Exercises For Part 2

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1 Effective action

Recall that for matrices:

$$\begin{aligned} 0 =& \frac{\partial}{\partial x}(M^{-1}M) = \frac{\partial M^{-1}}{\partial x}M + M^{-1}\frac{\partial M}{\partial x} \\ \longrightarrow \frac{\partial M^{-1}}{\partial x} =& M^{-1}\frac{\partial M}{\partial x}M^{-1} \end{aligned} \tag{1.1}$$

And also

$$\begin{aligned} \phi_{cl}(x) =& - \frac{\delta E}{\delta J(x)}, \\ \frac{\delta \Gamma}{\delta \phi_{cl}(x)} =& - J(x), \\ \frac{\delta \phi_{cl}}{\delta J(y)} =& - \frac{\delta^2 E}{\delta J(x) \delta J(y)} = iD(x,y) \end{aligned} \tag{1.2}$$

1. Derive the general expression for the three-point correlator $\langle \phi(x_1)\phi(x_2)\phi(x_3)\rangle$ in terms of propagators $D(x-y)$ and derivatives of the effective action $\Gamma[\phi_{cl}]$, similar to the expressions in (2.31) and (2.32) in the notes.

2. Derive the expression for the four-point correlator $\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle$

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2 Coleman-Weinberg effective potential

2.1 One scalar

Consider the theory

$$\mathcal{L}_{\text{light}} \supset \frac{1}{2}(\partial\phi_L)^2 - \frac{1}{2}m^2\phi_L^2 - \frac{\lambda}{24}\phi_L^4.$$

1. Replace $\phi_L \to \phi_{cl,L}$ in the lagrangian and take the classical field constant.
2. Take the second derivatives to obtain $m^2_{eff}(\phi_{cl,L})$.
3. Write down the one-loop effective potential.
4. Expand the effective potential to *second order* in ϕ_L.

2.2 Two scalars

Consider next the theory

$$\mathcal{L} \supset \frac{1}{2}(\partial_\mu\phi_L)^2 + \frac{1}{2}(\partial_\mu\phi_H)^2 - \frac{1}{2}M^2\phi_H^2 - \frac{1}{2}m^2\phi_L^2 - \frac{a}{2}\phi_H\phi_L^2 - \frac{\lambda}{24}\phi_L^4$$

1. Replace $\phi_H, \phi_L \to \phi_{cl,H}, \phi_{cl,L}$ in the lagrangian and take the classical fields constant.
2. Take the second derivatives to obtain the *matrix* $m^2_{eff}(\phi_{cl,L}, \phi_{cl,H})$.
3. Set $\lambda = 0$ and $\phi_H = 0$ and compute the eigenvalues (if we were matching the theories we would have to put $\phi_H = -\frac{a}{2M^2}\phi_L^2$; we are *not* doing that here).
4. Write down the one-loop effective potential.
5. Expand the eigenvalues to second order in ϕ_L.
6. Expand the effective potential to second order in ϕ_L.
7. You have computed a finite correction to the mass of the field ϕ_L; it is equivalent to $\Pi_{\phi_L\phi_L}(0)$.

3 EFT bases

Consider the lagrangian consisting of a single complex scalar field:

$$\mathcal{L} \supset |\partial\Phi|^2 - m^2|\Phi|^2 - \lambda|\Phi|^4. \tag{3.1}$$

1. Write down all possible effective operators at dimension 6.
2. Determine the set that are independent after integration-by-parts identities are applied: choose a minimam "Green Basis."
3. Apply equation-of-motion identities to find a minimal basis for the effective field theory at dimension 6.

4 Solutions

4.1 Effective action

Then

$$\frac{\delta^3 E}{\delta J(x_1)\delta J(x_2)\delta J(x_3)} = \frac{\delta}{\delta J(x_3)}\left(\frac{\delta^2\Gamma}{\delta\phi_{cl}(x_1)\delta\phi_{cl}(x_2)}\right)^{-1}$$
$$= \int d^d y_3 \frac{\delta\phi_{cl}(y_3)}{\delta J(x_3)}\frac{\delta}{\delta\phi_{cl}(y_3)}\left(\frac{\delta^2\Gamma}{\delta\phi_{cl}(x_1)\delta\phi_{cl}(x_2)}\right)^{-1}$$
$$= -\int d^d y_1 \int d^d y_2 \int d^d y_3 \frac{\delta E}{\delta J(y_3)\delta J(x_3)}\left(\frac{\delta^2\Gamma}{\delta\phi_{cl}(x_1)\delta\phi_{cl}(y_1)}\right)^{-1}\frac{\delta^3\Gamma}{\delta\phi_{cl}(y_1)\delta\phi_{cl}(y_2)\delta\phi_{cl}(y_3)}\left(\frac{\delta^2\Gamma}{\delta\phi_{cl}(x_2)\delta\phi_{cl}(y_2)}\right)^{-1}$$
$$= -\int d^d y_1 \int d^d y_2 \int d^d y_3 \frac{\delta E}{\delta J(y_3)\delta J(x_3)}\frac{\delta^2 E}{\delta J(x_1)\delta J(y_1)}\frac{\delta^3\Gamma}{\delta\phi_{cl}(y_1)\delta\phi_{cl}(y_2)\delta\phi_{cl}(y_3)}\frac{\delta^2 E}{\delta J(x_2)\delta J(y_2)}$$
$$= -\int d^d y_1 \int d^d y_2 \int d^d y_3 (-iD(x_1,y_1))(-iD(x_2,y_2))(-iD(x_3,y_3))\frac{\delta^3\Gamma}{\delta\phi_{cl}(y_1)\delta\phi_{cl}(y_2)\delta\phi_{cl}(y_3)}. \tag{4.1}$$

and then

$$\frac{\delta^4 E}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)} = \int d^d y_1 \int d^d y_2 \int d^d y_3 \int d^d y_4 (-iD(x_1,y_1))(-iD(x_2,y_2))(-iD(x_3,y_3))(-iD(x_4,y_4))$$
$$\times \frac{\delta^4\Gamma}{\delta\phi_{cl}(y_1)\delta\phi_{cl}(y_2)\delta\phi_{cl}(y_3)\delta\phi_{cl}(y_4)}$$
$$+ \Bigg[\int d^d y_1 \int d^d y_2 \int d^d y_3 \int d^d z_1 \int d^d z_2 \int d^d z_3 (-iD(x_1,y_1))(-iD(x_2,y_2))(-iD(y_3,z_1))(-iD(x_3,z_2))(-iD(x_4,z_3))$$
$$\times \frac{\delta^3\Gamma}{\delta\phi_{cl}(z_1)\delta\phi_{cl}(z_2)\delta\phi_{cl}(z_3)}\frac{\delta^3\Gamma}{\delta\phi_{cl}(y_1)\delta\phi_{cl}(y_2)\delta\phi_{cl}(y_3)}$$
$$+ \Big((x_3,y_3)\leftrightarrow(x_2,y_2)\Big)$$
$$+ \Big((x_3,y_3)\leftrightarrow(x_1,y_1)\Big)\Bigg] \tag{4.2}$$

4.2 Coleman-Weinberg

For the first part we have

$$m^2_{eff}(\phi_{cl,L}) = m^2 + \frac{\lambda}{2}\phi^2_{cl,L} \tag{4.3}$$

and

$$
\begin{aligned}
V^{(1)}_{eff} =& \frac{1}{64\pi^2}\left(m^2+\frac{\lambda}{2}\phi^2_{cl,L}\right)^2\left[\log\frac{m^2+\frac{\lambda}{2}\phi^2_{cl,L}}{\mu^2}-\frac{3}{2}\right] \\
=& \frac{1}{64\pi^2}\left(m^2+\frac{\lambda}{2}\phi^2_{cl,L}\right)^2\left[\log\frac{m^2}{\mu^2}+\log\left(1+\frac{\lambda\phi^2_{cl,L}}{2m^2}\right)-\frac{3}{2}\right] \\
=& \frac{1}{64\pi^2}\left(m^2+\frac{\lambda}{2}\phi^2_{cl,L}\right)^2\left[\log\frac{m^2}{\mu^2}+\frac{\lambda\phi^2_{cl,L}}{2m^2}-\frac{3}{2}+...\right] \\
=& \frac{1}{64\pi^2}\left[m^4\log\frac{m^2}{\mu^2}+\lambda m^2\phi^2_{cl,L}\left(\log\frac{m^2}{\mu^2}-1\right)+\mathcal{O}(\phi^4_{cl,L})\right]
\end{aligned} \tag{4.4}
$$

For the second part, we have

$$
\mathcal{M}^2 = \begin{pmatrix} m^2+a\phi_H+\frac{\lambda}{2}\phi^2_L & a\phi_L \\ a\phi_L & M^2 \end{pmatrix} \tag{4.5}
$$

This has characteristic equation (when we take $\phi_H = 0$ – not for matching therefore)

$$
0 = (m^2-x)(M^2-x)-a^2\phi^2_L = x^2-(M^2+m^2)x+(m^2M^4-a^2\phi^2_L)
$$

and so the eigenvalues are

$$
\begin{aligned}
x =& \frac{M^2+m^2}{2}\pm\frac{1}{2}\sqrt{(M^2-m^2)^2+4a^2\phi^2_L} \\
=& \frac{1}{2}\left[M^2+m^2\pm(M^2-m^2+2\frac{a^2\phi^2_L}{(M^2-m^2)}+...)\right]
\end{aligned} \tag{4.6}
$$

and therefore the effective potential is

$$
V^{(1)}_{eff} = \frac{1}{16\pi^2}\left[f(M^2+\frac{a^2\phi^2_L}{M^2-m^2})+f(m^2-\frac{a^2\phi^2_L}{M^2-m^2})\right], \tag{4.7}
$$

where I define

$$
f(x) \equiv \frac{1}{4}x^2(\log x/\mu^2-3/2), \tag{4.8}
$$

and we see

$$
f'(x) = \frac{1}{2}x(\log\frac{x}{\mu^2}-1) \equiv -\frac{1}{2}A_0(x). \tag{4.9}
$$

Therefore we find

$$
V^{(1)}_{eff} = \frac{1}{16\pi^2}\left[f(M^2)+f(m^2)-\frac{1}{2}\frac{a^2\phi^2_L}{M^2-m^2}(A_0(M^2)-A_0(m^2))+...\right] \tag{4.10}
$$

We therefore have

$$
\Pi_{\phi_L\phi_L}(0) = \frac{\partial^2 V^{(1)}_{eff}}{(\partial\phi_L)^2} = -\frac{1}{16\pi^2}\frac{a^2}{M^2-m^2}(A_0(M^2)-A_0(m^2)) \tag{4.11}
$$

5 EFT bases

$$\mathcal{L} \supset |\partial\Phi|^2 - m^2|\Phi|^2 - \lambda|\Phi|^4. \tag{5.1}$$

All operators at dimension 6:

$$\begin{aligned}
&|\Phi|^6, |\partial_\mu\Phi|^2|\Phi|^2,\\
&\overline{\Phi}^2\partial_\mu\Phi\partial^\mu\Phi, \Phi^2\partial_\mu\overline{\Phi}\partial^\mu\overline{\Phi}\\
&|\Phi|^2\overline{\Phi}\Box\Phi, |\Phi|^2\Phi\Box\overline{\Phi},\\
&|\Phi|^2\Box|\Phi|^2, \overline{\Phi}^2\Box\Phi^2, \overline{\Phi}^2\Box\Phi^2\\
&\overline{\Phi}\Box^2\Phi, \Phi\Box^2\overline{\Phi} + 4 - \text{derivative perms}
\end{aligned} \tag{5.2}$$

Obviously all the four-derivative terms are equivalent.

Applying integration by parts:

$$\begin{aligned}
\partial_\mu(\overline{\Phi}^2\Phi\partial_\mu\Phi) =& 2|\partial_\mu\Phi|^2|\Phi|^2 + \overline{\Phi}^2\partial_\mu\Phi\partial^\mu\Phi + \overline{\Phi}^2\Phi\Box\Phi\\
=& 2|\partial_\mu\Phi|^2|\Phi|^2 + \overline{\Phi}^2\partial_\mu\Phi\partial^\mu\Phi + |\Phi|^2\overline{\Phi}\Box\Phi
\end{aligned} \tag{5.3}$$

and we have

$$\begin{aligned}
|\Phi|^2\Box|\Phi|^2 =& |\Phi|^2\partial_\mu(\overline{\Phi}\partial^\mu\Phi + \Phi\partial^\mu\overline{\Phi})\\
=& |\Phi|^2|\partial_\mu\Phi|^2 + |\Phi|^2\overline{\Phi}\Box\Phi + |\Phi|^2\Phi\Box\overline{\Phi}
\end{aligned} \tag{5.4}$$

etc so those operators are redundant. Therefor a Greens basis could be

$$|\Phi|^6, |\partial_\mu\Phi|^2|\Phi|^2, |\Phi|^2\overline{\Phi}\Box\Phi, |\Phi|^2\Phi\Box\overline{\Phi}, \overline{\Phi}\Box^2\Phi \tag{5.5}$$

The equations of motion give $\Box\Phi \sim \Phi|\Phi|^2$ so our EFT basis becomes

$$|\Phi|^6, |\partial_\mu\Phi|^2|\Phi|^2. \tag{5.6}$$