot x, \qquad R \in O(d) \tag{2.1.13}$$

where R is an orthogonal matrix, *i.e.* $RR^t = 1$. The proof is left as an exercise to the reader. The corresponding Lie algebra is spanned by the vector fields

$$\xi^\mu = a^\mu + \omega^\mu{}_\nu x^\nu \tag{2.1.14}$$

where $\omega_{\mu\nu} = -\omega_{\nu\mu}$. From a previous exercise the reader will recognize the d generators of translations and the $\frac{d(d-1)}{2}$ generators of rotations (there are no reflections of the infinitesimal persuasion). All this is certainly very standard material, but we will use isometries as an excuse to prepare the way towards conformal transformations, and to introduce a few notions of differential geometry such as the pushforward and the pullback.

In order to make contact with isometries in the more general setting of Riemannian manifolds, we are going to give a slightly different but equivalent formulation. An isometry

[2] this is the analogue of the Poincaré group for Minkowski space, which is the group of transformation preserving spacetime intervals.

is a diffeomorphism that preserves the length of curves, in the sense that if $c : [t_0, t_1] \to \mathbb{R}^d$ is a smooth curve, then length$[c]$ = length$[f \circ c]$:

$$\int_{t_0}^{t_1} \|c'(t)\| dt = \int_{t_0}^{t_1} \|(f \circ c)'(t)\| dt \,. \tag{2.1.15}$$

where $c' = \frac{dc}{dt}$ is the velocity of the curve. Differentiating with respect to t_1, we find that this is equivalent to

$$\|c'(t)\| = \|(f \circ c)'(t)\| \tag{2.1.16}$$

In the above expression $(f \circ c)'(t)$ is the image of the tangent vector $c'(t)$ by the map f

$$(f \circ c)'(t) = df_{c(t)} \cdot c'(t) \tag{2.1.17}$$

where $df_{c(t)}$ is the Jacobian of f at $x = c(t)$

$$df_x = \begin{pmatrix} \partial_1 f^1(x) & \cdots & \partial_d f^1(s) \\ \vdots & & \vdots \\ \partial_1 f^d(x) & \cdots & \partial_d f^d(x) \end{pmatrix}. \tag{2.1.18}$$

So we have found that f is an isometry if and only if its Jacobian matrix is everywhere orthogonal $df_x df_x^t = 1$ or in coordinates

$$\boxed{\partial_\mu f^\rho(x) \delta_{\rho\sigma} \partial_\nu f^\sigma(x) = \delta_{\mu\nu}} \tag{2.1.19}$$

Let us introduce some terminology that will be useful later. The differential of f at x is the linear map $v \to df_x \cdot v$. Given a vector field $\xi(x)$, its image by a diffeomorphism f is called the pushforward $f_*\xi$

$$(f_*\xi)(x) = df_x \cdot \xi(f(x)) \tag{2.1.20}$$

Given a differential form α (say of degree p), the pullback by f is the p-form $f^*\alpha$ defined by

$$(f^*\alpha)_x(v_1, \cdots, v_p) = \alpha_{f(x)}(df_x \cdot v_1, \cdots, df_x \cdot v_p) \tag{2.1.21}$$

The relation (2.1.19) can be rewritten as

$$\boxed{f^*\eta = \eta} \tag{2.1.22}$$

where $\eta = \delta_{\mu\nu} dx^\mu \otimes dx^\nu$ is the Euclidean metric.

To sum up the above discussion about Iso(d), we have three equivalent characterizations of isometries. A diffeomorphism f is an isometry of the Euclidean space $\mathbb{R}^d$ iff

- f is distance preserving : $d(f(x), f(y)) = d(x, y)$
- f preserves the length of curves : length$[c]$ =length$[f \circ c]$
- f leaves the Euclidean metric invariant : $f^*\eta = \eta$

2.1.3 Isometries in Riemannian geometry, Killing vector fields

This discussion can be extended straightforwardly to Riemannian geometry, as soon as one has defined a notion of distance. The Riemannian metric allows one to define the length $\|v\| = \sqrt{g_p(v,v)}$ of a vector $v \in T_pM$ in the tangent space at p. Then one can define the length of a curve $c : [t_0, t_1] \to M$ as

$$L(c) = \int_{t_0}^{t_1} \|c'(t)\| dt \tag{2.1.23}$$

Finally the distance $d(x,y)$ between two points x and y is defined as the length of the shortest curve joining x and y. With this setting in place, we have again several equivalent characterizations of isometries.

Let $f : (M,g) \to (\tilde{M}, \tilde{g})$ be a diffeomorphism between two Riemannian manifolds, then the following are equivalent

- f is an isometry
- f is distance preserving : $d(f(x), f(y)) = d(x,y)$
- f preserves the length of curves : length$[c]$ =length$[f \circ c]$
- $f^*\tilde{g} = g$ *i.e.* in local coordinates $g_{\mu\nu}(x) = \tilde{g}_{\rho\sigma}(f(x))\partial_\mu f^\rho(x)\partial_\nu f^\sigma(x)$

The Riemannian manifolds (M,g) and $(\tilde{M},\tilde{g})$ are then completely equivalent as far as geometry is concerned. Indeed isometries leave everything metric-related (lengths, angles, curvature, parallel transport) invariant, the image of a geodesic is a geodesic, etc...

Before moving on to conformal transformations, let us consider infinitesimal isometries. The diffeomorphisms f_t induced by the flow of a vector field ξ on a Riemannian manifold (M,g) are isometries iff

$$\mathcal{L}_\xi g = 0 \tag{2.1.24}$$

where $\mathcal{L}_\xi g$ is the Lie derivative of g with respect to ξ, which reads in local coordinates

$$\mathcal{L}_\xi g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + g_{\nu\rho}\partial_\mu \epsilon^\rho + g_{\mu\rho}\partial_\nu \epsilon^\rho = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu \,. \tag{2.1.25}$$

So the condition for a vector field to be an infinitesimal isometry is that

$$\boxed{\nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu = 0\,.} \tag{2.1.26}$$

This is called the Killing equation, and infinitesimal generators of isometries are called Killing vector fields or Killing vectors.

▸ **Exercise :** recover the flat space solutions (2.1.14) from the Killing equation.

The isometries of a Riemannian manifold (M,g) form a Lie group of dimension at most $\frac{d(d+1)}{2}$. In that sense the Euclidean space is maximally symmetric, while a generic

Riemannian manifold (M, g) has no isometries (besides the trivial one $x \to x$). This should not come as a surprise : isometries leave curvature invariant. For instance a space of constant sectional curvature saturates the number $\frac{d(d+1)}{2}$ of (linearly independent) Killing vector fields (although this does not imply the existence of global isometries, as the flow might not be well defined). But it so happens that the infinitesimal transformations will prove sufficient for the purpose of deriving Ward identities. This will turn out to be a crucial point for conformal invariance in two dimensions. While the number of globally defined conformal transformations is always finite dimensional, the infinitesimal generators form an infinite dimensional Lie algebra in Euclidean space !

2.2 Conformal transformations

On a Riemannian manifold the metric allows one to define the angle θ between two vectors v_1 and v_2 tangent to the manifold (at the same point) through

$$\cos\theta = \frac{g(v_1, v_2)}{\|v_1\|\|v_2\|} \tag{2.2.1}$$

If two curves $c_1(t)$ and $c_2(t)$ intersect each other - say at $t = 0$ - with a non-zero velocity, then the angle between the two curves is defined as the angle between $v_1 = c_1'(0)$ and $v_2 = c_2'(0)$ (see Fig (2.4)). It is clear that isometries are angle-preserving, since we can

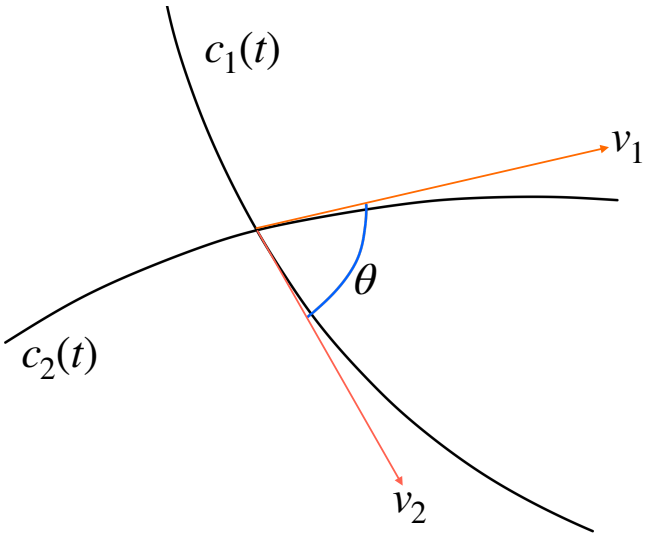


Figure 2.4: Two curves c_1 and c_2 intersect each other.

rewrite (2.2.1) as

$$\cos\theta = \frac{\|v_1 + v_2\|^2 - \|v_1\|^2\|v_2\|^2}{2\|v_1\|\|v_2\|} \tag{2.2.2}$$

but the reverse is not true : there exists angle-preserving maps that do not preserve length. The canonical example are scale transformations (or dilations) $x \to \lambda x$ in Euclidean space. More generally a diffeomorphism that preserve angles is called a *conformal transformation*. We can rephrase the above definition in terms of curve by saying that the angles as

measured by the metrics g and $f^*\tilde{g}$ must agree. This means

$$\boxed{f^*\tilde{g} = \Omega^2 g} \tag{2.2.3}$$

for some function Ω called the conformal factor. In coordinates this reads

$$g_{\rho\sigma}(f(x))\partial_\mu f^\rho \partial_\nu f^\sigma = \Omega^2(x) g_{\mu\nu}(x)\,. \tag{2.2.4}$$

This follows from the following fact : two inner products on a vector space induce the same notion of angle if and only if they are proportional. This is a simple exercise in linear algebra which is left to the reader.

Let $f : (M, g) \to (\tilde{M}, \tilde{g})$ be a diffeomorphism between two Riemannian manifolds, then the following are equivalent

- f is a conformal map
- f preserves angles between curves
- $f^*\tilde{g} = \Omega^2 g$ for some smooth function Ω

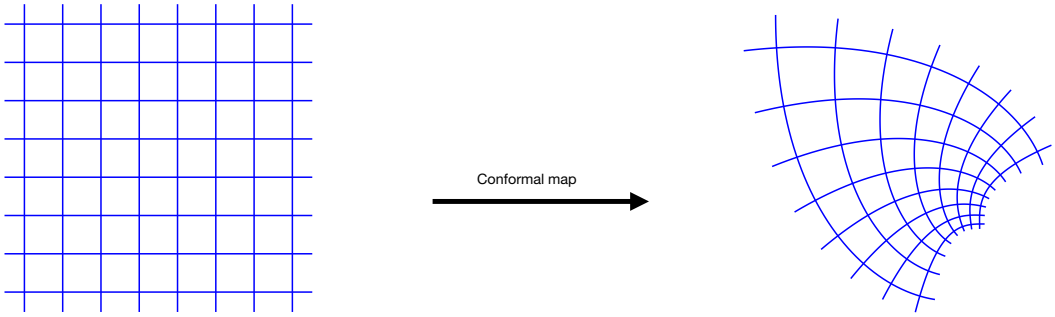


Figure 2.5: Image of a grid by a conformal map : lengths are not preserved, but angles between curves are.

Three important examples.

- **Inversion**

$$\begin{aligned} f : \mathbb{R}^d \setminus \{0\} &\to \mathbb{R}^d \setminus \{0\} \\ x &\to \frac{x}{\|x\|^2} \end{aligned} \tag{2.2.5}$$

The inversion can be continued into a conformal map on the sphere S^d. What is the image of the south pole ?

- The Euclidean space with the origin removed $\mathbb{R}^d \setminus \{0\}$ is conformally equivalent to the d-dimensional "cylinder" $\mathbb{R} \times S^{d-1}$ with the canonical metric, through the conformal map

$$f(x) = \left(\log\|x\|, \frac{x}{\|x\|} \right) \tag{2.2.6}$$

  In particular in two dimensions this yield the following map

$$\begin{aligned} f : \mathbb{C}^* &\to \mathbb{C}/2i\pi\mathbb{Z} \\ z &\to \log z \end{aligned} \tag{2.2.7}$$

- **Stereographic projection**
  The d-dimensional sphere S^d (with a point removed) is conformally equivalent to the Euclidean space, through the stereographic projection. To define this map, we first consider the canonical embedding $S^d = \{x \in \mathbb{R}^{d+1},\ \|x\|^2 = 1\}$, and we define

$$\begin{aligned} &f : S^d \setminus \{N\} \to \mathbb{R}^d \\ &(x_1, \cdots, x_{d+1}) \to \frac{1}{1 - x_{d+1}} (x_1, \cdots, x_d) \end{aligned} \tag{2.2.8}$$

  where N is the North pole $(0, \cdots, 0, 1)$. This means that the sphere S^d (with the round metric) is *conformally flat* : each point has a neighborhood that can be mapped to flat space by a conformal transformation.
  $f^*\tilde{g} = \sum_{i=1}^d d\left(\frac{x_i}{1-x_{d+1}}\right) \otimes d\left(\frac{x_i}{1-x_{d+1}}\right) = \frac{1}{(1-x_{d+1})^2} \sum_{i=1}^{d+1} dx_i \otimes dx_i = \frac{1}{(1-x_{d+1})^2} g$ using $\sum_{i=1}^{d+1} x_i dx_i = 0$

Some remarks.

Two Riemannian metrics g_1 and g_2 are called conformally equivalent if $g_1 = \Omega^2 g_2$ for some function Ω, *i.e.* when they define the same angles. An equivalence class of such metrics is called a *conformal structure* or *conformal class*. A change of metric within a given conformal class is called a Weyl transformation :

$$g_1 \to g_2 = \Omega^2 g_1 \,. \tag{2.2.9}$$

One can think of the Weyl transformation as a deformation of the Riemannian manifold, or as the following conformal map

$$\begin{aligned} f : (M, g_1) &\to (M, g_2) \\ x &\to x \,. \end{aligned} \tag{2.2.10}$$

Since conformal transformations only care about angles, not about lengths, Weyl transformations are a very natural deformation to consider. If $f : (M, g) \to (\tilde{M}, \tilde{g})$ is a conformal map, then it remains a conformal map after a Weyl rescaling of g and/or $\tilde{g}$. In that sense one does not need to specify a metric to talk about a conformal map, but only a conformal structure. It follows that if ξ is a conformal Killing vector field *w.r.t.* some metric, then it is a conformal Killing vector field *w.r.t.* all metrics in the same conformal class.

Equivalently, a conformal map $f : (M, g) \to (M, \tilde{g})$ such that $f^*\tilde{g} = \Omega^2 g$ can be decomposed into a Weyl rescaling $g \to \Omega^2 g$ followed by an isometry. For instance the inversion $x \to x/\|x\|^2$ is an isometry from $\mathbb{R}^d \setminus \{0\}$ with the metric $g_{\mu\nu} = \frac{1}{\|x\|^4}\delta_{\mu\nu}$ to $\mathbb{R}^d \setminus \{0\}$ with the Euclidean metric.

2.2.1 Conformal Killing vector fields

Let us analyse the condition $f^*g = \Omega^2 g$ for an infinitesimal transformation $f^\mu(x) = x^\mu + \epsilon\xi^\mu(x)$. Expanding the conformal factor $\Omega(x) = 1 + \epsilon\omega(x)$, we get

$$\mathcal{L}_\xi g = 2\omega g\,. \tag{2.2.11}$$

This is a constraint on the vector field ξ, which in local coordinates reads

$$\xi^\rho \partial_\rho g_{\mu\nu} + g_{\nu\rho}\partial_\mu \epsilon^\rho + g_{\mu\rho}\partial_\nu \epsilon^\rho = 2\omega(x) g_{\mu\nu}(x)\,, \tag{2.2.12}$$

or equivalently

$$\nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu = 2\omega(x) g_{\mu\nu}(x)\,. \tag{2.2.13}$$

for some *a priori* arbitrary function $\omega(x)$. However tracing the above equation (i.e. contracting with $g^{\mu\nu}$) yields $\omega(x) = \frac{1}{d}\nabla_\mu \xi^\mu = \frac{1}{d}\mathrm{div}(\xi)$, and we get

$$\boxed{\nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu = \frac{2}{d}\mathrm{div}(\xi)\, g_{\mu\nu}(x)} \tag{2.2.14}$$

or equivalently

$$\mathcal{L}_\xi g = \frac{2}{d}\mathrm{div}(\xi)\, g\,. \tag{2.2.15}$$

Such a vector field is called a conformal Killing vector field. We will see that for $d \geq 3$ there are finitely many solutions, but in two dimensions there are infinitely many conformal Killing vector fields !

In Euclidean space $\mathbb{R}^d$ with $d \geq 3$, conformal Killing vector fields form a $\frac{(d+1)(d+2)}{2}$-dimensional Lie algebra. In additions to translations and rotations, it contains

- dilations $\xi^\mu(x) = x^\mu$
- special conformal transformations $\xi^\mu(x) = 2(a\cdot x)x^\mu - x^2 a^\mu$

The derivation is a standard exercise (left to the reader) and can be found in any textbook about conformal field theory.

The only new thing here are special conformal transformations, whose flow is

$$f_t(x) = \frac{x - bx^2}{1 - 2b\cdot x + b^2 x^2}, \qquad b = ta \tag{2.2.16}$$

which is an inversion $x \to x/x^2$ followed by a translation $x \to x - b$, followed by another inversion. This map is not well-defined at $x = \frac{b}{b^2}$: the flow blows up. There are several ways out of this conundrum. The first one is to work on a domain of $\mathbb{R}^d$ that does not contain b/b^2. The other one is to consider the one-point compactification of $\mathbb{R}^d$ using the inverse stereographic projection. In this case one is really discussing the conformal transformations of the d-dimensional sphere S^d (endowed with the round metric).

It turns out that all the conformal maps we just mentioned, namely translations, rotations, dilations and special conformations, can be continued to the sphere S^d. In that sense the one-point compactification of Euclidean space via the stereographic projection is compatible with the action of conformal transformations. This is an example of a *conformal compactification*, and from now one we will often identify Euclidean space with the sphere.

‣ **Exercise :** Check that the (round) sphere S^d is also a conformal compactification of the (flat) cylinder $\mathbb{R} \times S^{d-1}$.

The conformal group of the d-dimensional sphere S^d turns out to be very much related to the Lorentz group $\mathrm{O}(d+1,1)$. In a nutshell, the conformal group is simply a projective version of the Lorentz group. This is the point of the exercise(s) below. But before jumping into the derivation, let first state the result.

Concretely to some $x \in S^d$, that is a point of $\mathbb{R}^{d+1}$ with $\|x\|^2 = 1$, one associates the point $X = (1, x)$ on the light-cone of $\mathbb{R}^{1,d+1}$. Acting on X with a Lorentz transformation yields another point $Y = A \cdot X$ on the light-cone, thus a point of the form $Y = (\|y\|, y)$ for some $y \in \mathbb{R}^{d+1}$. Then the map $x \to y/\|y\|$ is conformal ! Furthermore, this construction yields all possible conformal transformations on S^d (this part we will not prove[3]). At the end of the day, conformal maps on the sphere S^d are of the form

$$x \to \frac{A\begin{pmatrix} 1 \\ x \end{pmatrix}}{\left\| A\begin{pmatrix} 1 \\ x \end{pmatrix} \right\|}$$

form some matrix $A \in \mathrm{O}(1, d+1)$. Thus conformal transformations of the sphere can be expressed in terms of (indefinite) orthogonal transformations in a higher-dimensional space. This is quite remarkable, since the latter are linear !

Explaining where this identification comes from is the point of the following exercises. Differential geometry questions are denoted with a (*), and can be ignored by the reader unfamiliar with the notions involved.

‣ **Exercise : Lorentz group and light-cone**.

We start with a short reminder on the Lorentz group. Minkowski space $\mathbb{R}^{1,d+1}$ is simply $\mathbb{R}^{d+2}$ equipped with an inner product of signature $(-,+,\cdots,+)$. We will denote by $(X_0, \cdots, X_{d+1})$ the coordinates in $\mathbb{R}^{1,d+1}$ and the inner product by

$$\langle X, Y \rangle = -X_0 Y_0 + \sum_{i=1}^{d+1} X_i Y_i \,.$$

The Lorentz group $\mathrm{O}(1, d+1)$ is the subgroup of $\mathrm{GL}(d+2, \mathbb{R})$ that leaves the above inner product invariant, that is to say linear maps $X \to A \cdot X$ such that

$$\langle A \cdot X, A \cdot Y \rangle = \langle X, Y \rangle, \qquad \forall\, X, Y \in \mathbb{R}^{d+2} \,. \tag{2.2.17}$$

[3] In dimension $d \geq 3$ this follows from Liouville's theorem on conformal mappings (1850).

The null-cone (*a.k.a.* light-cone) is the subset of vectors with vanishing spacetime interval

$$C = \{X \in \mathbb{R}^{n+1},\ \langle X, X\rangle = 0\} \tag{2.2.18}$$

- Check that $C^* = C \setminus \{0\}$ is stable under the action of the Lorentz group.
- Argue that one can identify C^* with $\mathbb{R}^* \times S^d$ via the map $\Psi : \mathbb{R}^* \times S^d \to C^*$ defined by

$$\Psi(t, x) = (t, tx) \tag{2.2.19}$$

  What is the inverse map ?
- (*) Check that the induced induced metric on $\mathbb{R}^* \times S^d$ is

$$(\Psi^*\eta)_{(t,x)} = t^2 g_x \tag{2.2.20}$$

  where η is the pseudo-Riemannian metric on $\mathbb{R}^{1,d+1}$ (namely $\eta = -dX_0^2 + \sum_i dX_i^2$) and g is the round metric on S^d, that is $g = \sum_{i=1}^{d+1} dx_i^2$ (keeping in mind that on S^d the dx_i are not independent : they are subject to $\sum_i x_i dx_i = 0$ since $d\|x\|^2 = 1$).

$$(\Psi^*\eta)_{(x,t)} = -dt^2 + \sum_i d(tx_i)^2 = -dt^2 + |x\|^2 dt^2 + 2t \sum_i x_i dx_i dt + t^2 \sum_i dx_i^2$$

and the results follows using $\sum_i x_i dx_i = 0$ and $\|x\|^2 = 1$.

▸ **Exercise : projective space and the projective group**.

We have seen how Lorentz transformations act on the light-cone (with the origin removed) C^*, and that C^* is nothing but $\mathbb{R}^* \times S^d$. The idea is now to forget about the $\mathbb{R}^*$ factor, thus constructing transformations on the sphere S^d - transformations which will turn out to be conformal. However this projection is not completely trivial, and the proper setting to do so is that of projective geometry.

The (real) projective space $\mathbb{R}\mathrm{P}^{d+1}$ is the compact manifold obtained by quotienting the space $\mathbb{R}^{d+2} \setminus \{0\}$ by the equivalence relation $X \sim Y$ iff $X = \lambda Y$ for some $\lambda \in \mathbb{R}$. We denote by $[X]$ the equivalence class of X. Alternatively, one can think of $\mathbb{R}\mathrm{P}^{d+1}$ as the space of all lines (going through the origin) in $\mathbb{R}^{d+2}$.

- Argue that any invertible linear map $X \to A \cdot X$, *i.e.* with $A \in \mathrm{GL}(d+2, \mathbb{R})$, descends to the quotient, in the sense that it induces a well defined map on $\mathbb{R}\mathrm{P}^{d+1}$:

$$\varphi_A([X]) = [A \cdot X]\ .$$

  Check that $\varphi_A \circ \varphi_B = \varphi_{AB}$. (*) Using the standard atlas of the projective space, it is straighforward to check that the maps φ_A are smooth. Therefore there are diffeomorphisms.
- The group of diffeomorphisms so obtained is called the projective group and is denoted by $\mathrm{PGL}(d+2, \mathbb{R})$. Show that this group is isomorphic to $\mathrm{GL}(d+2)/\mathbb{R}^* I$.

- Check that the image of the light cone in projective space, namely

$$S = \{[X] \in \mathbb{R}^{d+1},\ \langle X, X\rangle = 0\} \tag{2.2.21}$$

is diffeomorphic to the sphere S^d (via the identification/diffeomorphism $x \to [1, x]$ for $x \in S^d$). In the following we will (abusively) identify S and S^d.

- Let $\mathrm{PO}(1, d+1)$ denote the subgroup of $\mathrm{PGL}(d+2, \mathbb{R})$ corresponding to the Lorentz group $\mathrm{O}(1, d+1)$. Argue that the above construction yields an action of $\mathrm{PO}(1, d+1)$ on the sphere S^d. Show that $\mathrm{PO}(1, d+1)$ is isomorphic to $\mathrm{O}(1, d+1)/\mathbb{Z}_2$, where $\mathbb{Z}_2$ stands for the subgroup $\{I, -I\}$ of $\mathrm{O}(1, d+1)$.

▸ **(*) Exercise : PO$(1, d+1)$ transformations are conformal.**
We are now going to establish that this group acts by conformal maps.

- If g denote the round metric on S^d, and $p : C^* \to S^d$ is the map $p(X_0, X_1, \cdots, X_{d+1}) = (X_1/X_0, \cdots, X_{d+1}/X_0)$, check that $p^* g = \frac{1}{X_0^2}\eta$.
  Correction : we can write $p = p_2 \circ \Psi^{-1}$ where $\Psi^{-1} : C^* \to \mathbb{R}^* \times S^d$ is the inverse of the map Ψ introduced above, namely $\Psi^{-1}(X_0, X_1, \cdots, X_{d+1}) = (X_0, X_1/X_0, \cdots, X_{d+1}/X_0)$, and $p_2 : \mathbb{R}^* \times S^d \to S^d$ is simply the projection $p_2(t, x) = x$. Then $p^* g = \Psi^{-1 *} g = \sum_i (d(X^i/X_0))^2 = 1/X_0^2(-dX_0^2 + \sum_i dX_i^2)$; using $X_0 dX_0 = \sum_i X_i dX_i$.

- (*) Let $\iota : S^d \hookrightarrow C$ be the inclusion map $\iota(x) = (1, x)$. By construction the map φ_A can be decomposed as

$$\varphi_A = p \circ A \circ \iota \tag{2.2.22}$$

where A stands for the map $X \to A \cdot X$. Deduce that

$$\varphi_A^* g = \lambda(x)^{-2} g, \qquad \lambda(x) = (A(\iota(x)))_0 \tag{2.2.23}$$

Thus we have established that φ_A is indeed a conformal map.

▸ **Exercise : recovering translations, rotations, inversion, dilations *etc*.**

It is a fact that the above construction yields all conformal transformations of the sphere. In dimension $d \geq 3$ this follows from Liouville's theorem on conformal mappings (1850). We will deal with the case $d = 2$ (for which this fact is also true) later.

We will not prove Liouville's theorem, but we will check that the conformal maps we constructed above using conformal Killing vector fields are indeed in $\mathrm{PO}(1, d+1)$.

First we have to go back to Euclidean space. First recall the stereographic projection (about the pole $x_0 = 1$) sending the sphere to the plane

$$x \to \frac{1}{1 - x_0}(x_1, \cdots, x_{d+1}) \tag{2.2.24}$$

The inverse map is the following conformal map

$$x = \frac{1}{1 + \|x\|^2}\left(\|x\|^2 - 1, 2x_1, \cdots, 2x_{d+1}\right) \tag{2.2.25}$$

Thus conformal maps on the Euclidean space are obtained as follow. To a given $x \in \mathbb{R}^d$, we associate the following point $X(x)$ in the light-cone of $\mathbb{R}^{1,d+1}$

$$x \to X(x) = \big(\|x\|^2+1, \|x\|^2-1, 2x\big) \tag{2.2.26}$$

Conformal maps act linearly on X, *i.e.* $X \to AX$ for some $A \in \mathrm{O}(1, d+1)$, and we must simply project back to Euclidean space $\mathbb{R}^d$ at the end. What is the matrix A corresponding to

- a rotation $x_i \to R_{ij}x_j$?
- inversion ?
- dilation ? $x \to e^{\mu}x$
- a translation $x \to x + a$?

2.2.2 Conformal maps on the complex plane

In two dimensions the story is quite different, and conformal geometry is very closely related to complex analysis. This is true on any (oriented) two-dimensional Riemannian manifold, but let us first see how this works on the Euclidean plane.

Let's denote by η the flat Euclidean metric ($\eta_{\mu\nu} = \delta_{\mu\nu}$), and recall that a diffeomorphism $f : (\mathbb{R}^2, \eta) \to (\mathbb{R}^2, \eta)$ from the Euclidean plane to itself is conformal iff $f^*\eta = \Omega^2\eta$ for some smooth function Ω, i.e.

$$\partial_\mu f^\rho \delta_{\rho\sigma} \partial_\nu f^\sigma = \Omega^2(x)\delta_{\mu\nu}\,. \tag{2.2.27}$$

This can be rephrased as $df_x^T df_x = \Omega^2(x)\mathbb{I}$, and furthermore taking the determinant of this equation we learn that $\Omega^2(x) = |\det df_x|$. Thus in flat two-dimensional space we have the following alternative characterization

$$df_x^T df_x = |\det df_x|\,1\,. \tag{2.2.28}$$

or equivalently

$$\mathrm{com}(df_x) = \pm df_x \tag{2.2.29}$$

where $\mathrm{com}(df_x)$ stands for the comatrix of f, and the $\pm$ sign is the sign of $\det df_x$. Note that this sign does not depend on x ($\det df_x$ cannot vanish by virtue of f being a diffeomorphism), and it simply tells us whether f is orientation preserving ($\det df_x > 0$) or orientation reversing ($\det df_x < 0$). Writing (2.2.29) explicitly yields

$$\begin{pmatrix} \partial_2 f^2 & -\partial_1 f^2 \\ -\partial_2 f^1 & \partial_1 f^1 \end{pmatrix} = \pm \begin{pmatrix} \partial_1 f^1 & \partial_2 f^1 \\ \partial_1 f^2 & \partial_2 f^2 \end{pmatrix}. \tag{2.2.30}$$

Depending on the sign, we recognise the Cauchy-Riemann equations or the anti-analytic version of Cauchy-Riemann

$$\begin{cases} \partial_1 f^1 = \pm\partial_2 f^2 \\ \partial_1 f^2 = \mp\partial_2 f^1 \end{cases} \tag{2.2.31}$$

Thus conformal maps from the Euclidean plane to itself are

- **holomorphic if they are orientation preserving,**
- **anti-holomorphic when they reverse orientation.**

Note that an orientation reversing conformal map is nothing but an orientation-preserving one followed by a reflection (complex conjugation). Thus it is sufficient to discuss orientation-preserving conformal maps.

At this stage it may look like there are many conformal maps in two-dimensions. In some sense this is true, but there is a caveat. So far we have only looked at the local constraint of being angle-conserving, namely $f^*\eta = \Omega^2\eta$. But let's not forget that the function f must be a diffeomorphism[4]. This global constraint is very sensitive to the domain of f.

Let's start with conformal maps $f : \mathbb{C} \to \mathbb{C}$, that is automorphisms of the complex plane $\mathbb{C}$. It is a fact that the (orientation-preserving) conformal maps $f : \mathbb{C} \to \mathbb{C}$ are of the form

$$z \to az + b, \qquad a \neq 0\,. \tag{2.2.32}$$

that is translations, rotations and dilations. If one adds a point at infinity $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, one gets the Möbius group

$$z \to \frac{az+b}{cz+d}, \qquad ad - bc \neq 0\,. \tag{2.2.33}$$

and this now includes special conformal transformations. Proving the above is a standard exercise in complex analysis, and is left to the reader. We recover the full conformal group of the previous section, and indeed at the level of globally defined conformal maps there is no difference between the two-dimensional case and the case $d \geq 3$. So in what sense are there more conformal maps in two-dimensions ?

To understand this, let us recall the inversion function theorem : a smooth map $f : \mathbb{R}^d \to \mathbb{R}^d$ is invertible in a neighborhood of some point x_0 as soon as $\det df_{x_0} \neq 0$. This means that - given a holomorphic map f, and a point x_0 such that $f'(x_0) \neq 0$ - there exist an open neighborhood U of x_0 such that $f : U \to f(U)$ is a diffeomorphism, and therefore a conformal map. So in this sense any holomorphic map yields a conformal map, albeit not on the whole complex plane. This is to be contrasted with the paucity of conformal maps in higher dimensions (Liouville's theorem): any conformal map on *any domain* of $\mathbb{R}^d$ is an element of the conformal group $\mathrm{PO}(1, d+1)$. Let us mention a striking and important result that illustrates the wealth of conformal maps in two-dimensions :

Riemann mapping theorem : any simply-connected open subset $U \subset \mathbb{C}$ (with $U \neq \varnothing, \mathbb{C}$) is conformally equivalent to the unit disk

$$\mathbb{D} = \{z \in \mathbb{C},\ |z| < 1\} \tag{2.2.34}$$

[4]A holomorphic diffeomorphism is called an automorphism

This result is remarkable and somehow counter-intuitive, as the domain U can be highly complicated, with a non-differentiable or even fractal boundary. Moreover the result is extremely simple. For non simply-connected domains, the story is more complicated. For instance let's consider the annulus $A_{r,R} = \{z \in \mathbb{C},\, r < |z| < R\}$. Schottky theorem states that two annuli A_{r_1,R_1} and A_{r_2,R_2} are conformally equivalent if and only if $R_1/r_1 = R_2/r_2$.

▸ **Exercise :** show that the unit disk $\mathbb{D}$ and the complex plane $\mathbb{C}$ are not conformally equivalent (hint : use Liouville's theorem : every bounded entire function is constant).

The abundance of conformal maps in two-dimensions is also manifest at the level of conformal-Killing vectors. In the Euclidean plane the conformal Killing condition reads for a vector field $\xi = \xi_1\partial_1 + \xi_2\partial_2$

$$\partial_1\xi_2 + \partial_2\xi_1 = 0, \qquad \partial_1\xi_1 = \partial_2\xi_2 \tag{2.2.35}$$

which is equivalent to

$$(\partial_1 + i\partial_2)\,(\xi_1 + i\xi_2) = 0 \tag{2.2.36}$$

and we find

$$\boxed{\xi = \xi(z)\partial + \overline{\xi(z)}\bar{\partial}} \tag{2.2.37}$$

where $\xi(z)$ is **any** holomorphic function, and $\partial = \frac{1}{2}(\partial_1 - i\partial_2)$. Of course the flow of most of these vector fields will not be globally defined, but this will not be an issue for quantum field theories.

Let us already mention a point that will have some importance later. At this stage the vector field ξ is required to be real, thus the component $\overline{\xi(z)}$ is the complex conjugate of $\xi(z)$. When dealing with conformal field theories it will be convenient to *complexify* this algebra, which amounts to relax this reality constraint and treat $\xi(z)$ and $\overline{\xi(z)}$ as independent complex functions. In turns this implies that we are dealing with a complex space-time, in the sense that z and $\bar{z}$ are independent variables. This is not an obviously innocuous assumption, and such a cavalier complexification can be motivated in QFT by doing a Wick rotation. Indeed in Lorentzian signature $z = x - t$ and $\bar{z} = x + t$ are independent. Lurking behind Wick rotation and space-time complexification is the idea/assumption that the quantities of interest in QFT can be analytically continued to the case where z and $\bar{z}$ are independent. This formal point of view often proves very convenient for calculations.

2.2.3 Conformal maps between Riemann surfaces

The above discussion can be extended to arbitrary two-dimensional Riemmanian manifolds, and in fact the main result remains.

Conformal maps from a surface (M, g) to another $(M, \tilde{g})$ are exactly the (anti-)holomorphic maps.

But for such a claim to make sense one first need to define what is meant by a holomorphic map on a generic surface. To do so, the relevant mathematical notion is that of a *complex structure*. Let us first briefly recall how one defines coordinates on a surface M. This is done through an *atlas*, which is a collection of *charts*. A chart is a homeomorphism φ from an open subset $U \subset M$ to an open subset of Euclidean space $\mathbb{R}^2 \simeq \mathbb{C}$. An atlas is a collection $\{(U_\alpha, \varphi_\alpha)\}$ of charts which covers M. In order to define the manifold through an atlas, one needs to know how to piece together the different charts. This is encoded in the *transition functions*. Whenever U_α and U_β overlaps, the transition function is simply

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha (U_\alpha \cap U_\beta) \to \varphi_\beta (U_\alpha \cap U_\beta) \tag{2.2.38}$$

On a generic (topological) manifold transition functions are homeomorphisms[5]. Morally

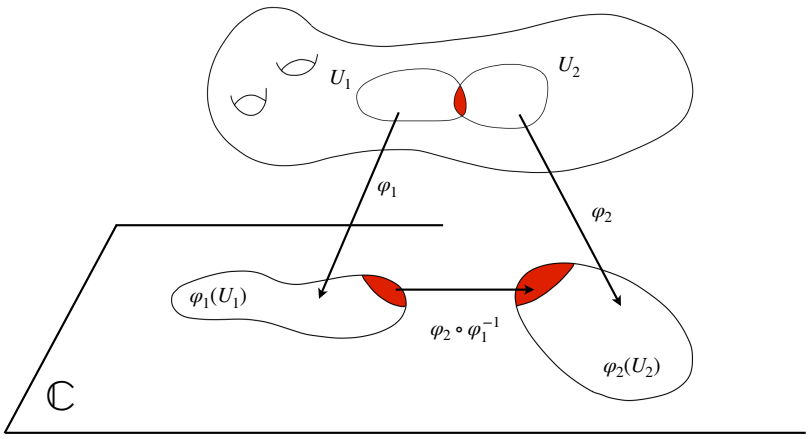


Figure 2.6: Transition function for a two-dimensional manifold.

this fixes the topology of the surface, thus the notion of a continuous function $f : M \to \mathbb{R}$ makes sense. But what does it mean for f to be smooth ? Since the surface is locally like $\mathbb{R}^2$, it would seem reasonable to declare f to be smooth on a chart U_1 as long as $f \circ \varphi_1^{-1}$ is smooth (notice that $f \circ \varphi_1^{-1}$ is map defined on a subset of $\mathbb{R}^2$, for which the notion of smoothness makes sense). However if U_2 is another chart overlapping U_1, it is not necessarily true that $f \circ \varphi_2^{-1}$ is also smooth, unless the transition function $\varphi_1 \circ \varphi_2^{-1}$ is smooth. Thus in order to talk about smooth functions on a manifold, a *smooth structure* is required, that is an atlas whose transition functions are smooth.

The very same line of reasoning applies to holomorphic functions. A *complex structure* is an atlas whose transition functions are holomorphic. Note that a manifold endowed with such an atlas is necessarily orientable (exercice : why ?).

But so far we are working on a smooth surface endowed with a metric. So where is the complex structure ? It turns out that - as long as the surface is oriented - the

[5]A homeomorphism is a continuous bijection whose inverse is also continuous.

metric induces a complex structure. This is based on the fact that two-dimensional Riemannian manifolds are conformally flat, *i.e.* there exists around any point a coordinate system (i.e. a chart) in which the metric is of the form $g_{\mu\nu}(x) = e^{2\sigma(x)}\delta_{\mu\nu}$. These are called isothermal coordinates, since each coordinate x^μ is harmonic (exercice : check that $\Delta(x^\mu) = 0$ using Appendix 8.5.1), thus a steady-state solution of the heat equation. For a proof of the existence of these isothermal coordinates (also known as "conformal gauge" in Lorentzian signature), the reader is invited to have a look at Donaldson's *Riemann surfaces*, or Schottenloher's *A Mathematical Introduction to Conformal Field Theory* for the Lorentzian case. So on a Riemannian surface there exists an atlas in which all charts are made of isothermal coordinates. But notice that transition functions between isothermal coordinate systems are conformal by construction. Indeed an isothermal chart (U, φ) is nothing but a conformal map φ from U to Euclidean space. Since the composition of conformal maps is conformal, clearly transition functions are also conformal. If the above explanation is not clear, the computationally inclined reader can also check that the pullback of the metric $e^{2\sigma_2}\delta_{\mu\nu}$ by the transition function $\varphi_2 \circ \varphi_1^{-1}$ is $e^{2\sigma_1}\delta_{\mu\nu}$:

$$\left(\varphi_2 \circ \varphi_1^{-1}\right)^* e^{2\sigma_2}\delta_{\mu\nu} = e^{2\sigma_1}\delta_{\mu\nu} \tag{2.2.39}$$

According to the analysis done in the previous section, this implies that transition functions between isothermal coordinates are either holomorphic or anti-holomorphic. Provided the surface is oriented, we can always choose our charts to be compatible with the orientation, in which case all transition functions are also orientation preserving, thus holomorphic functions. This provides a complex structure, which is canonical in the sense that it does not depend on the choice of isothermal coordinates. It depends only on the Riemannian metric g. This complex structure is said to be compatible with the Riemannian metric g.

A simple and explicit illustration of this construction is given by the two-sphere S^2 endowed with the round metric. Consider the canonical embedding $S^2 = \{x \in \mathbb{R}^3,\ \|x\|^2 = 1\}$, and denote by $N = (0, 0, 1)$ and $S = (0, 0, -1)$ the North and South pole, respectively. The standard atlas is made of two charts $U_1 = S^2 \setminus \{N\}$, and $U_2 = S^2 \setminus \{S\}$, with coordinates $\varphi_i : U_i \to \mathbb{C}$ given by the stereographic projections, namely :

$$\varphi_1(x) = \frac{1}{1 - x_3}(x_1 + ix_2) \qquad \varphi_2(x) = \frac{1}{1 + x_3}(x_1 + ix_2) \tag{2.2.40}$$

We have already seen in a previous exercise that these are isothermal coordinates (to be precise we have seen that stereographic projections are conformal maps from the round sphere to Euclidean space, which is the same thing). Is the transition function holomorphic ? A direct calculation[6] show that it is fact anti-holomorphic

$$\begin{aligned} \varphi_2 \circ \varphi_1^{-1} : \mathbb{C} \setminus \{0\} &\to \mathbb{C} \setminus \{0\} \\ z &\to \frac{1}{\bar{z}} \end{aligned} \tag{2.2.41}$$

This stems from the fact that the atlas we chose was not oriented : the two charts have opposite orientations. This is easily fixed, we simply change the orientation of say φ_2, while not spoiling the isothermal condition. To do so one simply composes φ_2 with any orientation-reversing conformal map of the plane. For instance $(x, y) \to (x, -y)$, *i.e.*

[6] One can first check that $\varphi_1^{-1}(x + iy) = \frac{1}{1+x^2+y^2}(2x, 2y, x^2 + y^2 - 1)$

complex conjugation. Thus we replace the chart (U_2, φ_2) by $(U_2, \overline{\varphi}_2)$, and the transition function is now holomorphic :

$$\begin{aligned} \overline{\varphi}_2 \circ \varphi_1^{-1} : \mathbb{C} \setminus \{0\} &\to \mathbb{C} \setminus \{0\} \\ z &\to \frac{1}{z} \end{aligned} \tag{2.2.42}$$

We have ourselves a complex structure on the sphere.

So we have seen how a Riemannian metric g on an oriented surface induces a natural complex structure, via isothermal coordinates. But of course two metrics in the same conformal class induce the same complex structure since they have the same isothermal coordinates. So what we have is that a *conformal structure* induces a unique *complex structure* on a two-dimensional oriented manifold. In fact this also goes the other way. Given a complex structure, *i.e.* an atlas whose transition functions are holomorphic, there is a unique compatible conformal structure. So we have the following

On a two-dimensional oriented manifold a choice of *conformal structure* is equivalent to a choice of *complex structure*.

A two-dimensional manifold equipped with a complex structure is called a *Riemann surface*. Now remember that conformal maps are insensitive to Weyl rescaling, we can work locally with the flat metric $dzd\bar{z}$, and the analysis done in the Euclidean case applies, thus :

A map between oriented Riemannian surfaces is conformal if and only if it is holomorphic or anti-holomorphic.

Some references for this chapter :

- Chapter 2 of *A Mathematical Introduction to Conformal Field Theory*, Martin Schottenloher
- *Riemann surfaces*, Simon Donaldson
- The exercise on the projective light-cone is inspired from exercise 16 in chapter 2 of *An Introduction to Differential Manifolds*, Jaques Lafontaine

Lecture 3

Conformal invariance in field theories

3.1 Conformal invariance in classical field theories

After this rather lengthy discussion about space transformations, let us get back to the subject of field theories and conformal invariance. We first describe the response of a classical field theory to infinitesimal space transformations, for which an important protagonist is the *stress-energy tensor* $T^{\mu\nu}$. We then move on to quantum field theories and discuss *Ward identities*.

3.1.1 The stress-energy tensor

For pedagogical purposes we will initially restrict ourselves to field theories defined on flat space, where traditionally the stress-energy tensor is presented as the conserved Noether current associated with spacetime translations, see Appendix 3.4.1. However this leads to ambiguities in defining the stress-energy tensor, and a more modern and powerful definition of the stress-energy tensor involves working in curved space, see Appendix 3.4.2.

We consider a field theory in flat Euclidean space $\mathbb{R}^d$, as characterized by a Lagrangian density $\mathcal{L}$

$$S[\Phi] = \int \mathcal{L}(\Phi(x), \partial_\mu \Phi(x))\, d^n x \tag{3.1.1}$$

which we assume depends only on the fields at x and their first derivatives. Here Φ stands for a collection of fields $\phi_1, \phi_2, \ldots$ which can be of different type (scalar, vector, spinor, etc). In these lecture notes we will only consider translation invariant theories.

The stress-energy tensor $T^{\mu\nu}$ encodes the response of the action to an arbitrary infinitesimal spacetime transformation (*i.e.* an infinitesimal diffeomorphism) regarded as an active transformation, that is, a displacement of the fields/configurations. Consider an arbitrary (smooth) vector field $\xi^\mu(x)$ and its corresponding flow $x \to f_t(x)$ as defined by

$$\frac{df_t(x)}{dt} = \xi(f_t(x))\,. \tag{3.1.2}$$

Imagine now that this flows carries the local degrees of freedom for some time t. If the initial configuration is Φ, we will denote by $\Phi' = f_t^* \Phi$ the new configuration. Infinitesimally (at time $t = \epsilon \ll 1$) the flow is given by

$$x^\mu \to x^\mu + \epsilon^\mu(x) + O(\epsilon^2) \tag{3.1.3}$$

where $\epsilon^\mu(x)$ stands for the infinitesimal vector field $\epsilon\xi^\mu(x)$. For a scalar field ϕ, the above transformation means $\phi'(x) = \phi(x + \epsilon(x))) \equiv \phi(x) + \epsilon^\mu \partial_\mu \phi(x)$:

$$\delta_\epsilon \phi(x) = \epsilon^\mu \partial_\mu \phi(x) + O(\epsilon^2) \ ,$$

but more generally fields transform covariantly. A gauge field for instance (such as the electromagnetic field) is a field which is locally represented by a differential 1-form $A(x) = A_\mu(x) dx^\mu$, for which $(f_t^* A)(x) = A_\mu(f_t(x)) df_t^\mu$, thus yielding[1]

$$\delta_\epsilon A_\mu = \epsilon^\rho \partial_\rho A_\mu + (\partial_\mu \epsilon^\rho) A_\rho + O(\epsilon^2) \ , \tag{3.1.5}$$

while the associated field strength $F = dA$ is a two-form, so

$$\delta_\epsilon F_{\mu\nu} = \epsilon^\rho \partial_\rho F_{\mu\nu} + (\partial_\mu \epsilon^\rho) F_{\rho\nu} + (\partial_\nu \epsilon^\rho) F_{\mu\rho} + O(\epsilon^2) \ . \tag{3.1.6}$$

More generally the change of a vector field along the flow of a vector field is called the Lie derivative. The Lie derivative is sometimes called fisherman's derivative: the flow carries all fields past the fisherman, and the fisherman sits there and differentiates them.

In a more geometric language, fields are sections of some fiber bundles, and prescribing how they transform under a given diffeomorphism amounts to lifting this diffeomorphism into the corresponding bundle. For tensor fields there is a natural way to do this, called the pullback, and the infinitesimal variation is given by the Lie derivative. For spinors the story is a bit more subtle since diffeomorphisms cannot be lifted globally (for instance spinors transform with a minus sign under a full rotation), but infinitesimal diffeomorphisms can be lifted by introducing a connection on the spinor bundle, called a *spin connection*.

In practice however we will not have to worry about how the fields appearing in the action transform under diffeomorphisms[2] since we will not even have an action to begin with ! All we will care about is that the transformation is local, in the sense that

$$\begin{aligned} \delta_\epsilon \phi(x) = & \epsilon^\rho(x) \partial_\rho \phi(x) + \text{finite number of terms involving} \\ & \text{derivatives of } \epsilon^\mu(x) \text{ and } \phi(x) \end{aligned} \tag{3.1.7}$$

and that there exists an object called the *stress-energy tensor* $T^{\mu\nu}$ that describes the infinitesimal variation of the action $\delta S = S[\Phi'] - S[\Phi]$ at first order in the vector field ϵ :

$$\boxed{\delta S = -\frac{1}{2\pi} \int T^\mu{}_\nu \partial_\mu \epsilon^\nu \, d^n x \, .} \tag{3.1.8}$$

[1]Here we ignore gauge invariance, which in principle adds some freedom in defining the action of infinitesimal diffeomorphisms. Indeed there is no reason not to couple space transformations to a gauge transformations, such as

$$\delta_\epsilon A_\mu = \epsilon^\rho \partial_\rho A_\mu + (\partial_\mu \epsilon^\rho) A_\rho - \partial_\mu(\epsilon^\rho A_\rho) + O(\epsilon^2) \ . \tag{3.1.4}$$

[2]All we will need is how they transform under conformal transformations.

The prefactor is conventional, and has been chosen for future convenience.

▸ **Exercise :** Compute the stress-energy tensor for

- the free scalar field

$$S = \frac{1}{4\pi}\int \left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right) d^n x\,.$$

- electromagnetism

$$S = -\frac{1}{8\pi}\int F_{\mu\nu}F^{\mu\nu}\, d^n x\,.$$

- Chern-Simons theory

$$S = \frac{1}{2\pi}\int \epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho}\, d^n x\,.$$

The stress-energy tensor is conserved on-shell (*i.e.* as long as the equation of motions are satisfied) by construction[3]

$$\partial_\mu T^{\mu\nu} = 0 \qquad \text{on-shell.} \tag{3.1.9}$$

▸ **Exercise :** Prove the above statement.

Physically the components of the stress-energy tensor describe the density and flux of energy and momentum in spacetime (strictly speaking this interpretation requires a Lorentzian metric), and the associated conserved charges are energy and momentum

$$E = \int T^{00}(t,x)d^{n-1}x, \qquad P^i = \int T^{0i}(t,x)d^{n-1}x \tag{3.1.10}$$

Usually this interpretation is At the classical level this interpretation follows from Noether's theorem (see Appendix 3.4.1). For a quantum field theory these charges will become the generators of spacetime translations. We will come back to this interpretation once we have derived the Ward identity (3.2.25)).

It must be stressed that the stress-energy tensor is a particularly subtle notion of field theory, with several more or less equivalent definitions available (see Appendices 3.4.1 and 3.4.2). Consider for instance the field theory of electromagnetism. The setup is $3+1$ dimensional Minkowski space with metric $\eta_{\mu\nu}$, and we choose the signature $(-+++)$. The Lagrangian density is

$$\mathcal{L}[A_\mu] = -\frac{1}{8\pi}F_{\mu\nu}F^{\mu\nu}, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.1.11}$$

[3]Indeed as long as the action is extremal, $\delta S = 0$ for all $\delta\Phi$, so in particular for $\delta\Phi = \delta_\epsilon\Phi$.

where the components of the field strength are related to the electric and magnetic fields through $F_{0i} = E_i$ and and $F_{ij} = \epsilon^{ijk} B_k$.

The correct stress-energy tensor is well known from Maxwell's equations. The energy density of the electromagnetic field is $T^{00} = \frac{1}{2}(E^2 + B^2)$, while the density of momentum is given by the Poynting vector $T^{0i} = (E \times B)_i$. Finally $T^{ij} = \frac{1}{2}(E^2 + B^2)\delta_{ij} - E_i E_j - B_i B_j$ is (minus the) Maxwell stress-tensor. More compactly

$$T^{\mu\nu} = F^{\mu\sigma} F^{\nu}{}_{\sigma} - \frac{1}{4}\eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \tag{3.1.12}$$

▸ **Exercise :** Compare the above stress-energy tensor with the various definitions of the stress-energy tensor, namely

- the canonical SET as defined through Noether's theorem (see eq. (3.4.8) in appendix 3.4.1)
- the definition used in this section, as in Eq.(3.1.8)
- the Hilbert SET (see eq. (3.4.30) in appendix 3.4.2)

$$T_{\mu\nu} = -4\pi \frac{\delta S}{\delta g^{\mu\nu}} = -4\pi \left(\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - \frac{1}{2} g_{\mu\nu} \mathcal{L} \right) \tag{3.1.13}$$

Exercise : Same question for the abelian Chern-Simons theory in 2+1 dimensions as defined by $\mathcal{L}[A_\mu] = \frac{1}{2\pi} \epsilon^{\mu\nu\rho} A_\mu F_{\nu\rho}$.

3.1.2 Ambiguities, symmetries and improvements

With equation (3.1.8) we have postulated the existence of the stress-energy tensor, but we have not commented on its unicity. In fact (3.1.8) does not define a unique stress-energy tensor but rather a class of stress-energy tensors that differ from each other by a divergence-free piece. Indeed one can always add to $T^{\mu}{}_{\nu}$ any tensor $\Theta^{\mu}{}_{\nu}$ such that

$$\int \Theta^{\mu}{}_{\nu} \partial_\mu \epsilon^\nu \, d^n x = 0 \tag{3.1.14}$$

for all vector fields $\epsilon^\mu(x)$. Integrating by parts, the above condition boils down to $\Theta^{\mu}{}_{\nu}$ being identically conserved (*i.e.* conserved off-shell)

$$\partial_\mu \Theta^{\mu}{}_{\nu} = 0 \tag{3.1.15}$$

For instance,

$$\Theta^{\mu}{}_{\nu} = \partial_\rho \Sigma^{\rho\mu}{}_{\nu}, \qquad \Sigma^{\rho\mu}{}_{\nu} = -\Sigma^{\mu\rho}{}_{\nu} \tag{3.1.16}$$

for an arbitrary $\Sigma^{\rho\mu}{}_{\nu}$. Adding such terms goes under the general name of *improving* the stress-energy tensor.

Such an ambiguity is not surprising. Think for instance of a one-dimensional such as the Heisenberg chain, with Hamiltonian

$$H = \sum_n h_n, \qquad h_n = \vec{\sigma}_n \cdot \vec{\sigma}_{n+1} \tag{3.1.17}$$

While the notion of total energy is clear (it is nothing but the Hamiltonian H), the notion of energy density is not. One could declare it to be $h_n = \vec{\sigma}_n \cdot \vec{\sigma}_{n+1}$. It is indeed local, hermitian, and their sum yield the total energy. But one could just as well add local terms of the form $X_{n+1} - X_n$ to h_n without spoiling any of these properties.

In fact defining a notion of energy density amounts to define how this theory couples to gravity ! More on this later.

Rotational invariance

It is possible to exploit this ambiguity to choose a "nice" form of the stress-energy tensor. We are now going to show that rotational invariance[4] is equivalent to the existence of a symmetric stress-energy tensor

$$T^{\mu\nu} = T^{\nu\mu} \tag{3.1.18}$$

A space transformation $\Phi \to \Phi + \delta_\epsilon \Phi$ is a symmetry when the variation of the action $\delta_\epsilon S$ vanishes, *i.e.* when the vector field ϵ^μ is such that

$$\int T^\mu{}_\nu \partial_\mu \epsilon^\nu \, d^n x = 0 \tag{3.1.19}$$

Since we want the above identity to be valid for all field configurations, this boils down to

$$T^\mu{}_\nu \partial_\mu \epsilon^\nu = \partial_\rho J^\rho \tag{3.1.20}$$

for some J^ρ. For the generators of rotations

$$\epsilon^\mu = \omega^\mu{}_\nu x^\nu \tag{3.1.21}$$

this means

$$T^{\mu\nu} - T^{\nu\mu} = \partial_\rho Y^{\rho\mu\nu} \tag{3.1.22}$$

for some tensor $Y^{\rho\mu\nu} = -Y^{\rho\nu\mu}$. If we now add an improvement term *a la* (3.1.16) where

$$\Sigma^{\rho\mu\nu} = \frac{1}{2}\left(Y^{\rho\mu\nu} - Y^{\mu\rho\nu} - Y^{\nu\rho\mu}\right) \tag{3.1.23}$$

then the new stress-energy tensor is symmetric.

Conformal invariance

Let us now consider a rotational invariant field theory, and choose a stress-energy tensor representative that is symmetric. Infinitesimal conformal invariance in flat space is tantamount to

$$\omega(x) T^\mu{}_\mu = \partial_\mu K^\mu \tag{3.1.24}$$

for all functions $\omega(x)$ susceptible to appear in the *r.h.s.* of the conformal Killing constraint

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = 2\omega(x)\delta_{\mu\nu} \, . \tag{3.1.25}$$

[4] We have already implicitly assumed and exploited translation invariance in (3.1.8).

In dimension $d \geq 3$, the function $\omega(x)$ can be any linear function $\omega(x) = a + b_\mu x^\mu$, in which the a term comes from pure dilations, and the $b_\mu x^\mu$ term from special conformal transformations. So we must have

$$T^\mu{}_\mu = \partial_\mu A^\mu \qquad \text{and} \qquad x^\nu T^\mu{}_\mu = \partial_\mu B^{\mu\nu} \tag{3.1.26}$$

And therefore $x^\nu \partial_\mu A^\mu = \partial_\mu B^{\mu\nu}$, yielding

$$A^\mu = \partial_\rho C^{\rho\mu}, \qquad C^{\mu\nu} = B^{\mu\nu} - x^\nu A^\mu \tag{3.1.27}$$

So infinitesimal conformal invariance in flat space with $d \geq 3$ is tantamount to

$$\boxed{T^\mu{}_\mu = \partial_\mu \partial_\nu C^{\mu\nu}} \tag{3.1.28}$$

In two-dimensions the above relation is necessary but by no means sufficient since $\omega(x)$ can now be any harmonic[5] function. Therefore for any harmonic function $h(x)$ the quantity

$$h(x) T^\mu{}_\mu \tag{3.1.29}$$

must be a total derivative. Following a similar line of reasoning as in the case $d \geq 3$, we reach the conclusion that $T^\mu{}_\mu$ must be of the form ΔC for some C and Δ is the Laplacian.

A two-dimensional, rotational invariant field theory in flat Euclidean space (for which a symmetric stress-energy tensor has been chosen) is invariant under conformal transformations if and only iff there exist a functional $C = C(\Phi, \partial_\mu \Phi, \cdots)$ of the fields such that

$$\boxed{T^\nu{}_\nu = \Delta C} \tag{3.1.30}$$

In both cases it is possible to improve the stress-energy tensor in order to make it traceless. A way to improve the stress-energy tensor without spoiling $T^{\mu\nu} = T^{\nu\mu}$ is to add terms of the form

$$\Theta^{\mu\nu} = \partial_\rho \partial_\sigma \Omega^{\rho\mu\sigma\nu} \tag{3.1.31}$$

for some tensor $\Omega^{\rho\mu\sigma\nu}$ such that

$$\Omega^{\rho\mu\sigma\nu} = -\Omega^{\mu\rho\sigma\nu}, \qquad \Omega^{\rho\mu\sigma\nu} = -\Omega^{\rho\mu\nu\sigma}, \qquad \Omega^{\rho\mu\sigma\nu} = \Omega^{\sigma\nu\rho\mu}\,. \tag{3.1.32}$$

Note that these are the same symmetries as the Riemann curvature tensor (see (8.5.15) in the Appendix). This is not an accident.

[5] A real function f of two real variables is harmonic (*i.e* its laplacian $\Delta\omega = \partial_1^2\omega + \partial_2^2\omega = 4\partial\bar{\partial}\omega$ vanishes) iff it is the real part of a holomorphic function : $\omega = g + \bar{g}$ for some g such that $\bar{\partial} g = 0$. This equivalence holds on any simply connected domain of $\mathbb{C}$.

In $d \geq 3$

$$\Omega^{\rho\mu\sigma\nu} = \frac{1}{d-2}\left(-\eta^{\mu\nu}C^{\rho\sigma} - \eta^{\rho\sigma}C^{\mu\nu} + \eta^{\rho\nu}C^{\mu\sigma} + \eta^{\mu\sigma}C^{\rho\nu}\right) \tag{3.1.33}$$

$$+ \frac{1}{(d-1)(d-2)}\left(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\rho\nu}\eta^{\mu\sigma}\right)C^{\lambda}{}_{\lambda} \tag{3.1.34}$$

does the trick, while for $d = 2$ we can use

$$\Omega^{\rho\mu\sigma\nu} = \left(\eta^{\rho\nu}\eta^{\mu\sigma} - \eta^{\mu\nu}\eta^{\rho\sigma}\right)C \tag{3.1.35}$$

A (classical) field theory in flat Euclidean space is invariant under conformal transformations if and only if it admits a symmetric and traceless stress-energy tensor.

▸ **Exercise :** Check that the massless scalar field is conformal invariant. What about electromagnetism ?

3.2 Ward identities

After having considered conformal invariance in classical field theories in the previous sections, we can now delve into quantum field theories. We are going to restrict our attention to two-dimensional theories, although large parts of the discussion in this section can be readily generalized to any dimension.

For the sake of argument let us assume that the theory is described by a Lagrangian density $\mathcal{L}$. Even though in practice we will rarely rely on an explicit action, and in fact it might not even exist, it can be useful at a formal and heuristic level to consider the path integral formulation of quantum field theories. Generically the partition function is given in terms of an action $S[\Phi]$ by

$$Z = \int e^{-S[\Phi]}\, D[\Phi]\,. \tag{3.2.1}$$

whereas in the previous section Φ stands for a collection of fields $\phi_1, \phi_2, \ldots$. From a statistical physics perspective Φ describes (a coarse-grained version of) the lattice degrees of freedom, and $e^{-S[\Phi]}$ corresponds to the Boltzmann weight of a given configuration Φ. Correlation functions are given by

$$\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_p(x_p)\rangle = \frac{1}{Z}\int \mathcal{O}_1(x_1)\cdots\mathcal{O}_p(x_p)e^{-S[\Phi]}\,[D\Phi]\,. \tag{3.2.2}$$

The truly new ingredient as compared to a classical field theory is the functional measure $D[\Phi]$, which typically is not mathematically sound and physically requires subtle regularisation schemes (see the tutorial on zeta regularization for instance). We will work on a formal level and ignore this issue, all the while assuming a few reasonable properties of the functional measure. Indeed we will merely use the path-integral approach as an intuitive and heuristic way to derive the Ward identities. These Ward identities are relations between correlation functions that follow from the continuous symmetries of the QFT. This is the quantum version of the classical current conservation of Noether's theorem. As such Ward identities are very generic, and they can be formulated independently from any

Lagrangian or functional measure. Later on we will bypass the path-integral formalism and simply postulate that the Ward identities hold (this amounts to assume that conformal invariance is not broken at the quantum level), and in this way we will construct two-dimensional CFTs without resorting to path-integrals or assuming the existence of a Lagrangian.

3.2.1 Schwinger-Dyson equation, redundant operators

In the classical field theory the equations of motion are obtained by demanding that the action $S[\Phi]$ is extremal, i.e. $\frac{\delta S}{\delta \Phi} = 0$. Under a small variation $\Phi + \delta\Phi$ (recall that Φ stands for a collection of fields $\varphi_1, \varphi_2, \dots$) the variation of the action is

$$\delta S = \int \left(\frac{\delta \mathcal{L}}{\delta \varphi_a} \delta\varphi_a(x) + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi_a)} \partial_\mu \delta\varphi_a(x) \right) d^2x \tag{3.2.3}$$

$$= \int \left(\frac{\delta \mathcal{L}}{\delta \varphi_a} - \partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi_a)} \right) \delta\varphi_a(x) d^2x \tag{3.2.4}$$

The equations of motion are precisely

$$\frac{\delta \mathcal{L}}{\delta \varphi_a} - \partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi_a)} = 0 \tag{3.2.5}$$

Let's denote by $R_a(x)$ the *l.h.s.* of the above equation. In the quantum theory these equations of motion also have a meaning. Consider the path-integral formulation of a correlation function

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_p(x_p) \rangle = \frac{1}{Z} \int \mathcal{O}_1(x_1) \cdots \mathcal{O}_p(x_p) e^{-S[\Phi]} D[\Phi] \,. \tag{3.2.6}$$

and let us perform a change of variable $\Phi \to \widetilde{\Phi} = \Phi + \delta\Phi$. The function $\delta\Phi(x)$ is arbitrary but independent of $\Phi(x)$ (the change of variable we consider is the analogue of a rigid translation in functional space : all configurations Φ are shifted by the *same* function $\delta\Phi$). In the quantum theory it is quite reasonable and natural to assume that the measure $D[\Phi]$ is invariant under such rigid translations :

$$D[\Phi + \delta\Phi] = D[\Phi] \tag{3.2.7}$$

while the variation of the action is

$$\delta S = \int R_a(x) \delta\varphi_a(x) d^2x \,. \tag{3.2.8}$$

This leads to the Schwinger-Dyson equation

$$\boxed{\sum_{i=1}^{p} \langle \mathcal{O}_1(x_1) \cdots \delta\mathcal{O}_i(x_i) \cdots \mathcal{O}_p(x_p) \rangle = \sum_a \int \delta\varphi_a(x) \langle R_a(x) X \rangle d^2x} \tag{3.2.9}$$

where X stands for the product of fields $\mathcal{O}_1(x_1) \cdots \mathcal{O}_p(x_p)$, and $\delta\mathcal{O}_i(x)$ is the variation of the composite field $\mathcal{O}_i(x)$ under the infinitesimal shift $\varphi_a \to \varphi_a + \delta\varphi_a$. By composite field

we mean any local functional of the field φ_a, such as $\partial_\mu\varphi_a$, φ_a^2 or $e^{i\alpha\varphi_a}$ for instance. For a generic functional we would have

$$\delta\mathcal{O}_i(x) = \int \frac{\delta\mathcal{O}_i(x)}{\delta\varphi_a(y)}\delta\varphi_a(y)d^dy + \int \frac{\delta\mathcal{O}_i(x)}{\delta\partial_\mu\varphi_a(y)}\partial_\mu\delta\varphi_a(y)d^dy + \cdots \tag{3.2.10}$$

but since we consider local fields, the above expression may only involve the functions $\delta\varphi_a$ and their derivatives taken at the point x. Therefore we find that

$$\langle R_a(x)\mathcal{O}_1(x_1)\cdots\mathcal{O}_p(x_p)\rangle = 0 \qquad \text{as long as } x \neq x_1, \cdots, x_p \tag{3.2.11}$$

At coincident points – when $x = x_i$ – there are *contact terms* (typically δ function and possibly their derivatives), but apart from these points the above expression vanishes. A field that vanishes in any correlation function away from coincident points is called a *redundant field*, and we write

$$\boxed{R_a(x) \equiv 0} \tag{3.2.12}$$

3.2.2 Quantum stress-energy tensor and Ward identities

Let us now repeat the above discussion for an infinitesimal spacetime transformation

$$\widetilde{\Phi} = f_\epsilon^*\Phi = \Phi + \delta_\epsilon\Phi + O(\epsilon^2) \tag{3.2.13}$$

as generated by an arbitrary infinitesimal vector field (with compact support) $\epsilon^\mu = \epsilon\xi^\mu$. In this case the measure $D[\Phi]$ has no reason to be invariant. Assuming translation invariance the variation must vanish if ϵ^ν is constant, so we expect that the measure changes according to the following local form

$$D[\widetilde{\Phi}] = \left(1 + \int \partial_\mu\epsilon^\nu(y)\tau^\mu{}_\nu(y)\, d^2y + O(\epsilon^2)\right) D[\Phi] \tag{3.2.14}$$

We can absorb such a term in a redefinition of the stress-energy tensor, and in a QFT the stress-energy tensor $T^\mu{}_\nu$ is defined as

$$e^{-S[\widetilde{\Phi}]}D[\widetilde{\Phi}] = \left(1 + \frac{1}{2\pi}\int \partial_\mu\epsilon^\nu(y)T^\mu{}_\nu(y)\, d^2y + O(\epsilon^2)\right) e^{-S[\Phi]}D[\Phi] \tag{3.2.15}$$

It contains, in addition to the classical part, quantum corrections coming from the variation of the measure. From now on, unless otherwise specified, whenever we mention the stress-energy tensor we will mean the quantum one.

It is not always possible to preserve all classical symmetries at the quantum level. When a classical symmetry is violated by quantum corrections the symmetry is said to be *anomalous*. Implicitly we have already assumed that the quantum theory remains translation invariant when writing (3.2.14). We will further assume that rotational invariance is also preserved, thus $T^{\mu\nu} = T^{\nu\mu}$ (up to possible improvements).

Upon performing the change of variable $\Phi \to \widetilde{\Phi} = \Phi + \delta_\epsilon\Phi$, we get

$$\langle X\rangle = \frac{1}{Z}\int X e^{-S[\Phi]}\, D[\Phi] = \frac{1}{Z}\int \widetilde{X} e^{-S[\widetilde{\Phi}]}\, D[\widetilde{\Phi}] \tag{3.2.16}$$

$$= \frac{1}{Z}\int (X + \delta_\epsilon X)\left(1 + \frac{1}{2\pi}\int \partial_\mu\epsilon^\nu(y)T^\mu{}_\nu(y)\, d^2y\right) e^{-S[\Phi]}D[\Phi]\,, \tag{3.2.17}$$

yielding

$$\sum_{i=1}^{p}\langle\mathcal{O}_1(x_1)\cdots\delta_\epsilon\mathcal{O}_i(x_i)\cdots\mathcal{O}_p(x_p)\rangle = -\frac{1}{2\pi}\int_{\mathbb{R}^2}\partial_\mu\epsilon^\nu(y)\langle T^\mu{}_\nu(y)X\rangle d^2y \tag{3.2.18}$$

or equivalently

$$\boxed{\sum_{i=1}^{p}\langle\mathcal{O}_1(x_1)\cdots\delta_\epsilon\mathcal{O}_i(x_i)\cdots\mathcal{O}_p(x_p)\rangle = \frac{1}{2\pi}\int_{\mathbb{R}^2}\epsilon^\nu(y)\langle\left(\partial_\mu T^\mu{}_\nu\right)(y)X\rangle d^2y} \tag{3.2.19}$$

This equation contains a lot of extremely interesting informations. First remember that fields transform in a local way, thus $\delta_\epsilon\mathcal{O}_i(x_i)$ may only involve the vector field $\epsilon^\mu(x)$ and finitely many of its derivatives taken at the point x_i. This implies

$$\boxed{\partial_\mu T^{\mu\nu} \equiv 0\,,} \tag{3.2.20}$$

which means that $T^{\mu\nu}$ is conserved in a QFT up to contact terms. Thus we can rewrite eq. (3.2.19) as

$$\sum_{j=1}^{p}\langle\mathcal{O}_1(x_1)\cdots\delta_\epsilon\mathcal{O}_j(x_j)\cdots\mathcal{O}_p(x_p)\rangle = \sum_{j=1}^{p}\frac{1}{2\pi}\int_{B_j}\epsilon^\nu(y)\langle\left(\partial_\mu T^\mu{}_\nu\right)(y)X\rangle d^2y \tag{3.2.21}$$

where $B_1,\cdots,B_p$ are non-overlapping neighborhoods of $x_1,\cdots,x_p$ as in Fig. 3.1. We will

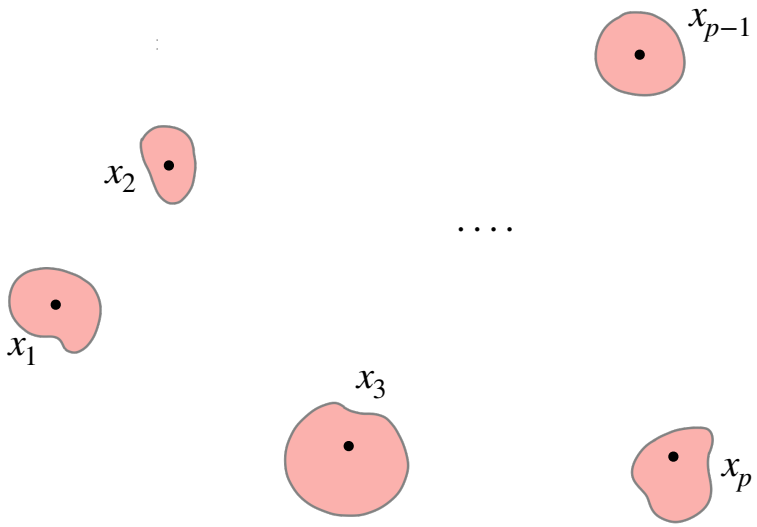


Figure 3.1: Non-overlapping neighborhoods of $x_1,\cdots,x_p$.

write

$$\delta_\epsilon\mathcal{O}_j(x_j) \equiv \frac{1}{2\pi}\int_{B_j}\epsilon^\nu(y)\left(\partial_\mu T^\mu{}_\nu\right)(y)\mathcal{O}_j(x_j)d^2y \tag{3.2.22}$$

as a short-hand notation for (3.2.21). The symbol ≡ is here as a reminder that the above identity is valid in correlation functions away from coincident points, and with the domain B_j containing no insertion other than $\mathcal{O}_j(x_j)$. Note that the above identity is insensitive to improvements of the stress-energy tensor.

Integrating by parts, we can also rewrite (3.2.22) as

$$\delta_\epsilon \mathcal{O}_j(x_j) \equiv -\frac{1}{2\pi}\int_{B_j} \partial_\mu \epsilon^\nu(y) T^\mu{}_\nu(y)\mathcal{O}_j(x_j) d^2y + \frac{1}{2\pi}\oint_{\partial B_j} \epsilon_{\mu\rho}\,\epsilon^\nu(y) T^\mu{}_\nu(y)\mathcal{O}_j(x_j)\, dy^\rho \tag{3.2.23}$$

in which the boundaries ∂B_j are oriented in the counterclockwise direction and $\epsilon_{\mu\rho}$ is the completely antisymmetric tensor ($\epsilon_{12} = -\epsilon_{12} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$).

While the above relation is valid for an arbitrary infinitesimal diffeomorphism, a particular role is played by symmetries and in particular by isometries. Consider the case in which the vector field ϵ^μ satisfies the Killing constraint inside all the B_j (and is arbitrary outside, while being smooth and compactly supported to avoid unpleasant issues). Then the integral over B_j in (3.2.23) vanishes (remember we have assumed that isometries are not anomalous, and that the stress-tensor is symmetric) and we get the Ward identity

$$\boxed{\delta_\epsilon \mathcal{O}_j(x_j) \equiv \frac{1}{2\pi}\oint_{\partial B_j} \epsilon_{\mu\rho}\,\epsilon^\nu(y) T^\mu{}_\nu(y)\mathcal{O}_j(x_j)\, dy^\rho} \tag{3.2.24}$$

for translations and rotations. In particular for translations $\delta_\epsilon \mathcal{O}_j = \epsilon^\nu \partial_\nu \mathcal{O}_j$, and we find

$$\partial_\nu \mathcal{O}_j(x_j) \equiv \frac{1}{2\pi}\oint_{\partial B_j} \epsilon_{\mu\rho} T^\mu{}_\nu(y)\mathcal{O}_j(x_j)\, dy^\rho \tag{3.2.25}$$

3.3 Conformal Ward identities and anomalous behavior

Following the discussion in section 2.2.1, an infinitesimal conformal transformation corresponds to a conformal Killing vector field ϵ^μ, *i.e.* such that

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \delta_{\mu\nu}\partial_\rho \epsilon^\rho\,. \tag{3.3.1}$$

For such a vector field Eq.(3.2.23) yields

$$\delta_\epsilon \mathcal{O}_j(x) \equiv -\frac{1}{4\pi}\int_{B_j} \partial_\rho \epsilon^\rho(y) T^\mu{}_\mu(y)\mathcal{O}_j(x_j) d^2y + \frac{1}{2\pi}\oint_{\partial B_j} \epsilon_{\mu\rho}\,\epsilon^\nu(y) T^\mu{}_\nu(y)\mathcal{O}_j(x_j)\, dy^\rho \tag{3.3.2}$$

Naively we would expect the first integral to vanish, as was the case for isometries. Indeed for a conformal invariant field theory, the classical stress-energy tensor can be chosen traceless, and we could expect the quantum stress-energy tensor to also be traceless. In flat space this turns out to be almost true, in the sense that $T^\mu{}_\mu$ is redundant :

$$T^\mu{}_\mu \equiv 0\,. \tag{3.3.3}$$

However there are contact terms. These are extremely important, as they encode the quantum corrections to the behavior of fields under conformal transformations, and in particular their anomalous dimension. We interpret the term

$$-\frac{1}{4\pi}\int_{B_j}\partial_\rho\epsilon^\rho(y)T^\mu{}_\mu(y)\mathcal{O}_j(x_j)d^2y \tag{3.3.4}$$

as a quantum correction to $\delta_\epsilon\mathcal{O}_j(x)$. What we are saying is that fields in a QFT might not transform as their classical counterpart. Technically these anomalous terms come from the need to introduce a length scale when regulating a QFT (see for instance the regularization of vertex operators in the tutorials). Following the same philosophy as for the stress-energy tensor (3.2.15), we absorb these quantum corrections in a redefinition of $\delta_\epsilon\mathcal{O}_j(x)$:

$$\widetilde{\delta}_\epsilon\mathcal{O}_j(x) = \delta_\epsilon\mathcal{O}_j(x) + \frac{1}{4\pi}\int_B\partial_\rho\epsilon^\rho(y)T^\mu{}_\mu(y)\mathcal{O}_j(x)d^2y \tag{3.3.5}$$

where B is a small neighborhood of x that does not contain any other field insertion. Notice that these quantum corrections do not spoil the local nature of the transformation law (3.1.7).

Consider for instance the behavior of fields under an infinitesimal dilatation $\epsilon^\mu = \epsilon x^\mu$. At the classical level the behavior of a field is completely fixed (for instance for scalar/vector/tensor field it is given by the Lie derivative) and is of the form

$$\delta_\epsilon\mathcal{O} = \epsilon\left(x^\rho\partial_\rho + \Delta_0\right)\mathcal{O} \tag{3.3.6}$$

where Δ_0 is the naive/canonical scaling dimension. Suppose the contact terms of $T^\mu{}_\mu$ with the field $\mathcal{O}$ are of the form

$$T^\mu{}_\mu(y)\mathcal{O}(x) = 2\pi\eta\delta(y-x)\mathcal{O}(x) + \text{higher derivatives of } \delta(y-x) \tag{3.3.7}$$

we find at the quantum level

$$\widetilde{\delta}_\epsilon\mathcal{O} = \epsilon\left(x^\rho\partial_\rho + \Delta\right)\mathcal{O}, \qquad \Delta = \Delta_0 + \eta \tag{3.3.8}$$

This violation of the naive scaling dimension is our first encounter with quantum anomalies, and the extra term η is called the *anomalous dimension*. These scaling dimensions are of tremendous importance in physics and are often directly accessible in experiments, but they are notoriously difficult to compute exactly. In statistical physics they correspond to critical exponents. One of the main achievements of two-dimensional CFT is to provide exact results for these scaling dimensions.

We can now rewrite the conformal Ward identity (3.2.23) in the following compact form

$$\boxed{\widetilde{\delta}_\epsilon\mathcal{O}_j(x) \equiv \frac{1}{2\pi}\oint_{\partial B}\epsilon_{\mu\rho}\,\epsilon^\nu(y)T^\mu{}_\nu(y)\mathcal{O}_j(x_j)\,dy^\rho\,.} \tag{3.3.9}$$

Of course the vector field ϵ^μ must satisfy the conformal Killing equation inside B, and the sign $\equiv$ is there to recall that the above equation is only valid when inserted in a correlation

Cartesian coordinates	Complex coordinates
$x = x^1 = \frac{1}{2}(z+\bar{z})$, $y = x^2 = \frac{1}{2i}(z-\bar{z})$ $\partial_x = \partial_z + \partial_{\bar{z}}$, $\partial_y = i(\partial_z - \partial_{\bar{z}})$ $g_{\mu\nu} = g^{\mu\nu} = \delta_{\mu\nu}$ $\epsilon_{12} = \epsilon^{12} = 1$	$z = x + iy$, $\bar{z} = x - iy$ $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$, $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$ $g_{z\bar{z}} = \frac{1}{2}$, $g^{z\bar{z}} = 2$ $\epsilon_{z\bar{z}} = \frac{i}{2}$, $\epsilon^{z\bar{z}} = -2i$
$T_{11} = T_{zz} + T_{z\bar{z}} + T_{\bar{z}z} + T_{\bar{z}\bar{z}}$ $T_{22} = -T_{zz} + T_{z\bar{z}} + T_{\bar{z}z} - T_{\bar{z}\bar{z}}$ $T_{12} = i(T_{zz} - T_{z\bar{z}} + T_{\bar{z}z} - T_{\bar{z}\bar{z}})$ $T_{21} = i(T_{zz} + T_{z\bar{z}} - T_{\bar{z}z} - T_{\bar{z}\bar{z}})$	$T_{zz} = \overline{T_{\bar{z}\bar{z}}} = \frac{1}{4}(T_{11} - T_{22} - iT_{12} - iT_{21})$ $T_{z\bar{z}} = \overline{T_{\bar{z}z}} = \frac{1}{4}(T_{11} + T_{22} + i(T_{12} - T_{21}))$

Table 3.1: From Cartesian to complex coordinates

function with no other insertion inside B (nor inside its closure : we do not want any contact terms with the stress-energy tensor). Equivalently

$$\sum_{i=1}^{p}\langle \mathcal{O}_1(x_1)\cdots\widetilde{\delta}_\epsilon\mathcal{O}_i(x_i)\cdots\mathcal{O}_p(x_p)\rangle = \sum_{i=1}^{p}\frac{1}{2\pi}\oint_{\partial B_i}\epsilon_{\mu\rho}\,\epsilon_\nu(y)\langle T^{\mu\nu}(y)\mathcal{O}_1(x_1)\cdots\mathcal{O}_p(x_p)\rangle\, dy^\rho \tag{3.3.10}$$

From now on we will drop the tilde on $\widetilde{\delta}_\epsilon\mathcal{O}_j(x)$.

3.3.1 Conformal Ward identities in complex coordinates

Needless to say the whole thing looks simpler in complex coordinates, see Table (3.1). For starters the vector field $\epsilon^z\partial_z + \epsilon^{\bar{z}}\partial_{\bar{z}}$ is conformal Killing iff $\epsilon^z = \epsilon(z)$ is holomorphic (it follows that $\epsilon^{\bar{z}} = \bar{\epsilon}$ is anti-holomorphic). Then consider the stress-energy tensor. In complex coordinates it has components

$$T_{zz} = \overline{T_{\bar{z}\bar{z}}} = \frac{1}{4}(T_{11} - T_{22} - iT_{12} - iT_{21}) \tag{3.3.11}$$

$$T_{z\bar{z}} = \overline{T_{\bar{z}z}} = \frac{1}{4}(T_{11} + T_{22} + i(T_{12} - T_{21})) \tag{3.3.12}$$

but since T is symmetric

$$T_{z\bar{z}} = T_{\bar{z}z} = \frac{1}{4}T^\mu{}_\mu \equiv 0\,. \tag{3.3.13}$$

Furthermore $\partial_\mu T^{\mu\nu} \equiv 0$, therefore

$$\partial_{\bar{z}}T_{zz} \equiv 0, \qquad \partial_z T_{\bar{z}\bar{z}} \equiv 0 \tag{3.3.14}$$

Note that in the *r.h.s.* of eq. (3.3.10) the position y of the SET never coincides with a field insertion, thus we can effectively forget about contact terms (they have already played their part). So as far as the conformal Ward identities are concerned the SET has only two components $T(z)$ and $\overline{T}(\bar{z})$:

$$\boxed{T(z) = T_{zz}, \qquad \overline{T}(\bar{z}) = T_{\bar{z}\bar{z}}, \qquad T_{z\bar{z}} = T_{\bar{z}z} = 0} \tag{3.3.15}$$

The conformal Ward identity (3.3.10) becomes

$$\sum_{i=1}^{p}\langle \mathcal{O}_1(z_1,\bar{z}_1)\dots\delta_\epsilon\mathcal{O}_i(z_i,\bar{z}_i)\dots\mathcal{O}_p(z_p,\bar{z}_p)\rangle$$
$$= \frac{1}{2\pi i}\sum_{i=1}^{p}\oint_{\partial B_i}\left[\langle\dots T(z)\mathcal{O}_i(z_i,\bar{z}_i)\dots\rangle\epsilon(z)dz - \langle\dots\overline{T}(\bar{z})\mathcal{O}_i(z_i,\bar{z}_i)\rangle\bar{\epsilon}(\bar{z})d\bar{z}\right], \quad (3.3.16)$$

where the integration over ∂B_i is performed in the counter-clockwise direction. We can rewrite the above Ward identity as

$$\boxed{\delta_\epsilon\mathcal{O}(z,\bar{z}) \equiv \frac{1}{2\pi i}\oint_{\circlearrowleft}\epsilon(\xi)T(\xi)\mathcal{O}(z,\bar{z})d\xi + \frac{1}{2\pi i}\oint_{\circlearrowright}\bar{\epsilon}(\bar{\xi})\overline{T}(\bar{\xi})\mathcal{O}(z,\bar{z})d\bar{\xi}\,,} \quad (3.3.17)$$

in which the integration contour circles around z. As usual the above relation is valid in correlation functions away from coincident points, and with the domain B containing no other insertion than $\mathcal{O}(z,\bar{z})$.

3.3.2 From Ward identities to OPEs

Recall that a field ϕ is called a *primary field* if it transforms as follows under a conformal transformation $z \to f(z)$

$$\phi'(z,\bar{z}) = (\partial f)^h(\bar{\partial}f)^{\bar{h}}\phi(f(z),\overline{f(z)}) \quad (3.3.18)$$

The numbers $(h,\bar{h})$ are the conformal dimensions of ϕ. Primary fields play a central role in CFTs. The reason will become clear when studying the representation theory of the conformal algebra (*a.k.a.* the Virasoro algebra).

Infinitesimally, for $f(z) = z + \epsilon(z)$, a primary field behaves as

$$\delta_\epsilon\phi(z,\bar{z}) = \left(\epsilon\partial_z + h\partial_z\epsilon\right)\phi(z,\bar{z}) + \left(\bar{\epsilon}\partial_{\bar{z}} + \bar{h}\partial_{\bar{z}}\bar{\epsilon}\right)\phi(z,\bar{z}) \quad (3.3.19)$$

Comparison with the Ward identity (3.3.17) yields the following Operator Product Expansion

$$\boxed{\begin{aligned} T(z)\phi(w,\bar{w}) &= \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\phi(w,\bar{w}) + \cdots \\ \overline{T}(\bar{z})\phi(w,\bar{w}) &= \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\phi(w,\bar{w}) + \cdots \end{aligned}} \quad (3.3.20)$$

Note that in deriving the above we heedlessly treated the functions $\epsilon(z)$ and $\bar{\epsilon}(\bar{z})$ as if they were independent, which in principle is not correct. This is what we alluded to earlier when we mentioned the *complexification* of the algebra of conformal Killing vectors. This point, namely the factorization of CFT correlation functions into left and right degrees of freedom, which amounts to treat z and $\bar{z}$ as being independent, will be commented more and to some extent justified in the next chapter.

For a generic scaling field, the OPEs $T\phi$ and $\overline{T}\phi$ are quite similar to (3.3.20) : the very same terms appear, but in general there are poles of higher degrees. For instance

$$T(z)\partial_w\phi(w,\bar{w}) = \frac{2h}{(z-w)^3}\phi(w,\bar{w}) + \frac{(h+1)}{(z-w)^2}\partial_w\phi(w,\bar{w}) + \frac{1}{z-w}\partial_w^2\phi(w,\bar{w}) + \cdots \quad (3.3.21)$$

While primary fields are quite special, recall the larger class of *quasi-primary fields*, for which one imposes the covariance relation (3.3.18) only under global conformal mappings of the sphere, namely

$$f(z) = \frac{az+b}{cz+d} \tag{3.3.22}$$

Exercise : show that a field ϕ is quasi-primary if and only if the OPEs $T(z)\phi(w,\bar{w})$ (and $\overline{T}(\bar{z})\phi(w,\bar{w})$) has no pole of order 3.

Let's now consider the behavior of the SET itself under conformal maps. At the classical level

$$\delta_\epsilon T_{\mu\nu} = \epsilon^\rho \partial_\rho T_{\mu\nu} + \partial_\mu \epsilon^\rho T_{\rho\nu} + \partial_\nu \epsilon^\rho T_{\mu\rho} \tag{3.3.23}$$

which in complex coordinates (recall that T is traceless) reads

$$\delta_\epsilon T = \epsilon \partial_z T + 2\partial_z \epsilon\, T, \tag{3.3.24}$$

$$\delta_\epsilon \overline{T} = \bar{\epsilon} \partial_{\bar{z}} \overline{T} + 2\partial_{\bar{z}} \bar{\epsilon}\, \overline{T} \tag{3.3.25}$$

The naive dimension of the SET is $\Delta = 2$. Because it remains conserved in the quantum theory, its dimension cannot be modified (this is in fact a generic properties of conserved current : they cannot acquire anomalous dimensions). However in the quantum theory the above transformation law under infinitesimal conformal transformations generically gets a quantum correction of the form

$$\delta_\epsilon T = \epsilon \partial_z T + 2\partial_z \epsilon\, T + \frac{c}{12}\partial_z^3 \epsilon, \tag{3.3.26}$$

$$\delta_\epsilon \overline{T} = \bar{\epsilon} \partial_{\bar{z}} \overline{T} + 2\partial_{\bar{z}} \bar{\epsilon}\, \overline{T} + \frac{c}{12}\partial_{\bar{z}}^3 \bar{\epsilon}\,. \tag{3.3.27}$$

where the number c is called the central charge. Note that the extra term does not spoil the scaling dimension of the SET. The origin of this anomalous term will be made clear in the next section. Comparison with (3.3.17) yields the following Operator Product Expansion

$$\boxed{T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots} \tag{3.3.28}$$

and likewise for $\overline{T}(\bar{z})\overline{T}(\bar{w})$. Notice that T and $\overline{T}$ are not primary fields (unless $c = 0$), but they are quasi-primary. In fact the transformation (3.3.26) is the infinitesimal version of

$$\boxed{T(z) \to \left(\frac{\partial w}{\partial z}\right)^2 T(w) + \frac{c}{12}\{w,z\}\,,} \tag{3.3.29}$$

where $\{w,z\}$ is the Schwarzian derivative of the conformal transformation $z \mapsto w(z)$, namely

$$\{w,z\} = \partial_z\left(\frac{\partial_z^2 w}{\partial_z w}\right) - \frac{1}{2}\left(\frac{\partial_z^2 w}{\partial_z w}\right)^2 = \frac{\partial_z^3 w}{\partial_z w} - \frac{3}{2}\left(\frac{\partial_z^2 w}{\partial_z w}\right)^2. \tag{3.3.30}$$

Exercise : Check that (3.3.29) is indeed the integrated version of the infinitesimal transformation (3.3.26). Let $S(f) = \{f, z\}$. Show that $S(f \circ g) = g'^2 S(f) \circ g + S(g)$. Check that $S(f)$ vanishes iff f is a Möbius transformation $z \to \frac{az+b}{cz+d}$.

Some references for this chapter :

- *Lectures on Liouville Theory and Matrix Models*, Alexei Zamolodchikov and Alexander Zamolodchikov

3.4 Appendix

3.4.1 Canonical stress-energy tensor

Noether's theorem

Consider a classical field theory on flat Euclidean d-dimensional spacetime with metric $\eta_{\mu\nu} = \delta_{\mu\nu}$. The dynamics of the various fields $(\phi_1, \phi_2, \cdots)$ which we will denote collectively by Φ is described by a Lagrangian density $\mathcal{L}$. For simplicity we will assume that $\mathcal{L} = \mathcal{L}(\Phi, \partial_\mu \Phi)$ depends only on the fields at x and their first derivatives. The equations of motion (*a.k.a.* the Euler-Lagrange equations) are obtained as usual from extremizing the action

$$S[\Phi] = \int \mathcal{L}(\Phi(x), \partial_\mu \Phi(x)) d^n x \tag{3.4.1}$$

Fields are called "on-shell" when they obey the equation of motions

$$\frac{\partial \mathcal{L}}{\partial \varphi_a} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \right) \tag{3.4.2}$$

An infinitesimal transformation $\Phi \to \Phi + \delta\Phi$ is said to be a symmetry if the infinitesimal variation of the Lagrangian is a pure divergence

$$\delta\mathcal{L} = \mathcal{L}[\Phi + \delta\Phi] - \mathcal{L}[\Phi] = \partial_\mu K^\mu \tag{3.4.3}$$

for **all** configurations $\phi(x)$ (not just on-shell[6])

Exercise Compute the equation of motion for

- the complex scalar field $\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - V(|\phi|^2)$
- the Proca action $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + m^2 A_\mu A^\mu$ with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$
- the Abelian $2+1$ Chern-Simons theory $\mathcal{L} = \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$.

and list their symmetries.

The corresponding Noether current is derived as follows. Assume the fields are on-shell, and consider an arbitrary variation $\delta\phi$. At first order the variation of the action is

$$\delta L = \partial_\mu \left(\delta\varphi_a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \right) + \delta\varphi_a \left(\frac{\partial \mathcal{L}}{\partial \varphi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \right) = \partial_\mu \left(\delta\varphi_a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \right) \tag{3.4.4}$$

where we have used the fact that the second term vanishes when ϕ satisfies Euler-Lagrange. If $\Phi \to \Phi + \delta\Phi$ is also a symmetry, comparing with (3.4.3), we find that

$$\partial_\mu j^\mu = 0, \qquad j^\mu = \delta\varphi_a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} - K^\mu \tag{3.4.5}$$

[6]While this definition of symmetry is perfectly sensible for a quantum field theory, it is not very natural for a classical one. Indeed a classical theory is fully characterized by its equations of motion, and it does not require a Lagrangian. It could happen that the equations of motion enjoy a symmetry that the Lagrangian does not. More importantly off-shell quantities are simply undefined/meaningless for a classical system : many Lagrangians can share the same equations of motion. This caveat will not be an issue in this lecture since we are just preparing the grounds for the quantum case.

provided the fields are on shell. The corresponding conserved charge is

$$Q = \int j^0(t,x) d^{n-1}x$$

Indeed

$$\dot{Q} = \int \partial_0 j^0(t,x) d^{n-1}x = -\int \partial_i j^i(t,x) d^{n-1}x = 0$$

by Stokes theorem, provided the current j^i vanishes sufficiently rapidly at infinity (or if space is compact, *e.g.* a torus).

Noether theorem associates a conservation law to each continuous symmetry of the system. Let us however note a potential pitfall. While the identification of the global charge Q is quite clear, there is an intrinsic ambiguity in defining the corresponding local current j^μ by simply demanding $\partial_\mu j^\mu = 0$. Indeed if j^μ is conserved, then so is $j^\mu + \partial_\nu b^{\mu\nu}$ for any antisymmetric tensor $b^{\mu\nu}$. In particular the charge density j^0 is ambiguous, as one can add a divergent-free term $j^0 \to j^0 + \partial_i b^{0i}$. On the other hand the total charge Q is well-defined, since

$$\int \partial_i b^{0i}\, d^{n-1}x = 0$$

provided b^{0i} vanishes at spatial infinity.

Spacetime translations

Under an infinitesimal translation of spacetime (most) fields transform as $\Phi(x) \to \Phi(x^\mu + \epsilon^\mu) = \Phi(x) + \epsilon^\mu \partial_\mu \Phi(x)$. It is a symmetry as soon as $\mathcal{L}$ does not depend explicitly on x^μ. Indeed the variation of the Lagrangian density $\mathcal{L}$ is a total derivative

$$\delta\mathcal{L} = \left(\epsilon^\mu \partial_\mu \varphi_a \frac{\partial \mathcal{L}}{\partial \varphi_a} + \epsilon^\mu \partial_\mu \partial_\nu \varphi_a \frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi_a)} \right) = \partial_\mu(\epsilon^\mu \mathcal{L}) \tag{3.4.6}$$

where we used the chain rule $\partial_\mu \mathcal{L}(\Phi, \partial_\nu \Phi) = \partial_\mu \varphi_a \frac{\partial \mathcal{L}}{\partial \varphi_a} + \partial_\mu(\partial_\nu \varphi_a) \frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi_a)}$.

According to Noether's theorem, the associated current is

$$j^\mu = \epsilon^\nu \left(\partial_\nu \varphi_a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} - \delta^\mu_\nu \mathcal{L} \right) = -\frac{1}{2\pi} \epsilon^\nu T^\mu{}_\nu \tag{3.4.7}$$

so the canonical energy-momentum tensor is

$$\boxed{T^{\mu\nu} = 2\pi \left(-\partial^\nu \varphi_a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} + \eta^{\mu\nu} \mathcal{L} \right)} \tag{3.4.8}$$

The prefactor 2π is conventional, but the overall sign is not. It must be chosen so that the energy density T^{00} is positive in Minkowski space with signature $(-,+,+,+)$. In signature $(+,-,-,-)$ one must change the sign of $T^{\mu\nu}$.

The conserved charges associated to spacetime translations are energy and momentum

$$E = \int T^{00}(t,x)d^{n-1}x, \qquad P^i = \int T^{0i}(t,x)d^{n-1}x \tag{3.4.9}$$

In the quantum setting these will become the generators of spacetime translations. For instance it is straightforward to check that E is (proportional to) the Hamiltonian

$$H = \int \left(\pi^a \dot{\phi}_a - \mathcal{L}\right) d^{n-1}x, \qquad \pi^a = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a} \tag{3.4.10}$$

However the naive canonical stress-energy tensor as defined above suffers from a number of severe issues and is in general unphysical :

- For gauge theories, it is in general not gauge invariant, as can be checked with electrodynamics
- It is not symmetric in general[7]
- It can fail to be traceless for scale invariant theories (more on this later)

All these issues are illustrated with free electrodymanics

$$\mathcal{L}[A_\mu] = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.4.11}$$

for which the canonical stress-energy tensor is

$$T^{\mu\nu} = F^{\mu\rho}\partial^\nu A_\rho - \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \tag{3.4.12}$$

It is manifestly not gauge invariant, nor symmetric, nor is it traceless in four-dimensions (in $4d$ this theory is scale invariant). An even more serious matter in $3+1$ dimensions is that T^{00} and T^{0i} fail to reproduce the well-known, experimentally tested expressions for the electromagnetic energy density $\frac{1}{2}(E^2+B^2)$ and energy flux density (the Poynting vector) $E \times B$. In this particular case these issues can be traced back to the gauge symmetry of the action.

Improved stress-energy tensor

The canonical SET suffers from the same ambiguity as one described in section 3.1.2. This can be exploited to improve the canonical SET and yield a partial cure to the above-mentioned issues. Indeed for electrodymanics choosing $\Sigma^{\rho\mu\nu} = F^{\rho\mu}A^\nu$ leads to the correct stress-energy tensor

$$\begin{aligned} T^{\mu\nu} &= F^{\mu\rho}F^{\nu}{}_{\rho} + A^\nu \partial_\rho F^{\rho\mu} - \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \\ &\equiv F^{\mu\rho}F^{\nu}{}_{\rho} - \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \qquad \text{on-shell} \end{aligned} \tag{3.4.13}$$

[7] This is required in a Lorentz invariant theory for which the angular momentum current $T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu$ should be conserved and it is also required to couple to curvature in general relativity (more on this to come).

where we have used the on-shell relation $\partial_\mu F^{\mu\nu} = 0$. This tensor is now symmetric, gauge-invariant, and traceless in four dimensions, **as long as we are on-shell**. In fact it is the correct one (up to a possible global sign depending on the choice of signature of $\eta^{\mu\nu}$), since $T^{00} = \frac{1}{2}(E^2 + B^2)$ and $T^{0i} = (E \times B)^i$.

There it however a better way to obtain the correct SET from Noether's theorem. Instead of $\delta A_\mu = \epsilon^\alpha \partial_\alpha A_\mu$, consider a transformation that combines a naive translation to a specific gauge transformation as follows :

$$\delta A_\mu = \epsilon^\alpha \partial_\alpha A_\mu - \partial_\mu(\epsilon^\alpha A_\alpha) = \epsilon^\alpha F_{\alpha\mu} \tag{3.4.14}$$

The variation of the Lagrangian density $\mathcal{L}$ is the same as for $\delta A_\mu = \epsilon^\alpha \partial_\alpha A_\mu$ since $\mathcal{L}$ is gauge invariant. So $K^\mu = \epsilon^\mu \mathcal{L}$. Noether's theorem now yields a different conserved SET, which turns out to be the correct one (and not just on-shell).

In general curing the canonical SE tensor by adding such an improvement term has to be done on a case by case basis, and is a rather ad hoc procedure. It should be stressed that improving the stress-energy tensor is not merely a formal manipulation. Improvements amount to a relocalization of the energy and momentum densities, which are observable quantities, and as such they must agree with experiments. Generically the canonical SE tensor is an unphysical object coming form a formal and rather ambiguous construction.

Exercise : Compute the canonical SET for the 2+1 dimensional Chern-Simons theory $\mathcal{L} = \frac{1}{2\pi}\epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$. Show that it can be improved to achieve $T^{\mu\nu} = 0$ on-shell (which is what we expect since for a topological field theory). Compare with the SET obtained using the "improved translation" (3.4.14).

3.4.2 Hilbert stress-energy tensor

Given a global symmetry, Noether theorem yields a well defined total charge, but the local distribution remains ambiguous. What is missing is a physically sound prescription of the notion of local charge and local currents. An elegant way to address this is to define how this charge couples to external fields.

Electric current as the source of the electromagnetic field

This is best illustrated for the electric charge. Consider for instance a complex scalar field with Lagrangian density

$$\mathcal{L} = \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi^* - V(|\phi|^2)$$

The global U(1) symmetry $\phi(x) \to e^{i\theta}\phi(x)$ yields the conservation of the total (electric) charge through the canonical Noether current

$$j^\mu = i\left(\phi \partial^\mu \phi^* - \phi^* \partial^\mu \phi\right) \tag{3.4.15}$$

but this current is only defined up to possible improvements. A way to circumvent this ambiguity is to couple this theory to an arbitrary external (electromagnetic) gauge field A_μ, for instance

$$\mathcal{L} = \eta^{\mu\nu} D_\mu \phi (D_\nu \phi)^* - V(|\phi|^2), \qquad D_\mu = \partial_\mu - iA_\mu \tag{3.4.16}$$

By prescribing how the complex scalar field couples to the background U(1) gauge field, we are promoting the global U(1) symmetry to a local one $\phi(x) \to e^{i\theta(x)}\phi(x)$, $A_\mu \to A_\mu + \partial_\mu\theta$. From the Lagrangian density (3.4.16) we can define unambiguously the local current as the response of the system to a variation of the external gauge field[8] :

$$\boxed{\delta S = -\int j^\mu \delta A_\mu \, d^n x, \qquad \text{or equivalently} \qquad j^\mu = -\frac{\delta S}{\delta A_\mu}} \tag{3.4.19}$$

and we recover the expression (3.4.15) when $A_\mu = 0$. That this current is conserved on-shell is a consequence of gauge invariance. Indeed the action is invariant under a simultaneous change $\delta\phi(x) = i\theta(x)\phi(x)$ and $\delta A_\mu(x) = \partial_\mu\theta(x)$, therefore

$$0 = \delta S = \int \left(\frac{\delta S}{\delta \phi}\delta\phi + \frac{\delta S}{\delta \phi^*}\delta\phi^* \right) d^n x - \int j^\mu \partial_\mu \theta \, d^n x \tag{3.4.20}$$

Since $\frac{\delta S}{\delta \phi} = \frac{\delta S}{\delta \phi^*} = 0$ on-shell, we get for an arbitrary function $\theta(x)$

$$\int j^\mu \partial_\mu \theta \, d^n x = 0 \qquad \text{on-shell} \tag{3.4.21}$$

which is equivalent to $\partial_\mu j^\mu = 0$ on-shell. Furthermore from

$$\int \left(\frac{\delta S}{\delta \phi}\delta\phi + \frac{\delta S}{\delta \phi^*}\delta\phi^* \right) d^n x = \int j^\mu \partial_\mu \theta \, d^n x \tag{3.4.22}$$

we find

$$\boxed{\delta S = \int j^\mu(x) \partial_\mu \theta(x) \, d^n x, \qquad \text{under} \qquad \delta\phi(x) = i\theta(x)\phi(x)} \tag{3.4.23}$$

which is also valid for the original problem of the theory that is **not** coupled to the gauge field (*i.e.* the case $A_\mu = 0$).

Linear response to spacetime deformations

The same approach can be used to define the stress-energy tensor. What plays the role of the external field is now the background metric : we are coupling the theory to gravity. Indeed in general relativity, the stress-energy tensor acts as the source of spacetime curvature.

One must prescribe how to extend the theory from flat space to an arbitrary curved space. Once our theory is defined on arbitrary curved background, we can define the stress

[8] Equivalently, upon considering a dynamical gauge field, the current j^μ can be seen as the source of the electromagnetic field. Indeed from

$$\mathcal{L} = \mathcal{L}_m + \mathcal{L}_A, \qquad \mathcal{L}_m = \eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi^* - V(|\phi|^2), \qquad \mathcal{L}_A = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{3.4.17}$$

the equation of motion of the electromagnetic field reads

$$\partial_\mu F^{\mu\nu} = j^\nu, \qquad j^\mu = -\frac{\delta S}{\delta A_\mu} \tag{3.4.18}$$

tensor as the susceptibility of the system with respect to the variations of the background metric :

$$\boxed{\delta S = \frac{1}{4\pi}\int T^{\mu\nu}\delta g_{\mu\nu}\sqrt{|g|}d^n x = -\frac{1}{4\pi}\int T_{\mu\nu}\delta g^{\mu\nu}\sqrt{|g|}d^n x} \tag{3.4.24}$$

where the minus sign arises because $\delta g^{\mu\nu} = -g^{\mu\sigma}\delta g_{\sigma\tau}g^{\tau\nu}$. This is equivalent to[9]

$$T_{\mu\nu} = -4\pi\frac{\delta S}{\delta g^{\mu\nu}} \tag{3.4.25}$$

This defines the so-called Hilbert stress-energy tensor. It suffers none of the unpleasant properties of the canonical one. To start with, it is well defined ! Moreover it is by construction gauge-invariant and symmetric.

In order to define the electric charge density, we promoted the U(1) symmetry to local gauge invariance. We now have defined the stress-energy tensor by promoting translation invariance to general (or diffeomorphism) covariance. What is left is to derive the analogue of (3.4.23), which will turn out to be nothing but the characterisation of the stress-energy tensor used in the main text, namely (3.1.8). To see this, consider once again an infinitesimal diffeomorphism $x^\mu \to x^\mu + \epsilon^\mu(x)$, acting this time both on the fields and the metric (that is to say an infinitesimal *isometry*) :

$$\delta g_{\mu\nu}(x) = \epsilon^\rho\partial_\rho g_{\mu\nu}(x) + g_{\mu\rho}(x)\partial_\nu\epsilon^\rho(x) + g_{\sigma\nu}(x)\partial_\mu\epsilon^\sigma(x) = \nabla_\mu\epsilon_\nu + \nabla_\nu\epsilon_\mu \tag{3.4.26}$$

where ∇ is the Levi-Civita connection and $\nabla_\mu\epsilon_\nu$ stands for $\left(\nabla_\mu\epsilon\right)_\nu$, namely

$$\left(\nabla_\mu\epsilon\right)_\nu = g_{\nu\lambda}(\nabla_\mu\epsilon)^\lambda = g_{\nu\lambda}\left(\partial_\mu\epsilon^\lambda + \Gamma^\lambda{}_{\rho\mu}\epsilon^\rho\right) = g_{\nu\lambda}\partial_\mu\epsilon^\lambda + \epsilon^\rho\Gamma_{\nu\rho\mu} \tag{3.4.27}$$

where $\Gamma_{\nu\rho\mu}$ is the Christoffel symbol

$$\Gamma_{\nu\rho\mu} = \frac{1}{2}\left(\partial_\mu g_{\nu\rho} + \partial_\rho g_{\nu\mu} - \partial_\nu g_{\mu\rho}\right) \tag{3.4.28}$$

In particular in flat space (and in flat coordinates, *i.e.* such that $g_{\mu\nu} = \eta_{\mu\nu}$)

$$\boxed{\delta g_{\mu\nu}(x) = \partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu} \tag{3.4.29}$$

But general covariance tells us that the theory remains unchanged under arbitrary diffeomorphisms (note that such diffeomorphisms correspond to symmetries only when they leave the metric unchanged, which infinitesimally means Killing vector fields). Therefore we have

$$\begin{aligned} 0 = \delta S &= S[\Phi + L_\epsilon\Phi, g + L_\epsilon g] - S[\Phi, g] \\ &= S[\Phi + L_\epsilon\Phi, g] - S[\Phi, g] + \int \left.\frac{\delta S}{\delta g_{\mu\nu}}\right|_{g=\eta} \delta g_{\mu\nu}\sqrt{|g|}d^n x \end{aligned}$$

[9] We recall our convention for functional derivatives : $\delta S = \int \frac{\delta S}{\delta g^{\mu\nu}(x)}\delta g^{\mu\nu}(x)\sqrt{|g|}d^n x$ and not $\int \frac{\delta S}{\delta g^{\mu\nu}(x)}\delta g^{\mu\nu}(x)d^n x$.

and we get

$$S[\Phi + L_\epsilon \Phi, g] - S[\Phi, g] = -\int \frac{\delta S}{\delta g_{\mu\nu}} \delta g_{\mu\nu} \sqrt{|g|} d^n x$$
$$= -\frac{1}{2\pi} \int T^{\mu\nu} \nabla_\mu \epsilon_\nu \sqrt{|g|} d^n x$$

In flat space and in flat coordinates we recover (3.1.8)

$$\boxed{S[\Phi + L_\epsilon \Phi] - S[\Phi] = \int T^{\mu\nu} \partial_\mu \epsilon_\nu \, d^n x, \qquad T^{\mu\nu} = -4\pi \frac{\delta S}{\delta g_{\mu\nu}}\bigg|_{g=\eta}} \tag{3.4.30}$$

Matter tells spacetime how to curve

Alternatively one can couple the theory (referred to as *matter* in this context, with action S_m) to a *dynamical* metric with the Einstein-Hilbert action

$$S_g = \frac{1}{2} \int R \sqrt{|g|} d^n x$$

where R is the Ricci scalar (see Appendix (8.5.1)). The total action is now

$$S = S_m + S_g$$

and the equation of motion of the metric is

$$0 = \frac{\delta S_g}{\delta g_{\mu\nu}} + \frac{\delta S_m}{\delta g_{\mu\nu}} \tag{3.4.31}$$

In (8.5.1) we compute

$$\frac{\delta S_g}{\delta g_{\mu\nu}} = -\frac{1}{2} \left(R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} \right) \tag{3.4.32}$$

while by definition

$$\frac{\delta S_m}{\delta g_{\mu\nu}} = -\frac{1}{2} T^{\mu\nu} \tag{3.4.33}$$

and we obtain the Einstein field equations

$$\boxed{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu}} \tag{3.4.34}$$

which is the analogue of $\partial_\mu F^{\mu\nu} = j^\nu$ for gravitation : the stress-energy tensor acts as the source of the gravitational field.

Lecture 4

Radial quantisation and the Virasoro algebra

In the previous lecture, we have worked in the Lagrangian formalism: implicitly or explicitly, we have assumed that the theory is governed by some action $\mathcal{A}[\Phi]$ defined on the local degrees of freedom $\{\Phi(r)\}$, so that the correlation functions are given by:

$$\langle \dots \rangle = \frac{1}{Z} \int [D\Phi]\, e^{-\mathcal{A}[\Phi]} (\dots)\,, \qquad Z = \int [D\Phi]\, e^{-\mathcal{A}[\Phi]}\,, \tag{4.0.1}$$

where the dots denote any functional of $\Phi(r)$. In the present lecture, we will show how to construct an equivalent viewpoint based on a Hilbert space. It is instructive to begin with the lattice analogs of these two viewpoints.

4.1 Critical model on the cylinder

We will now focus on conformal field theories defined on the cylinder, as depicted in Fig. 4.1

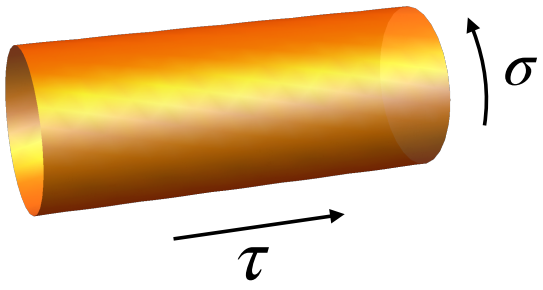


Figure 4.1: Cylinder of perimeter L. We denote by σ the compact coordinate (with σ and $\sigma + L$ identified) and by $\tau \in \mathbb{R}$ the non-compact direction.

This spacetime geometry is relevant for a variety of situations, critical or not, such as

- one-dimensional quantum systems on a circle (*i.e.* periodic boundary conditions) at zero temperature

- one-dimensional quantum systems on a an infinite line, at finite temperature
- transfer matrix approach to statistical physics

But for conformal field theories something special happens : the cylinder is conformally equivalent to the (punctured) flat plane, thus solving a CFT on the plane is equivalent to solving it on the cylinder. This will yield two important consequences, to be discussed in this chapter : *radial quantisation* and the *state-operator correspondence*.

4.1.1 Hamiltonian formalism

Consider a Euclidean CFT on a cylinder of circumference L and length K, with the space direction x along the circumference (so that $x \equiv x + L$), and the imaginary time direction τ along the axis of the cylinder. By convention, the origin of time is set to the middle of the cylinder. The quantum degrees of freedom are defined on a circle of circumference L, and we denote by H the Hamiltonian. The corresponding Hilbert space is called $\mathcal{V}$. Given a local observable $\widehat{\mathcal{O}}(x)$, one defines in the usual way the time-dependent operator

$$\widehat{\mathcal{O}}(x,\tau) = e^{\tau H}\widehat{\mathcal{O}}(x)e^{-\tau H}\,, \tag{4.1.1}$$

where we used the fact that $\exp(-\tau H)$ is the evolution operator for a time gap τ.

Remark: Beware that the Hamiltonian H, describing a quantum 1d system, should not be confused with the classical Hamiltonian $\mathcal{H}[S]$ of the 2d classical lattice model described in Chapter 1.

The generator of rotations around the cylinder is denoted P, so that a cyclic translation by x is performed by the operator $\exp(-ixP)$.

The boundary conditions on the edges of the cylinder are encoded into an in-state $|\text{in}\rangle$ and an out-state $|\text{out}\rangle$. With these notations, the partition function is

$$Z = \langle \text{out}|e^{-KH}|\text{in}\rangle\,. \tag{4.1.2}$$

We can express a two-point function $\langle \mathcal{O}(x_1,\tau_1)\mathcal{O}(x_2,\tau_2)\rangle$ as

$$\begin{aligned}\langle \mathcal{O}(x_1,\tau_1)\mathcal{O}(x_2,\tau_2)\rangle_{\text{cyl}} &= \frac{1}{Z}\langle \text{out}|e^{(-K/2+\tau_1)H}\widehat{\mathcal{O}}(x_1)e^{(\tau_2-\tau_1)H}\widehat{\mathcal{O}}(x_2)e^{(-K/2-\tau_2)H}|\text{in}\rangle \\ &= \frac{1}{Z}\langle \text{out}|e^{-KH/2}\widehat{\mathcal{O}}(x_1,\tau_1)\widehat{\mathcal{O}}(x_2,\tau_2)e^{-KH/2}|\text{in}\rangle\,,\end{aligned} \tag{4.1.3}$$

provided that $\tau_1 \geq \tau_2$. Assume that H admits a non-degenerate ground state $|0\rangle$, with energy E_0, and that the overlaps $\langle \text{out}|0\rangle$ and $\langle 0|\text{in}\rangle$ are nonzero. Then, as $K \to \infty$ with L fixed, we have

$$\langle \text{out}|e^{-KH/2} \sim \langle \text{out}|0\rangle\langle 0|\, e^{-KE_0/2}\,, \qquad e^{-KH/2}|\text{in}\rangle \sim e^{-KE_0/2}\,|0\rangle\langle 0|\text{in}\rangle\,, \tag{4.1.4}$$

where we have assumed that the ground state is normalised:

$$\langle 0|0\rangle = 1\,. \tag{4.1.5}$$

Hence, we get

$$Z \propto \exp(-KE_0)\,, \qquad \langle \mathcal{O}(x_1,\tau_1)\mathcal{O}(x_2,\tau_2)\rangle_{\text{cyl}} \to \langle 0|\widehat{\mathcal{O}}(x_1,\tau_1)\widehat{\mathcal{O}}(x_2,\tau_2)|0\rangle \tag{4.1.6}$$

This easily generalises to

$$\langle \mathcal{O}_1(x_1,\tau_1)\dots\mathcal{O}_1(x_n,\tau_n)\rangle_{\text{cyl}} = \langle 0|\widehat{\mathcal{O}}_1(x_1,\tau_1)\dots\widehat{\mathcal{O}}_n(x_n,\tau_n)|0\rangle\,, \tag{4.1.7}$$

on the infinite cylinder, provided the operators are time-ordered, *i.e.* $\tau_1 \geq \dots \geq \tau_n$.

4.1.2 Radial quantisation

Consider the conformal transformation from the plane to the cylinder:

$$z \mapsto w = \frac{L}{2\pi}\log z \tag{4.1.8}$$

If the point at infinity is included, we get a mapping from the compactified complex plane, also called the Riemann sphere, to the infinite cylinder. The points at $z = 0$ and $z = \infty$ are mapped to $w = -\infty$ and $w = +\infty$, respectively. Under the mapping $z \mapsto w$, a primary operator $\phi_{h,\bar{h}}$ transforms as

$$\phi_{h,\bar{h}}(z,\bar{z}) \to (2\pi z/L)^{-h}\,(2\pi\bar{z}/L)^{-\bar{h}}\,\phi_{h,\bar{h}}(w,\bar{w})\,. \tag{4.1.9}$$

Accordingly, we define the operator $\widehat{\phi}_{h,\bar{h}}(z,\bar{z})$ acting on the Hilbert space $\mathcal{V}$, as

$$\widehat{\phi}_{h,\bar{h}}(z,\bar{z}) := (2\pi z/L)^{-h}\,(2\pi\bar{z}/L)^{-\bar{h}}\,\widehat{\phi}_{h,\bar{h}}(w,\bar{w})\,. \tag{4.1.10}$$

A correlation function of primary operators on the plane is then given by

$$\boxed{\langle \phi_1(z_1,\bar{z}_1)\dots\phi_n(z_n,\bar{z}_n)\rangle_{\mathbb{C}} = \langle 0|\widehat{\phi}_1(z_1,\bar{z}_1)\dots\widehat{\phi}_n(z_n,\bar{z}_n)|0\rangle\,,} \tag{4.1.11}$$

if $|z_1| \geq \dots \geq |z_n|$.

We also know how the stress-energy tensor transforms under the map $w \mapsto z$:

$$T(z) \to z^{-2}\left[\left(\frac{L}{2\pi}\right)^2 T(w) + \frac{c}{24}\right], \qquad \bar{T}(\bar{z}) \to \bar{z}^{-2}\left[\left(\frac{L}{2\pi}\right)^2 \bar{T}(\bar{w}) + \frac{c}{24}\right], \tag{4.1.12}$$

which allows us to promote $T(z)$ and $\bar{T}(z)$ to linear operators acting on $\mathcal{V}$, like we have done with primary operators:

$$\widehat{T}(z) := z^{-2}\left[\left(\frac{L}{2\pi}\right)^2 \widehat{T}(w) + \frac{c}{24}\mathbf{1}\right], \qquad \widehat{\bar{T}}(\bar{z}) := \bar{z}^{-2}\left[\left(\frac{L}{2\pi}\right)^2 \widehat{\bar{T}}(\bar{w}) + \frac{c}{24}\mathbf{1}\right]. \tag{4.1.13}$$

As a result, we have, for instance,

$$\boxed{\langle \phi_1(z_1,\bar{z}_1)\dots\phi_n(z_n,\bar{z}_n)T(z)\rangle_{\mathbb{C}} = \langle 0|\widehat{\phi}_1(z_1,\bar{z}_1)\dots\widehat{\phi}_n(z_n,\bar{z}_n)\widehat{T}(z)|0\rangle\,,} \tag{4.1.14}$$

if $|z_1| \geq \dots \geq |z_n| \geq |z|$. More generally, the above relations can be extended to any set of scaling operators $\mathcal{O}_j(z_j,\bar{z}_j)$, whenever we know how they transform under conformal maps.

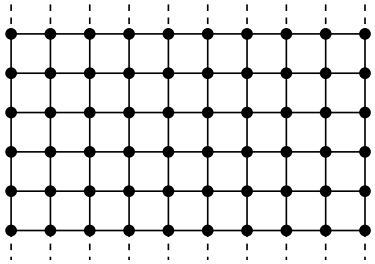


Figure 4.2: Square lattice on the cylinder, with $N = 6$ sites on the circumference, and $M = 10$ sites along the axis. The transfer matrix $\mathbb{T}_N$ acts in the horizontal direction.

4.1.3 The transfer matrix

What guarantees the existence of a non-degenerate ground state for the quantum Hamiltonian H ? For a CFT describing the scaling limit of a classical 2d model with local interaction, the justification comes from a simple analysis of the transfer matrix, *i.e.* the discrete imaginary-time evolution operator.

To illustrate the notion of transfer matrix, let us consider a spin model on a square lattice with periodic boundary conditions in the vertical direction. Each vertex carries a spin variable S_i, and the energy of a spin configuration is given by

$$\mathcal{H}[S] = \sum_{\langle ij \rangle} \varepsilon(S_i, S_j)\,, \tag{4.1.15}$$

where $\langle ij \rangle$ denotes a pair of adjacent sites. Let N be the number of sites in the vertical direction, and a the lattice spacing, so that the circumference of the cylinder is $L = aN$. For any pair of column spin configurations $S = (S_1, \dots, S_N)$ and $S = (S'_1, \dots, S'_N)$, we define the matrix element

$$(\mathbb{T}_N)_{S'S} = \prod_{n=1}^{N} \exp\left[-\frac{1}{2}\varepsilon(S_n, S_{n+1}) - \frac{1}{2}\varepsilon(S'_n, S'_{n+1}) - \varepsilon(S_n, S'_n) \right], \tag{4.1.16}$$

where $S_{N+1} := S_1$ and $S'_{N+1} := S'_1$. As a linear operator, the matrix $\mathbb{T}_N$ acts on *configuration vectors*. The linear space for these vectors is called the *quantum space* V_N. If each individual spin S_j can take q values, then $\dim V_N = q^N$.

The operator $\mathbb{T}_N$ plays the role of a discrete evolution operator. For instance, for any pair of column configurations α, β, the quantity

$$Z_{\alpha\beta} = \langle \alpha | (\mathbb{T}_N)^M | \beta \rangle \tag{4.1.17}$$

gives the partition function of the lattice model on a cylinder of size $M \times N$, with fixed boundary conditions given by α, β. Similarly, one may define time-dependent lattice operators acting on V_N through the relation

$$\widehat{\mathcal{O}}(n, m) = \mathbb{T}_N^{-m}\, \widehat{\mathcal{O}}(n)\, \mathbb{T}_N^{m} \tag{4.1.18}$$

The rotation operator by one lattice step is denoted Ω_N.

The physics of the model on the infinite cylinder is governed by the dominant eigenvalues of $\mathbb{T}_N$, *i.e.* those with maximal modulus. From (4.1.16), we see that $\mathbb{T}_N$ is a real

symmetric matrix with positive elements. Hence, the *Perron-Froebenius theorem* states that the dominant eigenvalue Λ_0 of $\mathbb{T}_N$ is positive and non-degenerate, and that the corresponding eigenvector $|v_0\rangle$ can be written with positive components.

Assume that the Boltzmann weights are invariant under a given transformation of the spin variables (for instance, the spin flip $S_j \to -S_j$ in the Ising model). Let G be the corresponding symmetry group. When acting on the quantum space, this group is represented by permutation matrices, which commute with the transfer matrix

$$\forall g \in G\,, \qquad [g, \mathbb{T}_N] = 0\,. \tag{4.1.19}$$

Since the dominant eigenvalue Λ_0 is non-degenerate, we see that $g|v_0\rangle$ must be a multiple of $|v_0\rangle$, and hence $|v_0\rangle$ is an eigenvector of g. Since g is unitary, the eigenvalue is of the form $e^{i\lambda}$ with real λ. The eigenvalue equation then reads

$$\sum_{k=1}^{\dim V_N} g_{jk}\, v_{0,k} = e^{i\lambda}\, v_{0,j}\,, \tag{4.1.20}$$

where the matrix elements $g_{jk} \in \{0, 1\}$, and the vector components $v_{0,j} > 0$. Hence, both sides of the eigenvalue equation are real and positive, and so the eigenvalue must be one. As a result, we obtain that the Perron-Froebenius vector is invariant under the full symmetry group:

$$\forall g \in G\,, \qquad g|v_0\rangle = |v_0\rangle\,. \tag{4.1.21}$$

Using a similar argument, we can show that the Perron-Froebenius vector is also invariant under a rotation around the cylinder:

$$\Omega|v_0\rangle = |v_0\rangle\,. \tag{4.1.22}$$

In the scaling limit, when $N \to \infty$ and $a \to 0$ with $L = Na$ fixed, the quantity

$$f_0 = \lim_{N\to\infty} \left(-\frac{1}{N} \log \Lambda_0\right) \tag{4.1.23}$$

is assumed to be finite, and gives the density of free energy per lattice site. Indeed, the free energy on a cylinder of size $K \times N$ is

$$F \sim -\log(\Lambda_0^K) \sim NK f_0\,. \tag{4.1.24}$$

We assume that the transfer matrix and the rotation operator take the form

$$\mathbb{T}_N \sim \exp(-aH)\,, \qquad \Omega \sim \exp(-iaP)\,. \tag{4.1.25}$$

The dominant eigenvalues of $\mathbb{T}_N$ thus correspond to the low-lying energies of H. The quantum space V_N becomes infinite-dimensional in the scaling limit. We consider the Hilbert space of the CFT as generated by the Perron-Froebenius vector $|v_0\rangle$, together with the subset of eigenstates of $\mathbb{T}_N$ whose eigenvalues scale as

$$-\log \frac{\Lambda_j}{\Lambda_0} \sim \frac{\mu_j}{N}\,, \tag{4.1.26}$$

where the μ_j's are positive constants. This can be quickly justified in the following way. The free energy on the $K \times N$ torus is given by

$$\begin{aligned} F_{\text{torus}} &= -\log \operatorname{Tr}(\mathbb{T}_N)^K = -\log\left(\sum_j \Lambda_j^K\right) \\ &= -\log(\Lambda_0^K) - \log\left[\sum_j (\Lambda_j/\Lambda_0)^K\right] \\ &\sim NKf_0 - \log\left[\sum_j \exp(-\mu_j K/N)\right], \end{aligned} \tag{4.1.27}$$

Hence (4.1.26) is the only scaling consistent with the expected form of the free energy of a scale-invariant system, namely the sum of an extensive term and a term depending only on the aspect ratio K/N of the torus.

The direct consequence of the above results are:

- CFTs describing the scaling limit of a critical 2d model possess a non-degenerate ground state $|0\rangle$, which can be viewed as the "limit" of the Perron-Froebenius vector $|v_0\rangle$. The corresponding left eigenvector $\langle 0|$ has an analogous relation to the conjugate Perron-Froebenius vector $\langle v_0|$, because the transfer matrix is real and symmetric.
- The ground state $|0\rangle$ has energy E_0, and momentum zero: $P|0\rangle = 0$.
- When the discrete model admits a symmetry under reparameterisation of spin variables (like $S_j \to -S_j$ in Ising), then $|0\rangle$ is invariant under this symmetry.
- All excited states of the CFT Hamiltonian H on a periodic system of size L have an energy gap $(E - E_0) \propto 1/L$.

4.1.4 Conserved charges associated to the stress-energy tensor

We will now argue that the Hamiltonian is related to the stress-energy tensor via

$$H = \frac{1}{2\pi}\int_0^L \left(T_{\text{cyl}}(x,\tau) + \overline{T}_{\text{cyl}}(x,\tau)\right) dx\,, \tag{4.1.28}$$

where T_{cyl} is the stress-energy tensor on the cylinder. This means that $T_{\tau\tau} = T + \overline{T}$ is the energy density. This identification follows from the Ward identity associated to time translations. We have for any scaling operator $\widehat{\mathcal{O}}$

$$\partial_\tau \widehat{\mathcal{O}}(x,\tau) = \left[H, \widehat{\mathcal{O}}(x,\tau)\right] \tag{4.1.29}$$

Meanwhile the conformal Ward identity under an infinitesimal conformal transformation $w \to w + \epsilon(w)$ reads

$$\delta_\epsilon \mathcal{O}(x,\tau) = \frac{1}{2i\pi}\oint \epsilon(w) T_{\text{cyl}}(w)\mathcal{O}(x,\tau)dw + \frac{1}{2i\pi}\oint \bar{\epsilon}(\bar{w})\overline{T}_{\text{cyl}}(\bar{w})\mathcal{O}(x,\tau)d\bar{w}\,, \tag{4.1.30}$$

where the contour integrals wind around the position $u = \tau + ix$. These integrals can be written as the difference of two integrals over the circumference of the cylinder, as in Fig. 4.3

$$\frac{1}{2i\pi}\oint \epsilon(w) T_{\text{cyl}}(w)\mathcal{O}(u,\bar{u})dw = \frac{1}{2i\pi}\left(\oint_{C_+} - \oint_{C_-}\right)\epsilon(w) T_{\text{cyl}}(w)\mathcal{O}(u,\bar{u})dw\,, \tag{4.1.31}$$

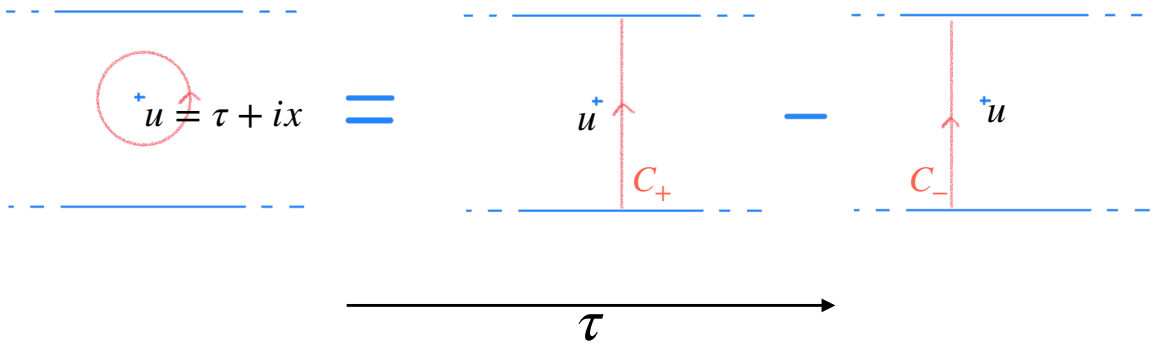


Figure 4.3: From Ward identities to commutators on the cylinder.

where C_+ (resp. C_-) is a circumference placed on the right (resp. left) of $O(u, \bar{u})$. Therefore the interpretation of the Ward identity Eq. 4.1.30 in Hamiltonian formalism is really that of a commutator :

$$\delta_\epsilon \widehat{\mathcal{O}}(u, \bar{u}) = [Q_\epsilon, \widehat{\mathcal{O}}(u, \bar{u})] \tag{4.1.32}$$

where

$$Q_\epsilon = \frac{1}{2i\pi} \oint \epsilon(w) T_{\text{cyl}}(w) dw + \frac{1}{2i\pi} \oint \bar{\epsilon}(\bar{w}) \bar{T}_{\text{cyl}}(\bar{w}) d\bar{w} \,, \tag{4.1.33}$$

$$= \frac{1}{2\pi} \int_0^L \left(\epsilon(x,\tau) T_{\text{cyl}}(x,\tau) + \bar{\epsilon}(x,\tau) \overline{T}_{\text{cyl}}(x,\tau) \right) dx \,. \tag{4.1.34}$$

Observe that the charge $Q_\epsilon(\tau) = Q_\epsilon$ is a conserved charge, in the sense that it does not depend on time. This follows from the fact that the integration contour can be shifted around freely, and ultimately boils down to the fact that both T and ϵ are holomorphic in $w = \tau + ix$.

A particular but important case of this construction is for ϵ real and constant, for which $\delta\mathcal{O} = \partial_\tau \mathcal{O}$, thus

$$\partial_\tau \widehat{\mathcal{O}} = [H, \widehat{\mathcal{O}}] \tag{4.1.35}$$

with H given by Eq. 4.1.28. Strictly speaking the argument above only allows to determine the Hamiltonian up to an additive constant. This is perhaps not too surprising, as in quantum mechanics energy is a priori only defined up to an arbitrary additive shift.

▸ **Exercise :** Show that the total momentum P, as characterized by $\partial_x \widehat{\mathcal{O}} = i[P, \widehat{\mathcal{O}}]$, is given by

$$P = \frac{1}{2\pi} \int_0^L \big(T_{\rm cyl}(x,t) - \overline{T}_{\rm cyl}(x,t) \big)\, dx \,. \tag{4.1.36}$$

4.2 The Virasoro algebra

4.2.1 Laurent expansion of the stress-energy tensor

Let us first describe the Fourier modes of the stress energy tensor on the cyliner. We use the coordinates on the cylinder

$$w = \tau + ix \,, \qquad \bar{w} = \tau - ix \,. \tag{4.2.1}$$

By invariance under translations around the cylinder, we have the periodicity

$$T_{\rm cyl}(x,\tau) = T_{\rm cyl}(x+L,\tau) \,, \qquad \bar{T}_{\rm cyl}(x,\tau) = \bar{T}_{\rm cyl}(x+L,\tau) \,. \tag{4.2.2}$$

Moreover, we have

$$\partial_{\bar{w}} T_{\rm cyl} = 0 \,, \qquad \partial_w \bar{T}_{\rm cyl} = 0 \,. \tag{4.2.3}$$

Hence, one easily shows that $T_{\rm cyl}$ and $\bar{T}_{\rm cyl}$ admit the decomposition

$$T_{\rm cyl}(w) = \sum_{n\in\mathbb{Z}} e^{-2\pi n w/L}\, q_n \,, \qquad \bar{T}_{\rm cyl}(\bar{w}) = \sum_{n\in\mathbb{Z}} e^{-2\pi n \bar{w}/L}\, \bar{q}_n \,, \tag{4.2.4}$$

where the Fourier coefficients are given by

$$q_n = \frac{1}{L} \int_0^L dx\, e^{2i\pi n x/L}\, T_{\rm cyl}(x,0) \,, \qquad \bar{q}_n = \frac{1}{L} \int_0^L dx\, e^{-2i\pi n x/L}\, \bar{T}_{\rm cyl}(x,0) \,. \tag{4.2.5}$$

We can now map the cylinder to the plane, through the conformal transformation

$$w \mapsto z = \exp(2\pi w/L) \,. \tag{4.2.6}$$

Under this mapping, the stress-energy tensor becomes

$$T(z) = z^{-2} \left[\left(\frac{L}{2\pi} \right)^2 T_{\rm cyl}(w) + \frac{c}{24} \right] , \qquad \bar{T}(\bar{z}) = \bar{z}^{-2} \left[\left(\frac{L}{2\pi} \right)^2 \bar{T}_{\rm cyl}(\bar{w}) + \frac{c}{24} \right] . \tag{4.2.7}$$

We can define

$$L_n = \left(\frac{L}{2\pi} \right)^2 q_n + \frac{c}{24} \delta_{n,0} \,, \qquad \bar{L}_n = \left(\frac{L}{2\pi} \right)^2 \bar{q}_n + \frac{c}{24} \delta_{n,0} \,, \tag{4.2.8}$$

so that the Fourier expansions of $T_{\rm cyl}$ and $\bar{T}_{\rm cyl}$ become Laurent expansions for T and $\bar{T}$:

$$\boxed{ T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}\, L_n \,, \qquad \bar{T}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{z}^{-n-2}\, \bar{L}_n \,. } \tag{4.2.9}$$

Conversely, the Laurent modes can be recovered as

$$\boxed{ L_n = \frac{1}{2i\pi} \oint dz\, z^{n+1}\, T(z) \,, \qquad \bar{L}_n = \frac{1}{2i\pi} \oint d\bar{z}\, \bar{z}^{n+1}\, \bar{T}(\bar{z}) \,, } \tag{4.2.10}$$

where the contours encircle the origin. Like for the primary operators, the currents $T, \bar{T}$ and the modes $L_n, \bar{L}_n$ can be promoted to operators acting on the quantum space $\mathcal{V}$. For instance, one defines

$$\widehat{L}_n = \left(\frac{L}{2\pi}\right)^2 \widehat{q}_n + \frac{c}{24}\delta_{n,0}\mathbf{1}\,, \qquad \widehat{\bar{L}}_n = \left(\frac{L}{2\pi}\right)^2 \widehat{\bar{q}}_n + \frac{c}{24}\delta_{n,0}\mathbf{1}\,. \tag{4.2.11}$$

In the following, we shall omit the symbol $\widehat{\ }$, in order to lighten the notation.

An important property, deduced directly from the above relations, is the expression of the Hermitian conjugates:

$$\boxed{L_n^\dagger = L_{-n}\,, \qquad \bar{L}_n^\dagger = \bar{L}_{-n}\,.} \tag{4.2.12}$$

4.2.2 Commutation relations

Let us now show that the OPEs $T.T$, $\bar{T}.\bar{T}$ and $T.\bar{T}$ determine the commutation relations:

$$\boxed{\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^2-1)m\delta_{m+n,0}\,, \\ [\bar{L}_m, \bar{L}_n] &= (m-n)\bar{L}_{m+n} + \frac{c}{12}(m^2-1)m\delta_{m+n,0}\,, \\ [L_m, \bar{L}_n] &= 0\,. \end{aligned}} \tag{4.2.13}$$

The modes L_n and $\bar{L}_n$ generate two independent Virasoro algebras Vir and $\overline{\text{Vir}}$.

We recall the OPE

$$T(y)T(z) = \frac{c/2}{(y-z)^4} + \frac{2T(z)}{(y-z)^2} + \frac{\partial T(z)}{y-z} + \text{reg}_{y\to z}\,. \tag{4.2.14}$$

Consider first the commutator

$$\begin{aligned} [L_m, T(z)] &= \frac{1}{2i\pi}\left(\oint_{C_{0,z}} - \oint_{C_0}\right) dy\, y^{m+1}\, T(y)T(z) \\ &= \frac{1}{2i\pi}\oint_{C_z} dy\, y^{m+1}\, T(y)T(z) \qquad (4.2.15) \\ &= \frac{1}{2i\pi}\oint_{C_z} dy\, y^{m+1}\left[\frac{c/2}{(y-z)^4} + \frac{2T(z)}{(y-z)^2} + \frac{\partial T(z)}{y-z}\right] \\ &= \frac{c}{2\times 3!}(m+1)m(m-1)z^{m-2} + 2(m+1)z^m\, T(z) + z^{m+1}\,\partial T(z)\,. \qquad (4.2.16) \end{aligned}$$

The crucial point is that only the singular terms of the OPE contribute in the contour integral. Integrating over z, we get

$$\begin{aligned} [L_m, L_n] &= \frac{1}{2i\pi}\oint_{C_0} dz\left[\frac{c}{12}(m^2-1)m\, z^{n+m-1} + 2mz^{m+n+1}\, T(z) + z^{m+n+2}\,\partial T(z)\right] \\ &= \frac{c}{12}(m^2-1)m\,\delta_{m+n,0} + 2(m+1)L_{m+n} - (m+n+2)L_{m+n} \\ &= (m-n)L_{m+n} + \frac{c}{12}(m^2-1)m\delta_{m+n,0}\,, \qquad (4.2.17) \end{aligned}$$

where we have performed an integration by parts on the last term.

The OPEs of T and $\bar{T}$ with a primary operator ϕ of conformal dimensions are

$$T(y)\phi(z,\bar{z}) = \frac{h\,\phi(z)}{(y-z)^2} + \frac{\partial\phi(z)}{y-z} + \mathrm{reg}_{y\to z}\,, \tag{4.2.18}$$

$$\bar{T}(\bar{y})\phi(z,\bar{z}) = \frac{\bar{h}\,\phi(z,\bar{z})}{(\bar{y}-\bar{z})^2} + \frac{\bar{\partial}\phi(z,\bar{z})}{\bar{y}-\bar{z}} + \mathrm{reg}_{y\to z}\,. \tag{4.2.19}$$

They yield the commutation relations:

$$\boxed{\begin{aligned}[L_n, \widehat{\phi}(z,\bar{z})] &= (n+1)hz^n\,\widehat{\phi}(z,\bar{z}) + z^{n+1}\,\partial\widehat{\phi}(z,\bar{z})\,,\\ [\bar{L}_n, \widehat{\phi}(z,\bar{z})] &= (n+1)\bar{h}\bar{z}^n\,\widehat{\phi}(z,\bar{z}) + \bar{z}^{n+1}\,\bar{\partial}\widehat{\phi}(z,\bar{z})\,.\end{aligned}} \tag{4.2.20}$$

4.2.3 Hamiltonian and momentum operators

Recall that from Ward identities on the cylinder, we have obtained

$$H = \frac{1}{2\pi}\int_0^L (T_{\mathrm{cyl}} + \overline{T}_{\mathrm{cyl}})\,dx\,, \qquad P = \frac{1}{2\pi}\int_0^L (T_{\mathrm{cyl}} - \overline{T}_{\mathrm{cyl}})\,dx\,. \tag{4.2.21}$$

In terms of the Virasoro modes, this gives

$$H = \frac{2\pi}{L}\left(L_0 + \bar{L}_0 - \frac{c}{12}\right), \qquad P = \frac{2\pi}{L}\left(L_0 - \bar{L}_0\right). \tag{4.2.22}$$

▸ **Exercise :** What is the charge Q_ϵ for $\epsilon(w) = e^{\frac{2\pi n}{L}w}$ and $\epsilon(w) = ie^{\frac{2\pi n}{L}w}$?

4.3 The quantum states of a CFT

4.3.1 The vacuum state

Consider the insertion of $T(z)$ into a correlation function of scaling operators:

$$\langle \mathcal{O}_1(z_1,\bar{z}_1)\dots\mathcal{O}_n(z_n,\bar{z}_n)T(z)\rangle = \langle 0|\widehat{\mathcal{O}}_1(z_1,\bar{z}_1)\dots\widehat{\mathcal{O}}_n(z_n,\bar{z}_n)\widehat{T}(z)|0\rangle\,, \tag{4.3.1}$$

with $|z_1|\geq\dots\geq|z_n|\geq|z|$. This function is regular as $z\to 0$, since no operator is inserted at the origin: this means that the state

$$\widehat{T}(z)|0\rangle = \sum_{m\in\mathbb{Z}} z^{-m-2}\,L_m|0\rangle \tag{4.3.2}$$

should be regular as $z\to 0$. A similar argument can be made for $\bar{T}|0\rangle$. Hence we get

$$\boxed{\begin{aligned}&\forall m\geq -1\,, \qquad L_m|0\rangle = \bar{L}_m|0\rangle = 0\,,\\ &\forall m\leq 1\,, \qquad \langle 0|L_m = \langle 0|\bar{L}_m = 0\,.\end{aligned}} \tag{4.3.3}$$

To obtain the second relation, we have used the fact that $L_m^\dagger = L_{-m}$ and $\bar{L}_m^\dagger = \bar{L}_{-m}$.

4.3.2 Primary states

Reasoning as we did above, on the correlation function

$$\langle \mathcal{O}_1(z_1,\bar{z}_1)\dots\mathcal{O}_n(z_n,\bar{z}_n)\phi(z,\bar{z})\rangle\,, \tag{4.3.4}$$

where $\widehat{\phi}(z,\bar{z})$ is a primary operator of conformal dimensions $(h,\bar{h})$, we see that the state $\widehat{\phi}(z,\bar{z})|0\rangle$ should be regular as $z\to 0$. We thus define:

$$|\phi\rangle := \lim_{z\to 0}\phi(z,\bar{z})|0\rangle\,, \tag{4.3.5}$$

and call it the primary state associated to $\phi(z,\bar{z})$. From this definition, we have the relation

$$\langle \mathcal{O}_1(z_1,\bar{z}_1)\dots\mathcal{O}_n(z_n,\bar{z}_n)\phi(0)\rangle = \langle 0|\widehat{\mathcal{O}}_1(z_1,\bar{z}_1)\dots\widehat{\mathcal{O}}_n(z_n,\bar{z}_n)|\phi\rangle\,. \tag{4.3.6}$$

Using the commutators $[L_m,\phi(z,\bar{z})]$ and $[\bar{L}_m,\phi(z,\bar{z})]$, and the properties of the ground state, we get

$$\boxed{\begin{aligned} L_0|\phi\rangle &= h\,|\phi\rangle\,, \qquad & L_{m>0}|\phi\rangle &= 0\,,\\ \bar{L}_0|\phi\rangle &= \bar{h}\,|\phi\rangle\,, \qquad & \bar{L}_{m>0}|\phi\rangle &= 0\,. \end{aligned}} \tag{4.3.7}$$

The left state $\langle\phi|$ will correspond to the insertion of $\phi(z,\bar{z})$ at infinity. We thus consider the correlation function

$$\langle \phi(z,\bar{z})\mathcal{O}_1(z_1,\bar{z}_1)\dots\mathcal{O}_n(z_n,\bar{z}_n)\rangle\,, \tag{4.3.8}$$

with $|z|\geq|z_1|\geq\dots\geq|z_n|$. We can use the inversion map $z\mapsto 1/z$. Under this map, the primary operator transforms as

$$\phi(z,\bar{z})\to z^{-2h}\,\bar{z}^{-2\bar{h}}\,\phi(1/z,1/\bar{z})\,, \tag{4.3.9}$$

and hence

$$\langle \phi(z,\bar{z})\mathcal{O}_1(z_1,\bar{z}_1)\dots\mathcal{O}_n(z_n,\bar{z}_n)\rangle = z^{-2h}\,\bar{z}^{-2\bar{h}}\,\langle \widetilde{\mathcal{O}}_n(1/z_n,1/\bar{z}_n)\dots\widetilde{\mathcal{O}}_1(1/z_1,1/\bar{z}_1)\phi(1/z,1/\bar{z})\rangle\,. \tag{4.3.10}$$

Here we do not specify the transformation $\mathcal{O}_j(z_j,\bar{z}_j)\to\widetilde{\mathcal{O}}_j(1/z_j,1/\bar{z}_j)$ for generic scaling operators, but the only important property is that it is independent of $z,\bar{z}$. This leads to the definition:

$$\langle \phi(\infty)\mathcal{O}_1(z_1,\bar{z}_1)\dots\mathcal{O}_n(z_n,\bar{z}_n)\rangle := \lim_{z\to\infty}\left[z^{2h}\,\bar{z}^{2\bar{h}}\,\langle \phi(z,\bar{z})\mathcal{O}_1(z_1,\bar{z}_1)\dots\mathcal{O}_n(z_n,\bar{z}_n)\rangle\right]. \tag{4.3.11}$$

Accordingly, we define the left primary state as:

$$\boxed{\langle\phi| := \lim_{z\to\infty} z^{2h}\,\bar{z}^{2\bar{h}}\langle 0|\widehat{\phi}(z,\bar{z})\,,} \tag{4.3.12}$$

so that we get the relation

$$\langle \phi(\infty)\mathcal{O}_1(z_1,\bar{z}_1)\dots\mathcal{O}_n(z_n,\bar{z}_n)\rangle = \langle\phi|\widehat{\mathcal{O}}_1(z_1,\bar{z}_1)\dots\widehat{\mathcal{O}}_n(z_n,\bar{z}_n)|0\rangle\,. \tag{4.3.13}$$

We also have the properties:

$$\boxed{\begin{aligned} \langle\phi|L_0 &= h\,\langle\phi|\,, \qquad & \langle\phi|L_{m<0} &= 0\,, \\ \langle\phi|\bar{L}_0 &= \bar{h}\,\langle\phi|\,, \qquad & \langle\phi|\bar{L}_{m<0} &= 0\,. \end{aligned}} \tag{4.3.14}$$

Remark 1. Note that the ground state is also a primary state – it is associated to the trivial identity operator $\phi(z,\bar{z}) = \mathbf{1}$, with conformal dimensions $h = \bar{h} = 0$.

Remark 2. Since the primary states are eigenvectors of the self-adjoint operators L_0 and $\bar{L}_0$, we have $\langle\phi_1|\phi_2\rangle = 0$ if $h_1 \neq h_2$ or $\bar{h}_1 \neq \bar{h}_2$. Also, from the above definitions, we easily get $\langle\phi|\phi\rangle = 1$. Hence, if all primary operators in the CFT have distinct dimensions, we recover the previous result on two-point functions

$$\langle\phi_1(\infty)\phi_2(0)\rangle = \langle\phi_1|\phi_2\rangle = \begin{cases} 1 & \text{if } (h_1,\bar{h}_1) = (h_2,\bar{h}_2)\,, \\ 0 & \text{otherwise.} \end{cases} \tag{4.3.15}$$

4.3.3 Descendant states

By acting with modes $L_{-m}, \bar{L}_{-\bar{m}}$ with $m, \bar{m} > 0$ on a primary state $|\phi\rangle$, one can create an infinity of non-zero descendant states: $L_{-1}|\phi\rangle, L_{-2}|\phi\rangle, L_{-1}^2|\phi\rangle, , L_{-2}\bar{L}_{-1}|\phi\rangle \ldots$ Using the commutation relations, a given descendant $L_{n_1}\ldots L_{n_\ell}|\phi\rangle$, with $n_j \in \mathbb{Z}$, can always be written as a linear combination of descendants of the form $L_{-m_1}\ldots L_{-m_k}|\phi\rangle$, with $m_1 \geq \cdots \geq m_k \geq 1$. Of course, this can be combined with descendants under the anti-holomorphic algebra $\overline{\text{Vir}}$. Hence we consider the generating set of descendants

$$|\phi^{[m,\bar{m}]}\rangle := L_{-m_1}\ldots L_{-m_k}\cdot L_{-\bar{m}_1}\ldots L_{-\bar{m}_{\bar{k}}}\cdot|\phi\rangle\,, \tag{4.3.16}$$

with $m_1 \geq \cdots \geq m_k \geq 1$ and $\bar{m}_1 \geq \cdots \geq \bar{m}_{\bar{k}} \geq 1$. The numbers

$$M = |m| = m_1 + \cdots + m_k\,, \qquad \bar{M} = |\bar{m}| = \bar{m}_1 + \cdots + \bar{m}_{\bar{k}} \tag{4.3.17}$$

are called the levels of the descendant state $|\phi^{[m,\bar{m}]}\rangle$. Using the Virasoro commutation relations, we get

$$L_0|\phi^{[m,\bar{m}]}\rangle = (h+M)\,|\phi^{[m,\bar{m}]}\rangle\,, \qquad \bar{L}_0|\phi^{[m,\bar{m}]}\rangle = (\bar{h}+\bar{M})\,|\phi^{[m,\bar{m}]}\rangle\,. \tag{4.3.18}$$

Hence, within the space of descendants of a given primary state, the levels label the eigenspaces of L_0 and $\bar{L}_0$.

Remark. The states $\{|\phi^{[m,\bar{m}]}\rangle\}$ generate the space of descendants of $|\phi\rangle$, but they are not always linearly independent. If a finite set of $|\phi^{[m,\bar{m}]}\rangle$ become linearly related, we say that ϕ is degenerate: this case will be studied in detail, later in this course.

4.3.4 The full Hilbert space

Recall the expression of the Hamiltonian operator, which governs the imaginary-time evolution on the cylinder:

$$H = \frac{2\pi}{L}\left(L_0 + \bar{L}_0 - \frac{c}{12}\right) \tag{4.3.19}$$

From the transfer matrix point of view, we expect that the CFT Hilbert space contains the primary states (among which the ground state $|0\rangle$ minimising the eigenvalue of H), and their descendants. Moreover, all these states should have an energy gap of order $1/L$. We also assume that the primary states form an orthonormal family. A simple way to construct a Hilbert space $\mathcal{V}$ with these properties, is to assume that this space is generated by the action of the algebra $\mathrm{Vir}\otimes\overline{\mathrm{Vir}}$ on a certain set of primary states $\{|\phi_j\rangle\}$ with distinct conformal dimensions.

If the set of primary operators is finite, we say that the CFT is rational. This condition entails strong constraints on the central charge and the possible conformal dimensions for the primary operators, which are satisfied for a discrete, infinite series of CFTs called the minimal models.

Some CFTs have an infinite (discrete or continuous) set of primary operators: they are called non-rational CFTs.

4.4 Representations of the Virasoro algebra

In this section, we present the basics of the representation theory of the Virasoro algebra. As we have just seen, the Hilbert space $\mathcal{V}$ of a CFT is a representation of the product of two commuting Virasoro algebras $\mathrm{Vir}\otimes\overline{\mathrm{Vir}}$, corresponding to the Laurent modes of the holomorphic and anti-holomorphic components of the stress-energy tensor. Here, we shall describe the representation theory of a single Virasoro algebra.

4.4.1 Verma modules

Condider a *lowest weight* state $|h\rangle$ such that

$$L_0|h\rangle = h|h\rangle\,, \qquad L_{n>0}|h\rangle = 0\,. \tag{4.4.1}$$

The representation generated by all the descendants of $|h\rangle$ is called the Verma module $\mathcal{V}_h$. As we have done for primary states $|\phi_{h,\bar{h}}\rangle$, we can define a generating set of the Verma module as

$$|h^{[m]}\rangle = L_{-m_1}\dots L_{-m_k}|h\rangle\,, \qquad m_1\geq\dots\geq m_k\geq 1\,. \tag{4.4.2}$$

Organising states by level, we get

$$\begin{array}{lllllll}
(L_0=h) & |h\rangle & & & & & \\
(L_0=h+1) & L_{-1}|h\rangle & & & & & \\
(L_0=h+2) & L_{-2}|h\rangle & L_{-1}^2|h\rangle & & & & \\
(L_0=h+3) & L_{-3}|h\rangle & L_{-2}L_{-1}|h\rangle & L_{-1}^3|h\rangle & & & \\
(L_0=h+4) & L_{-4}|h\rangle & L_{-3}L_{-1}|h\rangle & L_{-2}^2|h\rangle & L_{-2}L_{-1}^2|h\rangle & L_{-1}^4|h\rangle & \\
\dots & & & & & &
\end{array} \tag{4.4.3}$$

The number of independent states at a given level M is equal to $P(M)$, the number of partitions of the integer M into a sum of positive integers [by convention, $P(0)=1$].

The Virasoro character of the Verma module is the function defined as

$$\chi_h(\tau) := \mathrm{Tr}_{\mathcal{V}_h}\, q^{L_0-c/24}\,, \qquad q=\exp(2i\pi\tau)\,. \tag{4.4.4}$$

Using the Dedekind eta function

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty}(1-q^n) = \frac{q^{1/24}}{\sum_{M=0}^{\infty} P(M) q^M}\,, \tag{4.4.5}$$

we obtain the expression

$$\boxed{\chi_h(\tau) = \frac{q^{h+(1-c)/24}}{\eta(\tau)}\,.} \tag{4.4.6}$$

At this stage, this relation is just a compact way of encoding the degeneracies of L_0 in the module $\mathcal{V}_h$.

In the Verma module, we want to define a "scalar product" (more precisely, a bilinear form), which will be used for computing correlation functions. For compatibility with the scalar products between physical states, it is necessary to impose, in $\mathcal{V}_h$

$$\langle h|h\rangle \coloneqq 1\,, \qquad L_n^\dagger \coloneqq L_{-n}\,. \tag{4.4.7}$$

It is easy to realise that these relations actually determine completely the bilinear form on $\mathcal{V}_h$. We have the simple property:

$$\text{if} \quad \sum_j m_j \neq \sum_j m'_j\,, \qquad \text{then} \quad \langle h^{[m]}|h^{[m']}\rangle = 0\,, \tag{4.4.8}$$

namely, the levels are mutually orthogonal, due to the fact that L_0 is self-adjoint. However, the states $|h^{[m]}\rangle$ on a given level are not orthogonal to each other.

4.4.2 Degenerate Verma modules

For generic values of (h,c), a Verma module $\mathcal{V}_h$ is irreducible. However, for some special values of (h,c), $\mathcal{V}_h$ may become *reducible*, *i.e.* it may admit a non-trivial subspace $\mathcal{W}$ which is stable under the action of the L_n's. We then say that the module $\mathcal{V}_h$ (or the state $|h\rangle$) is degenerate. Let us quickly examine the consequences of reducibility, and then find for which values of (h,c) this may happen.

Suppose $\mathcal{V}_h$ is reducible: then, we call $|\chi\rangle$ a state with the minimal eigenvalue[1] for L_0 in the non-trivial submodule $\mathcal{W}$. We then have $L_0|\chi\rangle = (h+N)\,|\chi\rangle$ for some integer level $N>0$. Thus, the state $|\chi\rangle$ is of the form:

$$|\chi\rangle = (\#L_{-N} + \cdots + \#L_{-1}^N)|h\rangle\,. \tag{4.4.9}$$

For any $m>0$, we have $L_0 \cdot L_m|\chi\rangle = (h+N-m)L_m|\chi\rangle$, and hence $L_m|\chi\rangle$ must vanish (otherwise $h+N$ would not be the minimal eigenvalue for L_0 in $\mathcal{W}$). We say that $|h\rangle$ is degenerate at level N. We thus have

$$\forall m>0\,, \qquad L_m|\chi\rangle = 0\,. \tag{4.4.10}$$

This means that the descendant state $|\chi\rangle$ is itself a lowest-weight state. The action of the Virasoro algebra on $|\chi\rangle$ thus generates a submodule $\mathcal{W}_\chi$, isomorphic to $\mathcal{V}_{h+N}$. We may

[1] Throughout this argument, we assume that L_0 is diagonalisble in $\mathcal{V}_h$.

define the quotient space $\mathcal{V}'_h = \mathcal{V}_h/\mathcal{W}_\chi$. If $\mathcal{V}'_h$ is still reducible, then we need to identify a new lowest-weight vector $|\chi'\rangle$ at level $N' \geq N$, and repeat the quotient procedure, to obtain a space $\mathcal{V}''_h$, and so on. Once the subspaces generated by lowest-weight vectors (except $|h\rangle$) are quotiented out, one gets an irreducible module, called the Kac module $\mathcal{K}_h$.

The state $|\chi\rangle$ and all its descendants are *null-states*, namely:

$$\boxed{\forall |v\rangle \in \mathcal{V}_h\,, \quad \forall m_1 \geq \cdots \geq m_k \geq 1\,, \qquad \langle v|L_{-m_1}\dots L_{-m_k}|\chi\rangle = 0\,.} \tag{4.4.11}$$

This is proven for any $|v\rangle = |h^{[n]}\rangle$, by writing

$$\langle h^{[n]}|\chi^{[m]}\rangle = \langle h|L_{n_\ell}\dots L_{n_1}\cdot L_{-m_1}\dots L_{-m_k}|\chi\rangle\,,$$

which, using the commutation rules, can be written as a linear combination of terms of the form

$$\langle h|L_{p_1}\dots L_{p_r}|\chi\rangle\,,$$

with $p_1 \leq \cdots \leq p_r$ and $p_1 + \cdots + p_r = (n_1 + \cdots + n_\ell) - (m_1 + \cdots + m_k)$. Since $\langle h|$ and $|\chi\rangle$ are annihilated by negative and positive modes respectively, the result is zero.

Remark. In particular, $|\chi\rangle$ has norm zero: $\langle\chi|\chi\rangle = 0$.

Remark. The above result shows that all lowest-weight states and their descendants are null states. However, the reciprocal is not true: some null states are not lowest-weight. For instance, $L_{-1}|\chi\rangle$ is a null state, but it is not annihilated by the Virasoro positive modes. Indeed:

$$L_1\cdot L_{-1}|\chi\rangle = (L_{-1}L_1 + 2L_0)|\chi\rangle = 2(h+N)|\chi\rangle \neq 0\,, \qquad \text{if } h \neq -N\,. \tag{4.4.12}$$

Example 1. Let us consider a state $|h\rangle$ degenerate at level $N = 1$. This means that $|\chi\rangle = L_{-1}|h\rangle$ is a primary: $L_{n>0}|\chi\rangle = 0$. For $n > 1$, we always have $L_nL_{-1}|h\rangle = L_{-1}L_n|h\rangle + (n+1)L_{n-1}|h\rangle = 0$. Hence, the only non-trivial condition is

$$L_1L_{-1}|h\rangle = 0 \qquad \Leftrightarrow \qquad (L_{-1}L_1 + 2L_0)|h\rangle = 0 \qquad \Leftrightarrow \qquad 2h|h\rangle = 0\,. \tag{4.4.13}$$

The only lowest-weight state which is degenerate at level $N = 1$ is thus the one with $h = 0$, corresponding to the conformal dimension of the ground state.

Example 2. Let us do the same exercice for level $N = 2$. Writing $|\chi\rangle = (uL_{-2} + vL_{-1}^2)|h\rangle$, and imposing $L_1|\chi\rangle = L_2|\chi\rangle = 0$, we get the linear system

$$\begin{aligned} 3u + (4h+2)v &= 0 \\ \left(4h + \frac{c}{2}\right)u + 6hv &= 0\,. \end{aligned} \tag{4.4.14}$$

The system admits a non-trivial solution if and only if its determinant is zero, which yields the relation:

$$16h^2 + (2c-10)h + c = 0 \qquad \Leftrightarrow \qquad h = \frac{(5-c) \pm \sqrt{(1-c)(25-c)}}{16}\,. \tag{4.4.15}$$

For generic values of the central charge, there are two possible values for the dimension of an operator degenerate at level $N = 2$.

4.4.3 The Kac parametrisation of degenerate dimensions

For higher levels N, it becomes non-trivial to enumerate the degenerate dimensions. Let us state the result, which is known as the Kac parametrisation. We write the central charge

$$c = 1 - 6(b^{-1} - b)^2 \,, \tag{4.4.16}$$

with some nonzero parameter b. Then the degenerate states at level N are of the form $|h_{r,s}\rangle$ with $r, s \in \{1, 2, 3, \dots\}$ and $r \times s = N$, and their dimensions are given by

$$h_{rs} = \frac{(rb^{-1} - sb)^2 - (b^{-1} - b)^2}{4} \,. \tag{4.4.17}$$

In particular, we have $h_{12} = (3b^2 - 2)/4$, $h_{21} = (3b^{-2} - 2)/4$, and the corresponding null vectors are

$$|\chi_{12}\rangle = (L_{-2} - b^{-2}L_{-1}^2)|h_{12}\rangle \,, \qquad |\chi_{21}\rangle = (L_{-2} - b^2 L_{-1}^2)|h_{21}\rangle \,. \tag{4.4.18}$$

Note that the Kac parametrisation of degenerate dimensions holds for any value of the central charge. It tells us that only a finite number of dimensions h (namely, the h_{rs} with $rs = N$) can be degenerate at a given level N.

4.5 Descendant operators

Back to the physical setting, we can view the operators L_n as acting on the space of scaling operators. To do so, let us first remark that, for any scaling operator $\mathcal{O}$, the OPEs $T(z)\mathcal{O}(0)$ and $\bar{T}(\bar{z})\mathcal{O}(0)$ admit Laurent expansions in z and $\bar{z}$, respectively. Indeed, inserting $\mathcal{O}$ at the origin in the plane geometry corresponds to inserting it at $-\infty$ on the cylinder, which does not affect the Laurent expansions of $T(z)$ and $\bar{T}(\bar{z})$. Thus, we can write

$$T(z)\mathcal{O}(0) = \sum_{n\in\mathbb{Z}} z^{-n-2}\,(L_n \cdot \mathcal{O})(0) \,, \qquad (L_n \cdot \mathcal{O})(0) = \frac{1}{2i\pi} \oint dy\, y^{n+1}\, T(y)\mathcal{O}(0) \,, \tag{4.5.1}$$

$$\bar{T}(\bar{z})\mathcal{O}(0) = \sum_{n\in\mathbb{Z}} \bar{z}^{-n-2}\,(\bar{L}_n \cdot \mathcal{O})(0) \,, \qquad (\bar{L}_n \cdot \mathcal{O})(0) = \frac{1}{2i\pi} \oint d\bar{y}\, \bar{y}^{n+1}\, \bar{T}(\bar{y})\mathcal{O}(0) \,, \tag{4.5.2}$$

which defines the descendant operators $L_n \cdot \mathcal{O}$ and $\bar{L}_n \cdot \mathcal{O}$ at the origin. In the above expressions, the contour integrals enclose the origin. By applying a translation, we get the definition of descendant operators at any position on the plane:

$$\boxed{\begin{aligned} (L_n \cdot \mathcal{O})(z,\bar{z}) &:= \frac{1}{2i\pi} \oint dy\, (y-z)^{n+1}\, T(y)\mathcal{O}(z,\bar{z}) \,, \\ (\bar{L}_n \cdot \mathcal{O})(z,\bar{z}) &:= \frac{1}{2i\pi} \oint d\bar{y}\, (\bar{y}-\bar{z})^{n+1}\, \bar{T}(\bar{y})\mathcal{O}(z,\bar{z}) \,, \end{aligned}} \tag{4.5.3}$$

where the contours now enclose the point z. This corresponds to Laurent expansions around the point z:

$$\boxed{\begin{aligned} T(y)\mathcal{O}(z,\bar{z}) &= \sum_{n\in\mathbb{Z}} (y-z)^{-n-2}\,(L_n \cdot \mathcal{O})(z,\bar{z}) \,, \\ \bar{T}(\bar{y})\mathcal{O}(z,\bar{z}) &= \sum_{n\in\mathbb{Z}} (\bar{y}-\bar{z})^{-n-2}\,(\bar{L}_n \cdot \mathcal{O})(z,\bar{z}) \,. \end{aligned}} \tag{4.5.4}$$

In the case of a primary operator $\phi(z,\bar{z})$, the OPEs $T.\phi$ and $\bar{T}.\phi$ are equivalent to the conditions:

$$L_{-1}\cdot\phi = \partial\phi\,, \qquad L_0\cdot\phi = h\,\phi\,, \qquad L_{n>0}\cdot\phi = 0\,, \tag{4.5.5}$$

$$\bar{L}_{-1}\cdot\phi = \bar{\partial}\phi\,, \qquad \bar{L}_0\cdot\phi = \bar{h}\,\phi\,, \qquad \bar{L}_{n>0}\cdot\phi = 0\,. \tag{4.5.6}$$

For a generic scaling operator, recall the conformal Ward identity for an infinitesimal conformal transformation $(z,\bar{z}) \to (z+\epsilon, \bar{z}+\bar{\epsilon})$:

$$\delta_\epsilon \mathcal{O}(z,\bar{z}) = \frac{1}{2i\pi}\oint dy\,\epsilon(y)\,T(y)\mathcal{O}(z,\bar{z}) + \frac{1}{2i\pi}\oint d\bar{y}\,\bar{\epsilon}(\bar{y})\,\bar{T}(\bar{y})\mathcal{O}(z,\bar{z})\,, \tag{4.5.7}$$

where the contours are around the point z. A constant ϵ corresponds to an infinitesimal translation, whereas $\epsilon = \mu\, z$ corresponds to a dilatation if μ is real, and a rotation if μ is imaginary. Comparing with the definitions of $L_n\cdot\mathcal{O}$ and $\bar{L}_n\cdot\mathcal{O}$, we see that

$$L_{-1}\cdot\mathcal{O} = \partial\mathcal{O}\,, \qquad L_0\cdot\mathcal{O} = h\,\mathcal{O}\,, \tag{4.5.8}$$

$$\bar{L}_{-1}\cdot\mathcal{O} = \bar{\partial}\mathcal{O}\,, \qquad \bar{L}_0\cdot\mathcal{O} = \bar{h}\,\mathcal{O}\,. \tag{4.5.9}$$

However, the positive modes $L_{n>0}$ and $\bar{L}_{n>0}$ may act non-trivially on $\mathcal{O}$ if it is not primary.

Let us show the following property:

> For any scaling operator $\mathcal{O}_{h,\bar{h}}$ of conformal dimensions $(h,\bar{h})$, the descendant operators $L_{-n}\cdot\mathcal{O}_{h,\bar{h}}$ and $\bar{L}_{-n}\cdot\mathcal{O}_{h,\bar{h}}$ have conformal dimensions $(h+n,\bar{h})$ and $(h,\bar{h}+n)$, respectively.

Indeed, if we consider a scale transformation $z \to z' = z/\lambda$, with λ real and positive, recall that $\mathcal{O}_{h,\bar{h}}$ and T transform as

$$\mathcal{O}_{h,\bar{h}}(z,\bar{z}) \to \lambda^{-h-\bar{h}}\,\mathcal{O}_{h,\bar{h}}(z/\lambda,\bar{z}/\lambda)\,, \qquad T(y) \to \lambda^{-2}\,T(y/\lambda)\,, \tag{4.5.10}$$

and hence, the descendant operator $L_{-n}\cdot\mathcal{O}$ transforms as

$$(L_{-n}\cdot\mathcal{O}_{h,\bar{h}})(z,\bar{z}) \to \frac{1}{2i\pi}\oint dy\,(y-z)^{-n+1}\,\lambda^{-2-h-\bar{h}}\,T(y')\mathcal{O}_{h,\bar{h}}(z',\bar{z}') = \lambda^{-h-\bar{h}-n}(L_{-n}\cdot\mathcal{O}_{h,\bar{h}})(z',\bar{z}')\,. \tag{4.5.11}$$

Thus, the scaling dimension of $L_{-n}\cdot\mathcal{O}_{h,\bar{h}}$ is $h+\bar{h}+n$. Repeating the argument with the scale transformation replaced by a rotation, we get the conformal spin $h-\bar{h}+n$. These values of the scaling dimension and conformal spin indeed correspond to the conformal dimensions $(h+n,\bar{h})$ for $L_{-n}\cdot\mathcal{O}_{h,\bar{h}}$. The argument for $\bar{L}_{-n}\cdot\mathcal{O}_{h,\bar{h}}$ is identical, except that the conformal spin turns out to be $h-\bar{h}-n$.

Exercise: Consider a primary operator $\phi(z,\bar{z})$ of conformal dimensions $(h,\bar{h})$. Under a conformal mapping $z \mapsto w$, recall the transformations

$$\phi(z,\bar{z}) \to \left(\frac{\partial w}{\partial z}\right)^h \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{\bar{h}} \phi(w,\bar{w})\,, \tag{4.5.12}$$

$$T(z) \to \left(\frac{\partial w}{\partial z}\right)^2 T(w) + \frac{c}{12}\{w,z\}\,, \tag{4.5.13}$$

where $\{w, z\}$ denotes the Shwarzian derivative:

$$\{w, z\} = w'''/w' - 3/2(w''/w')^2 \,. \tag{4.5.14}$$

Compute the transformation of the descendant operators $(L_{-1}\cdot\phi)(z,\bar{z})$ and $(L_{-2}\cdot\phi)(z,\bar{z})$ under the mapping $z \mapsto w$.

4.6 Action of the L_n's in correlation functions

We would like to compute a correlation function where all operators are primary, except a descendant operator $\mathcal{O}^{[m]}_{h,\bar{h}} = L_{-m_1}\dots L_{-m_k}\mathcal{O}_{h,\bar{h}}$ inserted at the origin. Thus we consider the correlation function

$$\langle \phi_1(z_1,\bar{z}_1)\dots\phi_n(z_n,\bar{z}_n)\mathcal{O}^{[m]}(0)\rangle = \langle 0|\phi_1(z_1,\bar{z}_1)\dots\phi_n(z_n,\bar{z}_n)L_{-m_1}\dots L_{-m_k}|\mathcal{O}_{h,\bar{h}}\rangle \tag{4.6.1}$$

where $|z_1| \geq \dots \geq |z_n|$, and $\phi_1,\dots,\phi_n$ are primary operators, and $m_1 \geq \dots \geq m_k > 0$. In the above expressions, we have dropped the symbol $\widehat{\ }$, and we have simply written ϕ_j instead of $\widehat{\phi}_j$ in the expectation value, to lighten the notation. Recall the commutation relations:

$$[L_{-m}, \phi_j(z,\bar{z})] = (1-m)h_j z^{-m}\,\phi_j(z,\bar{z}) + z^{-m+1}\,\partial\phi_j(z,\bar{z})\,, \tag{4.6.2}$$

and the property

$$\forall m > 0\,, \qquad \langle 0|L_{-m} = 0\,. \tag{4.6.3}$$

Using these identities, we can commute $L_{-m_1},\dots L_{-m_k}$ to the left of the ϕ_j's, and we easily get

$$\begin{aligned}&\langle \phi_1(z_1,\bar{z}_1)\dots\phi_n(z_n,\bar{z}_n)\mathcal{O}^{[m]}_{h,\bar{h}}(0)\rangle \\ &\qquad = \mathcal{L}_{-m_1}\dots\mathcal{L}_{-m_k}\,\langle \phi_1(z_1,\bar{z}_1)\dots\phi_n(z_n,\bar{z}_n)\mathcal{O}_{h,\bar{h}}(0)\rangle\,,\end{aligned} \tag{4.6.4}$$

where $\mathcal{L}_{-m}$ is the first-order differential operator defined as

$$\mathcal{L}_{-m} = \sum_{j=1}^{n} z_j^{-m}\left[(m-1)h_j - z_j\partial_j\right]. \tag{4.6.5}$$

Remark: If the descendant operator $\mathcal{O}^{[m]}_{h,\bar{h}}$ is inserted at a generic position, by translation invariance we get

$$\begin{aligned}&\langle \phi_1(z_1,\bar{z}_1)\dots\phi_n(z_n,\bar{z}_n)\mathcal{O}^{[m]}_{h,\bar{h}}(w,\bar{w})\rangle \\ &\qquad = \mathcal{L}_{-m_1}(w)\dots\mathcal{L}_{-m_k}(w)\,\langle \phi_1(z_1,\bar{z}_1)\dots\phi_n(z_n,\bar{z}_n)\mathcal{O}_{h,\bar{h}}(w,\bar{w})\rangle\,,\end{aligned} \tag{4.6.6}$$

where

$$\mathcal{L}_{-m}(w) = \sum_{j=1}^{n}(z_j-w)^{-m}\left[(m-1)h_j - (z_j-w)\partial_j\right]. \tag{4.6.7}$$

Lecture 5

Fusion rules and minimal models

5.1 The Operator Product Expansion in CFTs

Consider two scaling operators $\mathcal{O}_i(r'), \mathcal{O}_j(r)$, and let $r' \to r$ inside a correlation function, where all the other operators are located at a larger distance to $\mathcal{O}_j(r)$. Then, for $r' = r + \epsilon$ with $\epsilon \to 0$, the product $\mathcal{O}_i(r+\epsilon), \mathcal{O}_j(r)$ may be viewed, at large distance, as a single local operator located at r. The latter may be, in turn, decomposed on the basis of scaling operators, with ε-dependant coefficients:

$$\mathcal{O}_i(r+\varepsilon)\,\mathcal{O}_j(r) \underset{\varepsilon\to 0}{\sim} \sum_k c_{ij}^k(\varepsilon)\,\mathcal{O}_k(r)\,. \tag{5.1.1}$$

This relation is called the Operator Product Expansion (OPE). It defines an algebra on operators, simply called the operator algebra. The coefficients $c_{ij}^k(\varepsilon)$ are the structure constants of this algebra.

In the case of a 2d CFT, the explicit space dependance of the c_{ij}^k's reads, in complex coordinates

$$\mathcal{O}_i(z,\bar z)\mathcal{O}_j(0,0) = \sum_{\mathcal{O}_k \text{ scaling op.}} \mathcal{C}_{ij}^k\, z^{-h_i-h_j+h_k}\,\bar z^{-\bar h_i-\bar h_j+\bar h_k}\,\mathcal{O}_k(0,0)\,. \tag{5.1.2}$$

In principle, any n-point correlation function could be computed by repeatedly applying the OPE relations, until one gets to a linear combination of two-point functions.

Let us focus on the OPE of two primary operators:

$$\phi_i(z,\bar z)\phi_j(w,\bar w) = \sum_{\substack{\phi_k \text{ primary} \\ [m,\bar m]}} C_{ij}^k([m],[\bar m])\,(z-w)^{-h_{ij}^k+M}\,(\bar z-\bar w)^{-\bar h_{ij}^k+\bar M}\phi_k^{[m],[\bar m]}(w,\bar w)\,, \tag{5.1.3}$$

where

$$h_{ij}^k := h_i + h_j - h_k\,, \qquad \bar h_{ij}^k := \bar h_i + \bar h_j - \bar h_k\,, \tag{5.1.4}$$

and $[m] = (m_1,\dots,m_\ell)$ with $m_1 \geq m_2 \geq \cdots \geq m_\ell \geq 1$, and similarly for $[\bar m]$, and we have used the short-hand notations

$$M = |m| := m_1 + \cdots + m_\ell\,, \qquad \bar M = |\bar m| := \bar m_1 + \cdots + \bar m_{\bar\ell}\,, \tag{5.1.5}$$

$$\phi_k^{[m],[\bar m]} := L_{-m_1}\cdots L_{-m_\ell}\cdot \bar L_{-\bar m_1}\cdots \bar L_{-\bar m_{\bar\ell}}\cdot \phi_k\,. \tag{5.1.6}$$

For a given pair ϕ_i, ϕ_j, only a subset of primary operators ϕ_k contribute to the r.h.s. of the OPE $\phi_i.\phi_j$. We denote this as

$$\phi_i \times \phi_j \to \sum_k N_{ij}^k \, \phi_k \,, \tag{5.1.7}$$

where the fusion integer $N_{ij}^k = 1$ if the fusion $\phi_i \times \phi_j \to \phi_k$ is allowed, and $N_{ij}^k = 0$ otherwise.

Descendant operators will be treated later in this course. At this point, we simply notice that, since the insertion of an operator L_{-m} or $\bar{L}_{-m}$ in a correlation function amounts to applying a differential operator, the OPE $\phi_i \cdot \phi_j^{[m,\bar{m}]}$ (including the structure constants) can be obtained directly from $\phi_i \cdot \phi_j$. Similarly, we shall see later that the fusion to a descendant operator, namely $\phi_i \times \phi_j \to \phi_k^{[m,\bar{m}]}$, is determined by the fusion $\phi_i \times \phi_j \to \phi_k$. Hence, in this lecture, we focus on fusion between primary operators.

5.2 Fusion with a degenerate operator

Consider the OPE of primary operators $\phi_i \cdot \phi_j$, in the case where ϕ_j is degenerate under the Virasoro algebra. We suppose that ϕ_j possesses a descendant χ_j at level N, which is itself a primary operator:

$$|\chi_j\rangle = (L_{-N} + \cdots + \#L_{-1}^N)|\phi_j\rangle \,, \qquad L_{m>0}|\chi_j\rangle = 0 \,. \tag{5.2.1}$$

Recall that, as a consequence, $|\chi_j\rangle$ is a null vector of the Hilbert space, namely it is orthogonal to any state, and thus every correlation function including $|\chi_j\rangle$ must vanish. In particular, we have

$$\langle\phi_k|\phi_i(z,\bar{z})|\chi_j\rangle = \langle\phi_k|\phi_i(z,\bar{z})(L_{-N} + \cdots + \#L_{-1}^N)|\phi_j\rangle = 0 \,. \tag{5.2.2}$$

As we have seen in the previous Lecture, each insertion of a mode L_{-m} can be expressed as the action of a differential operator, by the use of the commutators $[L_{-m}, \phi_i(z,\bar{z})]$. In the present case, the explicit expression is

$$\langle\phi_k|\phi_i(z,\bar{z})L_{-m_1}\dots L_{-m_\ell}|\phi_j\rangle = \mathcal{L}_{-m_1}\dots\mathcal{L}_{-m_\ell}\cdot\langle\phi_k|\phi_i(z,\bar{z})|\phi_j\rangle \,, \tag{5.2.3}$$

where $\mathcal{L}_{-m} \coloneqq z^{-m}\left[(m-1)h_i - z\partial_z\right]$. Moreover, the three-point function of primary operators is always of the form

$$\langle\phi_k|\phi_i(z,\bar{z})|\phi_j\rangle = \langle\phi_k(\infty)\phi_i(z,\bar{z})\phi_j(0)\rangle = \text{const}\times z^{-h_i-h_j+h_k}\,\bar{z}^{-\bar{h}_i-\bar{h}_j+\bar{h}_k} \,. \tag{5.2.4}$$

Comparing with the OPE relation, we see that the multiplicative constant is nothing but the OPE coefficient C_{ij}^k. Moreover, each differential operator $\mathcal{L}_{-m}$ has the following action on power functions:

$$\mathcal{L}_{-m}\cdot z^\alpha = \left[(m-1)h_i - \alpha\right] z^{\alpha-m} \,. \tag{5.2.5}$$

Thus, the condition $\langle\phi_k|\phi_i(z,\bar{z})|\chi_j\rangle = 0$ translates into a polynomial equation of order N, relating h_i, h_j, h_k. As a consequence:

For a given pair of primary operators (ϕ_i, ϕ_j), if ϕ_j is degenerate at level N under the Virasoro algebra Vir, with the null vector $|\chi_j\rangle$, then the fusion $\phi_i \times \phi_j \to \phi_k$ is only possible if the conformal dimension h_k is one of the N solutions of a polynomial equation parameterised by h_i and h_j.

Remark 1: The same result can be obtained for an operator ϕ_j degenerate under the algebra $\overline{\text{Vir}}$: in this case, the conformal dimension $\bar{h}_k$ must be a solution of a polynomial equation parameterised by $\bar{h}_i$ and $\bar{h}_j$. If ϕ_j is degenerate and scalar, then both h_k and $\bar{h}_k$ are constrained by the same type of equations, but they can be different, even when $h_i = \bar{h}_i$.

Remark 2: The above result is a *necessary* condition for the $\phi_i \times \phi_j \to \phi_k$ to be admissible, but it is not always sufficient. In particular, if ϕ_i is also degenerate, then some additional constraints can yield the possible values of h_k.

Let us treat the case of an operator ϕ_j degenerate at level $N = 2$ in more detail. Recall that, with the parameterisation of the central charge $c = 1 - 6(b^{-1} - b)^2$, the degenerate dimensions at level two are $h_{12} = (3b^2 - 2)/4$ and $h_{21} = (3b^{-2} - 2)/4$. Consider the case when the conformal dimensions of ϕ_j are $h_j = \bar{h}_j = h_{12}$ – we shall simply write $\phi_j = \phi_{12}$ in the following argument. The null vector at level two is $|\chi_{12}\rangle = (L_{-2} - b^{-2}L_{-1}^2)|\phi_{12}\rangle$, which yields the condition

$$\langle\phi_k|\phi_i(z,\bar{z})L_{-2}|\phi_{12}\rangle - b^{-2}\,\langle\phi_k|\phi_i(z,\bar{z})L_{-1}^2|\phi_{12}\rangle = 0\,, \tag{5.2.6}$$

and, in turn, the polynomial equation

$$(2h_i + h_{12} - h_k) - b^{-2}\,(h_i + h_{12} - h_k + 1)(h_i + h_{12} - h_k) = 0\,. \tag{5.2.7}$$

At this point, it is convenient to use the Kac parameterisation for h_i, even in the case when h_i is not degenerate. We write

$$h_i = h_{\lambda,\mu} = \frac{(\lambda b^{-1} - \mu b)^2 - (b^{-1} - b)^2}{4}\,, \tag{5.2.8}$$

with λ, μ real. Note that this parameterisation is not unique, because it is invariant under $(\lambda, \mu) \to (\lambda + \alpha b, \mu + \alpha b^{-1})$ for any real α. After some simple algebra, one finds that the solutions of the polynomial equation are

$$h_k = h_{\lambda,\mu+1} \qquad \text{or} \qquad h_k = h_{\lambda,\mu-1}\,. \tag{5.2.9}$$

Hence, in the Kac parameterisation, we get the fusion rule

$$\phi_{\lambda,\mu} \times \phi_{12} \to \phi_{\lambda,\mu+1} + \phi_{\lambda,\mu-1}\,. \tag{5.2.10}$$

The above case for $\phi_j = \phi_{12}$ has been fully treated with a simple argument, relying on the exact form of the null descendant χ_{12}. For degenerate operators at higher levels, more elaborate approaches, such as the "Coulomb Gas", are used to derive the fusion rules –

in this lecture, we state the results without proof. Recall that all operators degenerate at level N have a conformal dimension of the form

$$h_{r,s} = \frac{(rb^{-1} - sb)^2 - (b^{-1} - b)^2}{4}\,, \tag{5.2.11}$$

with r, s positive integers, and $r \times s = N$. As we have done in the above example, a generic conformal dimension h can also be written (in a non-unique way) in Kac parameterisation, namely $h = h_{\lambda,\mu}$ with λ, μ real. The fusion rule of a generic primary operator $\phi_{\lambda,\mu}$ with a degenerate primary operator $\phi_{r,s}$ is given by

$$\boxed{\phi_{\lambda,\mu} \times \phi_{r,s} = \sum_{k=-(r-1)/2}^{(r-1)/2} \sum_{\ell=-(s-1)/2}^{(s-1)/2} \phi_{\lambda+2k,\mu+2\ell}\,,} \tag{5.2.12}$$

where each sum has unit increments. Note that, when $\phi_{\lambda,\mu}$ is also degenerate (namely, when λ and μ are positive integers), some terms may be suppressed from the sum, due to additional fusion rules arising from the null-vector conditions of $\phi_{\lambda,\mu}$. Thus, strictly speaking, the right-hand side of the above relation should be read as the maximal set of admissible primary operators resulting from the OPE $\phi_{\lambda,\mu} \cdot \phi_{r,s}$.

5.3 Minimal models

5.3.1 Kac table and fusion rules

We want to build a "minimal" CFT model, namely a model with as few primary operators as possible, closing under the operator algebra. We suppose that the conformal symmetry encoded by the algebra $\mathrm{Vir} \otimes \overline{\mathrm{Vir}}$ describes completely the model, and hence each primary operator is completely specified by its conformal dimensions $(h, \bar{h})$.

Remark: For simplicity, in this lecture we shall study the case when all primary operators are scalars, namely they have conformal dimensions $h = \bar{h}$. There exist other series of Virasoro minimal models, with some non-scalar primary operators – we will deal with them in the lecture devoted to modular invariance.

For consistency, our CFT must include the identity operator $\mathbf{1}$. Suppose that the CFT also includes a non-trivial primary operator ϕ. Then, the fusion $\phi \times \phi$ will force our model to include other primary operators, which can semselves be fused with ϕ and so on. As we have seen above, a simple way to restrict the possible outcomes of the fusion, is to take ϕ to be a degenerate primary operator, $\phi = \phi_{rs}$, which will produce at most $N = rs$ new operators when fused with another operator. Hence, we suppose that $\phi = \phi_{12}$. By the fusion process:

$$\phi_{12} \times \phi_{12} \to \phi_{11} + \phi_{13}\,, \tag{5.3.1}$$

we see that a primary operator with dimension h_{13} should also be included. If we repeat this process, we produce the primary operators:

$$\phi_{11}\,, \phi_{12}\,, \phi_{13}\,, \phi_{14}\,, \ldots \tag{5.3.2}$$

The only condition under which this set of operators can be finite, is if there exists two positive integers k, ℓ such that $1 < k < \ell$ and $h_{1k} = h_{1\ell}$. From the explicit form of the

dimensions, this yields $b^2 = 2/(k+\ell)$. If we define the integer $p' = k + \ell$, then we have, for any m, the identity $h_{1m} = h_{1,p'-m}$. As a result, we obtain the set of primary operators

$$\{\phi_{11}\,, \phi_{12}\,, \ldots \phi_{1,p'-1}\}\,, \tag{5.3.3}$$

with the identification $\phi_{1m} = \phi_{1,p'-m}$. This defines the minimal model $\mathcal{M}_{2,p'}$, with a central charge $c = 1 - 3(p'-2)^2/p'$. Recall that $p' = k + \ell$ with $1 < k < \ell$, and hence the integer is in the range $p' \geq 3$. The model $\mathcal{M}_{2,3}$, with $c = 0$, contains only the identity operator $\phi_{11} = \phi_{12}$, and it is trivial.

Remark: It turns out that the above set of primary operators, for the even values $p' = 4, 6, 8, \ldots$, leads to an inconsistent CFT. For instance, if $p' = 4$, the identity operator is $\mathbf{1} = \phi_{11} = \phi_{13}$, and hence it obeys the two null-state equations:

$$L_{-1}\mathbf{1} = 0\,, \qquad (L_{-3} + \#L_{-2}L_{-1} + \#L_{-1}^3)\mathbf{1} = 0\,. \tag{5.3.4}$$

As a result, we have $L_{-1}\mathbf{1} = L_{-3}\mathbf{1} = 0$, which, in turns, yields

$$L_{-2}\mathbf{1} = \frac{1}{4}[L_1, L_{-3}]\mathbf{1} = 0\,. \tag{5.3.5}$$

Hence, in the model $\mathcal{M}_{2,4}$, the stress-energy tensor is $T = 0$, which is inconsistent with the central charge $c = -2$.

If we include not only the primary operator ϕ_{12}, but also ϕ_{21}, then by repeated fusion, for generic values of the central charge, we obtain all the ϕ_{rs} with r, s positive integers. Similarly to the previous case, the operator algebra closes on a finite set when the central charge is of the form $c = 1 - 6(b^{-1} - b)^2$, with $b^2 = p/p'$, where $p < p'$ are two coprime positive integers. The expressions for the central charge and the dimensions of degenerate operators take the form:

$$c = 1 - \frac{6(p'-p)^2}{pp'}\,, \qquad h_{rs} = \frac{(rp' - sp)^2 - (p'-p)^2}{4pp'}\,. \tag{5.3.6}$$

The operator algebra of this model closes on a *finite* number of primary operators, which are all degenerate:

$$\{\phi_{rs}\,, \quad r = 1, \ldots, p-1\,, \quad s = 1, \ldots, p'-1\}\,. \tag{5.3.7}$$

This set of operators is called the Kac table, and it defines the minimal model $\mathcal{M}_{p,p'}$. Each primary operator actually appears twice in this table, due to the relation:

$$h_{p-r,p'-s} = h_{rs} \qquad \Rightarrow \qquad \phi_{p-r,p'-s} \equiv \phi_{rs}\,. \tag{5.3.8}$$

As a consequence, in the minimal model $\mathcal{M}_{p,p'}$, the primary operator ϕ_{rs} is degenerate at level rs *and* at level $(p-r)(p'-s)$.

The total number of primary operators is $(p-1)(p'-1)/2$, and their fusion rules are given by

$$\phi_{r_1 s_1} \times \phi_{r_2 s_2} \to \sum_{\substack{k=1+|r_1-r_2| \\ k+r_1+r_2=1 \mod 2}}^{p-1-|p-r_1-r_2|} \sum_{\substack{\ell=1+|s_1-s_2| \\ \ell+s_1+s_2=1 \mod 2}}^{p'-1-|p'-s_1-s_2|} \phi_{k\ell}\,. \tag{5.3.9}$$

These fusion rules are of course compatible with the identification $\phi_{rs} \equiv \phi_{p-r,p'-s}$. Another important property of these fusion rules is that they obey a $\mathbb{Z}_2$ symmetry. We define the $\mathbb{Z}_2$ charge of a primary operator ϕ_{rs} as $(-1)^{m_{rs}}$, where

$$m_{rs} := \begin{cases} r-1 \mod 2 & \text{if } p \text{ is even,} \\ s-1 \mod 2 & \text{if } p' \text{ is even,} \\ r+s \mod 2 & \text{otherwise.} \end{cases} \tag{5.3.10}$$

With this definition, we have $m_{rs} = m_{p-r,p'-s}$, and the fusion rules conserve this $\mathbb{Z}_2$ charge:

$$\text{If} \quad \phi_{r_1 s_1} \times \phi_{r_2 s_2} \to \phi_{k\ell}\,, \qquad \text{then} \quad m_{r_1 s_1} + m_{r_2 s_2} = m_{k\ell} \mod 2\,. \tag{5.3.11}$$

Hence, all the minimal models encode an internal $\mathbb{Z}_2$ symmetry, on top of conformal invariance.

5.3.2 Unitary and non-unitary models

For generic values of the integers p, q, the model $\mathcal{M}_{p,p'}$ is "non-unitary", *i.e.* the "scalar product" defined by $\langle \phi_i | \phi_j \rangle = \delta_{ij}$ for primaries and $L_n^\dagger = L_{-n}$ is not positive (this scalar product is definite thanks to the quotienting with respect to null states, but some states can have negative norm). Alternatively, if one changes to a positive definite scalar product, then the Hamiltonian $\mathcal{H} = \frac{2\pi}{L}(L_0 + \bar{L}_0 - c/12)$ is no longer Hermitian. This is clearly seen on the dimensions of the Kac table for the minimal model $\mathcal{M}_{p,p'}$. In a unitary CFT, all primary dimensions should be positive, because, for any primary state $|\phi\rangle$:

$$\|L_{-1}|\phi\rangle\|^2 = \langle \phi | L_1 L_{-1} | \phi \rangle = \langle \phi | (L_{-1}L_1 + 2L_0) | \phi \rangle = 2h\,.$$

This is consistent with the relation with eigenvalues of the transfer matrix.

$$\log(\lambda_j/\lambda_0) = -\frac{2\pi}{N}(h_j + \bar{h}_j)\,. \tag{5.3.12}$$

Now let us find the minimal dimension in the Kac table (5.3.7). From the Bézout theorem, one can find two positive integers u, v such that $uq - vp = 1$. Let $k = \lfloor u/p \rfloor = \lfloor v/p' \rfloor$, and $r_0 := u - kp, s_0 := v - kp'$. These two indices satisfy $1 \le r_0 \le p-1$ and $1 \le s_0 \le p'-1$, and the corresponding dimension is

$$h_0 = h_{r_0,s_0} = \frac{1-(p'-p)^2}{4pp'}\,. \tag{5.3.13}$$

If $p' = p+1$ then $h_0 = h_{11} = 0$, and all the other dimensions h_{rs} are positive. One can show that the models $\mathcal{M}_{p,p+1}$ are in fact unitary. In contrast, if $p' > p+1$ then $h_0 < 0$, and the model cannot be unitary. This is in contradiction with the interpretation of the vacuum state $|0\rangle$ as the (scaling limit of the) Perron-Froebenius vector of the transfer matrix. Actually, non-unitary CFTs are associated to local lattice models with non-positive Boltzmann weights, where the transfer matrix cannot be chosen real and symmetric in the canonical basis of row configurations.

Critical exponent	Value for Ising
specific heat $C \propto \lvert t\rvert^{-\alpha}$	$\alpha = 0$
spontaneous magnetisation $\langle\sigma\rangle \propto \lvert t\rvert^{\beta}$	$\beta = 1/8$
zero-field susceptibility $\chi \propto \lvert t\rvert^{-\gamma}$	$\gamma = 7/4$
magnetisation at critical temperature $\langle\sigma\rangle \propto \lvert h\rvert^{1/\delta}$	$\delta = 15$
correlation length $\xi \propto \lvert t\rvert^{-\nu}$	$\nu = 1$
anomalous dimension $\langle\sigma(0)\sigma(r)\rangle \propto \lvert r\rvert^{-\eta}$	$\eta = 1/4$

Table 5.1: Critical exponents of the Ising universality class

5.3.3 Example: the Ising model

Consider the two-dimensional Ising model in a magnetic field h, with Boltzmann weight

$$\exp -\left(J \sum_{\langle i,j\rangle} \sigma_i \sigma_j + h \sum_i \sigma_i \right), \tag{5.3.14}$$

and the lattice spins σ_i are ± 1. Criticality is achieved for a particular value $J = J_c$ of the spin coupling and in the absence of magnetic field $h = 0$. Euclidean quantum field theory provides a good description of the universal behavior of the scaling region, *i.e.* the vicinity of the critical point in which the correlation length is much larger than the lattice spacing, with an effective action of the form

$$S = S_0 + t \int d^2x\, \epsilon(x) + h \int d^2x\, \sigma(x)\,, \tag{5.3.15}$$

where S_0 denotes the unperturbed, conformal invariant action describing the fluctuations of the Ising model at criticality. The two perturbations account for change of J (or equivalently temperature) and magnetic field, respectively. While the field σ is simply the scaling limit of the lattice operator σ_i, the energy ϵ is the scaling limit of the composite operator $\sigma_i\sigma_j$, for i, j neighbors. In the CFT language this is expressed via the fusion rules

$$\sigma \times \sigma = 1 + \epsilon \tag{5.3.16}$$

The scaling dimensions of these fields are $\Delta_\epsilon = 1$ and $\Delta_\sigma = \frac{1}{8}$. These values follow from the critical exponents, as defined in section 1.1.3, which for the Ising model are known exactly[1]. They are listed in table 5.1, where $t \coloneqq (J - J_c)$ is the (reduced) temperature and h the magnetic field. The RG exponent y_t (respectively y_h) of the scaling field ϵ (respectively σ) is readily obtained from the critical exponents, via

$$\beta = \frac{2 - y_h}{y_t}, \qquad \gamma = \frac{2y_h - 2}{y_t}\,. \tag{5.3.17}$$

Thus $y_t = 1$ and $y_h = 15/8$. The scaling dimensions follow, as described in section 1.1.3.

[1]The Ising model at zero magnetic is one of the most celebrated integrable models and was solved on the square lattice in 1944 by Onsager. In the late eighties A. Zamolodchikov solved the model in a magnetic field at the critical temperature within the framework of integrable quantum field theory.

The central charge of the Ising model can be extracted from the system-size dependence of the ground-state energy, which is also known exactly. For the Ising model one finds

$$c = \frac{1}{2} \tag{5.3.18}$$

The critical behavior of the two-dimensional Ising model is described by the minimal model $\mathcal{M}_{3,4}$, which contains three admissible chiral primary fields:

$$\phi_{11} = \phi_{23}\,, \qquad \phi_{13} = \phi_{21}\,, \qquad \phi_{12} = \phi_{22}\,, \tag{5.3.19}$$

with conformal dimensions

$$h_{11} = 0\,, \qquad h_{13} = \frac{1}{2}\,, \qquad h_{12} = \frac{1}{16}\,, \tag{5.3.20}$$

as listed in the Kac Table, Fig.(5.1). Their (chiral) fusion rules are

$$\phi_{12} \times \phi_{12} \to \phi_{11} + \phi_{13}\,, \qquad \phi_{13} \times \phi_{13} \to \phi_{11}\,, \qquad \phi_{12} \times \phi_{13} \to \phi_{12}\,. \tag{5.3.21}$$

Of course the physical operators occurring in the Ising model can be arranged into representations of $\mathrm{Vir} \otimes \overline{\mathrm{Vir}}$. In some sense one can think of primary operators as "products" of holomorphic and anti-holormophic fields, and thus they are labelled by their left and right conformal dimensions $(h, \bar{h})$. In the Ising model there are three local primary operators

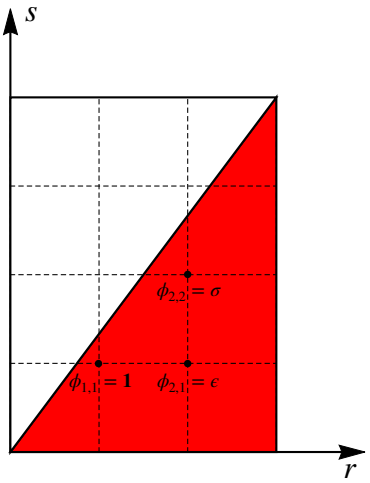


Figure 5.1: Kac Table for the minimal model $\mathcal{M}_{3,4}$ associated with the Ising model.

- the identity operator $\mathbf{1}$ with $(h, \bar{h}) = (0, 0)$,
- the energy operator ϵ with $(h, \bar{h}) = (1/2, 1/2)$,
- the spin operator σ with $(h, \bar{h}) = (1/16, 1/16)$.

with $\mathbb{Z}_2$ charges

$$(-1)^{m_{\mathbf{1}}} = +1\,, \qquad (-1)^{m_\epsilon} = +1\,, \qquad (-1)^{m_\sigma} = -1\,, \tag{5.3.22}$$

and fusion rules

$$\sigma \times \sigma \to \mathbf{1} + \epsilon\,, \qquad \epsilon \times \epsilon \to \mathbf{1}\,, \qquad \sigma \times \epsilon \to \sigma\,. \tag{5.3.23}$$

This corresponds to the smallest unitary model with $\mathbb{Z}_2$ symmetry, and a single odd operator σ. Hence, it can be identified with the continuum limit of the critical Ising model on a regular lattice.

5.3.4 Kac modules and Virasoro characters

The character of a representation $\mathcal{W}$ of the Virasoro algebra is defined as

$$\mathrm{Tr}_{\mathcal{W}}(q^{L_0-c/24})\,, \tag{5.3.24}$$

where q is a formal parameter.

Let us first recall the character of the Verma module $\mathcal{V}_h$:

$$\begin{aligned}\mathrm{Tr}_{\mathcal{V}_h}(q^{L_0-c/24}) &= q^{h-c/24}\sum_{N=0}^{\infty}\dim_N(\mathcal{V}_h)q^N = q^{h-c/24}\sum_{N=0}^{\infty}P(N)q^N \\ &= \frac{q^{h+(1-c)/24}}{\eta(q)}\,,\end{aligned} \tag{5.3.25}$$

where $\dim_N(\mathcal{V}_h)$ is the dimension of level N in the representation $\mathcal{V}_h$. In a Verma module, this dimension is simply equal to $P(N)$, the number of partitions of the integer N. In the final expression, we have used $\eta(q)$, the Dedekind eta function, defined for $|q|<1$ by

$$\eta(q) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)\,, \qquad \frac{1}{\eta(q)} = q^{-1/24}\prod_{n=1}^{\infty}\sum_{k=0}^{\infty}q^{nk} = \sum_{N=0}^{\infty}P(N)q^N\,. \tag{5.3.26}$$

For $h=h_{rs}$, we shall use the short-hand notation for Verma modules $\mathcal{V}_{rs}\coloneqq\mathcal{V}_{h_{rs}}$.

The primary operator ϕ_{rs} has null vectors of dimensions

$$h_{rs}+rs = h_{p+r,p'-s} = h_{p'-r,q+s}\,, \qquad \text{and} \qquad h_{rs}+(p-r)(p'-s) = h_{r,2p'-s} = h_{2p-r,s}\,.$$

The Kac module $\mathcal{K}_{rs}$ is obtained by quotienting the Verma module $\mathcal{V}_{rs}$ by the two corresponding submodules of null vectors, which are themselves isomorphic to two Verma modules:

$$\mathcal{K}_{rs} = \mathcal{V}_{rs}/(\mathcal{V}_{p+r,p'-s}+\mathcal{V}_{r,2p'-s})\,. \tag{5.3.27}$$

It turns out that the submodules $\mathcal{V}_{p+r,p'-s}$ and $\mathcal{V}_{r,2p'-s}$ of $\mathcal{V}_{rs}$ are *not* in direct sum. Let us first identify the submodules of $\mathcal{V}_{p+r,p'-s}$. Since $\phi_{p+r,p'-s}$ is degenerate at level $(p+r,p'-s)$, the representation $\mathcal{V}_{p+r,p'-s}$ admits a submodule isomorphic to $\mathcal{V}_{2p+r,s}$. This same operator $\phi_{p+r,p'-s}\equiv\phi_{p-r,p'+s}$ is also degenerate at level $(p-r,p'+s)$, and thus it has a submodule $\mathcal{V}_{r,2p'+s}$. With the same reasoning, we find that $\mathcal{V}_{r,2p'-s}$ also has $\mathcal{V}_{2p+r,s}$ and $\mathcal{V}_{r,2p'+s}$ as submodules. Hence, the intersection is given by

$$\mathcal{V}_{p+r,q-s}\cap\mathcal{V}_{r,2p'-s} = \mathcal{V}_{2p+r,s}+\mathcal{V}_{r,2p'+s}\,, \tag{5.3.28}$$

and the character of the Kac module is

$$\chi_{rs}(q) \coloneqq \mathrm{Tr}_{\mathcal{K}_{rs}}(q^{L_0-c/24}) = (\mathrm{Tr}_{\mathcal{V}_{rs}}-\mathrm{Tr}_{\mathcal{V}_{p+r,p'-s}+\mathcal{V}_{r,2p'-s}})(q^{L_0-c/24}) \tag{5.3.29}$$

$$= \left[\mathrm{Tr}_{\mathcal{V}_{rs}}-(\mathrm{Tr}_{\mathcal{V}_{p+r,p'-s}}+\mathrm{Tr}_{\mathcal{V}_{r,2p'-s}})+\mathrm{Tr}_{\mathcal{V}_{2p+r,s}+\mathcal{V}_{r,2p'+s}}\right](q^{L_0-c/24}) \tag{5.3.30}$$

This process repeats recursively:

$$\mathcal{V}_{2p+r,s}\cap\mathcal{V}_{r,2p'+s} = \mathcal{V}_{3p+r,p'-s}+\mathcal{V}_{r,4p'-s}\,, \tag{5.3.31}$$

and so on. This yields the expression for the Kac character:

$$\chi_{rs}(q) = \frac{q^{(1-c)/24}}{\eta(q)}\left[q^{h_{rs}}+\sum_{k=1}^{\infty}(-1)^k\left(q^{h_{kp+r,s_k}}+q^{h_{r,kp'+s_k}}\right)\right], \tag{5.3.32}$$

where

$$s_k := \begin{cases} s & \text{for } k \text{ even}, \\ p' - s & \text{for } k \text{ odd}. \end{cases}$$

Using the notation

$$\Delta_{rs} := h_{rs} + \frac{1-c}{24} = \frac{(p'r - ps)^2}{4pp'}\,, \tag{5.3.33}$$

we can write

$$\chi_{rs}(q) = \frac{1}{\eta(q)}\left[q^{\Delta_{rs}} + \sum_{n=1}^{\infty}(q^{\Delta_{2np+r,s}} + q^{\Delta_{r,2np'+s}})\right] - \frac{1}{\eta(q)}\sum_{n=1}^{\infty}(q^{\Delta_{(2n-1)p+r,p'-s}} + q^{\Delta_{r,2np'-s}})\,, \tag{5.3.34}$$

which can be re-arranged as the difference:

$$\boxed{\chi_{rs}(q) = K_{rs}(q) - K_{r,-s}(q)\,, \qquad \text{where} \quad K_{rs}(q) := \frac{1}{\eta(q)}\sum_{n\in\mathbb{Z}} q^{(p'r-ps+2npp')^2/4pp'}\,.} \tag{5.3.35}$$

Lecture 6

Correlation functions

6.1 Descendants in the Operator Product Expansion

In the previous lecture, we have discussed the OPE between two primary operators:

$$\phi_i(z,\bar{z})\phi_j(w,\bar{w}) = \sum_{\substack{\phi_k \text{ primary} \\ [m],[\bar{m}]}} C_{ij}^k([m],[\bar{m}])\,(z-w)^{-h_{ij}^k+|m|}\,(\bar{z}-\bar{w})^{-\bar{h}_{ij}^k+|\bar{m}|}\phi_k^{[m,\bar{m}]}(w,\bar{w})\,, \tag{6.1.1}$$

where

$$h_{ij}^k := h_i + h_j - h_k\,, \qquad \bar{h}_{ij}^k := \bar{h}_i + \bar{h}_j - \bar{h}_k\,. \tag{6.1.2}$$

So far, we have focused on the terms in the right-hand side where $[m] = [\bar{m}] = \cdot$, namely the primary operators ϕ_k. Let us discuss here the terms corresponding to descendants of ϕ_k.

First, recall that the descendants of different primary operators or different levels are orthogonal:

$$\text{if} \quad (k,|m|,|\bar{m}|) \neq (\ell,|n|,|\bar{n}|)\,, \qquad \text{then} \quad \langle\phi_k^{[m,\bar{m}]}|\phi_\ell^{[n,\bar{n}]}\rangle = 0\,. \tag{6.1.3}$$

However, at a given level M, the descendant states $|\phi_k^{[m,\bar{m}]}\rangle$ do not form, in general, an orthogonal basis. Hence, we introduce a dual basis $\langle\widetilde{\phi}_k^{[m,\bar{m}]}|$ of descendants at level M, such that

$$\langle\widetilde{\phi}_k^{[m,\bar{m}]}|\phi_k^{[n,\bar{n}]}\rangle = \delta_{m,n}\,\delta_{\bar{m},\bar{n}}\,. \tag{6.1.4}$$

With this definition, it is straightforward to relate the structure constants $C_{ij}^k([m],[\bar{m}])$ to expectation values:

$$\langle\widetilde{\phi}_k^{[m,\bar{m}]}|\phi_i(z,\bar{z})|\phi_j\rangle = C_{ij}^k([m],[\bar{m}])z^{-h_{ij}^k+|m|}\,\bar{z}^{-\bar{h}_{ij}^k+|\bar{m}|}\,. \tag{6.1.5}$$

Let us approach the computation of these expectation values in two steps. First, we consider a simpler object $\langle\phi_k^{[n,\bar{n}]}|\phi_i(z,\bar{z})|\phi_j\rangle$, and compute it using the commutation relations $[L_{-n},\phi_i]$ and $[\bar{L}_{-\bar{n}},\phi_i]$:

$$\langle\phi_k^{[n,\bar{n}]}|\phi_i(z,\bar{z})|\phi_j\rangle = \langle\phi_k|L_{n_\ell}\dots L_{n_1}\,\bar{L}_{\bar{n}_{\bar{\ell}}}\dots\bar{L}_{\bar{n}_1}\,\phi_i(z,\bar{z})|\phi_j\rangle \tag{6.1.6}$$

$$= U_{n_1}\dots U_{n_\ell}\cdot\bar{U}_{\bar{n}_1}\dots\bar{U}_{\bar{n}_{\bar{\ell}}}\cdot\langle\phi_k|\phi_i(z,\bar{z})|\phi_j\rangle \tag{6.1.7}$$

$$= U_{n_1}\dots U_{n_\ell}\cdot\bar{U}_{\bar{n}_1}\dots\bar{U}_{\bar{n}_{\bar{\ell}}}\cdot C_{ij}^k\,z^{-h_{ij}^k}\,\bar{z}^{-\bar{h}_{ij}^k} \tag{6.1.8}$$

where

$$U_n = z^n \left[(n+1)h_i + z\partial_z\right], \qquad \bar{U}_n = \bar{z}^n \left[(n+1)\bar{h}_i + \bar{z}\bar{\partial}_{\bar{z}}\right]. \tag{6.1.9}$$

These differential operators have a simple action on power functions:

$$U_n \cdot z^\alpha \bar{z}^{\bar{\alpha}} = \left[(n+1)h_i + \alpha\right] z^{\alpha+n} \bar{z}^{\bar{\alpha}}, \qquad \bar{U}_n \cdot z^\alpha \bar{z}^{\bar{\alpha}} = \left[(n+1)\bar{h}_i + \bar{\alpha}\right] z^\alpha \bar{z}^{\bar{\alpha}+n}. \tag{6.1.10}$$

As a result, the expectation value under study takes the factorised form:

$$\langle \phi_k^{[n,\bar{n}]} | \phi_i(z,\bar{z}) | \phi_j \rangle = C_{ij}^k \, \gamma_{ij}^k[n] \, \bar{\gamma}_{ij}^k[\bar{n}] \, z^{-h_{ij}^k + |n|} \, \bar{z}^{-\bar{h}_{ij}^k + |\bar{n}|}, \tag{6.1.11}$$

where C_{ij}^k is the structure constant associated to the fusion of primary operators $\phi_i \times \phi_j \to \phi_k$, and $\gamma_{ij}^k[n]$ is a polynomial in (h_i, h_j, h_k) of degree $|n|$, and $\bar{\gamma}_{ij}^k[\bar{n}]$ is a polynomial in $(\bar{h}_i, \bar{h}_j, \bar{h}_k)$ of degree $|\bar{n}|$.

Let us now argue that the structure constants $C_{ij}^k([m],[\bar{m}])$ obey a similar factorisation:

$$\boxed{C_{ij}^k([m],[\bar{m}]) = C_{ij}^k \, \beta_{ij}^k[m] \, \bar{\beta}_{ij}^k[\bar{m}].} \tag{6.1.12}$$

The main idea is to show that the dual basis elements $\langle \widetilde{\phi}_k^{[m,\bar{m}]}|$ have a factorised form. We write $|\phi_k\rangle = |h_k\rangle \otimes |\bar{h}_k\rangle$, and we consider separately the representations of Vir and $\overline{\text{Vir}}$ generated by $|h_k\rangle$ and $|\bar{h}_k\rangle$. Let Q be the matrix of the scalar product in the space of descendants of $|h_k\rangle$, at level M, and A be the inverse of Q. The dual basis of the $|h_k^{[m]}\rangle$'s is constructed as

$$\langle \widetilde{h}_k^{[m]}| = \sum_{[n]} a_{mn} \langle h_k^{[n]}|, \tag{6.1.13}$$

where the sum runs over the canonical basis of descendants of level $M = |m|$. Similarly, we have a dual basis

$$\langle \widetilde{\bar{h}}_k^{[\bar{m}]}| = \sum_{[\bar{n}]} \bar{a}_{\bar{m}\bar{n}} \langle \bar{h}_k^{[\bar{n}]}|, \tag{6.1.14}$$

at level $\bar{M}$. Because of the property

$$\langle \phi_k^{[m,\bar{m}]} | \phi_k^{[m',\bar{m}']} \rangle = \langle h_k^{[m]} | h_k^{[m']} \rangle \times \langle \bar{h}_k^{[\bar{m}]} | \bar{h}_k^{[\bar{m}']} \rangle, \tag{6.1.15}$$

it is then straighforward to see that the dual basis elements $\langle \phi_k^{[m,\bar{m}]}|$ are of the form

$$\langle \widetilde{\phi}_k^{[m,\bar{m}]}| = \sum_{n,\bar{n}} a_{mn} \, \bar{a}_{\bar{m}\bar{n}} \langle \phi_k^{[n,\bar{n}]}|, \tag{6.1.16}$$

where the sum runs over the canonical basis of descendants of level $(M, \bar{M})$. Hence, we get

$$\begin{aligned} C_{ij}^k([m],[\bar{m}]) &= \langle \widetilde{\phi}_k^{[m,\bar{m}]} | \phi_i(1) | \phi_j \rangle & (6.1.17)\\ &= \sum_{n,\bar{n}} a_{mn} \, \bar{a}_{\bar{m}\bar{n}} \langle \phi_k^{[n,\bar{n}]} | \phi_i(1) | \phi_j \rangle & (6.1.18)\\ &= \sum_{n,\bar{n}} a_{mn} \, \bar{a}_{\bar{m}\bar{n}} \, C_{ij}^k \, \gamma_{ij}^k[n] \, \bar{\gamma}_{ij}^k[\bar{n}], & (6.1.19) \end{aligned}$$

which is the expected result, with

$$\beta_{ij}^k[m] = \sum_n a_{mn} \, \gamma_{ij}^k[n], \qquad \bar{\beta}_{ij}^k[\bar{m}] = \sum_{\bar{n}} a_{\bar{m}\bar{n}} \, \bar{\gamma}_{ij}^k[\bar{n}]. \tag{6.1.20}$$

The above argument can be easily generalised to obtain a family of expectation values which will be useful in this lecture:

$$\langle\widetilde{\phi}_k^{[m,\bar{m}]}|\phi_i(z,\bar{z})|\phi_j^{[n,\bar{n}]}\rangle = C_{ij}^k\,\beta_{ij}^k[m,n]\,\bar{\beta}_{ij}^k[\bar{m},\bar{n}]\,z^{-h_{ij}^k+|m|-|n|}\,\bar{z}^{-\bar{h}_{ij}^k+|\bar{m}|-|\bar{n}|}\,. \tag{6.1.21}$$

In this expression, the constants $\beta_{ij}^k[m,n]$ and $\bar{\beta}_{ij}^k[\bar{m},\bar{n}]$ are obtained by using only the Virasoro commutation relations, like for the $\gamma_{ij}^k[m]$ and $\beta_{ij}^k[m]$ discussed previously.

6.2 Conformal blocks

Let us consider the correlation function of four primary operators:

$$\langle\phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\phi_3(z_3,\bar{z}_3)\phi_4(z_4,\bar{z}_4)\rangle\,. \tag{6.2.1}$$

The Moebius transformation

$$z \mapsto \frac{z_2-z_1}{z_2-z_4}\times\frac{z-z_4}{z-z_1} \tag{6.2.2}$$

maps the points z_i as

$$(z_1,z_2,z_3,z_4)\mapsto(\infty,1,z,0)\,,\qquad \text{where}\quad z=\frac{(z_2-z_1)(z_3-z_4)}{(z_2-z_4)(z_3-z_1)}\,. \tag{6.2.3}$$

Hence, the above correlation function is related by conformal covariance, to:

$$G(z,\bar{z}) = \langle\phi_1(\infty,\infty)\phi_2(1,1)\phi_3(z,\bar{z})\phi_4(0,0)\rangle\,. \tag{6.2.4}$$

In the range $|z|<1$, $G(z,\bar{z})$ can be written in Heisenberg formalism:

$$G(z,\bar{z}) = \langle\phi_1|\phi_2(1,1)\phi_3(z,\bar{z})|\phi_4\rangle\,. \tag{6.2.5}$$

In the Hilbert space, we can write the identity operator as:

$$\mathbf{1} = \sum_k\sum_{[m]}\sum_{[\bar{m}]}|\phi_k^{[m,\bar{m}]}\rangle\langle\widetilde{\phi}_k^{[m,\bar{m}]}|\,, \tag{6.2.6}$$

where the sum is over all possible primary states ϕ_k, and we have used the dual basis of descendants derived before. We insert this resolution of identity into the expression for $G(z,\bar{z})$, between $\phi_2(1,1)$ and $\phi_3(z,\bar{z})$:

$$G(z,\bar{z}) = \sum_{k,[m],[\bar{m}]}\langle\phi_1|\phi_2(1,1)|\phi_k^{[m,\bar{m}]}\rangle\langle\widetilde{\phi}_k^{[m,\bar{m}]}|\phi_3(z,\bar{z})|\phi_4\rangle\,. \tag{6.2.7}$$

Recall from the previous calculations that we have:

$$\langle\phi_k^{[m,\bar{m}]}|\phi_2(1,1)|\phi_1\rangle = C_{21}^k\,\gamma_{21}^k[m]\bar{\gamma}_{21}^k[\bar{m}]\,, \tag{6.2.8}$$

$$\langle\widetilde{\phi}_k^{[m,\bar{m}]}|\phi_3(z,\bar{z})|\phi_4\rangle = C_{34}^k\,\beta_{34}^k[m]\bar{\beta}_{34}^k[\bar{m}]\,z^{-h_{34}^k+|m|}\,\bar{z}^{-\bar{h}_{34}^k+|\bar{m}|}\,. \tag{6.2.9}$$

This gives:

$$G(z,\bar{z}) = \sum_{k,[m],[\bar{m}]}(C_{21}^k)^*\,C_{34}^k\,(\gamma_{21}^k[m])^*\beta_{34}^k[m]z^{-h_{34}^k+|m|} \times(\bar{\gamma}_{21}^k[\bar{m}])^*\bar{\beta}_{34}^k[\bar{m}]\bar{z}^{-\bar{h}_{34}^k+|\bar{m}|}\,, \tag{6.2.10}$$

and hence the decomposition:

$$\boxed{G(z,\bar{z}) = \sum_k (C_{21}^k)^* \, C_{34}^k \, \mathcal{F}_{12}^{34}(k|z) \, \bar{\mathcal{F}}_{12}^{34}(k|\bar{z}) \,,} \tag{6.2.11}$$

where

$$\boxed{\begin{aligned} \mathcal{F}_{12}^{34}(k|z) &:= z^{-h_{34}^k} \sum_{[m]} (\gamma_{21}^k[m])^* \beta_{34}^k[m] \, z^{|m|} \,, \\ \bar{\mathcal{F}}_{12}^{34}(k|\bar{z}) &:= \bar{z}^{-\bar{h}_{34}^k} \sum_{[\bar{m}]} (\bar{\gamma}_{21}^k[\bar{m}]))^* \bar{\beta}_{34}^k[\bar{m}] \, \bar{z}^{|\bar{m}|} \,. \end{aligned}} \tag{6.2.12}$$

These functions are called the *conformal blocks* of the correlation function $G(z,\bar{z})$. Note that they have the form of a power function multiplied by a power series. Also, by construction of the coefficients $\gamma_{ij}^k[m]$ and $\beta_{ij}^k[m]$, the dominant coefficient of these power series is always equal to one. For example:

$$\mathcal{F}_{12}^{34}(k|z) = z^{-h_{34}^k} \left(1 + \alpha_1 z + \alpha_2 z^2 + \dots\right) , \tag{6.2.13}$$

where α_1 contains the contribution from $L_{-1}\phi_k$, α_2 contains contributions from $\{L_{-2}\phi_k, L_{-1}^2\phi_k\}$, and so on.

In the regime $|1-z| < 1$, we can use the invariance of $G(z,\bar{z})$ under the conformal map $u \mapsto 1-u$, to get :

$$\begin{aligned} G(z,\bar{z}) &= \langle \phi_1(\infty,\infty)\phi_2(0,0)\phi_3(1-z,1-\bar{z})\phi_4(1,1)\rangle & (6.2.14)\\ &= \langle \phi_1 | \phi_4(1,1)\phi_3(1-z,1-\bar{z})|\phi_2\rangle \,, & (6.2.15) \end{aligned}$$

which yields the decomposition:

$$\boxed{G(z,\bar{z}) = \sum_k (C_{41}^k)^* \, C_{32}^k \, \mathcal{F}_{14}^{32}(k|1-z) \, \bar{\mathcal{F}}_{14}^{32}(k|1-\bar{z}) \,.} \tag{6.2.16}$$

Similarly, for $|z| > 1$, if we use the map $u \mapsto 1/u$, we get:

$$\begin{aligned} G(z,\bar{z}) &= z^{-2h_3} \, \bar{z}^{-2\bar{h}_3} \, \langle \phi_1(0,0)\phi_2(1,1)\phi_3(1/z,1/\bar{z})\phi_4(\infty,\infty)\rangle & (6.2.17)\\ &= z^{-2h_3} \, \bar{z}^{-2\bar{h}_3} \, \langle \phi_4 | \phi_2(1,1)\phi_3(1/z,1/\bar{z})|\phi_1\rangle \,, & (6.2.18) \end{aligned}$$

which yields the decomposition:

$$\boxed{G(z,\bar{z}) = z^{-2h_3} \, \bar{z}^{-2\bar{h}_3} \sum_k (C_{24}^k)^* \, C_{31}^k \, \mathcal{F}_{42}^{31}(k|1/z) \, \bar{\mathcal{F}}_{42}^{31}(k|1/\bar{z}) \,.} \tag{6.2.19}$$

Hence, for any four-point function of primary operators, we have three possible *channels* for the decomposition into conformal blocks. The consistency conditions between these three possible decompositions is often called *crossing symmetry*, and it is the basis for the conformal bootstrap.

6.3 Chiral primary operators

The conformal blocks are not physical correlation functions. Instead, they are algebraic objects belonging in the representation theory of a single Virasoro algebra. As a tool to study the conformal blocks, we introduce the *chiral* primary operators $\phi_i(z)$, acting on representations of a single Virasoro algebra. First, recall the form of the expectation value for *physical* states and operators:

$$\langle \widetilde{\phi}_k^{[m,\bar{m}]} | \phi_i(z,\bar{z}) | \phi_j^{[n,\bar{n}]} \rangle = C_{ij}^k \, \beta_{ij}^k[m,n] \, \bar{\beta}_{ij}^k[\bar{m},\bar{n}] \, z^{-h_{ij}^k+|m|-|n|} \, \bar{z}^{-\bar{h}_{ij}^k+|\bar{m}|-|\bar{n}|} \,. \tag{6.3.1}$$

We thus define the chiral primary operators $\phi_i(z)$ and $\bar{\phi}_i(\bar{z})$ through their matrix elements:

$$\langle \widetilde{h}_k^{[m]} | \phi_i(z) | h_j^{[n]} \rangle := \beta_{ij}^k[m,n] \, z^{-h_{ij}^k+|m|-|n|} \,, \tag{6.3.2}$$

$$\langle \widetilde{\bar{h}}_k^{[\bar{m}]} | \bar{\phi}_i(\bar{z}) | \bar{h}_j^{[\bar{n}]} \rangle := \bar{\beta}_{ij}^k[\bar{m},\bar{n}] \, \bar{z}^{-\bar{h}_{ij}^k+|\bar{m}|-|\bar{n}|} \,, \tag{6.3.3}$$

with the same constants $\beta_{ij}^k[m,n]$ and $\bar{\beta}_{ij}^k[\bar{m},\bar{n}]$ as in the physical matrix elements.

Using the dual bases of chiral descendants, the orthogonal projectors onto the modules generated by $|h_k\rangle$ and $|\bar{h}_k\rangle$ are given by

$$\mathcal{P}_k := \sum_{[m]} |h_k^{[m]}\rangle\langle \widetilde{h}_k^{[m]}| \,, \qquad \bar{\mathcal{P}}_k := \sum_{[\bar{m}]} |\bar{h}_k^{[\bar{m}]}\rangle\langle \widetilde{\bar{h}}_k^{[\bar{m}]}| \,. \tag{6.3.4}$$

In terms of chiral primary operators, the conformal blocks can be written

$$\mathcal{F}_{12}^{34}(k|z) = \sum_{[m]} \langle h_1 | \phi_2(1) h_k^{[m]} \rangle \langle \widetilde{h}_k^{[m]} | \phi_3(z) | h_4 \rangle \,, \tag{6.3.5}$$

$$\bar{\mathcal{F}}_{12}^{34}(k|\bar{z}) = \sum_{[\bar{m}]} \langle \bar{h}_1 | \bar{\phi}_2(1) \bar{h}_k^{[\bar{m}]} \rangle \langle \widetilde{\bar{h}}_k^{[\bar{m}]} | \bar{\phi}_3(\bar{z}) | \bar{h}_4 \rangle \,, \tag{6.3.6}$$

and hence we get:

$$\boxed{\begin{aligned} \mathcal{F}_{12}^{34}(k|z) &= \langle 0 | \phi_1(\infty) \phi_2(1) \mathcal{P}_k \phi_3(z) \phi_4(0) | 0 \rangle \,, \\ \bar{\mathcal{F}}_{12}^{34}(k|\bar{z}) &= \langle 0 | \bar{\phi}_1(\infty) \bar{\phi}_2(1) \bar{\mathcal{P}}_k \bar{\phi}_3(\bar{z}) \bar{\phi}_4(0) | 0 \rangle \,. \end{aligned}} \tag{6.3.7}$$

The conformal block $\mathcal{F}_{12}^{34}(k|z)$ thus appears as a correlator of chiral primary operators, with the insertion of a projector $\mathcal{P}_k$ on a "virtual" intermediary sector, namely the Virasoro module with lowest weight h_k.

Through a reversed computation, one can show that the chiral primary operators defined by the above matrix elements satisfy the commutation relations with the L_n's

$$[L_n, \phi_i(w)] = w^n [(n+1)h_i + w\partial_w] \phi_i(w) \,. \tag{6.3.8}$$

In turn, the chiral OPE $T \cdot \phi_i$ can be derived from the commutation relations:

$$T(z)\phi_i(w) = \frac{h_i \phi_i(w)}{(z-w)^2} + \frac{\partial \phi_i(w)}{z-w} + \mathrm{reg}_{z\to w} \,, \qquad \text{for } |z| > |w| \,, \tag{6.3.9}$$

and the same can be done for the chiral OPE $\phi_i \cdot \phi_j$

$$\mathcal{P}_k \phi_i(z) \phi_j^{[n]}(w) = \sum_{[m]} \beta_{ij}^k[m,n] \, (z-w)^{-h_{ij}^k+|m|-|n|} \, \phi_k^{[m]}(w) \,, \qquad \text{for } |z| > |w| \,. \tag{6.3.10}$$

By construction, it is obvious that the projectors satisfy:

$$[L_n, \mathcal{P}_k] = 0\,, \qquad \mathcal{P}_k\mathcal{P}_\ell = \mathcal{P}_\ell\mathcal{P}_k = \delta_{k\ell}\,. \tag{6.3.11}$$

Now let us consider conformal blocks with primary operators inserted at generic positions

$$\langle 0|\phi_1(z_1)\phi_2(z_2)\mathcal{P}_k\phi_3(z_3)\phi_4(z_4)|0\rangle\,, \tag{6.3.12}$$

with $|z_1| \geq |z_2| \geq |z_3| \geq |z_4|$. From the above properties of chiral primary operators and projectors, we are now able to study the transformation of these conformal blocks under conformal mappings. First, note that, for the rescalings $z \mapsto z/\lambda$, the chiral OPE $\phi_i \cdot \phi_j$ behaves as

$$\mathcal{P}_k\phi_i(z)\phi_j^{[n]}(w) = \lambda^{-h_i-h_j-|n|}\sum_{[m]}\beta_{ij}^k[m,n]\,(z/\lambda - w/\lambda)^{-h_{ij}^k+|m|-|n|}\,\lambda^{h_k+|m|}\phi_k^{[m]}(w)\,. \tag{6.3.13}$$

Using repeatedly this relation to decompose the conformal block, one gets the scaling covariance relation:

$$\begin{aligned}&\langle 0|\phi_1(z_1)\phi_2(z_2)\mathcal{P}_k\phi_3(z_3)\phi_4(z_4)|0\rangle\\ &\qquad = \prod_{j=1}^4 \lambda^{-h_j}\,\langle 0|\phi_1(z_1/\lambda)\phi_2(z_2/\lambda)\mathcal{P}_k\phi_3(z_3/\lambda)\phi_4(z_4/\lambda)|0\rangle\,.\end{aligned} \tag{6.3.14}$$

A similar argument yields the behaviour under rotations:

$$\begin{aligned}&\langle 0|\phi_1(z_1)\phi_2(z_2)\mathcal{P}_k\phi_3(z_3)\phi_4(z_4)|0\rangle\\ &\qquad = \prod_{j=1}^4 e^{-ih_j\theta}\,\langle 0|\phi_1(e^{-i\theta}z_1)\phi_2(e^{-i\theta}z_2)\mathcal{P}_k\phi_3(e^{-i\theta}z_3)\phi_4(e^{-i\theta}z_4)|0\rangle\,.\end{aligned} \tag{6.3.15}$$

In fact, since the chiral OPE $\phi_i \cdot \phi_j$ completely determines, recursively, the conformal block, it also yields its covariance property under any global conformal transformation $z \mapsto z'$:

$$\begin{aligned}&\langle 0|\phi_1(z_1)\phi_2(z_2)\mathcal{P}_k\phi_3(z_3)\phi_4(z_4)|0\rangle\\ &\qquad = \prod_{j=1}^4 \left(\frac{\partial z_j'}{\partial z_j}\right)^{h_j}\,\langle 0|\phi_1(z_1')\phi_2(z_2')\mathcal{P}_k\phi_3(z_3')\phi_4(z_4')|0\rangle\,,\end{aligned} \tag{6.3.16}$$

provided the radial ordering is preserved, namely $|z_1'| \geq |z_2'| \geq |z_3'| \geq |z_4'|$.

For infinitesimal conformal maps $z \mapsto z + \epsilon(z)$, we can derive Ward identities for conformal blocks as follows. On one hand, the variation of the conformal block is given by the above covariance property, and on the other hand, any closed contour integral with an insertion of $T(w)$ is determined by the chiral OPEs $T \cdot \phi_j$ derived above – the presence of $\mathcal{P}_k$ does not affect the contour deformation, because it commutes with $T(w)$. Recall that, for physical primary operators, the OPE $T \cdot \phi_j$ was derived from the conformal Ward identity, and hence its singular terms exactly correspond to the holomorphic part of the

variation of the physical ϕ_j. Since the chiral OPEs mimick the physical ones, we get the conformal Ward identity:

$$\begin{aligned}\delta_\epsilon \langle 0|\phi_1(z_1)\phi_2(z_2)\mathcal{P}_k\phi_3(z_3)\phi_4(z_4)|0\rangle \\ = \frac{1}{2i\pi}\oint dw\,\epsilon(w)\,\langle 0|T(w)\phi_1(z_1)\phi_2(z_2)\mathcal{P}_k\phi_3(z_3)\phi_4(z_4)|0\rangle\,,\end{aligned} \tag{6.3.17}$$

where the integration contour encloses all points z_1, z_2, z_3, z_4.

Remark: Let us stress again the logic we have used here to derive this Ward identity for conformal blocks. In contrast to the Ward identities for physical correlation functions, the derivation does *not* rely on the definition of $T(z)$ as the stress-energy tensor associated to a fundamental action. Instead, we have defined the chiral primary operators through their matrix elements, chosen as to mimick the holomorphic part of their physical counterparts. Then, we have argued that, through a sequence of reversed computations, we can recover the chiral OPEs $T\cdot\phi_j$ and $\phi_i\cdot\phi_j$, which yield in turn the transformation properties of chiral correlators: the latter are thus also given by the holomorphic part of their physical counterparts.

6.4 Correlation functions of degenerate operators

So far, we have discussed generic conformal blocks, and described their main properties under conformal transformations. In order to make more advanced analysis, we need some additional knowledge on the admissible values of the internal dimensions h_k: for this reason, we shall now consider the case when some of the external ϕ_j's are degenerate primary operators. To illustrate the general argument, let us focus on four-point functions of the form

$$G(z,\bar z) = \langle \phi_1(\infty)\phi_2(1)\phi_{12}(z,\bar z)\phi_4(0)\rangle\,. \tag{6.4.1}$$

For this analysis, it will be convenient to use the following form of the differential action of the L_{-m}'s in the correlation function (or in the conformal block). For any set of primary operators $\phi_1,\phi_2,\phi_3,\phi_4$, we have:

$$\begin{aligned}\langle \phi_1(\infty)\phi_2(1)(L_{-m_1}\dots L_{-m_k}\phi_3)(z,\bar z)\phi_1(0)\rangle \\ = \mathcal{U}_1^{[m]}\dots\mathcal{U}_k^{[m]}\,\langle \phi_1(\infty)\phi_2(1)\phi_3(z,\bar z)\phi_1(0)\rangle\,,\end{aligned} \tag{6.4.2}$$

where the differential operators $\mathcal{U}_j^{[m]}$ are given by

$$\mathcal{U}_j^{[m]} = \frac{\alpha_{m_j-1}(z)}{z^{m_j-1}(1-z)^{m_j-1}}\partial_z + \frac{\beta_j^{[m]}(z)}{z^{m_j}(1-z)^{m_j}}\,, \tag{6.4.3}$$

and the polynomials $\alpha_m(z)$ and $\beta_j^{[m]}(z)$ are:

$$\alpha_{-1}(z) = 0\,,\qquad \alpha_0(z) = 1\,, \tag{6.4.4}$$

$$\alpha_m(z) = z^{m+1} - (z-1)^{m+1} = \sum_{\ell=0}^{m} z^\ell (z-1)^{m-\ell}\,, \tag{6.4.5}$$

$$\begin{aligned}\beta_j^{[m]}(z) &= (m_j-1)[h_2 z^{m_j} + h_4(z-1)^{m_j}] \\ &\quad + (h_1 - h_2 - h_4 - h_3 - m_{j+1} - \dots - m_k)z(z-1)\alpha_{m_j-2}(z)\,,\end{aligned} \tag{6.4.6}$$

which canbe proven using the commutation relations $[L_n, \phi_j(z_j, \bar{z}_j)]$.

In particular, we have:

$$\langle \phi_1(\infty)\phi_2(1)(L_{-1}^2\phi_3)(z,\bar{z})\phi_4(0)\rangle = \partial_z^2 \langle \phi_1(\infty)\phi_2(1)\phi_3(z,\bar{z})\phi_4(0)\rangle\,, \tag{6.4.7}$$

$$\begin{aligned}\langle \phi_1(\infty)\phi_2(1)(L_{-2}\phi_3)(z,\bar{z})\phi_4(0)\rangle = \\ \left[\frac{2z-1}{z(1-z)}\partial_z + \frac{h_2 z^2 + h_4(1-z)^2 - (h_1-h_2-h_4-h_3)z(1-z)}{z^2(1-z)^2}\right] \\ \times \langle \phi_1(\infty)\phi_2(1)\phi_3(z,\bar{z})\phi_4(0)\rangle\,.\end{aligned} \tag{6.4.8}$$

Recall the null-vector conditions for $\phi_3 = \phi_{12}$:

$$(L_{-2} - b^{-2}L_{-1}^2)\phi_{12} = 0\,, \qquad (\bar{L}_{-2} - b^{-2}\bar{L}_{-1}^2)\phi_{12} = 0\,. \tag{6.4.9}$$

They yield the two ordinary differential equations (ODEs) for the correlation function $G(z,\bar{z}) = \langle \phi_1(\infty)\phi_2(1)\phi_{12}(z,\bar{z})\phi_4(0)\rangle$:

$$\boxed{\begin{aligned}&\left[b^{-2}\,\partial_z^2 - \frac{2z-1}{z(1-z)}\partial_z - \frac{h_2 z^2 + h_4(1-z)^2 - (h_1-h_2-h_4-h_{12})z(1-z)}{z^2(1-z)^2}\right]G(z,\bar{z}) = 0\,,\\ &\left[b^{-2}\,\partial_{\bar{z}}^2 - \frac{2\bar{z}-1}{\bar{z}(1-\bar{z})}\partial_{\bar{z}} - \frac{h_2 \bar{z}^2 + h_4(1-\bar{z})^2 - (h_1-h_2-h_4-h_{12})\bar{z}(1-\bar{z})}{\bar{z}^2(1-\bar{z})^2}\right]G(z,\bar{z}) = 0\,.\end{aligned}} \tag{6.4.10}$$

An important property of this pair of differential equations is that their invariance under the change of parameters and variables:

$$(h_1,h_2,h_4) \leftrightarrow (h_1,h_4,h_2)\,, \qquad (z,\partial_z) \leftrightarrow (1-z,-\partial_z)\,, \qquad (\bar{z},\partial_{\bar{z}}) \leftrightarrow (1-\bar{z},-\partial_{\bar{z}})\,, \tag{6.4.11}$$

which reflects the identity

$$\langle \phi_1(\infty)\phi_2(1)\phi_3(z,\bar{z})\phi_4(0)\rangle = \langle \phi_1(\infty)\phi_4(1)\phi_3(1-z,1-\bar{z})\phi_2(0)\rangle\,. \tag{6.4.12}$$

To discuss the conformal blocks, for simplicity, we restrict to the case

$$G(z,\bar{z}) = \langle \phi_{rs}(\infty)\phi_{12}(1)\phi_{12}(z,\bar{z})\phi_{rs}(0)\rangle\,. \tag{6.4.13}$$

We have the fusion rules:

$$\phi_{12}\times\phi_{12} \to \mathbf{1} + \phi_{13}\,, \qquad \phi_{12}\times\phi_{rs} \to \phi_{r,s+1} + \phi_{r,s-1}\,, \tag{6.4.14}$$

and hence the two conformal blocks in the $z\to 0$ channel are

$$\begin{aligned}I_1(z) &= \langle \phi_{rs}(\infty)\phi_{12}(1)\mathcal{P}_{h_{r,s+1}}\phi_{12}(z)\phi_{rs}(0)\rangle\,,\\ I_2(z) &= \langle \phi_{rs}(\infty)\phi_{12}(1)\mathcal{P}_{h_{r,s-1}}\phi_{12}(z)\phi_{rs}(0)\rangle\,,\end{aligned} \tag{6.4.15}$$

and similarly in the $z\to 1$ channel:

$$\begin{aligned}J_1(z) &= \langle \phi_{rs}(\infty)\phi_{rs}(1)\mathcal{P}_{\mathbf{1}}\phi_{12}(1-z)\phi_{12}(0)\rangle\,,\\ J_2(z) &= \langle \phi_{rs}(\infty)\phi_{rs}(1)\mathcal{P}_{h_{13}}\phi_{12}(1-z)\phi_{12}(0)\rangle\,.\end{aligned} \tag{6.4.16}$$

The differential equations obeyed by $G(z,\bar{z})$ were derived using the commutators $[L_n, \phi_j(z,\bar{z})]$. Since the commutators with chiral operators $\phi_j(z)$ and $\bar{\phi}_j(\bar{z})$ have the same form, and

the L_n's commute with the projectors $\mathcal{P}_h$, we get the same differential equations for the conformal blocks I_k and $\bar{I}_k$. From the invariance of these equations under (6.4.11), the conformal blocks J_k and $\bar{J}_k$ are also solutions of these equations. Let us write the holomorphic and anti-holomorphic differential equations with $h_1 = h_4 = h_{rs}$ and $h_2 = h_{12}$ as

$$\left[b^{-2}\,\partial_z^2 - \frac{2z-1}{z(1-z)}\partial_z - \frac{h_{rs}(1-z)^2 + h_{12}z(2-z)}{z^2(1-z)^2}\right]F(z) = 0\,, \tag{6.4.17}$$

$$\left[b^{-2}\,\partial_{\bar{z}}^2 - \frac{2\bar{z}-1}{\bar{z}(1-\bar{z})}\partial_{\bar{z}} - \frac{h_{rs}(1-\bar{z})^2 + h_{12}\bar{z}(2-\bar{z})}{\bar{z}^2(1-\bar{z})^2}\right]\bar{F}(\bar{z}) = 0\,. \tag{6.4.18}$$

Let us discuss the holomorphic differential equation (6.4.17). The two sets of conformal blocks $\{I_1, I_2\}$ and $\{J_1, J_2\}$ are two bases of holomorphic solutions of this equation. By construction, the block $I_k(z)$ is a function of the form:

$$I_k(z) = z^\lambda \sum_{n=0}^{\infty} a_n\, z^n\,. \tag{6.4.19}$$

Inserting this form into the differential equation, and keeping only the dominant term in the limit $z \to 0$, we get the characteristic equation for the *local exponent* λ:

$$b^{-2}\lambda(\lambda-1) + \lambda - h_{rs} = 0\,. \tag{6.4.20}$$

The two solutions are

$$\lambda_1 = -h_{12} - h_{rs} + h_{r,s+1}\,, \qquad \lambda_2 = -h_{12} - h_{rs} + h_{r,s-1}\,, \tag{6.4.21}$$

corresponding to the solutions I_1, I_2. The differential equation turns into a linear recursion system for the series coefficients a_n. A convenient way of treating this problem is to introduce the operator $\theta = z\partial_z$, which satisfies

$$\theta \cdot z^\lambda = \lambda\, z^\lambda\,. \tag{6.4.22}$$

Using the property $z^2\partial_z^2 = \theta(\theta-1)$, we rewrite the equation as

$$\left\{b^{-2}(1-z)^2\,\theta(\theta-1) - (1-z)(2z-1)\theta - \left[h_{rs}(1-z)^2 + h_{12}z(2-z)\right]\right\}F(z) = 0\,, \tag{6.4.23}$$

which is of the form

$$\left[R_0(\theta) + z\,R_1(\theta) + z^2\,R_2(\theta)\right]F(z) = 0\,, \tag{6.4.24}$$

where R_0, R_1, R_2 are polynomials of degree two. Inserting the power series, we get the recursion

$$R_0(\lambda)a_0 = 0\,, \tag{6.4.25}$$

$$R_0(\lambda+1)a_1 + R_1(\lambda)a_0 = 0\,, \tag{6.4.26}$$

$$R_0(\lambda+n)a_n + R_1(\lambda+n-1)a_{n-1} + R_2(\lambda+n-2)a_{n-2} = 0 \qquad n \geq 2\,. \tag{6.4.27}$$

The explicit form of R_0 is

$$R_0(\theta) = b^{-2}\theta(\theta-1) + \theta - h_{rs}\,, \tag{6.4.28}$$

and hence, since we have set λ to one of the solutions $\{\lambda_1, \lambda_2\}$ of the characteristic equation, we have $R_0(\lambda) = 0$, and we can set $a_0 = 1$ by convention. Moreover, we assume

that the local exponents are non-degenerate, namely $(\lambda_1 - \lambda_2)$ is *not* an integer, so that for any positive integer n, we have $R_0(\lambda + n) \neq 0$. As a result, all the coefficients a_n are completely determined by the recursion. This way of solving a holomorphic differential equation is known as the Froebenius method. A similar approach can be used to determine the conformal blocks in the $z \to 1$ channel:

$$J_k(z) = (1-z)^\mu \sum_{n=0}^{\infty} b_n (1-z)^n \,. \tag{6.4.29}$$

The physical correlation function is reconstructed from the conformal blocks as

$$G(z,\bar{z}) = (C_{\phi_{12},\phi_{rs}}^{\phi_{r,s+1}})^2 \, |I_1(z)|^2 + (C_{\phi_{12},\phi_{rs}}^{\phi_{r,s-1}})^2 \, |I_2(z)|^2 \tag{6.4.30}$$

$$= C_{\phi_{12},\phi_{12}}^{\mathbf{1}} C_{\phi_{rs},\phi_{rs}}^{\mathbf{1}} |J_1(z)|^2 + C_{\phi_{12},\phi_{12}}^{\phi_{13}} C_{\phi_{rs},\phi_{rs}}^{\phi_{13}} |J_2(z)|^2 \,. \tag{6.4.31}$$

6.5 Crossing symmetry

This identity between the two expansions of the physical correlation function is known as *crossing symmetry*. An important additional ingredient is the linear relation between the two bases of conformal blocks

$$I_k(z) = \sum_{\ell=1}^{2} P_{k\ell} \, J_\ell(z) \,. \tag{6.5.1}$$

Recall that $\{I_1, I_2\}$ and $\{J_1, J_2\}$ are bases of holomorphic solutions to the same differential equation, which is why they are related by a matrix P, known as the fusion matrix.

If we generalise to an expansion of the form:

$$G(z,\bar{z}) = \sum_{i,j=1}^{2} X_{ij} \, \bar{I}_i(\bar{z}) \, I_j(z) \,, \tag{6.5.2}$$

after the change of basis, we get:

$$G(z,\bar{z}) = \sum_{k,\ell=1}^{2} Y_{k\ell} \, \bar{J}_k(\bar{z}) \, J_\ell(z) \,, \qquad Y = P^\dagger X P \,. \tag{6.5.3}$$

The particular choice

$$X_{ij} = X_i \, \delta_{ij} \,, \qquad Y_{k\ell} = Y_k \, \delta_{k\ell} \,, \tag{6.5.4}$$

with

$$X_1 = (C_{\phi_{12},\phi_{rs}}^{\phi_{r,s+1}})^2 \,, \quad X_2 = (C_{\phi_{12},\phi_{rs}}^{\phi_{r,s-1}})^2 \,, \quad Y_1 = C_{\phi_{12},\phi_{12}}^{\mathbf{1}} \, C_{\phi_{rs},\phi_{rs}}^{\mathbf{1}} \,, \quad Y_2 = C_{\phi_{12},\phi_{12}}^{\phi_{13}} \, C_{\phi_{rs},\phi_{rs}}^{\phi_{13}} \,, \tag{6.5.5}$$

corresponds to our physical correlation function. Different forms for the matrices X and Y may arise if we allow G to have non-trivial monodromies as $\phi_{12}(z,\bar{z})$ winds around $\phi_{rs}(0)$ or $\phi_{12}(1)$. The conditions $Y_{k\ell} = 0$ (resp. $X_{k\ell} = 0$) for $k \neq \ell$ lead to the linear system for the X_i's (resp. Y_i's):

$$\boxed{\forall k \neq \ell \,, \qquad \sum_{i=1}^{2} \bar{P}_{ik} \, P_{i\ell} \, X_i = 0 \,, \qquad \sum_{i=1}^{2} (\bar{P}^{-1})_{ik} \, (P^{-1})_{i\ell} \, Y_i = 0 \,.} \tag{6.5.6}$$

The crucial ingredient in the determination of $\{X_i\}$ and $\{Y_k\}$ is the fusion matrix P relating the two bases of conformal blocks. For the case of a differential equation of order N, a general method, known as the Coulomb-Gas approach, and based on the representation of I_k and J_k as contour integrals over $(N-1)$ auxiliary variables, may be used to calculate the matrix elements of A. However, in this lecture, we are focusing on the simplest case

$$G(z,\bar{z}) = \langle \phi_h(\infty,\infty)\phi_{12}(1,1)\phi_{12}(z,\bar{z})\phi_h(0,0)\rangle\,, \tag{6.5.7}$$

where the ODE is second order, and can be reduced to the *hypergeometric differential equation*. Using the Kac parameterisation $c = 1-6Q^2$, and $h_\alpha = \alpha(\alpha-2Q)$, with $2Q = b^{-1}-b$, we have the fusion :

$$\phi_{12} \times \phi_{h_\alpha} = \phi_{h_{\alpha+b/2}} + \phi_{h_{\alpha-b/2}}\,. \tag{6.5.8}$$

From the simple identity

$$h_{\alpha+\beta} = h_\alpha + h_\beta + 2\alpha\beta\,,$$

we see that the conformal blocks at $z = 0$ with internal dimensions $h_{\alpha+b/2}$ and $h_{\alpha-b/2}$, respectively, are of the form:

$$I_1(z) = z^{b\alpha}(1 + a_1 z + \dots)\,, \qquad I_2(z) = z^{b(2Q-\alpha)}(1 + b_1 z + \dots)\,. \tag{6.5.9}$$

In the channel $z \to 1$, we have the fusion

$$\phi_{12} \times \phi_{12} = \mathbf{1} + \phi_{13}\,, \tag{6.5.10}$$

and the conformal blocks at $z = 1$ are given by:

$$J_1(z) = z^{-2h_{12}}(1 + c_1 z + \dots)\,, \qquad J_2(z) = z^{h_{13}-2h_{12}}(1 + d_1 z + \dots)\,. \tag{6.5.11}$$

To study the local exponents at $z \to \infty$, let us first use conformal invariance under $z \mapsto 1/z$, which yields the relation:

$$G(z,\bar{z}) = |z|^{-4h_{12}}\, G(1/z, 1/\bar{z})\,. \tag{6.5.12}$$

The local exponents at $z \to \infty$ are thus $2h_{12} + b\alpha$ and $2h_{12} + b(2Q-\alpha)$. The Riemann scheme for the ODE satisfied by G is:

$$\begin{array}{ccc} 0 & 1 & \infty \\ \hline b\alpha & -2h_{12} & 2h_{12}+b\alpha \\ b(2Q-\alpha) & h_{13}-2h_{12} & 2h_{12}+b(2Q-\alpha) \end{array}$$

We define the function $g(z,\bar{z})$ as

$$G(z,\bar{z}) := |z|^{2b\alpha}\,|1-z|^{-4h_{12}}\, g(z,\bar{z})\,. \tag{6.5.13}$$

This function g satisfies a second-order differential equation, with local exponents

$$\begin{array}{ccc} 0 & 1 & \infty \\ \hline 0 & 0 & 2b\alpha \\ 1-b^2-2b\alpha & 2b^2-1 & 1-b^2 \end{array} \;=\; \begin{array}{ccc} 0 & 1 & \infty \\ \hline 0 & 0 & A \\ 1-C & C-A-B & B \end{array} \tag{6.5.14}$$

with

$$A = 2b\alpha\,, \qquad B = 1-b^2\,, \qquad C = b^2+2b\alpha\,. \tag{6.5.15}$$

The differential equation for g takes the explicit form

$$\left[\partial_z^2 + \frac{C-(A+B+1)z}{z(1-z)}\partial_z - \frac{AB}{z(1-z)}\right] g = 0\,, \tag{6.5.16}$$

which is known as the hypergeometric differential equation. The fusion matrix for the holomorphic solutions of this equation are known (see Appendix), and we have:

$$\begin{aligned} P &= \begin{bmatrix} \frac{\Gamma(C)\Gamma(D)}{\Gamma(C-A)\Gamma(C-B)} & \frac{\Gamma(C)\Gamma(-D)}{\Gamma(A)\Gamma(B)} \\ \frac{\Gamma(2+C)\Gamma(D)}{\Gamma(1-A)\Gamma(1-B)} & \frac{\Gamma(2-C)\Gamma(-D)}{\Gamma(1-C+A)\Gamma(1-C+B)} \end{bmatrix}, \\ P^{-1} &= \begin{bmatrix} \frac{\Gamma(1-C)\Gamma(1-D)}{\Gamma(1-C+A)\Gamma(1-C+B)} & \frac{\Gamma(C-1)\Gamma(1-D)}{\Gamma(A)\Gamma(B)} \\ \frac{\Gamma(1-C)\Gamma(1+D)}{\Gamma(1-A)\Gamma(1-B)} & \frac{\Gamma(C-1)\Gamma(1+D)}{\Gamma(C-A)\Gamma(C-B)} \end{bmatrix}, \end{aligned} \tag{6.5.17}$$

with $D = C - A - B = 2b^2 - 1$. Using $P = \bar{P}$, the linear system for Y_1, Y_2 yields the ratio:

$$\frac{Y_2}{Y_1} = \frac{\gamma(C-A)\gamma(C-B)\gamma(-D)}{\gamma(A)\gamma(B)\gamma(D)} = \frac{\gamma^2(b^2)\gamma(2-2b^2)\gamma(1-2b^2)}{\gamma(2b\alpha)\gamma[2b(2Q-\alpha)]}\,, \tag{6.5.18}$$

where $\gamma(x) := \Gamma(x)/\Gamma(1-x)$. The coefficients Y_1, Y_2 are related to the structure constants:

$$Y_1(\alpha) = C^{\mathbf{1}}_{\phi_{12},\phi_{12}} \times C^{\mathbf{1}}_{\phi_{rs},\phi_{rs}} = 1\,, \qquad Y_2(\alpha) = C^{\phi_{13}}_{\phi_{12},\phi_{12}} \times C^{\phi_{13}}_{\phi_{rs},\phi_{rs}}\,, \tag{6.5.19}$$

with $h = \alpha(\alpha - 2Q)$. We obtain the non-trivial structure constant by combining the equations for generic $\alpha = \alpha_{rs} = [(1-r)/b - (1-s)b]/2$ and $\alpha = \alpha_{12} = b/2$:

$$\boxed{C^{\phi_{13}}_{\phi_{rs},\phi_{rs}} = \frac{Y_2(\alpha)}{\sqrt{Y_2(b/2)}} = \frac{\sqrt{\gamma^3(b^2)\gamma(2-2b^2)\gamma(1-2b^2)\gamma(2-3b^2)}}{\gamma(2b\alpha)\gamma[2b(2Q-\alpha)]}\,.} \tag{6.5.20}$$

We can then compute the coefficients X_1, X_2:

$$\begin{aligned} X_1 &= (P^{-1})^2_{11}\, Y_1 + (P^{-1})^2_{21}\, Y_2 = \frac{\gamma(C-A)\gamma(C-B)}{\gamma(C)\gamma(D)}\,, \\ X_2 &= (P^{-1})^2_{12}\, Y_1 + (P^{-1})^2_{22}\, Y_2 = \frac{\gamma(C-1)\gamma(D)}{\gamma(A)\gamma(B)}\,. \end{aligned} \tag{6.5.21}$$

This yields the OPE structure constants:

$$\boxed{\begin{aligned} C^{\phi_{r,s+1}}_{\phi_{12},\phi_{rs}} &= \sqrt{\frac{\gamma(b^2)\gamma(2-2b^2)}{\gamma[2b(\alpha+b/2)]\gamma[2b(2Q-\alpha)]}}\,, \\ C^{\phi_{r,s-1}}_{\phi_{12},\phi_{rs}} &= \sqrt{\frac{\gamma(b^2)\gamma(2-2b^2)}{\gamma[2b(2Q-\alpha+b/2)]\gamma(2b\alpha)}}\,. \end{aligned}} \tag{6.5.22}$$

6.6 Appendix: Fuchsian differential equations

6.6.1 Regular singularities

Consider a linear differential equation of the form

$$\left[\partial_z^N + \frac{p_1(z)}{z(1-z)}\partial_z^{N-1} + \cdots + \frac{p_N(z)}{z^N(1-z)^N}\right] G(z,\bar{z}) = 0\,, \tag{6.6.1}$$

where $p_j(z)$ is a polynomial of degree j or smaller.

At $z = 0$, the term $\partial_z^{N-j} G$ has a pole of order at most j: we say that $z = 0$ is a *regular singularity* of the ODE. Similarly, the point $z = 1$ is a regular singularity.

To study the solutions of (6.6.1) in the vicinity of a singular point $w \in \{0, 1, \infty\}$, it will be convenient to express the ODE in terms of the differential operator

$$\theta_w := (z - w)\partial_z \,,$$

with the property

$$(z - w)^k \partial_z^k = \theta_w(\theta_w - 1) \dots (\theta_w - k + 1) \,. \tag{6.6.2}$$

For instance, at $w = 0$, after multiplying the ODE by z^N, we get an equation of the form:

$$\left[\theta^N + q_1(z)\theta^{N-1} + \dots + q_{N-1}(z)\theta + q_N(z)\right] G = 0 \,, \tag{6.6.3}$$

where we have denoted simply $\theta := \theta_0$, and the functions $q_j(z)$ are linear combinations of $p_\ell(z)/(1 - z)^\ell$, and thus, they are regular as $z \to 0$:

$$q_j(z) := \sum_{\ell=1}^{N} \frac{\alpha_{j\ell}\, p_\ell(z)}{(1 - z)^\ell} \,, \qquad p_\ell(z) = \sum_{n=0}^{j} p_{\ell,n}\, z^n \tag{6.6.4}$$

For $w = \infty$, we use the variable $u = 1/z$, and define $\widehat{\theta} := u\partial_u = -z\partial_z$. Equation (6.6.3) takes the form:

$$\left[\widehat{\theta}^N + \widehat{q}_1(u)\widehat{\theta}^{N-1} + \dots + \widehat{q}_{N-1}(u)\widehat{\theta} + \widehat{q}_N(u)\right] G = 0 \,, \tag{6.6.5}$$

where

$$\widehat{q}_j(u) := \sum_{\ell=1}^{N} \frac{\alpha_{j\ell}\, \widehat{p}_\ell(u)}{(1 - u)^\ell} \,, \qquad \widehat{p}_\ell(u) = \sum_{n=0}^{\ell} p_{\ell,\ell-n}\, u^n \,. \tag{6.6.6}$$

Since the coefficients $\widehat{q}_j(u)$ are regular as $u \to 0$, the point $z = \infty$ is also a regular singularity.

Hence the three singular points $\{0, 1, \infty\}$ of the ODE (6.6.1) are *regular* singularities: this type of complex differential equation is called a *Fuchsian differential equation*.

6.6.2 Basis of solutions in the vicinity of a singularity

Let us look for solutions which behave as a power law in the z variable in the vicinity of the singularity $z = 0$. Inserting $G(z, \bar{z}) \sim z^\lambda$ into the ODE (6.6.3), and using $\theta . z^\lambda = \lambda z^\lambda$, we find that the exponent λ should be a solution of the characteristic equation:

$$R(\lambda) := \lambda^N + \sum_{j=0}^{N-1} q_{N-j}(0)\, \lambda^j = 0 \,. \tag{6.6.7}$$

The roots of R are called the local exponents of the ODE at $z = 0$. Now we consider a local exponent λ, and we shall construct a solution of the form

$$z^\lambda f(z) = z^\lambda \sum_{n=0}^{\infty} a_n\, z^n \,, \qquad a_0 = 1 \,, \tag{6.6.8}$$

for $0 < |z| < \rho$, and $-\pi < \operatorname{Arg} z < \pi$, where ρ is the radius of convergence of the solution ($\rho \leq 1$, because $z = 1$ is a singular point). Let us write the series expansion of the coefficients q_j as

$$q_j(z) = \sum_{n=1}^{\infty} q_{j,n}\, z^n \qquad \text{for } |z| < 1\,. \tag{6.6.9}$$

Then, inserting $G(z) = z^\lambda f_\lambda(z)$ into (6.6.3), we get

$$\sum_{n=0}^{\infty} \left[(\lambda+n)^N a_n + \sum_{j=0}^{N-1} \sum_{k=0}^{n} (\lambda+k)^j q_{N-j,n-k}\, a_k \right] z^{\lambda+n} = 0\,. \tag{6.6.10}$$

Introducing the polynomials

$$Q_k(\lambda) := \sum_{j=0}^{N-1} q_{N-j,k}\, \lambda^j \qquad \text{for} \quad k \geq 1\,,$$

This can be rewritten as

$$z^\lambda \sum_{n=0}^{\infty} \left[R(\lambda+n)\, a_n + \sum_{k=0}^{n-1} Q_{n-k}(\lambda+k)\, a_k \right] z^n = 0\,. \tag{6.6.11}$$

For any n, the coefficients $a_0, \dots a_n$ of the function $f_\lambda(z)$ are determined by an $(n+1) \times (n+1)$ lower-triangular linear system:

$$\begin{aligned} &R(\lambda)a_0 &&= 0 \\ &Q_1(\lambda)a_0 + R(\lambda+1)a_1 &&= 0 \\ &\dots \\ &Q_n(\lambda)a_0 + \dots + Q_1(\lambda+n-1)a_{n-1} + R(\lambda+n)a_n &&= 0 \end{aligned} \tag{6.6.12}$$

Since $R(\lambda) = 0$, the leading coefficient can always be set to $a_0 = 1$. Let us consider the "generic" case, when the roots of R *do not* differ by an integer, so that, for any integer $n \geq 1$, $R(\lambda+n) \neq 0$. The entire function $f_\lambda(z)$ is then defined uniquely by the recursion relation:

$$a_0 = 0\,, \qquad a_n = -\frac{1}{R(\lambda+n)} \sum_{k=0}^{n-1} Q_{n-k}(\lambda+n-k)\, a_k \qquad \text{for } n \geq 1\,. \tag{6.6.13}$$

From this procedure, called the *Froebenius method*, we get a basis of solutions to the ODE:

$$\{ z^{\lambda_1} f_1(z)\,, z^{\lambda_2} f_2(z)\,, \dots\,, z^{\lambda_N} f_N(z) \}\,, \tag{6.6.14}$$

where $\lambda_1, \dots, \lambda_N$ are the roots of the characteristic equation. It is possible to show that each entire function $f_j(z)$ admits an analytic continuation on $\mathbb{C}\backslash\{0, 1, \infty\}$.

6.6.3 The case of degenerate exponents

If R admits two roots of the form $(\lambda_1, \lambda_2) = (\lambda, \lambda + r)$ with $r \in \mathbb{N}$, we say that the local exponent λ is degenerate. Let us first treat the case $r \geq 1$. The recursion system (6.6.12) for λ_2 has a unique solution, which defines the function $z^{\lambda+r} f_2(z)$, with $f_2(0) = 1$. For the solution associated to λ_1, after setting $a_0 = 1$, the coefficients $a_1, \dots a_{r-1}$ are fixed uniquely, but a_r can be set to any value $a_r = \alpha$, which defines the one-parameter family of solutions

$z^\lambda f_1^{(\alpha)}(z)$, and we have $f_1^{(\alpha)}(z) = f_1^{(0)}(z) + \alpha z^r f_2(z)$. For any value of α, the functions $z^\lambda f_1^{(\alpha)}(z)$ and $z^{\lambda+r} f_2(z)$ are independent solutions of the form (6.6.8). In the case of a multiply-degenerate exponent λ, where the roots of R are of the form

$$(\lambda_1, \dots, \lambda_m) = (\lambda, \lambda + r_1, \dots, \lambda + r_{m-1}), \tag{6.6.15}$$

with $0 < r_1 < r_2 < \cdots < r_{m-1}$, using the same method, one can construct a set of m independent solutions

$$\{z^\lambda f_1(z), z^{\lambda+r_1} f_2(z), \dots, z^{\lambda+r_{m-1}} f_m(z)\} \tag{6.6.16}$$

with $f_j(0) = 1$. The function f_m is unique, whereas, for the entire function f_j associated to $\lambda + r_j$ with $1 \le j < m-1$, we can choose freely the values of the coefficients $a_j, \dots, a_{m-1}$.

In the case of a double root $\lambda_1 = \lambda_2 = \lambda$, if no other root is of the form $\lambda + \mathbb{Z}$, the recursion system (6.6.12) gives a unique solution $z^\lambda f_\lambda(z)$. To build intuitively a second, independent solution, it is useful to think of this situation as the limit $\epsilon \to 0$ of an ODE with exponents $\lambda, \lambda + \epsilon$. Under this hypothesis, the difference of the two independent solutions $z^\lambda f_\lambda(z), z^{\lambda+\varepsilon} f_{\lambda+\varepsilon}(z)$ would be of the form:

$$z^{\lambda+\varepsilon} \sum_{n=0}^{\infty} a_n(\lambda+\varepsilon)\, z^n - z^\lambda \sum_{n=0}^{\infty} a_n(\lambda)\, z^n \underset{\epsilon\to 0}{\sim} \epsilon\, z^\lambda \left[\ln z \sum_{n=0}^{\infty} a_n\, z^n + \sum_{n=0}^{\infty} b_n\, z^n \right], \tag{6.6.17}$$

where $a_n = a_n(\lambda)$ and $b_n = \partial_\lambda a_n(\lambda)$. Hence, we look for a solution of the form

$$z^\lambda \left[g_\lambda(z) + \ln z\, f_\lambda(z) \right], \qquad \text{where} \quad g_\lambda(z) = \sum_{n=0}^{\infty} b_n\, z^n, \tag{6.6.18}$$

and $z^\lambda f_\lambda(z)$ is the "regular" solution. Inserting the above form into the ODE, and using

$$\theta^k.(z^\lambda \ln z) = \lambda^k z^\lambda \ln z + k\lambda^{k-1} z^\lambda,$$

we get the relations for $n \ge 0$:

$$R(\lambda+n)a_n + \sum_{k=0}^{n-1} Q_{n-k}(\lambda+k)a_k = 0, \tag{6.6.19}$$

$$\left[R(\lambda+n)b_n + \sum_{k=0}^{n-1} Q_{n-k}(\lambda+k)b_k \right] + \left[R'(\lambda+n)a_n + \sum_{k=0}^{n-1} Q'_{n-k}(\lambda+k)a_k \right] = 0. \tag{6.6.20}$$

From the first relation, we recover the regular solution if we set $a_0 = 1$. In the second relation, since $R(\lambda) = R'(\lambda) = 0$, we can set b_0 to any value $b_0 = \alpha$, and this defines uniquely the function $g_\lambda^{(\alpha)}(z) = g_\lambda^{(0)}(z) + \alpha f_\lambda(z)$. For any value of α, the functions

$$\{z^\lambda f_\lambda(z), \quad z^\lambda [g_\lambda^{(0)}(z) + (\alpha + \ln z) f_\lambda(z)]\} \tag{6.6.21}$$

are independent solutions.

Similarly, for a root λ of multiplicity m, we can construct linearly independent solutions $\{\varphi_1(z), \dots \varphi_m(z)\}$ of the form

$$\varphi_1(z) = z^\lambda f_1(z), \tag{6.6.22}$$

$$\varphi_2(z) = z^\lambda \left[\ln z\, f_1(z) + f_2(z) \right], \tag{6.6.23}$$

$$\dots \tag{6.6.24}$$

$$\varphi_k(z) = z^\lambda \sum_{r=0}^{k-1} \binom{k-1}{r} (\ln z)^{k-1-r} f_{r+1}(z), \tag{6.6.25}$$

To describe the determination of the entire functions $f_1(z), \ldots, f_m(z)$, let us introduce the polynomial of degree N in t:

$$\mathcal{D}(z,t) = t^N + q_1(z)t^{N-1} + \cdots + q_N(z)\,, \tag{6.6.26}$$

so that the ODE reads $\mathcal{D}(z,\theta)G = 0$. Using the identity

$$\theta^p.[(\ln z)^r\,\varphi(z)] = \sum_{\ell=0}^{\min(r,p)} \binom{r}{\ell} (\ln z)^{r-\ell}\,\frac{p!}{(p-\ell)!}\theta^{p-\ell}.\varphi(z)\,, \tag{6.6.27}$$

we get

$$\mathcal{D}(z,\theta).[(\ln z)^r\,\varphi(z)] = \sum_{\ell=0}^{r} \binom{r}{\ell} (\ln z)^{r-\ell}\,\mathcal{D}^{(\ell)}(z,\theta).\varphi(z)\,, \tag{6.6.28}$$

where $\mathcal{D}^{(\ell)}(z,t) := (\partial_t)^\ell \mathcal{D}(z,t)$ is also a polynomial in t. The function $f_1(z)$ is determined by the homogeneous ODE:

$$\mathcal{D}(z,\theta).[z^\lambda f_1(z)] = 0\,. \tag{6.6.29}$$

Then, the condition $\mathcal{D}(z,\theta).\varphi_2 = 0$ yields:

$$\mathcal{D}^{(1)}(z,\theta).[z^\lambda f_1(z)] + \mathcal{D}(z,\theta).[z^\lambda f_2(z)] = 0\,, \tag{6.6.30}$$

which is an ODE for $f_2(z)$, with a source term given by the knowledge of $f_1(z)$. The functions $f_k(z)$ are determined recursively by the coefficient of $(\ln z)^0$ in the ODE $\mathcal{D}(z,\theta).\varphi_k = 0$, which reads:

$$\sum_{r=0}^{k-1} \binom{k-1}{r} \mathcal{D}^{(r)}(z,\theta).[z^\lambda f_{k-r}(z)] = 0\,. \tag{6.6.31}$$

The value of the polynomials at $z = 0$:

$$\mathcal{D}^{(\ell)}(0,\lambda) = R^{(\ell)}(\lambda) = 0 \qquad \text{for} \quad \ell = 0, \ldots, m-1\,, \tag{6.6.32}$$

allow us to set $f_k(0) = 1$. Then, the higher coefficients in the power series for $f_k(z)$ are uniquely determined by the above ODE.

6.6.4 Riemann scheme of the differential equation

For each singular point $w \in \{0, 1, \infty\}$, we can apply the Froebenius method, and construct a basis with local exponents given by the roots of the characteristic polynomial $R_w(\lambda)$. We denote $\{\lambda_1, \ldots, \lambda_N\}$ the local exponents at $z = 0$, $\{\mu_1, \ldots, \mu_N\}$ those at $z = 1$, and $\{\nu_1, \ldots, \nu_N\}$ those at $z = \infty$. These data are stored in the Riemann scheme:

0	1	∞
λ_1	μ_1	ν_1
λ_2	μ_2	ν_2
$\vdots$	$\vdots$	$\vdots$
λ_N	μ_N	ν_N

Using the characteristic equations (6.6.7), one can prove Fuchs' relation:

$$\sum_{j=1}^{N} (\lambda_j + \mu_j + \nu_j) = \frac{N(N-1)}{2}\,. \tag{6.6.33}$$

If we change the unknown function to $g(z)$ with $G(z) = z^\alpha\, g(z)$, then from the relation

$$\theta.[z^\alpha \varphi(z)] = z^\alpha(\theta+\alpha).\varphi(z)\,, \tag{6.6.34}$$

we get

$$\left[(\theta+\alpha)^N + \sum_{j=0}^{N-1} q_{N-j}(z)\,(\theta+\alpha)^j\right] g(z) = 0\,, \tag{6.6.35}$$

and hence, the local exponents for g at $z=0$ are $\{\lambda_1-\alpha,\dots\lambda_N-\alpha\}$, and those at $z=\infty$ are $\{\nu_1+\alpha,\dots\nu_N+\alpha\}$. We can similarly introduce a factor $(1-z)^\beta$, to shift the exponents μ_j and ν_j by $-\beta$ and $+\beta$, respectively. Overall, if we make the change of function

$$G(z) = z^\alpha\,(1-z)^\beta\, g(z)\,, \tag{6.6.36}$$

the new function $g(z)$ satisfies a Fuchsian equation with Riemann scheme

0	1	∞
$\lambda_1-\alpha$	$\mu_1-\beta$	$\nu_1+\alpha+\beta$
$\lambda_2-\alpha$	$\mu_2-\beta$	$\nu_2+\alpha+\beta$
$\vdots$	$\vdots$	$\vdots$
$\lambda_N-\alpha$	$\mu_N-\beta$	$\nu_N+\alpha+\beta$

Of course, the degeneracies of exponents, as well as the Fuchs relation are preserved by this operation.

6.6.5 Monodromy of solutions

Let $\{\varphi_1(z),\dots\varphi_N(z)\}$ be a basis of solutions of the Fuchsian ODE (6.6.1). Then, for any singular point w, if we apply a rotation of 2π around w, we get the functions $\{\varphi_1(w+e^{2i\pi}(z-w)),\dots\varphi_N(w+e^{2i\pi}(z-w))\}$, which are again solutions of the ODE. Hence, there exists an invertible matrix $M^{(w)}$ (the monodromy matrix around w) such that

$$\varphi_j(w+e^{2i\pi}(z-w)) = \sum_{k=1}^{N} M^{(w)}_{jk}\,\varphi_k(z)\,. \tag{6.6.37}$$

In the case when the characteristic polynomial $R(\lambda)$ at $z=0$ has no multiple roots, the monodromy matrix for the basis $\{z^{\lambda_1}f_1(z),\dots z^{\lambda_N}f_N(z)\}$ constructed above is the diagonal matrix:

$$M^{(0)} = \mathrm{diag}(e^{2i\pi\lambda_1},\dots,e^{2i\pi\lambda_N})\,.$$

If $P(\lambda)$ has multiple roots, then $M^{(0)}$ has some additional off-diagonal terms, and is no longer diagonalisable.

Given a basis of solutions $\{\varphi_1(z),\dots\varphi_N(z)\}$ expressed in terms of series at $z=0$, it is a harder task to determine its monodromy around $z=1$. A way to do this, is to use the *fusion matrix* from $\{\varphi_1(z),\dots\varphi_N(z)\}$ to a basis $\{\psi_1(z),\dots\psi_N(z)\}$ of solutions defined in the vicinity of $z=1$:

$$\varphi_j(z) = \sum_{k=1}^{N} P_{jk}\,\psi_k(z)\,. \tag{6.6.38}$$

Then, the monodromy of the basis $\{\varphi_1(z),\dots\varphi_N(z)\}$ around $z=1$ is given by:

$$M^{(1)} = P.\mathrm{diag}(e^{2i\pi\mu_1},\dots,e^{2i\pi\mu_N}).P^{-1}\,. \tag{6.6.39}$$

6.6.6 Example: the hypergeometric equation

Consider a second-order Fuchsian equation with the Riemann scheme:

$$\begin{array}{ccc} 0 & 1 & \infty \\ \hline 0 & 0 & A \\ 1-C & C-A-B & B \end{array}$$

By identifying the coefficients of the characteristic polynomials at $z = 0, 1, \infty$, one can show that the corresponding ODE must be given by the *hypergeometric equation*:

$$\left[\partial_z^2 + \frac{C-(A+B+1)z}{z(1-z)}\partial_z - \frac{AB}{z(1-z)}\right] g = 0\,. \tag{6.6.40}$$

As long as $C \notin \mathbb{Z}$, the Froebenius method at $z = 0$ and $z = 1$ respectively provides the bases of solutions:

$$I_1(z) = {}_2\mathrm{F}_1(A,B;C|z)\,, \qquad I_2(z) = z^{1-C}\,{}_2\mathrm{F}_1(1+B-C,1+A-C;2-C|z)\,, \tag{6.6.41}$$

$$J_1(z) = {}_2\mathrm{F}_1(A,B;1-D|1-z)\,, \quad J_2(z) = (1-z)^D\,{}_2\mathrm{F}_1(C-A,C-B;1+D|1-z)\,, \tag{6.6.42}$$

where $D = C - A - B$. The function ${}_2\mathrm{F}_1(A,B;C|z)$ is Gauss's hypergeometric series

$${}_2\mathrm{F}_1(A,B;C|z) = \sum_{n=0}^{\infty} \frac{(A)_n\,(B)_n}{(C)_n\,n!}\,z^n\,, \tag{6.6.43}$$

where we have used the Pochhammer symbol:

$$(u)_n = u(u+1)\ldots(u+n-1) = \frac{\Gamma(u+n)}{\Gamma(u)}\,. \tag{6.6.44}$$

Using the Euler Beta function

$$\mathbf{B}(u,v) = \int_0^1 t^{u-1}\,(1-t)^{v-1}\,dt = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} \qquad \text{for} \quad \mathrm{Re}(u), \mathrm{Re}(v) > 0\,, \tag{6.6.45}$$

One can write the hypergeometric function as

$${}_2\mathrm{F}_1(A,B;C|z) = \frac{1}{\mathbf{B}(B,C-B)}\int_0^1 x^{B-1}(1-x)^{C-B-1}(1-zx)^{-A}\,dx\,, \tag{6.6.46}$$

for $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$ and $z \in \mathbb{C}\backslash[1,+\infty[$. Suppose $z \in]0,1[$. Using the change of variables $x = 1/t$ and $x = t/z$, respectively, we get

$$I_1(z) = \frac{1}{N_1}\int_1^{\infty} t^{A-C}(t-z)^{-A}(t-1)^{C-B-1}dt\,, \tag{6.6.47}$$

$$I_2(z) = \frac{1}{N_2}\int_0^{z} t^{A-C}(z-t)^{-A}(1-t)^{C-B-1}dt\,, \tag{6.6.48}$$

where $N_1 = \mathbf{B}(B,C-B)$ and $N_2 = \mathbf{B}(1-A,1+A-C)$. Similarly, with the changes $x = 1/(1-t)$ and $x = (1-t)/(1-z)$, we get

$$J_1(z) = \frac{1}{M_1}\int_{-\infty}^{0} (-t)^{A-C}(z-t)^{-A}(1-t)^{C-B-1}dt\,, \tag{6.6.49}$$

$$J_2(z) = \frac{1}{M_2}\int_z^{1} t^{A-C}(t-z)^{-A}(1-t)^{C-B-1}dt\,, \tag{6.6.50}$$

where $M_1 = \mathbf{B}(B, 1+A-C)$ and $M_2 = \mathbf{B}(1-A, C-B)$.

If we define the holomorphic function

$$f(t) = t^{A-C}(t-z)^{-A}(t-1)^{C-B-1} \tag{6.6.51}$$

for $t \in \mathbb{C}\backslash]-\infty, 0]$, the closed-contour relations

$$\int_{-\infty+i\epsilon}^{+\infty+i\epsilon} f(t)dt = 0\,, \tag{6.6.52}$$

for $\epsilon \to 0^{\pm}$ we get the two equations:

$$e^{-i\pi(B+1)}\widehat{J}_1(z) + e^{i\pi(D-1)}\widehat{I}_2(z) + e^{i\pi(C-B-1)}\widehat{J}_2(z) + \widehat{I}_1(z) = 0\,, \tag{6.6.53}$$

$$e^{+i\pi(B+1)}\widehat{J}_1(z) + e^{i\pi(1-D)}\widehat{I}_2(z) + e^{i\pi(1-C+B)}\widehat{J}_2(z) + \widehat{I}_1(z) = 0\,, \tag{6.6.54}$$

where $\widehat{I}_k = N_k I_k(z)$ and $\widehat{J}_k = M_k J_k(z)$ are the unnormalised integrals. We obtain the relations

$$\widehat{I}_1(z) = \frac{1}{\sin\pi D}\left[\sin\pi(C-A)\,\widehat{J}_1(z) - \sin\pi A\,\widehat{J}_2(z)\right], \tag{6.6.55}$$

$$\widehat{I}_2(z) = \frac{1}{\sin\pi D}\left[\sin\pi B\,\widehat{J}_1(z) - \sin\pi(C-B)\,\widehat{J}_2(z)\right]. \tag{6.6.56}$$

From the identity $\Gamma(x)\Gamma(1-x) = \pi/\sin\pi x$, we get the change of basis:

$$\begin{bmatrix} I_1(z) \\ I_2(z) \end{bmatrix} = \begin{bmatrix} \frac{\Gamma(C)\Gamma(D)}{\Gamma(C-A)\Gamma(C-B)} & \frac{\Gamma(C)\Gamma(-D)}{\Gamma(A)\Gamma(B)} \\ \frac{\Gamma(2+C)\Gamma(D)}{\Gamma(1-A)\Gamma(1-B)} & \frac{\Gamma(2-C)\Gamma(-D)}{\Gamma(1-C+A)\Gamma(1-C+B)} \end{bmatrix} \cdot \begin{bmatrix} J_1(z) \\ J_2(z) \end{bmatrix}. \tag{6.6.57}$$

We have just proven these relations for $z \in]0,1[$, but by analytic continuation, they extend to $z \in \mathbb{C}\backslash(]-\infty, 0] \cup [1, +\infty[)$.

Lecture 7

CFT on the torus and the modular bootstrap

7.1 Complex tori

So far we have mostly focused on conformal field theories on the Riemann sphere. It's time to consider the torus. From a statistical physics point of view, this is quite a natural thing to do. Indeed as soon as one works with periodic boundary condition we are in fact working on the torus, see Fig. 7.1. Likewise in condensed matter physics, the torus appears naturally upon considering a periodic one-dimensional quantum system at finite temperature. At a more abstract level, having a consistent formulation of conformal field

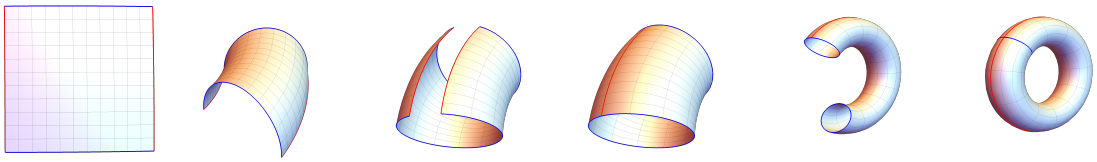


Figure 7.1: The square grid with periodic boundary condition is a torus

theories on higher genus surfaces, and in particular on the torus, yields deep insights into their structure.

7.1.1 Topological classification of closed surfaces

A closed[1] surface refers to a two-dimensional manifold that is compact and without boundary. As is probably well known to many readers, the topology of a connected closed surface is fully determined by two topological invariants : its genus g (or equivalently its Euler characteristic χ), and whether it is orientable or not. Given a closed surface, how can one figure out its genus g ? A very concrete and elementary way is to compute its Euler characteristic χ. Given a triangulation of the surface, χ is defined as

$$\chi = V - E + F \tag{7.1.1}$$

where V, E and F are the total number of vertices, edges and faces of the triangulation, respectively. It is a fact that χ does not depend on the triangulation, and is indeed a

[1] not to be confused with *closed* in the sense of topology.

topological invariant of the surface. The Euler characteristic is related to the genus as follows

- if the surface is orientable, $\chi = 2 - 2g$
- if the surface is unorientable, $\chi = 2 - g$

For an informal discussion about the topological classification of closed surfaces including a sketch of proof, the reader is invited to read chapter 2 of Donaldson's *Riemann Surfaces*. This classification also holds for *smooth* surfaces : any two smooth manifolds with the same genus and that are both orientable (or both non-orientable) are equivalent, in the sense that there exists a diffeomorphism between them. This is because two-dimensional smooth manifolds are diffeomorphic if and only if they are homeomorphic (this is non trivial, and not valid in higher dimensions). From now on we will restrict our attention to oriented surfaces, for which the first few topological classes are depicted in fig. (7.2).

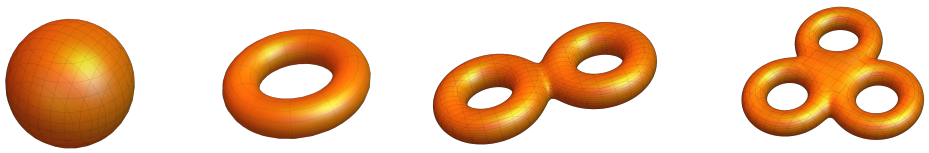


Figure 7.2: Oriented closed surfaces of genus $g = 0, 1, 2$ and 3.

While the topological classification of surfaces is all well and good, in order to put a CFT on a surface one needs to endow the surface with a Riemannian metric. As soon as extra structure is added (beyond the smoothness structure), such as a Riemannian metric or a conformal structure, the topological classification is clearly no longer sufficient. As depicted in fig. (7.3), a two-dimensional sphere can have many different shapes ! Given

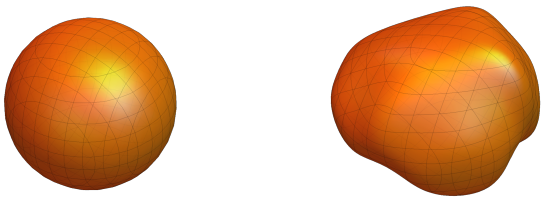


Figure 7.3: Two surfaces with the topology (and smooth structure) of the sphere S^2. They are clearly not isometric : one displays constant curvature while the other does not. They are however conformally equivalent since the moduli space of the sphere is trivial.

the simple manner in which CFT correlation functions behave under Weyl rescaling, what

we really care about is the complex structure induced by the metric[2]. Thus within the framework of CFT one is concerned about the classification of (oriented) surfaces endowed with a complex structure, that is the classification of *Riemann surfaces*. The *moduli space* describes the various inequivalent complex structures (compatible with the given orientation) one can put on a surface of fixed genus. In the case of a genus 0 (topological sphere), the moduli space turns out to be trivial : all metrics are in the same conformal class (up to diffeomorphism). To say things differently, given two spheres with arbitrary metric g_1 and g_2, there always exist a conformal map between them. For surfaces of higher genus this is no longer the case. We will describe in detail the moduli space of the torus (genus $g = 1$) below. Let us simply mention that for genus $g \geq 2$ the moduli space is parametrized by $3g - 3$ complex parameters.

7.1.2 Curvature, genus and the uniformization theorem

One could wonder whether there is a notion of a canonical metric in a given conformal class. The existence of isothermal coordinates tells us that every metric is **locally** conformally flat : for each point x there exists a metric (in the same conformal class) which is flat in some neighborhood of x (*i.e.* there exist coordinates such that $g_{\mu\nu} = \delta_{\mu\nu}$). If we forget about the complex structure, this is rather intuitive. Imagine that the surface is made from dough. Then we can flatten it from the sides pushing all curvature away from x. While it is less clear that this can be done while preserving angles, it turns out to be true. However the Gauss-Bonnet theorem provides an obstruction to removing all the curvature and having a globally flat metric. Indeed the total curvature is related to the genus through

$$\int_M R\, dV = 8\pi(1 - g)\,. \tag{7.1.2}$$

So unless $g = 1$, it is not possible to make the metric globally flat. The next best thing is constant curvature, and this is the essence of the uniformization theorem : every smooth Riemannian metric on a two-dimensional surface is conformal to one with constant curvature. Whether this curvature is positive, null, or negative depends on the genus, as dictated by the Gauss-Bonnet theorem. In practice this means that the sphere ($g = 0$) admits a metric with constant positive curvature. This is the usual round metric as inherited from the canonical embedding in $\mathbb{R}^3$. The torus ($g = 1$) can be endowed with a flat metric (zero curvature), while for higher genus there exist a metric of constant negative curvature.

> Two oriented Riemannian surfaces (M_1, g_1) and (M_2, g_2) are conformally equivalent (*i.e.* related through a Weyl rescaling) if and only if they have the same genus and same moduli. Any Riemannian surface (M, g) is conformally equivalent to a surface of constant curvature.

[2]Recall that two Riemannian metrics are called conformally equivalent when they differ by a Weyl transformation, *i.e.* when they define the same angles. An equivalence class of such metrics is called a *conformal structure* or *conformal class*. Furthermore if the surface is oriented a conformal structure induces a unique complex structure (and vice-versa) via isothermal coordinates.

7.1.3 Complex tori and moduli space

Topologically a (two-dimensional) torus $\mathbb{T}$ is simply the cartesian product of two circles $S^1 \times S^1$, thus we may write

$$\mathbb{T} \simeq \mathbb{R}^2/\mathbb{Z}^2 \tag{7.1.3}$$

where $\simeq$ stands for diffeomorphic. This means that as a smooth surface, the torus is equivalent to the plane quotiented by the group $\mathbb{Z}^2$ of translations generated by $(x, y) \to (x+1, y)$ and $(x, y) \to (x, y+1)$. This is nothing but a square with opposite side identified (*a.k.a.* periodic boundary conditions). If we consider instead a torus endowed with a complex structure (or equivalently a conformal class of Riemannian metrics), we are now looking at a *complex torus*, that is a genus 1 Riemann surface. The moduli space of the torus is not trivial, which amounts to saying that two complex tori are in general not conformally equivalent[3]. This is strikingly different from the sphere !

A convenient and simple way to describe a complex torus is as a quotient of the plane by a lattice. Given two complex numbers ω_1 and ω_2 such that ω_1/ω_2 is not real, the lattice Λ generated by ω_1 and ω_2 is the subset

$$\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z} \tag{7.1.4}$$

as depicted in fig. (7.4). The complex numbers ω_1 and ω_2 are called *periods* of the lattice. Given a lattice Λ, the quotient space $\mathbb{C}/\Lambda$ defines a complex torus. Concretely this means

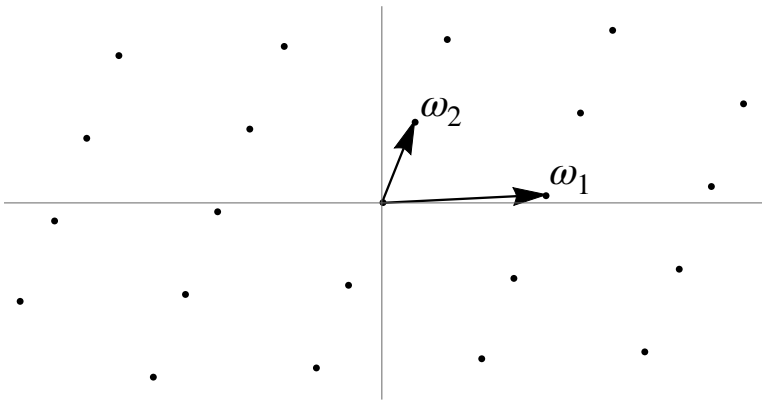


Figure 7.4: The lattice $\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$ generated by ω_1 and ω_2.

identifying $z \equiv z + \omega_1 \equiv z + \omega_2$. As a Riemannian surface the torus inherits the flat metric $g = dx \otimes dx + dy \otimes dy$ of Euclidean space $\mathbb{R}^2$. As a Riemann surface, the complex structure (or conformal class) of $\mathbb{C}$ induces a complex structure on $\mathbb{T}$ (the only one compatible with the above flat metric), and this complex structure depends on the lattice Λ.

[3]Recall that two Riemann surfaces are said to be (conformally) equivalent if there exist a biholomorphic isomorphism between them, *i.e.* a holomorphic map which is one-one and onto, and whose inverse is also holomorphic.

It is a standard result of the theory of Riemann surfaces that the above construction exhausts all possible complex tori, in the sense that any complex torus is conformally equivalent to such a quotient (the reader is invited to read Donaldson chapter 6). But not all lattices yield inequivalent complex tori :

Fact[4] : two complex tori $\mathbb{C}/\Lambda$ and $\mathbb{C}/\Lambda'$ are conformally equivalent if and only if the two lattices Λ and Λ' differ by a rotation/dilation, *i.e.* $\Lambda' = \alpha\Lambda$ for some $\alpha \in \mathbb{C}^*$.

In particular the lattice $\omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$ is equivalent to

$$\Lambda = \mathbb{Z} + \tau\mathbb{Z}, \qquad \tau = \frac{\omega_2}{\omega_1} \tag{7.1.5}$$

Note that we can assume without loss of generality that $\mathrm{Im}(\tau) > 0$ (if it is negative, one can simply interchange ω_1 and ω_2, or change $\omega_1 \to -\omega_1$). We will denote by $\mathcal{H}$ the upper half-plane

$$\mathcal{H} = \{\tau \in \mathbb{C},\, \mathrm{Im}\ \tau > 0\}\,. \tag{7.1.6}$$

Furthermore given a lattice Λ, the choice of periods is not unique. Two bases (ω_1, ω_2) and (ω_1', ω_2') generate the same lattice if an only if they are related by a unimodular matrix A, *i.e.* a matrix with integer coefficients and determinant ± 1. Namely

$$\begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix}, \qquad a, b, c, d \in \mathbb{Z}, \qquad ad - bc = \pm 1 \tag{7.1.7}$$

In terms of the parameter τ, this means

$$\tau' = \frac{a\tau + b}{c\tau + d}, \qquad a, b, c, d \in \mathbb{Z}, \qquad ad - bc = 1 \tag{7.1.8}$$

Note that $ad - bc = -1$ is no longer allowed once we restrict $\tau \in \mathcal{H}$. Indeed

$$\mathrm{Im}\left(\frac{a\tau + b}{c\tau + d}\right) = \frac{(ad - bc)}{|c\tau + d|^2}\,\mathrm{Im}(\tau) \tag{7.1.9}$$

We have reached a full description of the moduli space of the torus as

$$\mathcal{H}/\mathrm{SL}(2, \mathbb{Z})\,, \tag{7.1.10}$$

namely the quotient of the upper half-plane $\tau = \tau_1 + i\tau_2 \in \mathcal{H}$ by the group $\mathrm{SL}(2, \mathbb{Z})$ acting as in eq. (7.1.8). The standard choice for the fundamental domain $\mathcal{F}$ is depicted in Fig. 7.5. In fact staring at (7.1.8) reveals that $\mathrm{SL}(2, \mathbb{Z})$ does not have a faithful (or effective) action, in the sense that the matrices A and $-A$ have identical actions. Thus one introduces the *modular group* (*a.k.a.* the projective special linear group) $\mathrm{PSL}(2, \mathbb{Z}) = \mathrm{SL}(2, \mathbb{Z})/\mathbb{Z}_2$ of matrices with integer coefficients and unit determinant, in which the matrices A and $-A$ are identified.

[4]The proof is rather elementary given a little knowledge about covering spaces and lifting of maps (for which an excellent reference is *A. Hatcher, Algebraic topology* chapter 1) and rests on the two following facts

(i) any bi-holomorphic map between $\mathbb{C}/\Lambda$ and $\mathbb{C}/\Lambda'$ lifts into a conformal automorphism of $\mathbb{C}$

(ii) conformal automorphisms of $\mathbb{C}$ are of the form $z \to \alpha z + \beta$

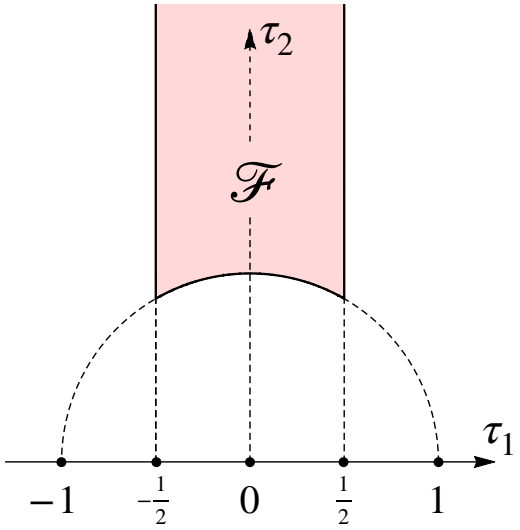


Figure 7.5: Fundamental domain $\mathcal{F}$ of the torus moduli space.

Complex tori are realized as quotient spaces

$$\mathbb{T}_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}) \tag{7.1.11}$$

where τ is in the upper half-plane. Furthermore $\mathbb{T}_\tau$ and $\mathbb{T}_{\tau'}$ are conformally equivalent if and only τ and τ' are related by the modular group $\mathrm{PSL}(2,\mathbb{Z})$, namely if there exists a, b, c, d integers with $ad - bc = 1$ such that

$$\tau' = \frac{a\tau + b}{c\tau + d} \tag{7.1.12}$$

In the following we will need the following fact about the modular group: it is generated by the so-called modular T- and S-transformations :

$$T : \tau \mapsto \tau + 1\,, \qquad S : \tau \mapsto -1/\tau\,. \tag{7.1.13}$$

The proof of this statement is left as the following exercice.

Exercice : Let S, T and U be the following matrices

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{7.1.14}$$

Check that $U = TST$. Show that for two coprime integers a, c, Euclid's algorithm yields

a sequence of integers $n_1, \cdots, n_{2p}$ such that

$$T^{n_1} U^{n_2} T^{n_3} \ldots U^{n_{2p}} \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{7.1.15}$$

Since $U = TST$ and $S \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, the above implies that there exists $m_1, \cdots, m_k$ such that

$$T^{m_1} S T^{m_2} \cdots S T^{m_k} \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{7.1.16}$$

Deduce that S and T indeed generate $\mathrm{SL}(2, \mathbb{Z})$.

7.2 Torus partition function and modular invariance

Geometrically the complex torus $\mathbb{T}_\tau$ is a parallelogram spanned by 1 and $\tau = \tau_1 + i\tau_2$ with its opposite sides identified. Identifying the red pair of sides in Fig. 7.6 amounts to identifying z and $z + 1$. This rolls the parallelogram into a cylinder of height τ_2.

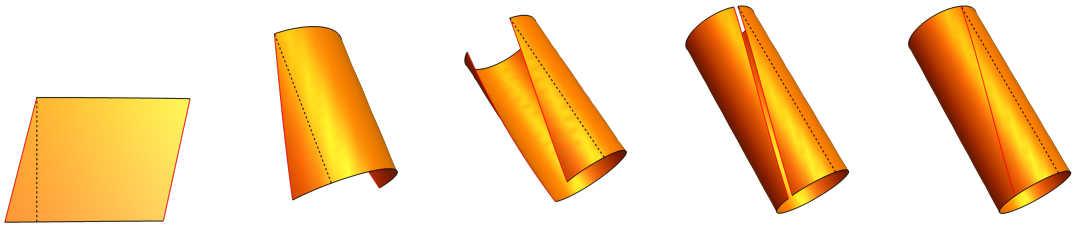


Figure 7.6: The cylinder obtained from the unit cell after the identification of the red edges.

We now need to close up our torus, by identifying z and $z + \tau$. Since we are on a cylinder, this is conveniently done in Hamiltonian formalism as follows. Decomposing $z = x + it$ we interpret x as the space coordinate (on a circle) and t as the (imaginary) time. If τ was pure imaginary, say $\tau = i\tau_2$, the partition function would coincide exactly with that of a one-dimensional quantum system at inverse temperature $\beta = \tau_2$, namely

$$Z(i\tau_2) = \mathrm{Tr}\left(e^{-\tau_2 H}\right),$$

where H is the Hamiltonian of the 1d quantum system on a circle of unit length. However the real part τ_1 of τ requires an additional translation of space on top of the Euclidean time evolution before sewing up (*i.e.* before taking the trace), therefore

$$Z(\tau) = \mathrm{Tr}\left(e^{-\tau_2 H} e^{-i\tau_1 P}\right).$$

The cylinder Hamiltonian H and total momentum P can be obtained easily via conformal mapping to the plane through $w = e^{-i2\pi z}$, as in Fig. 7.7, where we can use the formalism of radial quantization. Recall that P and H are characterized (in Euclidean time) by

$$\partial_t O = [H, O], \qquad -i\partial_x O = [P, O]$$

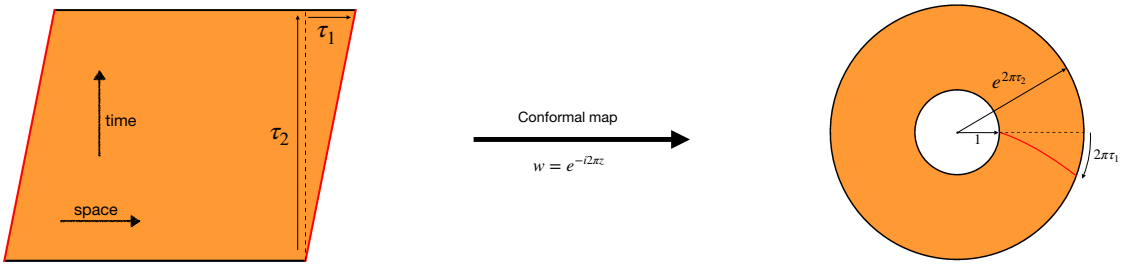


Figure 7.7: The flat torus is conformally equivalent to an annulus with edges identified.

where the slightly unusual form of the time translation stems from the fact that we are working in imaginary time, *i.e.* in Euclidean signature. Comparing with the Ward identity (3.2.25) we get

$$P = -\frac{i}{2\pi}\int_0^1 T_{21}(x,0)dx, \qquad H = \frac{1}{2\pi}\int_0^1 T_{22}(x,0)dx\,, \tag{7.2.1}$$

While the above relations holds for any QFT, for conformal field theories we can go further. First since T is traceless (in flat space), we have $T_{21} = i(T-\overline{T})$ and $T_{22} = -(T+\overline{T})$ (see Table 3.1). Furthermore $T_{\text{cyl}}(z) = -(2\pi)^2\left(w^2 T_{\text{plane}}(w) - c/24\right)$, yielding

$$\frac{1}{2\pi}\int_0^1 T dx = i\oint_{\mathcal{C}}\left(wT(w) - \frac{c}{24}\frac{1}{w}\right)dw = -2\pi\left(L_0 - \frac{c}{24}\right), \tag{7.2.2}$$

leading to[5]

$$P = 2\pi(\bar{L}_0 - L_0)\,, \qquad H = 2\pi\left(L_0 + \bar{L}_0 - \frac{c}{12}\right). \tag{7.2.3}$$

Looking back at the annulus in Fig. 7.7, this is not surprising: P implements (clock-wise) rotations, while H is the generator of dilations. The only non-trivial part is the shift of energy proportional to $-\frac{c}{12}$, which comes from the Weyl transformation when going from the cylinder to the annulus.

We end up with the following expression for the partition function

$$\boxed{Z(\tau) = \text{Tr}\left(q^{L_0-\frac{c}{24}}\,\bar{q}^{\bar{L}_0-\frac{c}{24}}\right), \qquad q = e^{2\pi i\tau}\,,} \tag{7.2.4}$$

in which the trace is taken over the whole Hilbert space of the CFT. To be more precise, this is the space of states for the system with periodic boundary conditions (*i.e.* a circle), that is to say the Hilbert space relevant for radial quantization (as opposed, for instance, to the Hilbert space of a system with open boundary conditions, which is relevant for boundary CFT).

7.2.1 Modular invariance

We have just obtained an expression of the partition function on the torus $\mathbb{T}_\tau$ that depends explicitly on the modulus τ. However we have seen that acting on τ with the modular

[5]Note the sign of the total momentum, which is the opposite of the one obtained in Chapter 4. This is correct, and is due to the choice of orientation for the space coordinate x.

group $\mathrm{PSL}(2,\mathbb{Z})$ yields the same torus. Thus the partition function must be *modular invariant*:

$$Z(\tau) = Z\left(\frac{a\tau+b}{c\tau+d}\right), \qquad \text{for all} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})\,. \tag{7.2.5}$$

From the above discussion about the modular group, modular invariance boils down to invariance under the modular T- and S-transformations :

$$\boxed{Z(\tau) = Z(\tau+1)\,, \qquad \text{and} \qquad Z(\tau) = Z(-1/\tau)}\,. \tag{7.2.6}$$

Let us illustrate modular invariance with two examples: the (non-compact, free massless) scalar field and the Ising model.

Scalar field

The free scalar field provides an interesting first example. The torus partition function can be evaluated using the path-integral approach, *i.e.* by computing the zeta-regularized determinant of the Laplacian on the torus. This calculation can be found for instance in *Lectures on Conformal Field Theory* by Krzysztof Gawedzki (problem 4, page 10) or in *Conformal Field Theory* by Di Francesco *et al.*, section 10.2. Here we will not follow this path-integral approach, but rather the Hamiltonian formalism as in (7.2.4).

To do so, all we need to know is the spectrum of $(L_0, \bar{L}_0)$. Recall that the Hilbert space is spanned by the states

$$|\{n\},\{\overline{n}\},\alpha\rangle = a_{-n_1}\cdots a_{-n_p}\overline{a}_{-\bar{n}_1}\cdots\overline{a}_{-\bar{n}_{\bar{p}}}|\alpha\rangle \tag{7.2.7}$$

where $n_i, \bar{n}_i > 0$, and the state $|\alpha\rangle$ is characterized by

$$a_0|\alpha\rangle = \alpha|\alpha\rangle, \qquad a_n|\alpha\rangle = \overline{a}_n|\alpha\rangle = 0 \quad \text{for } n>0\,, \tag{7.2.8}$$

where α can be any real number. The state $|\{n\},\{\overline{n}\},\alpha\rangle$ has conformal dimensions

$$h = \frac{\alpha^2}{2} + \sum_{j=1}^{p} n_j\,, \qquad \bar{h} = \frac{\alpha^2}{2} + \sum_{j=1}^{\bar{p}} \bar{n}_j\,. \tag{7.2.9}$$

Thus

$$Z(\tau) = \mathrm{Tr}\left(q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}}\right) = \int_{\mathbb{R}} d\alpha \sum_{\{n,\bar{n}\}} q^{\alpha^2+\sum_j n_j}\bar{q}^{\alpha^2+\sum_j \bar{n}_j} \tag{7.2.10}$$

$$= \int_{\mathbb{R}} d\alpha\ (q\bar{q})^{\frac{\alpha^2}{2}} \prod_{n=1}^{\infty}\left(\sum_{k=0}^{\infty} q^{kn}\right)\prod_{\bar{n}=1}^{\infty}\left(\sum_{k=0}^{\infty}\bar{q}^{k\bar{n}}\right) = \int_{\mathbb{R}} d\alpha\, e^{-2\pi\alpha^2\tau_2}\prod_{n=1}^{\infty}\frac{1}{1-q^n}\frac{1}{1-\bar{q}^n} \tag{7.2.11}$$

As long as $\mathrm{Im}(\tau) > 0$ this means

$$\boxed{Z(\tau) = \frac{1}{\sqrt{2\mathrm{Im}(\tau)}\,|\eta(\tau)|^2}\,,} \tag{7.2.12}$$

where $\eta(\tau)$ is the Dedekind eta function defined (for $\mathrm{Im}(\tau) > 0$) by

$$\eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n=1}^{\infty}(1 - e^{2n\pi i\tau}) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1 - q^n)\,. \tag{7.2.13}$$

It is not completely trivial that the above partition function is indeed modular invariant. While it is straightforward to check that $Z(\tau + 1) = Z(\tau)$, checking that $Z(-1/\tau) = Z(\tau)$ is less trivial. It is however a consequence of the identity

$$\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau)\,, \tag{7.2.14}$$

where we choose the principal branch of the square-root.

The Ising model

The critical Ising model in two dimensions exhibits a central charge $c = \frac{1}{2}$, and is described by the minimal model $\mathcal{M}_{3,4}$, as explained in section 5.3.3. We have already encountered the following operators :

- the identity operator $\Phi_{1,1} = \mathbf{1}$ with $(h, \bar{h}) = (0, 0)$,
- the energy operator $\Phi_{2,1} = \epsilon$ with $(h, \bar{h}) = (1/2, 1/2)$,
- the spin operator $\Phi_{2,2} = \sigma$ with $(h, \bar{h}) = (1/16, 1/16)$.

Let us assume that there are no further local primary fields[6]. Thus the Hilbert space is spanned by the three above primary states together with their descendants,

$$\mathcal{H} = \left(\mathcal{K}_{1,1} \otimes \overline{\mathcal{K}}_{1,1}\right) \bigoplus \left(\mathcal{K}_{2,1} \otimes \overline{\mathcal{K}}_{2,1}\right) \bigoplus \left(\mathcal{K}_{2,2} \otimes \overline{\mathcal{K}}_{2,2}\right)\,, \tag{7.2.15}$$

where $\mathcal{K}_{r,s}$ stands for the Kac module (*i.e.* the Verma module quotiented by the null-vectors), with highest weight $h_{r,s}$, as described in Lecture 5. The expression of the partition function follows

$$\boxed{Z(\tau, \bar{\tau}) = \chi_{1,1}(q)\chi_{1,1}(\bar{q}) + \chi_{2,1}(q)\chi_{2,1}(\bar{q}) + \chi_{2,2}(q)\chi_{2,2}(\bar{q})} \tag{7.2.16}$$

in terms of the characters $\chi_{r,s}(q)$

$$\chi_{r,s}(q) := \mathrm{Tr}_{\mathcal{K}_{r,s}}(q^{L_0 - c/24}) \tag{7.2.17}$$

In order to derive the above partition function, we have made a strong conjecture about the operator content of the Ising CFT. Modular invariance provides a highly non trivial consistency check. It turns out that the three characters $\chi_1 = \chi_{1,1}, \chi_2 = \chi_{2,1}$ and $\chi_3 = \chi_{2,2}$ transform linearly under the modular group:

$$\chi_i(\tau + 1) = \sum_{j=1}^{3} T_{ij}\chi_j(\tau)\,, \qquad \chi_i(-1/\tau) = \sum_{j=1}^{3} S_{ij}\chi_j(\tau)\,,$$

[6]We already know that there are other primary fields : the disorder operator μ, the chiral fermions Ψ and $\overline{\Psi}$, but these are not strictly local. In terms of Hilbert space, they correspond to anti-periodic boundary conditions, as will be established in the next exercise.

where the so-called modular T and S matrix are

$$T = e^{-i\frac{\pi}{24}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\pi} & 0 \\ 0 & 0 & e^{\frac{i\pi}{8}} \end{pmatrix}, \qquad S = \frac{1}{2} \begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}. \tag{7.2.18}$$

Invariance of the partition function (7.2.16) under $\tau \to \tau + 1$ holds trivially. Checking that the partition function (7.2.16) is indeed invariant $\tau \to -1/\tau$ is a straightforward exercise left to the reader. Since these two transformations generate the whole modular group, it follows that (7.2.16) is indeed modular invariant.

In fact a much stronger argument can be given in favor of our tentative partition function (7.2.16). Not only is (7.2.16) modular invariant, it is in fact the only modular invariant partition function that one can build out of the the chiral primary fields $\phi_{1,1}, \phi_{2,1}$ and $\phi_{2,2}$! Indeed the most generic partition function is of the form

$$Z(\tau, \bar{\tau}) = \sum_{i,j} N_{i,j} \chi_i(q) \chi_j(\bar{q}), \tag{7.2.19}$$

where $N_{i,j}$ are non negative integers and $N_{1,1} = 1$. Modular invariance boils down to

$$S^\dagger N S = N, \qquad T^\dagger N T = N \tag{7.2.20}$$

In the Ising model all the conformal dimensions being distinct modulo 1, the constraint $T^\dagger N T = N$ is tantamount to N being diagonal. Solving $S^\dagger N S = N$ yields $N \propto I$. Finally $N_{0,0} = 1$ yields $N = I$ as the unique solution. Thus the only modular invariant partition function that one can build is indeed (7.2.16).

Exercise : So far we considered the Ising model on the torus with periodic boundary conditions. However we can play around with the $\mathbb{Z}_2$ symmetry to twist the boundary conditions, and these twisted partition functions will involve the quasi-local fields μ, Ψ and $\bar{\Psi}$. Let $Z_{n_1,n_2}(\tau)$ denote the torus partition function with the following boundary conditions

$$\sigma(z+1) = (-1)^{n_1} \sigma(z), \qquad \sigma(z+\tau) = (-1)^{n_2} \sigma(z) \tag{7.2.21}$$

The numbers of $\mathbb{Z}_2$ defects n_1 and n_2 are naturally defined modulo 2.

▸ Under a modular transformation $\tau \to \frac{a\tau+b}{c\tau+d}$ argue that $\begin{pmatrix} n_2 \\ n_1 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} n_2 \\ n_1 \end{pmatrix}$, in the sense that

$$Z_{n_1,n_2}\left(\frac{a\tau+b}{c\tau+d}\right) = Z_{cn_2+dn_1, an_2+bn_1}(\tau) \tag{7.2.22}$$

In particular

$$Z_{n_1,n_2}(\tau+1) = Z_{n_1,n_2+n_1}(\tau), \qquad Z_{n_1,n_2}(-1/\tau) = Z_{n_2,n_1}(\tau) \tag{7.2.23}$$

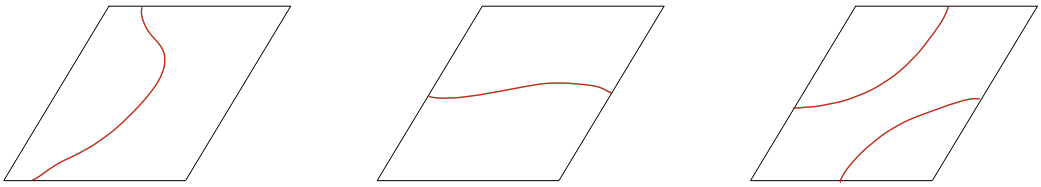
▸ Show that each partition function Z_{n_1,n_2} is invariant under the congruence subgroup $\Gamma(2)$ defined as

$$\Gamma(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}), \quad \text{with } a,d \text{ odd and } b,c \text{ even} \right\} \tag{7.2.24}$$

It is a fact that $\Gamma(2)$ is generated by T^2 and ST^2S. Argue that the most generic $\Gamma(2)$ invariant partition function is a linear combination of

$$|\chi_{1,1}|^2 + |\chi_{2,1}|^2\,, \qquad |\chi_{2,2}|^2 \quad \text{and} \quad \chi_{1,1}\overline{\chi}_{2,1} + \chi_{2,1}\overline{\chi}_{1,1} \tag{7.2.25}$$

▸ The twisted partition functions can equivalently be interpreted as partition functions with periodic boundary conditions, but with a closed (non-contractible) line of disorder operator wrapping around the torus. Argue that Z_{10}, Z_{01} and Z_{11} correspond respectively to the three following situations, in which the red line represents the disorder line :



▸ The partition function $Z_{0,1}(\tau)$ is

$$Z_{0,1}(\tau) = \mathrm{Tr}_{\mathcal{H}}\left(\Theta q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}}\right) \tag{7.2.26}$$

where $\mathcal{H}$ is the Hilbert space (7.2.15) corresponding to periodic boundary condition, and Θ is an operator that implements the $\mathbb{Z}_2$ twist. Argue that Θ is characterized by

$$\Theta|0\rangle = |0\rangle, \qquad \Theta|\epsilon\rangle = |\epsilon\rangle, \qquad \Theta|\sigma\rangle = -|\sigma\rangle, \qquad \text{and} \qquad [\Theta, L_n] = 0 \tag{7.2.27}$$

Deduce that

$$Z_{0,1}(\tau) = |\chi_{1,1}|^2 + |\chi_{2,1}|^2 - |\chi_{2,2}|^2 \tag{7.2.28}$$

▸ Deduce $Z_{0,1}$ and $Z_{1,1}$

$$Z_{1,0}(\tau) = Z_{0,1}(-1/\tau) = \chi_{1,1}\overline{\chi}_{2,1} + \chi_{2,1}\overline{\chi}_{1,1} + |\chi_{2,2}|^2 \tag{7.2.29}$$

$$Z_{1,0}(\tau+1) = Z_{1,1}(\tau) = -\chi_{1,1}\overline{\chi}_{2,1} - \chi_{2,1}\overline{\chi}_{1,1} + |\chi_{2,2}|^2 \tag{7.2.30}$$

The partition function $Z_{1,0}$ and $Z_{1,1}$ can be written as

$$Z_{1,0}(\tau) = \mathrm{Tr}_{\mathcal{H}'}\left(q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}}\right), \qquad Z_{1,1}(\tau) = \mathrm{Tr}_{\mathcal{H}'}\left(\Theta q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}}\right) \tag{7.2.31}$$

where $\mathcal{H}'$ is the Hilbert space of the Ising model with anti-periodic boundary conditions. The above expressions imply that

$$\mathcal{H}' = \left(V_{1,1}\otimes\overline{V_{2,1}}\right)\bigoplus\left(V_{2,1}\otimes\overline{V_{1,1}}\right)\bigoplus\left(V_{2,2}\otimes\overline{V_{2,2}}\right) \tag{7.2.32}$$

Thus the primary fields in the anti-periodic sector are $\bar{\Psi}$, Ψ and μ. Under the $\mathbb{Z}_2$ symmetry Θ, the fermions Ψ and $\bar{\Psi}$ are odd, while μ is even.

$$\Theta|\Psi\rangle = -|\Psi\rangle, \qquad \Theta|\bar{\Psi}\rangle = -|\bar{\Psi}\rangle, \qquad \Theta|\mu\rangle = |\mu\rangle\,. \tag{7.2.33}$$

This is consistent with the lattice construction of Ψ and $\bar{\Psi}$ as products of a σ and a μ, or equivalently with the fusion rule

$$\sigma \times \mu = \Psi + \bar{\Psi} \tag{7.2.34}$$

7.2.2 Rational CFTs

The infinite-dimensional symmetry algebra of local conformal transformations[7] is isomorphic to two copies of the Virasoro algebra. Thus the Hilbert space can be decomposed into irreducible representations of $\mathrm{Vir} \otimes \overline{\mathrm{Vir}}$. Typically

$$\bigoplus_{i,j} N_{i,j} V_i \otimes \overline{V_j} \tag{7.2.35}$$

where $N_{i,j}$ is the multiplicity of the occurence of the representation $V_i \otimes \overline{V_j}$. In some cases, such as the free scalar field, the discrete sum is replaced by an integral. Broadly speaking there are two types of CFTs :

(i) theories with a discrete spectrum

(ii) theories with a continuous spectrum

where it should be understood that we are talking about the spectrum of $(L_0 + \bar{L}_0)$, or equivalently the energy spectrum on a circle, and not the energy spectrum on the real line (indeed, the latter is always continuous for a critical system as there is no energy gap). From now on we will restrict our attention to CFTs with a discrete spectrum, or equivalently a discrete set of primary fields. In this case the decomposition (7.2.35) is valid, and it follows that the partition function can be written as

$$\boxed{Z(\tau, \bar{\tau}) = \sum_{i,j} N_{i,j} \chi_i(q) \chi_j(\bar{q})} \tag{7.2.36}$$

where $\chi_i(q)$ are the Virasoro characters

$$\chi_i(q) = \mathrm{Tr}_{V_i} \left(q^{L_0 - \frac{c}{24}} \right) \tag{7.2.37}$$

as introduced in previous lectures. Note that $N_{i,j}$ are non-negative integers, and that the vacuum being non degenerate we must have $N_{0,0} = 1$ (in a unitary CFT the vacuum is the only state with a vanishing conformal dimension, therefore $V_0 \otimes \overline{V_0}$ can only appear once).

We will further restrict our attention to *rational* CFTs, that is CFTs with *finitely many* chiral primary fields $\{\phi_i, i \in \mathcal{I}\}$ labeled by some finite set $\mathcal{I}$ (*e.g.* the entries of the Kac table in the case of minimal models). In this case the *r.h.s.* of (7.2.36) is a finite sum. Very much like the conformal blocks on the sphere transform linearly under monodromy operations, the characters of a rational CFT form a finite-dimensional representation of the modular group :

$$\chi_i(\tau + 1) = \sum_{j \in \mathcal{I}} T_{ij} \, \chi_j(\tau), \qquad \chi_i(-1/\tau) = \sum_{j \in \mathcal{I}} S_{ij} \, \chi_j(\tau) \, . \tag{7.2.38}$$

Modular invariance of the partition function (7.2.36) is tantamount to

$$S^\dagger N S = N \, , \qquad T^\dagger N T = N \, . \tag{7.2.39}$$

[7] strictly speaking we are dealing with the complexification of this algebra (recall the discussion at the end of section 2.2.2) with a central extension (the central charge).

The modular bootstrap amounts to finding all matrices $N_{i,j}$ with non-negative integer coefficients, subject to $N_{1,1} = 1$, and satisfying the above constraints.

Modular invariance of the partition function is analogous to crossing symmetry of four-point functions on the sphere : it provides a constraint on how to piece together the holomorphic and anti-holomorphic parts of the CFT. One can think of the characters $\chi_i(q)$ as conformal blocks on the torus (notice that they are holomorphic in τ), and the modular group on the torus is the analog of the braid group on the four-punctured sphere. In fact these are two examples of a more general problem, that of computing n-point correlation functions on a genus g surface. The braid group and modular group are particular instances of mapping class groups (see *Conformal field theory and mapping class groups* by T. Gannon).

7.3 Minimal models and ADE classification

Minimal models are a prime example of rational CFTs, where the modular bootstrap program described above can be applied. This leads to the complete classification of modular invariant partition functions. This classification, known as the *ADE classification*, was obtained in 1987 by Cappelli, Itzykson and Zuber. The proof of this classification goes in two steps:

- find all invariants without demanding $N_{i,j}$ integer,
- impose $N_{i,j}$ to be non negative integers.

The details are very technical and go well beyond the scope of this lecture. We will not go further into the ADE classification, and invite the reader to look at section 3.2. in *A-D-E Classification of Conformal Field Theories* for the classification itself. We will simply illustrate the modular bootstrap with some examples. We have already solved the problem for $\mathcal{M}_{3,4}$, for which there is only one modular invariant. This corresponds to the universality class of the Ising model. We will also discuss the minimal model $\mathcal{M}_{5,6}$. We will find that there are *two* modular invariants, one of which being relevant for the three-state Potts model.

Recall that minimal models $\mathcal{M}_{p,p'}$ are labelled by two coprime integers p, p', with $2 \leq p < p'$, with central charge

$$c(p,p') = 1 - \frac{6(p-p')^2}{pp'}\,. \tag{7.3.1}$$

For a given minimal model $\mathcal{M}_{p,p'}$, the allowed representations of the Virasoro algebra are labelled by a two integers (r,s), with conformal dimension

$$h_{r,s} = \frac{(p'r-ps)^2-(p-p')^2}{4pp'} \tag{7.3.2}$$

The operator algebra of this model closes on a *finite* number of primary operators, which are all degenerate, and correspond to

$$\begin{cases} 1 \leq r \leq p\ -1 \\ 1 \leq s \leq p'-1 \end{cases} \tag{7.3.3}$$

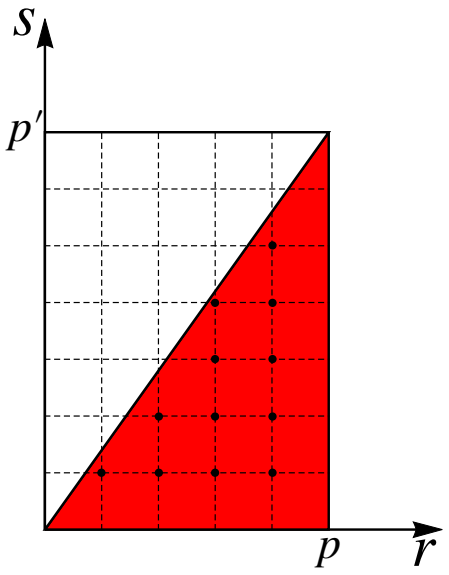


Figure 7.8: Fundamental domain $ps < p'r$ of the Kac Table for the minimal model $\mathcal{M}_{5,7}$.

and are subject to the identification $(r,s) = (p-r, p'-s)$. Thus we can restrict (r,s) to be in the range $\mathcal{I}$

$$\begin{cases} 1 \le r \le p\ -1 \\ 1 \le s \le p'-1 \\ \quad ps < p'r \end{cases} \tag{7.3.4}$$

as in figure (7.8), for a total of $\frac{(p-1)(p'-1)}{2}$ chiral primary fields.

The corresponding characters are

$$\chi_{r,s}(\tau) = \frac{1}{\eta(\tau)} \left(\sum_{n\in\mathbb{Z}} q^{(p'r-ps+2npp')^2/4pp'} - \sum_{n\in\mathbb{Z}} q^{(p'r+ps+2npp')^2/4pp'} \right). \tag{7.3.5}$$

Note that the characters are linearly independent[8]functions of τ. Thus the T and S matrices are unequivocally defined through (7.2.38).

Modular T matrix for minimal models

The T matrix is particularly simple as it is diagonal, namely

$$T_{ij} = \delta_{ij} e^{2i\pi(h_i - \frac{c}{24})}$$

This is straightforward since χ_i is of the form

$$\chi_i(\tau) = e^{2i\pi\tau(h_i - \frac{c}{24})} \sum_{n=0}^{\infty} N_i(n) e^{2i\pi n\tau}$$

[8]By construction all conformal dimensions $h_{r,s}$ are distincts. Indeed $h_{r_1,s_1} = h_{r_2,s_2}$ iff $p'r_1 - ps_1 = \pm(p'r_2 - ps_2)$, which we can rewrite as $p'(r_1 \mp r_2) = p(s_1 \mp s_2)$. Thus $r_1 \mp r_2 = 0 \bmod p$ and $s_1 \mp s_2 = 0 \bmod p'$. In the range (7.3.3) the only solutions are $(r_1,s_1) = (r_2,s_2)$ and $(r_1,s_1) = (p-r_2, p'-s_2)$. More generally for a given central charge, two irreducible representations of the Virasoro algebra with the same conformal dimension are necessarily isomorphic, that is they are in fact the same representation.

although care has to be taken into defining the phase $e^{2i\pi(h_i-\frac{c}{24})}$. What we really mean by the modular T-transformation $\tau \to \tau+1$ is the transformation of the characters χ_i under analytic continuation as we follow a path from τ to $\tau+1$ in the fundamental domain of the moduli space. Thus invariance of the partition function

$$Z(\tau) = \sum_{i,j} N_{i,j}\chi_i(q)\chi_j(\bar{q}) \tag{7.3.6}$$

under $\tau \to \tau+1$ is achieved as soon as $N_{i,j} = 0$ when $h_i \neq h_j \bmod 1$. This means that all[9] primary fields must have integer spin.

Modular S matrix for minimal models

Invariance under $\tau \to -1/\tau$ is far less trivial. For a generic minimal model $\mathcal{M}_{p,p'}$ the S matrix is known (see Appendix 7.4.3 for a derivation), and is given by

$$S_{(r_1,s_1),(r_2,s_2)} = 2\sqrt{\frac{2}{pp'}}(-1)^{1+s_1r_2+s_2r_1}\sin\pi\frac{p'r_1r_2}{p}\,\sin\pi\frac{ps_1s_2}{p'}\,. \tag{7.3.7}$$

Note that $S = S^\dagger$, thus for any minimal model the following partition function

$$Z(\tau) = \sum_{i,j\in\mathcal{I}} |\chi_i(q)|^2 \tag{7.3.8}$$

is modular invariant. For obvious reasons, this is called the *diagonal modular invariant* of the minimal model $\mathcal{M}_{p,p'}$. In such a diagonal theory, the physical primary fields are labelled by the Kac table (in principle the Kac table labels chiral primary fields, not physical ones). Thus there are exactly $(p-1)(p'-1)/2$ physical primary fields, and they are all scalar. This is what happens for instance for the Ising model. But we will see with the three-state Potts model that (7.3.8) may not be the only possible modular invariant partition function.

The minimal model $\mathcal{M}_{5,6}$ and the three-state Potts model

The three-state Potts model has Boltzmann weight

$$\exp\left(-\frac{J}{3}\sum_{\langle i,j\rangle}(1+\sigma_i\overline{\sigma_j}+\overline{\sigma_i}\sigma_j)\right) = \exp\left(-J\sum_{\langle i,j\rangle}\delta_{\sigma_i,\sigma_j}\right) \tag{7.3.9}$$

where the lattice spins σ_i take values in $\{1, e^{\frac{2i\pi}{3}}, e^{-\frac{2i\pi}{3}}\}$. The Boltzmann weight is invariant under global $\mathbb{Z}_3$ rotations

$$\sigma_i \to e^{\frac{2i\pi}{3}}\sigma_i \tag{7.3.10}$$

and reflection

$$\sigma_i \to \overline{\sigma_i}\,. \tag{7.3.11}$$

[9] at least the local ones, that is those appearing in the partition function. For instance we have seen that the Ising model contains primary fields (Ψ and $\bar{\Psi}$) with spin $\pm 1/2$, but those are not strictly local and do not appear in the Ising partition function.

Together these transformations generate the dihedral group $\mathbb{D}_3$ (which is isomorphic to the group of permutations $\mathfrak{S}_3$).

The lattice three-state Potts model is exactly solvable, and the critical exponents and central charge are known. In particular the central charge is $c = 4/5$. This corresponds to the minimal model $\mathcal{M}_{5,6}$, whose Kac table contains 10 chiral primary fields (or equivalently 10 degenerate representations of the Virasoro algebra), with conformal dimensions listed in Fig.(7.9).

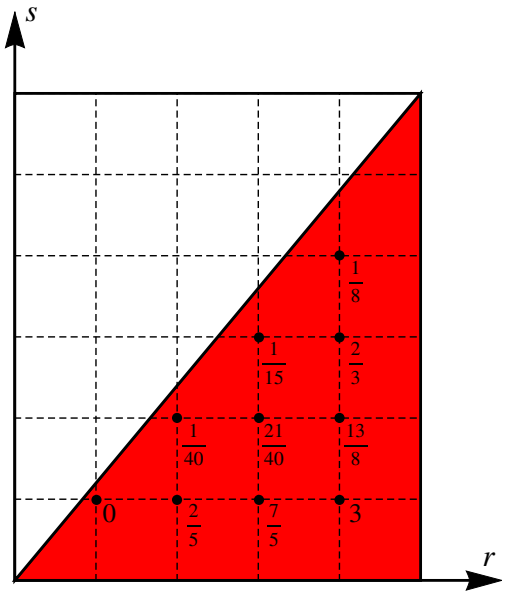


Figure 7.9: Kac Table for the minimal model $\mathcal{M}_{5,6}$.

Let us solve the modular bootstrap for this model, *i.e.* classify all possible modular invariant partition functions one can build with these representations. An important difference with the minimal model $\mathcal{M}_{3,4}$ is that some of the conformal dimensions appearing in the Kac table differ by an integer, namely

$$h_{4,1} = h_{1,1} + 3\,, \qquad h_{3,1} = h_{2,1} + 1\,. \tag{7.3.12}$$

Thus T-invariance does not preclude the appearance of the following non-diagonal sectors

$$V_{1,1} \otimes \overline{V_{4,1}}, \quad V_{4,1} \otimes \overline{V_{1,1}}, \quad V_{2,1} \otimes \overline{V_{3,1}}, \qquad V_{3,1} \otimes \overline{V_{2,1}} \tag{7.3.13}$$

in the Hilbert space. Further demanding S-invariance[10], thus achieving full modular invariance, we are left with *two* two linearly independent solutions. This means that there are two distinct universality classes with a central charge $c = \frac{4}{5}$. Of course one of them is the diagonal modular invariant (7.3.8), in which the above-mentioned non

[10] Already in this simple case this requires a little bit for work (or a computer). To better appreciate the tour de force achieved by the ADE classification, and the computational difficulty of solving the modular

diagonal sectors are forbidden. But these sectors do appear in the the other modular invariant, namely :

$$Z = |\chi_{1,1} + \chi_{4,1}|^2 + |\chi_{2,1} + \chi_{3,1}|^2 + 2|\chi_{4,3}|^2 + 2|\chi_{3,3}|^2 \tag{7.3.14}$$

Equivalently a CFT with such a modular invariant has the following Hilbert space (on the circle)

$$\begin{aligned}\mathcal{H} = (V_{1,1} \oplus V_{4,1}) \otimes \left(\overline{V}_{1,1} \oplus \overline{V}_{4,1}\right) \bigoplus (V_{2,1} \oplus V_{3,1}) \otimes \left(\overline{V}_{2,1} \oplus \overline{V}_{3,1}\right) \\ \bigoplus 2V_{4,3} \otimes \overline{V}_{4,3} \bigoplus 2V_{3,3} \otimes \overline{V}_{3,3}\end{aligned} \tag{7.3.15}$$

Let's conclude this chapter with some notable features about this modular invariant.

- **Some chiral fields are forbidden.** Not all possible chiral fields $\phi_{r,s}$ are involved in (7.3.14), only those with s odd. Note that this is consistent with fusion rules, in the sense that this subset is stable under fusion. Thus there are no (local) physical operators with conformal dimension (left or right) in the set $\{1/40, 21/40, 1/8, 13/8\}$.

- **Multiplicities and $\mathbb{D}_3$ symmetry.** Some operators appear with multiplicities. These degeneracies are related to the $\mathbb{D}_3$ symmetry of the three-state Potts model. For instance there are *two* primary fields with conformal dimensions $(1/15, 1/15)$. They correspond to two spin fields σ_1 and σ_2 which are respectively the scaling limit of the lattice spin operators σ and $\sigma^2 = \overline{\sigma}$. These two fields are distinguished by their behavior under $\mathbb{Z}_3$ rotations

  $$\sigma_1 \to e^{\frac{2i\pi}{3}}\sigma_1, \qquad \sigma_2 \to e^{-\frac{2i\pi}{3}}\sigma_2 \tag{7.3.16}$$

  Furthermore $\mathbb{Z}_3$ symmetry requires them to be dual to each other, in the sense that

  $$\langle\sigma_1(r_1)\sigma_2(r_2)\rangle = \frac{1}{|r_1 - r_2|^{\frac{4}{15}}}, \qquad \langle\sigma_1(r_1)\sigma_1(r_2)\rangle = \langle\sigma_2(r_1)\sigma_2(r_2)\rangle = 0\,. \tag{7.3.17}$$

- **Non-scalar primary fields and W-algebra.** The modular invariant (7.3.14) involves sectors for which the highest weight state has $h \neq \bar{h}$. In particular the field $W = \phi_{4,1} \otimes \bar{\phi}_{1,1}$ has spin 3, and it is holomorphic (since $\bar{h} = 0$, $\bar{\partial}W = \overline{L}_{-1}W$ is a null-vector and thus decouples from the theory). Such a chiral field signals the

bootsrap, here is the S-matrix for $\mathcal{M}_{5,6}$:

$$S = \frac{1}{\sqrt{5!}}\begin{pmatrix}
a & b & \sqrt{3}b & b & \sqrt{3}b & 2b & a & \sqrt{3}a & 2a & \sqrt{3}a \\
b & -a & \sqrt{3}a & -a & \sqrt{3}a & -2a & b & -\sqrt{3}b & 2b & -\sqrt{3}b \\
\sqrt{3}b & \sqrt{3}a & \sqrt{3}a & -\sqrt{3}a & -\sqrt{3}a & 0 & -\sqrt{3}b & -\sqrt{3}b & 0 & \sqrt{3}b \\
b & -a & -\sqrt{3}a & -a & -\sqrt{3}a & -2a & b & \sqrt{3}b & 2b & \sqrt{3}b \\
\sqrt{3}b & \sqrt{3} & -\sqrt{3}a & -\sqrt{3}a & \sqrt{3}a & 0 & -\sqrt{3}b & \sqrt{3}b & 0 & -\sqrt{3}b \\
2b & -2a & 0 & -2a & 0 & 2a & 2b & 0 & -2b & 0 \\
a & b & -\sqrt{3}b & b & -\sqrt{3}b & 2b & a & -\sqrt{3}a & 2a & -\sqrt{3}a \\
\sqrt{3}a & -\sqrt{3}b & -\sqrt{3}b & \sqrt{3}b & \sqrt{3}b & 0 & -\sqrt{3}a & -\sqrt{3}a & 0 & \sqrt{3}a \\
2a & 2b & 0 & 2b & 0 & -2b & 2a & 0 & -2a & 0 \\
\sqrt{3}a & -\sqrt{3}b & \sqrt{3}b & \sqrt{3}b & -\sqrt{3}b & 0 & -\sqrt{3}a & \sqrt{3}a & 0 & -\sqrt{3}a
\end{pmatrix}$$

where $a = \sqrt{5 - \sqrt{5}}$ and $b = \sqrt{5 + \sqrt{5}}$. The entries corresponds to the fields $\phi_{r,s}$ in the Kac table given in figure 7.9, arranged according to lexicographic order on (r, s).

presence of an additional symmetry (besides Virasoro) for which its modes W_n are the generators

$$W(z) = \sum_{n\in\mathbb{Z}} z^{-n-3} W_n \,. \tag{7.3.18}$$

These modes, together with the Virasoro modes, generate the so-called $\mathcal{W}_3$ algebra (of which the Virasoro is a subalgebra), which we mention here without derivation

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0} \\
[L_n, W_m] &= (2n-m)\, W_{n+m} \\
[W_n, W_m] &= \frac{16}{22+5c}(n-m)\Lambda_{n+m} + \frac{c}{360} n(n^2-1)(n^2-4)\delta_{n+m,0} \\
&\quad + (n-m)\left[\frac{1}{15}(n+m+2)(n+m+3) - \frac{1}{6}(n+2)(m+2)\right] L_{n+m}
\end{aligned}$$

where

$$\Lambda_n = \sum_{m=-\infty}^{\infty} :L_m L_{n-m}: + d_n L_n \,, \qquad d_{2m} = \frac{(1-m^2)}{5} \,, \qquad d_{2m-1} = \frac{(1+m)(2-m)}{5} \,.$$

This is a particular case of a (quantum) W-algebra.

Since we have a larger symmetry algebra at our disposal, it is a good idea to organise the Virasoro representations into (irreducible) representations of the extended $\mathcal{W}_3$ algebra. For instance the state highest state of $V_{4,1}$, $|\Phi_{4,1}\rangle = |W\rangle$, is *not* primary with respect to the $\mathcal{W}_3$ algebra, since it is descendant of a the vacuum $|W\rangle = W_{-3}|0\rangle$, very much like $|T\rangle = L_{-2}|0\rangle$. Thus it is very natural to gather the Virasoro representations $V_{1,1}$ and $V_{4,1}$ into a single representation of $\mathcal{W}_3$. Let's call it $\mathcal{V}_1$

$$\mathcal{V}_1 = V_{1,1} \oplus V_{4,1} \,. \tag{7.3.19}$$

How do we know that $\mathcal{V}_1$ is a representation a $\mathcal{W}_3$, *i.e.* that it is stable under action of the modes W_n ? Simply from the fusion rules $W \times 1 = W$ and $W \times W = 1$, which can be recast as

$$W(z)V_{1,1} \subset V_{4,1}, \quad \text{and} \quad W(z)V_{4,1} \subset V_{1,1} \tag{7.3.20}$$

Thus

$$W(z)\mathcal{V}_1 \subset \mathcal{V}_1 \tag{7.3.21}$$

Furthermore $\mathcal{V}_1$ is an *irreducible* representation of $\mathcal{W}_3$, since $V_{1,1}$ and $V_{4,1}$ are irreducible *w.r.t.* Virasoro. Likewise, $|\Phi_{3,1}\rangle = W_{-1}|\Phi_{2,1}\rangle$, and we have another irreducible $\mathcal{W}_3$ representation

$$\mathcal{V}_2 = V_{2,1} \oplus V_{3,1} \,. \tag{7.3.22}$$

Finally the $\mathcal{W}_3$-algebra discriminates between the two copies of the representation $V_{4,3}$ (and likewise $V_{3,3}$), in the sense that σ_1 and σ_2 have opposite eigenvalue under W_0. Thus the two copies of the Virasoro representation $V_{4,3}$ become two inequivalent representations $\mathcal{V}_3$ and $\mathcal{V}_3^*$

$$V_{3,3} \to \mathcal{V}_3 \text{ and } \mathcal{V}_3^* \,. \tag{7.3.23}$$

Likewise for $V_{3,4}$:

$$V_{3,4} \to \mathcal{V}_4 \text{ and } \mathcal{V}_4^* \,. \tag{7.3.24}$$

In terms of these $\mathcal{W}_3$ representations, the modular invariant of the three-state Potts model is simpler : it is diagonal, in the sense that the Hilbert space is

$$\mathcal{H} = \left(\mathcal{V}_1 \otimes \overline{\mathcal{V}}_1\right) \bigoplus \left(\mathcal{V}_2 \otimes \overline{\mathcal{V}}_2\right) \bigoplus \left(\mathcal{V}_3 \otimes \overline{\mathcal{V}}_3\right) \bigoplus \left(\mathcal{V}_3^* \otimes \overline{\mathcal{V}}_3^*\right) \bigoplus \left(\mathcal{V}_4 \otimes \overline{\mathcal{V}}_4\right) \bigoplus \left(\mathcal{V}_4^* \otimes \overline{\mathcal{V}}_4^*\right)$$

It turns out that the CFT underlying the three-state Potts model is the first of a sequence of minimal models of the $\mathcal{W}_3$ algebra.

Some references for this chapter:

- *Introduction to Conformal Field Theory*, Ralph Blumenhagen and Erik Plauschinn. In particular chapter 4 deals with modular invariance. A presentation of W-algebras and more generally extended symmetries can be found in chapter 3.
- section 10.7 in *Conformal Field Theory*, Di Francesco *et al.*. In particular section 10.7.2 contains a much more detailed description of the field content of the three-state Potts model.
- *A-D-E Classification of Conformal Field Theories*, scholarpedia, Andrea Cappelli and Jean-Bernard Zuber

7.4 Appendix

7.4.1 Some useful identities

The Poisson summation formula

The Poisson summation formula is the following identity between distributions

$$\sum_{n\in\mathbb{Z}} \delta(x-n) = \sum_{k\in\mathbb{Z}} e^{i2\pi kx} \tag{7.4.1}$$

Note that both sides of the above equation are periodic in x, so we can think of x as living on the circle $\mathbb{R}/\mathbb{Z}$. The Poisson summation formula is simply the decomposition of the Dirac delta on the circle (the left hand side) in Fourier series (the right hand side).

The Poisson summation formula implies that for appropriate functions $f : \mathbb{R} \to \mathbb{C}$

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{k\in\mathbb{Z}} \hat{f}(k) \tag{7.4.2}$$

where $\hat{f}$ is the Fourier transform of f, namely

$$\hat{f}(k) = \int_{\mathbb{R}} e^{-i2\pi kx} f(x) dx \tag{7.4.3}$$

Indeed formally

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \hat{f}(k) &= \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} e^{-i2\pi kx} f(x) dx = \int_{\mathbb{R}} \left(\sum_{k\in\mathbb{Z}} e^{-i2\pi kx}\right) f(x) dx \\ &= \int_{\mathbb{R}} \sum_{n\in\mathbb{Z}} \delta(x-n) f(x) dx = \sum_{n\in\mathbb{Z}} f(n)\end{aligned}$$

In particular for $f(x) = \frac{1}{\sqrt{\alpha}} e^{-\frac{\pi}{\alpha}(x+\beta)^2}$ we get

$$\boxed{\sum_{k\in\mathbb{Z}} e^{-\pi\alpha k^2 + 2\pi i\beta k} = \frac{1}{\sqrt{\alpha}} \sum_{n\in\mathbb{Z}} e^{-\frac{\pi}{\alpha}(n+\beta)^2}.} \tag{7.4.4}$$

Jacobi's triple product identity

For complex numbers $z \neq 0$ and $|x| < 1$,

$$\boxed{\prod_{m=1}^{\infty} \left(1 - x^{2m}\right)\left(1 + zx^{2m-1}\right)\left(1 + z^{-1}x^{2m-1}\right) = \sum_{n=-\infty}^{\infty} x^{n^2} z^n} \tag{7.4.5}$$

Many proofs of this result are known. Some are combinatorial/enumerative (having to do with counting partitions), some are number-theoretic. The curious reader can find a very readable computational proof in George E. Andrews *The Theory of Partitions*. There is also a very elegant combinatorial proof in *Conformal Field Theory* by Di Francesco *et al*, in appendix 10.A.1.

Let's just mention that it is fairly easy to show that

$$\prod_{m=1}^{\infty} \left(1 - x^{2m}\right)\left(1 + zx^{2m-1}\right)\left(1 + z^{-1}x^{2m-1}\right) = a_0(x) \sum_{n=-\infty}^{\infty} x^{n^2} z^n \tag{7.4.6}$$

up to an unknown function $a_0(x)$. Indeed the function

$$F(z) = \prod_{m=1}^{\infty} \left(1 - x^{2m}\right)\left(1 + zx^{2m-1}\right)\left(1 + z^{-1}x^{2m-1}\right) \tag{7.4.7}$$

is holomorphic on $\mathbb{C} \setminus \{0\}$, so it is can be represented as a Laurent series

$$F(z) = \sum_{n\in\mathbb{Z}} a_n(x) z^n \tag{7.4.8}$$

Now notice that F obeys

$$zxF(zx^2) = F(z) \tag{7.4.9}$$

In terms of the Laurent series, this means $a_{n+1} = x^{2n+1} a_n$, thus $a_n = x^{n^2} a_0$ and

$$F(z) = a_0(x) \sum_{n=-\infty}^{\infty} x^{n^2} z^n \tag{7.4.10}$$

To finish the proof one would have to evaluate $a_0(x)$. This is more technical, and it turns out that $a_0 = 1$.

7.4.2 Theta functions

Jacobi's theta function $\vartheta(z;\tau)$ is defined as

$$\vartheta(z;\tau) = \sum_{n=-\infty}^{\infty} e^{(\mathrm{i}\pi \mathrm{n}^2\tau + \mathrm{i}2\pi \mathrm{n}z)} = \sum_{n=-\infty}^{\infty} q^{\frac{n^2}{2}} e^{\mathrm{i}2\pi \mathrm{n}z} \tag{7.4.11}$$

for $z \in \mathbb{C}$ and τ in the upper half plane ($\mathrm{Im}(\tau) > 0$). Jacobi's triple product identity states that

$$\vartheta(z;\tau) = \prod_{n=1}^{\infty} \left(1 - e^{2n\pi \mathrm{i}\tau}\right)\left(1 + e^{(2n-1)\pi \mathrm{i}\tau + 2\pi \mathrm{i}z}\right)\left(1 + e^{(2n-1)\pi \mathrm{i}\tau - 2\pi \mathrm{i}z}\right). \tag{7.4.12}$$

Some auxiliary theta functions

In some cases it may be useful to consider the following auxiliary theta functions

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z|\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i\tau(n+a)^2} e^{2\pi i(n+a)(z+b)} \tag{7.4.13}$$

$$= \sum_{k\in a+\mathbb{Z}} e^{\pi i\tau k^2} e^{2\pi ik(z+b)} \tag{7.4.14}$$

$$= e^{i\pi\tau a^2} e^{2i\pi(z+b)a} \vartheta\big(z+b+a\tau;\tau\big) \tag{7.4.15}$$

In particular is is customary to introduce

$$\theta_1 = \vartheta\begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}(z|\tau) = -\vartheta\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}(z|\tau) \tag{7.4.16}$$

$$\theta_2 = \vartheta\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}(z|\tau) \tag{7.4.17}$$

$$\theta_3 = \vartheta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(z|\tau) \tag{7.4.18}$$

$$\theta_4 = \vartheta\begin{bmatrix} 0 \\ \pm 1/2 \end{bmatrix}(z|\tau) \tag{7.4.19}$$

Using Jacobi's triple product identity, this can be rewritten as

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z|\tau) = e^{i\pi\tau a^2} e^{2i\pi(z+b)a} \prod_{m\geq 1} \left(1-e^{2i\pi\tau m}\right)\left(1+e^{i\pi\tau(2m-1)}e^{2i\pi(z+b+a\tau)}\right)\left(1+e^{i\pi\tau(2m-1)}e^{-2i\pi(z+b+a\tau)}\right) \tag{7.4.20}$$

from which we can see that the zeroes are located at

$$z = \left(\frac{1}{2}-b\right) + \tau\left(\frac{1}{2}-a\right) + \mathbb{Z} + \tau\mathbb{Z} \tag{7.4.21}$$

We have

$$\vartheta\begin{bmatrix} a+1 \\ b \end{bmatrix}(z|\tau) = \vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z|\tau) = \vartheta\begin{bmatrix} -a \\ -b \end{bmatrix}(-z|\tau) \tag{7.4.22}$$

from which we can always choose $0 \leq a < 1$ (or at least for the real part of a if a is not real). Similarly

$$\vartheta\begin{bmatrix} a \\ b+c \end{bmatrix}(z|\tau) = \vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z+c|\tau) \tag{7.4.23}$$

The two boundary conditions are

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z+n|\tau) = e^{2i\pi an}\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z|\tau), \qquad n\in\mathbb{Z} \tag{7.4.24}$$

and

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z+c\tau|\tau) = e^{-2i\pi c(z+b)}e^{-i\pi\tau c^2}\vartheta\begin{bmatrix} a+c \\ b \end{bmatrix}(z|\tau), \qquad c\in\mathbb{R} \tag{7.4.25}$$

Under the "easy" modular transformation we have

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z|\tau+n) = e^{-i\pi a(1+a)n}\vartheta\begin{bmatrix} a \\ na+\frac{n}{2}+b \end{bmatrix}(z|\tau), \qquad n\in\mathbb{Z} \tag{7.4.26}$$

The non trivial modular transformation can be obtained from the Poisson resummation formula

$$\sum_{n\in\mathbb{Z}} e^{-\pi\alpha n^2+2\pi i\beta n} = \frac{1}{\sqrt{\alpha}}\sum_{n\in\mathbb{Z}} e^{-\frac{\pi}{\alpha}(n+\beta)^2} \tag{7.4.27}$$

from which we get

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}\left(-\frac{z}{\tau}\middle|-\frac{1}{\tau}\right) = \sqrt{-i\tau}e^{-\frac{i\pi a^2}{\tau}}e^{2\pi i a\left(b-\frac{z}{\tau}\right)}\sum_{n\in\mathbb{Z}} e^{\pi i\tau\left(n-b+\frac{a+z}{\tau}\right)^2} \tag{7.4.28}$$

$$= \sqrt{-i\tau}e^{-\frac{i\pi a^2}{\tau}}e^{2\pi i a\left(b-\frac{z}{\tau}\right)}e^{\frac{\pi i}{\tau}(a+z)^2}\sum_{n\in\mathbb{Z}} e^{\pi i\tau(n-b)^2}e^{2\pi i(n-b)(z+a)} \tag{7.4.29}$$

$$= \sqrt{-i\tau}\, e^{\pi i\left(2ab+\frac{z^2}{\tau}\right)}\vartheta\begin{bmatrix} -b \\ a \end{bmatrix}(z|\tau) \tag{7.4.30}$$

There is also a simple transformation under Fourier sums

$$\sum_{n=1}^{N} e^{\frac{2\pi i n m}{N}}\vartheta\begin{bmatrix} a+\frac{n}{N} \\ b \end{bmatrix}(z|\tau) = e^{-2\pi i a m}\vartheta\begin{bmatrix} Na \\ \frac{b+m}{N} \end{bmatrix}\left(\frac{z}{N}\middle|\frac{\tau}{N^2}\right) \tag{7.4.31}$$

The theta functions relevant for the Dirac fermion

When studying the characters of the Dirac fermions on the torus, the following theta functions turn out to be useful.

$$\theta_1(z|\tau) = -i\sum_{r\in\mathbb{Z}+1/2}(-1)^{r-1/2}y^r q^{r^2/2} \tag{7.4.32}$$

$$\theta_2(z|\tau) = \sum_{r\in\mathbb{Z}+1/2} y^r q^{r^2/2} \tag{7.4.33}$$

$$\theta_3(z|\tau) = \sum_{n\in\mathbb{Z}} y^n q^{n^2/2} \tag{7.4.34}$$

$$\theta_4(z|\tau) = \sum_{n\in\mathbb{Z}}(-1)^n y^n q^{n^2/2} \tag{7.4.35}$$

where z is a complex variable and τ a complex parameter living on the upper-half place. We introduced $q = e^{2i\pi\tau}$ and $y = e^{2i\pi z}$. Theta functions have a single zero, located at $z = 0, 1/2, 1/2(1+\tau)$ and $\tau/2$, respectively. They have no pole. Using Jacobi's triple product identity one can rewrite them as

$$\theta_1(z|\tau) = -iy^{1/2}q^{1/8}\prod_{n=1}^{\infty}(1-q^n)\prod_{n=0}^{\infty}(1-yq^{n+1})(1-y^{-1}q^n) \tag{7.4.36}$$

$$\theta_2(z|\tau) = y^{1/2}q^{1/8}\prod_{n=1}^{\infty}(1-q^n)\prod_{n=0}^{\infty}(1+yq^{n+1})(1+y^{-1}q^n) \tag{7.4.37}$$

$$\theta_3(z|\tau) = \prod_{n=1}^{\infty}(1-q^n)\prod_{r\in\mathbb{N}+1/2}^{\infty}(1+yq^r)(1+y^{-1}q^r) \tag{7.4.38}$$

$$\theta_4(z|\tau) = \prod_{n=1}^{\infty}(1-q^n)\prod_{r\in\mathbb{N}+1/2}^{\infty}(1-yq^r)(1-y^{-1}q^r) \tag{7.4.39}$$

Theta functions are all related by shifting their arguments.

$$\theta_4(z|\tau) = \theta_3(z+1/2|\tau) \tag{7.4.40}$$
$$\theta_2(z|\tau) = \theta_1(z+1/2|\tau) \tag{7.4.41}$$
$$\theta_1(z|\tau) = -ie^{i\pi z}q^{1/8}\theta_4(z+1/2\tau|\tau) \tag{7.4.42}$$

Theta functions are the building blocks of doubly periodic functions on the complex plane. They are not periodic under $z \to z+1$ or $z \to z+\tau$, but they behave in the following simple fashion

$$\theta_1(z+1|\tau) = -\theta_1(z|\tau) \qquad \theta_1(z+\tau|\tau) = -\frac{1}{yq^{1/2}}\theta_1(z|\tau) \tag{7.4.43}$$
$$\theta_2(z+1|\tau) = -\theta_2(z|\tau) \qquad \theta_2(z+\tau|\tau) = \frac{1}{yq^{1/2}}\theta_2(z|\tau) \tag{7.4.44}$$
$$\theta_3(z+1|\tau) = \theta_3(z|\tau) \qquad \theta_3(z+\tau|\tau) = \frac{1}{yq^{1/2}}\theta_3(z|\tau) \tag{7.4.45}$$
$$\theta_4(z+1|\tau) = \theta_4(z|\tau) \qquad \theta_4(z+\tau|\tau) = -\frac{1}{yq^{1/2}}\theta_4(z|\tau) \tag{7.4.46}$$

We have

$$\theta_1(z|\tau+1) = e^{i\pi/4}\theta_1(z|\tau), \qquad \theta_1(z|-1/\tau) = -i\sqrt{-i\tau}e^{i\pi z^2\tau}\theta_1(-\tau z|\tau) \tag{7.4.47}$$

7.4.3 Modular S-matrix for the minimal model $\mathcal{M}_{p,p'}$

The characters are a representation of the modular group. One can prove that they are the solutions of a differential equation in τ (of order $\frac{(p-1)(p'-1)}{2}$) that follows from the null-vector at level $(p-1)(p'-1)$ in the module of the identity. In order to compute the S-matrix, we follow closely Di Francesco et al, chapter 10, section 10.6 (up to some change of notations). Let us first rewrite the characters as

$$\chi_{r,s}(\tau) = K_{\lambda_{r,s}}(\tau) - K_{\lambda_{r,-s}}(\tau)\,, \tag{7.4.48}$$

where $\lambda_{r,s} = p'r - ps$ and $\lambda_{r,-s} = p'r + ps$ and

$$K_\lambda(q) := \frac{1}{\eta(\tau)} \sum_{k=\lambda \bmod N} q^{\frac{k^2}{2N}}, \qquad N = 2pp'\,. \tag{7.4.49}$$

Since $K_\lambda = K_{\lambda+N}$, λ only matters mod N. The parameter $\lambda \in \mathbb{Z}/N\mathbb{Z}$ will soon turn out to be a simpler parametrization of the characters than (r,s). But first we need to understand the reflexion $(r,s) \to (r,-s)$ at the level of $\mathbb{Z}/N\mathbb{Z}$.

Fact : let (r_0, s_0) be two integers such that $p'r_0 - ps_0 = 1$, and let $\omega_0 = p'r_0 + ps_0 \bmod N$. Then the reflexion $(r,s) \to (r,-s)$ is implemented in $\mathbb{Z}/N\mathbb{Z}$ by multiplication by ω_0 :

$$\lambda_{r,-s} = \omega_0\lambda_{r,s} \tag{7.4.50}$$

as follows easily from the identity

$$(p'r_0 - ps_0)(p'r + ps) = (p'r_0 + ps_0)(p'r - sps) \mod N \tag{7.4.51}$$

Thus the characters can be conveniently written as

$$\chi_\lambda(\tau) = K_\lambda(\tau) - K_{\omega_0\lambda}(\tau) \tag{7.4.52}$$

At this stage a straightforward application of the Poisson resummation formula yields

$$K_\lambda(-1/\tau) = \frac{1}{\sqrt{N}} \sum_{\mu=0}^{N-1} e^{-i2\pi\frac{\lambda\mu}{N}} K_\mu(\tau) \tag{7.4.53}$$

and thus (since $\omega_0^2 = 1 \bmod N$)

$$K_{\omega_0\lambda}(-1/\tau) = \frac{1}{\sqrt{N}} \sum_{\mu=0}^{N-1} e^{-i2\pi\frac{\omega_0\lambda\mu}{N}} K_\mu(\tau) = \frac{1}{\sqrt{N}} \sum_{\mu=0}^{N-1} e^{-i2\pi\frac{\lambda\mu}{N}} K_{\omega_0\mu}(\tau) \tag{7.4.54}$$

It follows that

$$\chi_\lambda(-1/\tau) = \frac{1}{\sqrt{N}} \sum_{\mu=0}^{N-1} e^{-i2\pi\frac{\lambda\mu}{N}} \chi_\mu(\tau) \tag{7.4.55}$$

However this is not the end of the story, since

- we need to go back to the label (r, s) instead of λ
- the characters appearing in the *r.h.s.* are not linearly independent.

To resolve these two issues, we use the following facts.

Fact 1: The map

$$(r, s) \to \lambda_{r,s} = p'r - ps \bmod N \tag{7.4.56}$$

is a bijection from $\{(r, s),\ 0 \le r \le p-1,\ 1-p' \le s \le p'\}$ to $\mathbb{Z}_N$. Indeed since p and p' are coprime this map is injective[11], and therefore a bijection since both sets are of size N. Thus we may write

$$\chi_\lambda(-1/\tau) = \frac{1}{\sqrt{N}} \sum_{r=0}^{p-1} \sum_{s=1-p'}^{p'} e^{-i2\pi\frac{\lambda\lambda_{r,s}}{N}} \chi_{r,s}(\tau) \tag{7.4.57}$$

Fact 2: For all values of λ such that $\lambda = \pm\omega_0\lambda \bmod N$, we have $\chi_\lambda = 0$, so we can remove them from the above sum. But $\lambda_{r,s} = \pm\omega_0\lambda_{r,s}$ iff $r = 0 \bmod p$ or $s = 0 \bmod p'$. Therefore we can exclude $r = 0$ and $s = 0$, yielding

$$\chi_\lambda(-1/\tau) = \frac{1}{\sqrt{N}} \sum_{r=1}^{p-1} \sum_{s=1}^{p'-1} \left(e^{-i2\pi\frac{\lambda\lambda_{r,s}}{N}} \chi_{r,s}(\tau) + e^{-i2\pi\frac{\lambda\lambda_{r,-s}}{N}} \chi_{r,-s}(\tau) \right) \tag{7.4.58}$$

$$= \frac{1}{\sqrt{N}} \sum_{r=1}^{p-1} \sum_{s=1}^{p'-1} \left(e^{-i2\pi\frac{\lambda\lambda_{r,s}}{N}} - e^{-i2\pi\frac{\lambda\lambda_{r,-s}}{N}} \right) \chi_{r,s}(\tau) \tag{7.4.59}$$

[11] Suppose $\lambda_{r_1,s_1} = \lambda_{r_2,s_2} \bmod N$. This mean $p'(r_1 - r_2) = p(s_1 - s_2) \bmod 2pp'$, and in particular $p'(r_1 - r_2) = 0 \bmod p$. Given the range of r, this implies $r_1 = r_2$ and thus $s_1 - s_2 = 0 \bmod 2p'$, which in turns means $s_1 = s_2$.

Finally we exploit $\chi_{p-r,p'-s} = \chi_{r,s}$, together with $\lambda_{p-r,p'-s} = -\lambda_{r,s}$ to restrict the sum to the fundamental domain of the Kac table (see Fig. (7.8))

$$\chi_\lambda(-1/\tau) = \frac{2}{\sqrt{N}} \sum_{r,s} \left(\cos 2\pi \frac{\lambda\lambda_{r,s}}{N} - \cos 2\pi \frac{\lambda\lambda_{r,-s}}{N} \right) \chi_{r,s}(\tau) \tag{7.4.60}$$

$$= \frac{4}{\sqrt{N}} \sum_{r,s} \sin 2\pi \frac{\lambda p' r}{N} \sin 2\pi \frac{\lambda p s}{N} \chi_{r,s}(\tau) \tag{7.4.61}$$

and we get the S-matrix

$$S_{(r_1,s_1),(r_2,s_2)} = \frac{4}{\sqrt{2pp'}} \sin \pi \frac{(p'r_1 - ps_1)p'r_2}{pp'} \sin \pi \frac{(p'r_1 - ps_1)ps_2}{pp'} \tag{7.4.62}$$

i.e .

$$\boxed{S_{(r_1,s_1),(r_2,s_2)} = 2\sqrt{\frac{2}{pp'}}(-1)^{1+s_1r_2+s_2r_1} \sin \pi \frac{p'r_1r_2}{p} \sin \pi \frac{ps_1s_2}{p'}} \tag{7.4.63}$$

In the above derivation we took (r_2, s_2) in the range (7.3.4), *i.e.* with $ps_2 < p'r_2$, but for obvious reasons the above S matrix has to be invariant under $(r_2, s_2) \to (p - r_2, p' - s_2)$. This elementary sanity check is left to the reader.

This S matrix enjoys

$$S^2 = 1, \qquad S = \overline{S} = S^t \tag{7.4.64}$$

In particular S is unitary.

Lecture 8

CFT in curved space

So far we have discussed conformal invariance for a quantum field theory on the complex plane, and we now consider a field theory living on an arbitrary two-dimensional curved space (M, g). This essentially brings two new features in the game

- the *topology* of space-time (*i.e.* the smooth surface M itself, as characterized by its genus[1]),
- the *geometry* of space-time, as described by the Riemannian metric g.

One must prescribe how to extend the theory from flat Euclidean space to an arbitrary curved space, through an action that now depends explicitly on the metric g (and on the manifold M)

$$S[\Phi, g] = \int_M \mathcal{L}(\Phi(x), \partial_\mu \Phi(x), g_{\mu\nu}(x))\, dV(x) \tag{8.0.1}$$

where $dV(x) = \sqrt{|g(x)|}\, d^2x$ is the Riemannian volume element, and $|g(x)|$ stands for the determinant of the matrix $g_{\mu\nu}(x) = g(\partial_\mu, \partial_\nu)$. Besides allowing us to work on curved space, introducing an arbitrary metric g has an important benefit : the action now takes the same mathematical form in all coordinate systems (even in flat space, one could decide to work in polar coordinates for instance). One says that such a theory is *generally covariant*, and it means that the theory does not require the choice of a coordinate system in order to be well-defined. For instance the free scalar field can be coupled to the metric as follow

$$S[\phi, g] = \int g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \, dV(x) \tag{8.0.2}$$

where $g^{\mu\nu} = g(dx^\mu, dx^\nu)$ is the inverse matrix of $g_{\mu\nu}$. It is not very difficult to check that the above expression remains the same in all coordinate systems.

On a generic curved manifold there is no longer a preferred coordinate system (nor a globally defined one), so having a coordinate-independent formulation of physics is a very natural requirement indeed. One can push this idea even further and use a coordinate-free formalism in which fields become purely geometric objects. For instance the action (8.0.2) is just the expression in local coordinates of the intrinsic quantity

$$S = \int d\phi \wedge \star d\phi = \int g(d\phi, d\phi) dV = \|d\phi\|_g^2 \tag{8.0.3}$$

[1]For now we will restrict ourselves to compact, oriented manifolds without boundary. Conformal field theories can also be defined on unorientable manifolds and in the presence of boundaries, but this is a more advanced topic which we will not address at this stage.

At the classical level general covariance means that the action is invariant under isometries, in the sense that for any diffeomorphism $f : M \to M$

$$S[f^*\Phi, f^*g] = S[\Phi, g] \tag{8.0.4}$$

which essentially means that the Lagrangian density $\mathcal{L}$ is a scalar : $f^*\mathcal{L} = \mathcal{L} \circ f$.

We assume that general covariance still holds at the quantum level. In the path integral this amounts to

$$e^{-S[f^*\Phi, f^*g]} D_{f^*g}[f^*\Phi] = e^{-S[\Phi,g]} D_g[\Phi] \tag{8.0.5}$$

where we write $D_g[\Phi]$ to emphasise that the functional measure does in general depend on the metric. Why is that ? Naively it does not, but the fact that it has to be regularized (for instance through a short-distance cut-off) makes the story more subtle. For instance a short-distance cut-off - such as a lattice constant - introduces a UV length, de facto making the integration measure scale-dependent. In practice assuming general covariance means finding a way to regularize the QFT in such a way as to preserve this invariance. We will see in the next tutorial how this is done for the free boson. If we denote by Z_g the partition function of the QFT on (M, g), we have for any diffeomorphism $f : M \to M$

$$\boxed{Z_g = Z_{f^*g}} \tag{8.0.6}$$

where we dropped the subscript M. Furthermore for a correlation function

$$\langle O \rangle_{f^*g} = \frac{1}{Z_{f^*g}} \int O e^{-S[\Phi, f^*g]} D_{f^*g}[\Phi] = \frac{1}{Z_g} \int (f^*O)\, e^{-S[f^*\Phi, f^*g]} D_{f^*g}[f^*\Phi] \tag{8.0.7}$$

$$= \frac{1}{Z_g} \int (f^*O)\, e^{-S[\Phi,g]} D_g[\Phi] = \langle f^*O \rangle_g \tag{8.0.8}$$

or more explicitly

$$\boxed{\langle (f^*O_1)(x_1) \cdots (f^*O_p)(x_p) \rangle_g = \langle O_1(x_1) \cdots O_p(x_p) \rangle_{f^*g}} \tag{8.0.9}$$

Another way to think about (8.0.6) and (8.0.9) is in term of active versus passive transformation. An elementary but useful example is that of a scale transformation $f(x) = \lambda x$ on the Euclidean plane (with metric $\eta_{\mu\nu} = \delta_{\mu\nu}$). The active transformation amounts to send the point x to λx, hence multiplying the length of curves by a factor λ. In the passive transformation points do not move, but the definition of distance is rescaled by changing the metric to $f^*\eta = \lambda^2 \eta$. Again the length of all curves is multiplied by λ. Clearly these two transformations are equivalent. For instance in a CFT the two point function of a scalar primary field ϕ_Δ with scaling dimension Δ obeys

$$\langle \phi_\Delta(x_1) \phi_\Delta(x_2) \rangle_{\lambda^2 \eta} = \lambda^{-2\Delta} \frac{1}{\|x_1 - x_2\|^{2\Delta}} = \langle \phi_\Delta(\lambda x_1) \phi_\Delta(\lambda x_2) \rangle_\eta \,. \tag{8.0.10}$$

In the first equality we used the behavior of a correlation function under a Weyl rescaling (8.2.12) which we'll derive shortly.

8.1 Stress-energy tensor revisited

In section 3.1 we have defined the classical stress-energy tensor through (3.1.8), that is as the response of the theory to an infinitesimal diffeomorphism *while leaving the (flat) metric unchanged*. The generalization to curved space is immediate :

$$S[\Phi + \delta_\epsilon \Phi, g] - S[\Phi, g] = -\frac{1}{2\pi} \int_M T^{\mu\nu} \nabla_\mu \epsilon_\nu \, dV(x) \tag{8.1.1}$$

but just as in flat space this leaves the freedom to add to $T^{\mu\nu}$ any covariantly conserved tensor. General covariance allows to bypass this ambiguity by defining the stress-energy tensor as the response of the theory to an infinitesimal change of the metric (see appendix (3.4.2))

$$\boxed{S[\Phi, g + \delta g] - S[\Phi, g] = \frac{1}{4\pi} \int_M T^{\mu\nu} \delta g_{\mu\nu} \, dV(x) = -\frac{1}{4\pi} \int_M T_{\mu\nu} \delta g^{\mu\nu} \, dV(x)} \tag{8.1.2}$$

or equivalently

$$T_{\mu\nu}(x) = -4\pi \frac{\delta S}{\delta g^{\mu\nu}(x)} \, . \tag{8.1.3}$$

Note that we work with the following convention for functional derivatives :

$$\delta F = \int_M \frac{\delta F}{\delta f(x)} \delta f(x) \, dV(x)$$

for a generic functional F of a function f defined on M. This means

$$\frac{\delta f(x)}{\delta f(y)} = \delta_y(x) \tag{8.1.4}$$

where $\delta_y(x)$ is the delta-function normalized with respect to the volume form, in the sense that

$$\int_M \delta_y(x) h(x) \sqrt{|g(x)|} d^2x = h(y) \, . \tag{8.1.5}$$

for any test function h.

Equation (8.1.7) defines the so-called Hilbert stress-energy tensor, and the fact that it coincides with our previous definition (8.1.1) in flat space is a simple consequence of general covariance. This stress-energy tensor is symmetric by construction. The reader might wonder what happened to the intrinsic ambiguity underlying the definition of the stress-energy tensor in flat space. It is not really gone, and it hides in the fact that the extension of the theory from flat space to curved space may not be unique. For instance we could decide that the action of a free scalar field in curved space is

$$S[\phi, g] = \int \left(g^{\mu\nu} \partial^\mu \phi \partial_\nu \phi + \alpha R \phi \right) \, dV(x) \, . \tag{8.1.6}$$

Both theories (8.0.2) and (8.1.6) are indistinguishable in the flat space limit since the scalar curvature R vanishes, but even in flat space the corresponding Hilbert stress-energy

tensors as defined by (8.1.7) do not coincide : they differ by an improvement term.

At the quantum level there may be contributions coming from the integration measure, thus we define

$$\begin{aligned}&\langle T_{\mu_1\nu_1}(y_1)\cdots T_{\mu_r\nu_r}(y_r)O_1(x_1)\cdots O_p(x_p)\rangle_g\\ =&\frac{(4\pi)^r}{Z_g}\frac{\delta^r}{\delta g^{\mu_1\nu_1}(y_1)\cdots\delta g^{\mu_r\nu_r}(y_r)}Z_g\langle O_1(x_1)\cdots O_p(x_p)\rangle_g\end{aligned} \tag{8.1.7}$$

This allows to write the infinitesimal change of the partition function and correlation functions under a change of metric $g \to g+\delta g$ as

$$\frac{\delta Z_g}{Z_g} = \frac{1}{4\pi}\int \langle T_{\mu\nu}(x)\rangle \delta g^{\mu\nu}(x)\sqrt{|g(x)|}d^2x \tag{8.1.8}$$

and

$$\delta\langle O\rangle_g = \frac{1}{4\pi}\int \left(\langle T_{\mu\nu}(x)O\rangle_g - \langle T_{\mu\nu}(x)\rangle_g\langle O\rangle_g\right)\delta g^{\mu\nu}(x)\sqrt{|g(x)|}d^2x \tag{8.1.9}$$

where O stands for $O_1(x_1)\cdots O_p(x_p)$.

The terminology *tensor* is now fully justified, since general covariance implies that the Hilbert stress-energy tensor indeed transforms as a tensor under isometries :

$$\langle (f^*T)_{\mu\nu}(x)(f^*O_1)(x_1)\cdots(f^*O_p)(x_p)\rangle_g = \langle T_{\mu\nu}(x)O_1(x_1)\cdots O_p(x_p)\rangle_{f^*g} \tag{8.1.10}$$

where

$$(f^*T)_{\mu\nu}(x) = T_{\rho\sigma}(f(x))\frac{\partial f^\rho}{\partial x^\mu}\frac{\partial f^\sigma}{\partial x^\nu}\,. \tag{8.1.11}$$

Another benefit of defining $T_{\mu\nu}$ through (8.1.7) is that the Ward identity becomes an elementary consequence of general covariance. Indeed for an infinitesimal diffeomorphism $f^\mu(x) = x^\mu + \epsilon^\mu(x)$ the relation (8.0.9) together with (8.1.9) yields

$$\sum_j \langle O_1(x_1)\cdots\delta_\epsilon O_j(x_j)\cdots O_p(x_p)\rangle_g = -\frac{1}{4\pi}\int \delta_\epsilon g_{\mu\nu}\langle T^{\mu\nu}(x)O\rangle_g\, dV(x) \tag{8.1.12}$$

$$= -\frac{1}{2\pi}\int \nabla_\mu\epsilon_\nu(x)\langle T^{\mu\nu}(x)O\rangle_g\, dV(x) \tag{8.1.13}$$

and we recover the Ward identity (3.2.18) in flat space.

8.2 Conformal/Weyl covariance

As explained in section (2.2), a conformal transformation can be split into an isometry and a Weyl rescaling. Since we have assumed invariance under isometries (this is the content of general covariance) conformal invariance is then equivalent to Weyl invariance. Under an infinitesimal Weyl rescaling $\delta g_{\mu\nu}(x) = 2\sigma(x)g_{\mu\nu}(x)$, one has from (8.1.7)

$$\delta Z = -\left(\frac{1}{2\pi}\int_M \langle T^\mu{}_\mu\rangle\sigma\, dV\right)Z \tag{8.2.1}$$

Naïvely conformal symmetry requires $T^{\mu}{}_{\mu} \simeq 0$, but for a QFT in curved space there is an anomaly. From general covariance, the trace of the stress-energy tensor must be invariant under isometries, and it must vanish in the flat space limit. Furthermore, it must be of scaling dimension 2. In two-dimensions this only leaves one possibility : $T^{\mu}{}_{\mu}$ can only be proportional to the scalar curvature R. The usual convention is to write the numerical prefactor as $-c/12$, where c is called the *central charge* :

$$\boxed{T^{\mu}{}_{\mu} \simeq -\frac{c}{12} R} \tag{8.2.2}$$

This is called the Weyl anomaly, and under an infinitesimal Weyl rescaling we have

$$Z_{e^{2\delta\sigma} g} = \left(1 + \frac{c}{24\pi} \int_M \delta\sigma(x) R_g(x)\, dV_g(x) + O(\delta\sigma^2)\right) Z_g \,. \tag{8.2.3}$$

where R_g is the scalar curvature associated to the metric g, and $dV = dV_g = \sqrt{|g|} d^2x$. This can be rephrased as

$$\left.\frac{\delta}{\delta\sigma(x)}\right|_{\sigma=0} \log Z_{e^{2\sigma} g} = \frac{c}{24\pi} R_g(x) \tag{8.2.4}$$

Starting from some partition function Z_{g_0}, (8.2.4) determines the partition function for all metrics in the same conformal class, *i.e.* of the form $g = e^{2\sigma} g_0$. Indeed from (8.2.3) we have

$$Z_{e^{2\delta\sigma} g} = \left(1 + \frac{c}{24\pi} \int_M \delta\sigma(x) \left[R_0 - 2\Delta_0 \sigma(x)\right] dV_0(x) + O(\delta\sigma^2)\right) Z_g \,. \tag{8.2.5}$$

where we used $R_{e^{2\sigma} g_0} = e^{-2\sigma}(R_0 - 2\Delta_0\sigma)$, with R_0 and Δ_0 begin the curvature and Laplacian of the reference metric g_0. Thus

$$\frac{\delta}{\delta\sigma(x)} \log Z_{e^{2\sigma} g_0} = \frac{c}{24\pi} \left(R_0(x) - 2\Delta_0\sigma\right) \tag{8.2.6}$$

Upon integrating (8.2.6) we find the following behavior of the partition function under a finite Weyl rescaling (as long as the function σ has compact support)

$$\boxed{Z_{e^{2\sigma} g} = \exp\left(\frac{c}{24\pi} \int_M \left(g^{\mu\nu} \partial_\mu\sigma \partial_\nu\sigma + R_g \sigma\right) dV_g\right) Z_g} \tag{8.2.7}$$

Exercise : Wess-Zumino consistency condition. Check that (8.2.7) is consistent under the composition of Weyl transformations, in the sense that $g \to e^{2\sigma_1 + 2\sigma_2} g$ yields the same answer as $g \to e^{2\sigma_1} g$ followed by $g \to e^{2\sigma_2} g$.

Exercise : For a given background metric g_0, consider the functional differential equation

$$\frac{\delta}{\delta\sigma(x)} F[\sigma] = R_0(x) - 2\Delta_0\sigma \tag{8.2.8}$$

with boundary condition $F[0] = 0$. It is clear that the solution, if it exists, is unique. Check that

$$F[\sigma] = \int_M \left(g_0^{\mu\nu} \partial_\mu\sigma \partial_\nu\sigma + R_0\sigma\right) dV_0 \tag{8.2.9}$$

is a solution, as long as the function σ has compact support.

We now know how the partition function of a conformal field theory behaves under Weyl rescaling. What about correlation functions ? The definition (8.1.7) of the stress-energy tensor *a la* Hilbert tells us that under an infinitesimal Weyl rescaling $\delta g_{\mu\nu}(x) = 2\sigma(x)g_{\mu\nu}(x)$ a correlation function $\langle O\rangle = \langle O_1(x_1)\cdots O_p(x_p)\rangle$ behaves as

$$\delta\left(Z\langle O\rangle\right) = -\left(\frac{1}{2\pi}\int_M \langle T^{\mu}{}_{\mu} O\rangle \sigma\, dV\right) Z \tag{8.2.10}$$

i.e.

$$\delta\langle O\rangle = -\frac{1}{2\pi}\int_M \langle\left(T^{\mu}{}_{\mu} O\rangle - \langle T^{\mu}{}_{\mu}\rangle\langle O\rangle\right)\sigma\, dV\,. \tag{8.2.11}$$

The Weyl anomaly cancels out in $\langle T^{\mu}{}_{\mu} O\rangle - \langle T^{\mu}{}_{\mu}\rangle\langle O\rangle$, and only the contact terms of $T^{\mu}{}_{\mu}$ can contribute : as in flat space, they may modify the naive classical transformation of fields. While classical fields are invariant under Weyl transformation, quantum fields have to be regularized, and hence become sensitive to change of scales. Vertex operators in the free scalar theory provide a good and explicit example. For instance for scalar primary fields

$$\boxed{\langle O_1(x_1)\cdots O_p(x_p)\rangle_{e^{2\sigma}g} = \prod_{i=1}^{p} e^{-\gamma_i\sigma(x_i)}\langle O_1(x_1)\cdots O_p(x_p)\rangle_g\,.} \tag{8.2.12}$$

which we will write as

$$\left(O_i\right)_{e^{2\sigma}g} = e^{-\gamma_i\sigma}\left(O_i\right)_g \tag{8.2.13}$$

where γ_i is the anomalous dimension of the field O_i.

Exercise : By computing $\frac{4\pi}{Z}\frac{\delta}{\delta g^{\mu\nu}} Z\langle T^{\rho}{}_{\rho} O\rangle$ using (8.2.2), show that the contact term between $T^{\rho}{}_{\rho}$ and $T_{\mu\nu}$ is

$$T^{\rho}{}_{\rho}(x)T_{\mu\nu}(y) = 4\pi\frac{c}{12}\frac{\delta R(x)}{\delta g^{\mu\nu}(y)} \tag{8.2.14}$$

and evaluate the functional derivative using eq. (8.5.25). Deduce that in a locally flat metric

$$T^{\rho}{}_{\rho}(x)T_{zz}(y) = -4\pi\frac{c}{12}\partial_z^2\delta(x-y) \tag{8.2.15}$$

and recover (3.3.26).

8.3 Behavior of the stress-energy tensor under Weyl rescaling

In a classical field theory enjoying Weyl invariance, the stress tensor behaves as

$$\left(T_{\mu\nu}\right)_{e^{2\sigma}g} = \left(T_{\mu\nu}\right)_g \tag{8.3.1}$$

Indeed Weyl invariance of the action implies $S[\Phi, g+\delta g] - S[\Phi, g] = S[\Phi, e^{2\sigma}(g+\delta g)] - S[\Phi, e^{2\sigma} g]$, yielding

$$\int_M (T_{\mu\nu})_g \, \delta g^{\mu\nu} \, dV = \int_M (T_{\mu\nu})_{e^{2\sigma} g} \, \delta g^{\mu\nu} \, dV \tag{8.3.2}$$

since $\delta \tilde{g}^{\mu\nu} = e^{-2\sigma} \delta g^{\mu\nu}$ and $d\tilde{V} = e^{2\sigma} dV$ for $\tilde{g} = e^{2\sigma} g$.

At the quantum level however the anomalous behavior (8.2.7) means that the quantum stress tensor transforms as

$$\boxed{(T_{\mu\nu})_{e^{2\sigma} g} = (T_{\mu\nu})_g + \frac{c}{6}\left(\partial_\mu \sigma \partial_\nu \sigma - \frac{1}{2} g_{\mu\nu} \partial^\rho \sigma \partial_\rho \sigma + g_{\mu\nu} \Delta \sigma - \nabla_\mu \nabla_\nu \sigma\right)} \tag{8.3.3}$$

Indeed under $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$ we have

$$\frac{1}{Z_{e^{2\sigma} g}} \delta \left(Z_{e^{2\sigma} g} \langle O \rangle_{e^{2\sigma} g} \right) = \frac{1}{4\pi} \int \langle T_{\mu\nu} O \rangle_{e^{2\sigma} g} \delta \tilde{g}^{\mu\nu} \, d\tilde{V} \tag{8.3.4}$$

$$= \frac{1}{4\pi} \int \langle T_{\mu\nu} O \rangle_{e^{2\sigma} g} \delta g^{\mu\nu} \, dV \tag{8.3.5}$$

Thus

$$\langle T_{\mu\nu} O \rangle_{e^{2\sigma} g} = \frac{4\pi}{Z_{e^{2\sigma} g}} \frac{\delta}{\delta g^{\mu\nu}} \left(Z_{e^{2\sigma} g} \langle O \rangle_{e^{2\sigma} g} \right) \tag{8.3.6}$$

When the functional derivative $\frac{\delta}{\delta g^{\mu\nu}}$ acts on O, it only produces contact terms. So away from coinciding points we can write

$$(T_{\mu\nu})_{e^{2\sigma} g} = \frac{4\pi}{Z_{e^{2\sigma} g}} \frac{\delta}{\delta g^{\mu\nu}} Z_{e^{2\sigma} g} \tag{8.3.7}$$

Plugging in (8.2.7) we get

$$(T_{\mu\nu})_{e^{2\sigma} g} = \frac{4\pi}{Z_{e^{2\sigma} g}} \frac{\delta}{\delta g^{\mu\nu}} Z_{e^{2\sigma} g} \tag{8.3.8}$$

$$= (T_{\mu\nu})_g + \frac{c}{6} \frac{\delta}{\delta g^{\mu\nu}} \int_M \left(g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma + R_g \sigma \right) dV \tag{8.3.9}$$

Computing the functional derivatives (see (8.5.22) and (8.5.25) in the Appendix) yields (8.3.3).

Exercise : using equation (8.3.3), check that $T^\mu{}_\mu$ transforms as

$$\left(T^\mu{}_\mu\right)_{e^{2\sigma} g} = e^{-2\sigma} \left(\left(T^\mu{}_\mu\right)_g + \frac{c}{6} \Delta \sigma \right) \tag{8.3.10}$$

and in particular we recover $(T^\mu{}_\mu)_{e^{2\sigma} dz d\bar{z}} = \frac{c}{6} e^{-2\sigma} \Delta \sigma = -\frac{c}{12} R$.

We are now going to work in isothermal coordinates according to which we can define complex coordinates. We now have

$$g_{\mu\nu}(x) = e^{2\sigma(x)} \delta_{\mu\nu}, \qquad R_g = -2\Delta_g \sigma = -2e^{-2\omega} \left(\partial_1^2 + \partial_2^2\right) \sigma = -8 e^{-2\sigma} \partial \bar{\partial} \sigma \tag{8.3.11}$$

We get

$$\boxed{(T_{zz})_{e^{2\sigma} dz d\bar{z}} = (T_{zz})_{dz d\bar{z}} + \frac{c}{6} \left((\partial \sigma)^2 - \partial^2 \sigma \right)} \tag{8.3.12}$$

8.4 Behavior of the stress-energy tensor under conformal maps

Since we know how the stress-tensor transforms under isometries and Weyl transformation, we know how it behaves under conformal transformations. Consider a conformal map

$$f : (M, g) \to (M, \tilde{g}) \tag{8.4.1}$$
$$z \to f(z) \tag{8.4.2}$$

In practice we can always exploit Weyl transformations to make the metric locally flat around all insertions. So we consider a point z and its image $w = f(z)$ with metric (locally, *i.e.* in some neighborhood) of the form $g = dzd\bar{z}$ and $\tilde{g} = dwd\bar{w}$, so that $f^*\tilde{g} = e^{2\sigma}g$ with $2\sigma = \log\frac{\partial f}{\partial z} + \log\frac{\partial \bar{f}}{\partial \bar{z}}$, at least in some neighborhood of z. From general covariance

$$\langle (f^*T)(z)\cdots\rangle_{\tilde{g}} = \langle T(z)\cdots\rangle_{f^*\tilde{g}} = \langle T(z)\cdots\rangle_{e^{2\sigma}g} \tag{8.4.3}$$

Now remember that T stands for T_{zz}, and therefore $(f^*T)(z) = (\partial_z f)^2 T(f(z)) = (\partial_z w)^2 T(w)$. Now (8.3.12) gives the behavior under a Weyl rescaling :

$$\langle T(z)\cdots\rangle_{e^{2\sigma}g} = \langle \left(T(z) - \frac{c}{12}S(f)(z)\right)\cdots\rangle_g \tag{8.4.4}$$

where $S(f)$ is the Schwarzian derivative as defined in eq. (3.3.30)

$$S(f) = \partial_z\left(\frac{\partial_z^2 f}{\partial_z f}\right) - \frac{1}{2}\left(\frac{\partial_z^2 f}{\partial_z f}\right)^2 = \frac{\partial_z^3 f}{\partial_z f} - \frac{3}{2}\left(\frac{\partial_z^2 f}{\partial_z f}\right)^2 \tag{8.4.5}$$

Thus we recover the anomalous transformation eq. (3.3.29)

$$\langle \left((\partial_z f)^2 T(f(z)) + \frac{c}{12}S(f)(z)\right)\cdots\rangle_{\tilde{g}} = \langle T(z)\cdots\rangle_g \tag{8.4.6}$$

We are going to write the above slightly abusively as

$$\boxed{T(w) = \left(\frac{\partial w}{\partial z}\right)^{-2}\left[T(z) - \frac{c}{12}\{w, z\}\right]} \tag{8.4.7}$$

keeping in mind however that we are working in locally flat metrics (hence the metric need *not* be the same in the l.h.s. and the r.h.s.).

Some references for this chapter:

- *Lectures on Conformal Field Theory*, Krzysztof Gawedzki (chapter 2)
- *String Theory: Volume 1*, Joseph Polchinski

8.5 Appendix

8.5.1 Some basic notions in Riemannian geometry

While Riemannian geometry does not play an essential role throughout this lecture, notions such as metric, Levi-Civita connection and curvature are mentioned at times. This section - aimed at the reader already familiar with these concepts - is simply a reminder of basic definitions and an excuse to fix our notations.

Covariant derivative, Levi-Civita connection, Christoffel symbols

We consider a Riemannian manifold (M, g) where g is the metric. Given a local chart (coordinate system) x^μ, the components of the metric are

$$g_{\mu\nu} = g(\partial_\mu, \partial_\nu), \qquad g^{\mu\nu} = g(dx^\mu, dx^\nu)\,, \tag{8.5.1}$$

and these two matrices are inverses of each other

$$g_{\mu\nu} g^{\nu\rho} = \delta^\rho_\mu\,. \tag{8.5.2}$$

The Levi-Civita connection (or covariant derivative) ∇ is the only torsion-free, metric-compatible connection on the tangent bundle of Riemannian manifold. Torsion-free means

$$\nabla_X Y - \nabla_Y X = [X, Y] \tag{8.5.3}$$

while metric-compatibility is $\nabla g = 0$, *i.e.*

$$\nabla_Z \left(g(X, Y)\right) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \tag{8.5.4}$$

In local coordinates the Christoffel symbols $\Gamma^\rho{}_{\mu\nu}$ are defined by

$$\nabla_{\partial_\mu} \partial_\nu = \Gamma^\rho{}_{\mu\nu} \partial_\rho \tag{8.5.5}$$

and they can be expressed in terms of the metric as

$$\Gamma^\rho{}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\nu g_{\rho\mu} + \partial_\mu g_{\rho\nu} - \partial_\rho g_{\mu\nu}\right) \tag{8.5.6}$$

The Levi-Civita connection has a natural extension to the dual bundle (*i.e.* the cotangent bundle) and more generally to arbitrary tensor bundles :

$$\begin{aligned}
(\nabla_\rho T)^{\nu_1 \dots \nu_s}{}_{\mu_1 \dots \mu_r} &= \frac{\partial}{\partial \rho} T^{\nu_1 \dots \nu_s}{}_{\mu_1 \dots \mu_r} && (8.5.7)\\
&+ \Gamma^{\nu_1}{}_{\sigma\rho} T^{\sigma \nu_2 \dots \nu_s}{}_{\mu_1 \dots \mu_r} + \dots + \Gamma^{\nu_s}{}_{\sigma\rho} T^{\nu_1 \dots \nu_{s-1} \sigma}{}_{\mu_1 \dots \mu_r} && (8.5.8)\\
&- \Gamma^{\sigma}{}_{\mu_1 \rho} T^{\nu_1 \dots \nu_s}{}_{\sigma \mu_2 \dots \mu_r} - \dots - \Gamma^{\sigma}{}_{\mu_r \rho} T^{\nu_1 \dots \nu_s}{}_{\mu_1 \dots \mu_{r-1} \sigma}. && (8.5.9)
\end{aligned}$$

Following standard notations we will often drop the parenthesis and write $\nabla_\rho T^{\nu_1 \dots \nu_s}{}_{\mu_1 \dots \mu_r}$ in place of $(\nabla_\rho T)^{\nu_1 \dots \nu_s}{}_{\mu_1 \dots \mu_r}$. One should be careful about this abuse of notation. For instance the metric-compatibility reads

$$\nabla_\rho g_{\mu\nu} = 0 \tag{8.5.10}$$

being understood that $\nabla_\rho g_{\mu\nu}$ stands for $(\nabla_\rho g)_{\mu\nu}$ and not for $\nabla_\rho(g_{\mu\nu})$. Indeed the latter is in general non vanishing since

$$\nabla_\rho(g_{\mu\nu}) = \partial_\rho g_{\mu\nu}\,. \tag{8.5.11}$$

A last remark is that this extension to arbitrary tensors is natural in the sense that it is compatible with Leibniz rule and with trace operations (tensor contraction). For instance

$$g^{\mu\nu} \nabla_\rho T_{\mu\nu} = \nabla_\rho(g^{\mu\nu} T_{\mu\nu}) \quad \left(= \partial_\rho T^\mu{}_\mu\right)\,. \tag{8.5.12}$$

Curvature

The Riemann curvature tensor is defined by

$$R(u,v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w \tag{8.5.13}$$

In local coordinates, we work with the convention

$$R_{\mu\nu\rho\sigma} = g(R(\partial_\mu, \partial_\nu)\partial_\sigma, \partial_\rho) \tag{8.5.14}$$

This tensor enjoys the following properties

$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma}, \qquad R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho}, \qquad R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}\,. \tag{8.5.15}$$

In terms of the Christoffel symbols we have

$$R^\rho{}_{\mu\sigma\nu} = \partial_\sigma \Gamma^\rho{}_{\nu\mu} - \partial_\nu \Gamma^\rho{}_{\sigma\mu} + \Gamma^\rho{}_{\sigma\lambda}\Gamma^\lambda{}_{\nu\mu} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\sigma\mu} \tag{8.5.16}$$

The Ricci tensor is given by

$$Ric(u,v) = \mathrm{Tr}_g(x \to R(x,u)v) \tag{8.5.17}$$

which means in local coordinates

$$R_{\mu\nu} = Ric(\partial_\mu, \partial_\nu) = g^{\rho\sigma} R_{\rho\mu\sigma\nu} = R^\rho{}_{\mu\rho\nu} \tag{8.5.18}$$

This is a symmetric tensor, whose local expression in terms of the Christoffel symbols is

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho{}_{\nu\mu} - \partial_\nu \Gamma^\rho{}_{\rho\mu} + \Gamma^\rho{}_{\rho\lambda}\Gamma^\lambda{}_{\nu\mu} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\rho\mu} \tag{8.5.19}$$

Finally the Ricci scalar is simply $R = R^\mu{}_\mu$.

Variation of the Einstein-Hilbert action

We want to compute the variation δS_g of the Einstein-Hilbert action

$$S_g = \frac{1}{2}\int R\sqrt{|g|}d^n x \tag{8.5.20}$$

under an infinitesimal change of metric $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$. The answer is

$$\boxed{\delta S_g = \frac{1}{2}\int G_{\mu\nu}\delta g^{\mu\nu}\sqrt{|g|}d^n x} \tag{8.5.21}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor.

A first useful formula is

$$\boxed{\delta\sqrt{|g|} = -\frac{1}{2}\sqrt{|g|}g_{\mu\nu}\delta g^{\mu\nu} = \frac{1}{2}\sqrt{|g|}g^{\mu\nu}\delta g_{\mu\nu}} \tag{8.5.22}$$

Indeed for any invertible matrix A one has

$$\det(A+H) = \det A(1+A^{-1}H) = \det A + \det A\,\mathrm{Tr}(A^{-1}H) + O(H^2) \tag{8.5.23}$$

which can be written as $\frac{\partial \det A}{\partial A_{\mu\nu}} = \det A A^{\nu\mu}$, where $A^{\mu\nu}$ stands for $(A^{-1})_{\mu\nu}$. So under $A \to A + \delta A$ we get

$$\delta \det A = \det A A^{\nu\mu}\delta A_{\mu\nu} \tag{8.5.24}$$

and (8.5.22) follows.

The variation of the Ricci curvature R is in itself of interest and will prove useful :

$$\boxed{\delta R = \left(R_{\mu\nu} - \nabla_\mu\nabla_\nu + g_{\mu\nu}\nabla_\rho\nabla^\rho\right)\delta g^{\mu\nu}} \tag{8.5.25}$$

where $\delta g^{\mu\nu} = -g^{\mu\rho}g^{\nu\sigma}\delta g_{\rho\sigma}$. Deriving this identity is a rather lengthy but standard calculation. We start with

$$\delta R = (\delta g^{\mu\nu})R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu} \tag{8.5.26}$$

and

$$\begin{aligned}
\delta R^\rho{}_{\sigma\mu\nu} &= \partial_\mu \delta\Gamma^\rho{}_{\nu\sigma} - \partial_\nu\delta\Gamma^\rho{}_{\mu\sigma} + \delta\Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma} && (8.5.27)\\
&\quad + \Gamma^\rho{}_{\mu\lambda}\delta\Gamma^\lambda{}_{\nu\sigma} - \delta\Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\sigma} - \Gamma^\rho{}_{\nu\lambda}\delta\Gamma^\lambda{}_{\mu\sigma} && (8.5.28)\\
&= \nabla_\mu\delta\Gamma^\rho{}_{\nu\sigma} - \nabla_\nu\delta\Gamma^\rho{}_{\mu\sigma} && (8.5.29)
\end{aligned}$$

Indeed $\delta\Gamma^\rho{}_{\nu\sigma}$ being the difference between two connections, it is a tensor, and it makes sense to take its covariant derivative. This yields the *Palatini identity*

$$\delta R_{\mu\nu} = \nabla_\rho\delta\Gamma^\rho{}_{\nu\mu} - \nabla_\nu\delta\Gamma^\rho{}_{\rho\mu}\,. \tag{8.5.30}$$

Since covariant derivative commutes with contraction we get

$$\begin{aligned}
\delta R &= (\delta g^{\mu\nu})R_{\mu\nu} + \nabla_\rho\left(g^{\mu\nu}\delta\Gamma^\rho{}_{\nu\mu}\right) - \nabla_\nu\left(g^{\mu\nu}\delta\Gamma^\rho{}_{\rho\mu}\right) && (8.5.31)\\
&= (\delta g^{\mu\nu})R_{\mu\nu} + \nabla_\rho\left(g^{\mu\nu}\delta\Gamma^\rho{}_{\nu\mu} - g^{\mu\rho}\delta\Gamma^\sigma{}_{\sigma\mu}\right) && (8.5.32)
\end{aligned}$$

This is enough to get (8.5.21), since the term in ∇_ρ above yields a pure divergence in δS_g and therefore does not contribute. But to get (8.5.25) we must further massage this divergence term.

$$g^{\mu\nu}\delta\Gamma^\rho{}_{\nu\mu} - g^{\mu\rho}\delta\Gamma^\sigma{}_{\sigma\mu} = \delta\left(g^{\mu\nu}\Gamma^\rho{}_{\mu\nu}\right) - (\delta g^{\mu\nu})\Gamma^\rho{}_{\mu\nu} - g^{\mu\rho}\delta\partial_\mu\log\sqrt{|g|} \tag{8.5.33}$$

where we have used

$$\Gamma^\sigma{}_{\mu\sigma} = \frac{1}{2}g^{\rho\sigma}\partial_\mu g_{\rho\sigma} = \frac{1}{2g}\partial_\mu g = \partial_\mu\log\sqrt{|g|} \tag{8.5.34}$$

We now use

$$g^{\mu\nu}\Gamma^\rho{}_{\nu\mu} = -\frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}g^{\mu\rho}\right) = -\partial_\mu g^{\mu\rho} - g^{\mu\rho}\partial_\mu\log\sqrt{|g|} \tag{8.5.35}$$

and we get

$$g^{\mu\nu}\delta\Gamma^{\rho}{}_{\nu\mu} - g^{\mu\rho}\delta\Gamma^{\sigma}{}_{\sigma\mu} = -\partial_\mu\delta g^{\mu\rho} - \Gamma^{\sigma}{}_{\mu\sigma}\delta g^{\mu\rho} \tag{8.5.36}$$

$$- \Gamma^{\rho}{}_{\mu\nu}\delta g^{\mu\nu} - 2g^{\mu\rho}\partial_\mu\delta\log\sqrt{|g|} \tag{8.5.37}$$

$$= -\nabla_\mu\delta g^{\mu\rho} - g^{\mu\rho}\partial_\mu\delta\log|g| \tag{8.5.38}$$

$$= -\nabla_\mu\delta g^{\mu\rho} - g^{\mu\rho}\partial_\mu\frac{\delta g}{g} \tag{8.5.39}$$

$$= -\nabla_\mu\delta g^{\mu\rho} + g^{\mu\rho}\partial_\mu(g_{\nu\sigma}\delta g^{\nu\sigma}) \tag{8.5.40}$$

$$= -\nabla_\mu\delta g^{\mu\rho} + g^{\mu\rho}\nabla_\mu(g_{\nu\sigma}\delta g^{\nu\sigma}) \tag{8.5.41}$$

$$= -\nabla_\mu\delta g^{\mu\rho} + g^{\mu\rho}g_{\nu\sigma}\nabla_\mu\delta g^{\nu\sigma} \tag{8.5.42}$$

and this yields (8.5.25).

Peculiarities in two dimensions

In two-dimensions, due to the symmetries (8.5.15) of the Riemann curvature tensor, there is only one independent component :

$$R_{\mu\nu\rho\sigma} = \frac{R}{2}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right) \tag{8.5.43}$$

and the Einstein tensor vanishes[2]

$$R_{\mu\nu} = \frac{R}{2}g_{\mu\nu} \tag{8.5.45}$$

In fact one can say much more : any two-dimensional Riemannian manifold is (locally) conformally flat, in the sense that there exist local coordinates in which the metric is conformal to the Euclidean metric :

$$g_{\mu\nu}(x) = e^{\sigma(x)}\delta_{\mu\nu} \tag{8.5.46}$$

Such coordinates are called *isothermal coordinates*. To say things differently, in two dimensions each point has a neighborhood that can be mapped to flat space by a conformal map.

Furthermore if the surface under consideration is oriented, then it is naturally a Riemann surface, *i.e.* a one-dimensional complex manifold. This is a simple consequence of the existence of isothermal coordinates. The main observation is that transition functions between isothermal coordinates are conformal maps (and therefore holomorphic or anti-holomorphic). If the surface under consideration is oriented, then we can choose our atlas of isothermal coordinates to be oriented, in which case the transition functions - being orientation preserving - are holomorphic. What we have on our hands is an atlas whose

[2] As a cultural remark, this makes two-dimensional gravity quite peculiar. In fact the Einstein-Hilbert action becomes a topological invariant since on a compact surface of genus g the Gauss-Bonnet theorem states that

$$\int R\sqrt{|g|}d^2x = 8\pi(1-g)\,. \tag{8.5.44}$$

Deforming the surface (i.e. changing the metric) leaves the Einstein-Hilbert action unchanged, which is another way to see that the Einstein tensor must vanish (from (8.5.21)).

transition functions are holomorphic, *i.e.* a Riemann surface. To say things differently (oriented) isothermal coordinates are complex coordinates.

Notice also that two oriented Riemannian manifolds that are conformally equivalent yield the same complex structure, so in this sense Weyl transformations are compatible with complex structure. In layman terms : if a function is holomorphic for the metric g, it is also holomorphic for $e^\sigma g$.

Finally a map between two oriented Riemannian manifolds is conformal if and only if it is holomorphic or anti-holomorphic (in the sense that $(\partial_1 \mp i\partial_2)f = 0$ in isothermal coordinates).

It should be clear from the discussion above that using (local) complex coordinates is going to prove rather beneficial. We list below some related basic formulas.

Let (x^1, x^2) be isothermal coordinates, and define $z = x^1 + ix^2$, $\bar{z} = x^1 - ix^2$. A local frame of the (complexified) cotangent space is given by

$$dz = dx^1 + idx^2, \qquad d\bar{z} = dx^1 - idx^2 \tag{8.5.47}$$

and the dual frame in the tangent space reads

$$\partial = \frac{1}{2}\left(\partial_1 - i\partial_2\right), \qquad \bar{\partial} = \frac{1}{2}\left(\partial_1 + i\partial_2\right) \tag{8.5.48}$$

In particular the metric

$$g = e^{\sigma(x)}\left(dx^1 \otimes dx^1 + dx^2 \otimes dx^2\right) \tag{8.5.49}$$

reads in complex coordinates

$$g = \frac{1}{2}e^{\sigma(x)}\left(dz \otimes d\bar{z} + d\bar{z} \otimes dz\right) \tag{8.5.50}$$

which is to say in components

$$g_{zz} = g_{\bar{z}\bar{z}} = 0, \qquad g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}e^{\sigma} \tag{8.5.51}$$

The inverse metric is given by

$$g^{zz} = g^{\bar{z}\bar{z}} = 0, \qquad g^{z\bar{z}} = g^{\bar{z}z} = 2e^{-\sigma} \tag{8.5.52}$$

A real vector field $\xi = \xi^1\partial_1 + \xi^2\partial_2$ can be decomposed in the basis $(\partial, \bar{\partial})$ as

$$\xi = \xi^z\partial + \xi^{\bar{z}}\bar{\partial}, \qquad \xi^z = \overline{\xi^{\bar{z}}} = \xi^1 + i\xi^2 \tag{8.5.53}$$

Complex indices are raised and lowered according to

$$\xi_z = g_{z\bar{z}}\xi^{\bar{z}} = \frac{1}{2}e^{\sigma}\xi^{\bar{z}}, \qquad \xi^z = 2e^{-\sigma}\xi_{\bar{z}} \tag{8.5.54}$$

The stress-energy tensor $T_{\mu\nu}$ has components

$$T_{zz} = \overline{T_{\bar{z}\bar{z}}} = \frac{1}{4}\left(T_{11} - T_{22} - iT_{12} - iT_{21}\right) \tag{8.5.55}$$

$$T_{z\bar{z}} = \overline{T_{\bar{z}z}} = \frac{1}{4}\left(T_{11} + T_{22} + i(T_{12} - T_{21})\right) \tag{8.5.56}$$

The Riemannian volume form $\omega = \sqrt{|g|}dx^1 \wedge dx^2$ is

$$\omega = e^{\sigma} \frac{d\bar{z} \wedge dz}{2i} \tag{8.5.57}$$

The covariant derivative becomes particularly simple: the only non-zero Christoffel symbols are

$$\Gamma^{z}{}_{zz} = \partial\sigma, \qquad \Gamma^{\bar{z}}{}_{\bar{z}\bar{z}} = \bar{\partial}\sigma \tag{8.5.58}$$

as well as the scalar curvature

$$R = -4e^{-\sigma}\partial\bar{\partial}\sigma = -\Delta\sigma \tag{8.5.59}$$

where Δ is the Laplacian.