Solutions to exercises in lectures 1 and 2

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1 Cyclotrons

1. *What limits the maximum energy of a cyclotron?*
   Answer: The size! For a classic cyclotron, when the speed becomes relativistic the time per revolution is no longer independent of the speed. After synchronising the oscillating field (to obtan a synchrocyclotron) the size of the device is the main problem.

2 Synchrotrons

For units,

$$\begin{aligned}
\hbar c =& 0.1973269602(77)\mathrm{GeV\,fm} \\
eV =& 1.6021766209(98) \times 10^{-19} J \\
\mathrm{kg} =& J\,\mathrm{s}^2/\mathrm{m}^2 = \mathrm{eV}/(1.6021766209(98) \times 10^{-19}) \times c^2 \\
=& 5.60959 \times 10^{35}\mathrm{eV} \\
e\mathrm{T} =& 1.602 \times 10^{-19}\mathrm{kg.s}^{-1} \\
=& 1.602 \times 10^{-19} \times 5.60959 \times 10^{35} \times 6.582119569 \times 10^{-16}\ (\mathrm{eV})^2 \\
=& 5.91571 \times 10^{-17}(\mathrm{GeV})^2.
\end{aligned} \tag{2.1}$$

For a synchrotron,

$$E \approx eBr$$

The LHC/LEP tunnel has a circumference of 26.7km. For the LHC the "bending radius" of the magnets is only 2.812km. Their magnetic field is 8.3T. The classic Larmor equation for power emitted in instantaneous rest frame of particle is

$$P^* = \frac{2}{3}\frac{1}{4\pi\epsilon_0}\frac{e^2(a^*)^2}{c^3}$$

Recall that the electric charge in natural units can be derived from $\alpha \simeq \frac{1}{137} \equiv \frac{e^2}{4\pi\epsilon_0\hbar c} \to \frac{e^2}{4\pi}$.

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1. *Calculate the maximum energy of the LHC beams*
   Answer: $E = 8.3 \times 5.91 \times 10^{-17} \times \frac{2812}{0.197\times10^{-15}}\text{GeV} \simeq 7000$ GeV

2. *LEP2 magnets had a bending radius of 3.1km. At 100 GeV per beam, what magnetic field did that require?*
   Answer: $eB = \frac{E}{r} \to B = \frac{100\times\hbar c}{3100} \times \frac{1}{5.91\times10^{-17}} \simeq 0.1\,T$

3. *Show that the power in the lab frame can be written as* $P = \frac{2}{3}\frac{1}{4\pi\epsilon_0}\frac{e^2c}{r^2}\beta^4\gamma^4$.

4. *What fraction of the electron energy was lost per revolution at LEP?*

5. *What fraction of the proton energy is lost per revolution at the LHC at 6.5 TeV/beam?*

For question 3, first we note that $P = P^*$ because E/t is Lorentz invariant. Then we note that the acceleration is perpendicular to the velocity, so

$$\begin{aligned} d\tau =& \gamma(dt - \beta dx) = \gamma(dt - \beta^2 dt) = \frac{dt}{\gamma} \qquad (2.2) \\ dz^* =& dz \\ \frac{d^2z^*}{d\tau^2} =& \gamma^2 a = \gamma^2 \frac{v^2}{r} \qquad (2.3) \end{aligned}$$

and hence

$$\begin{aligned} P =& \frac{2}{3}\frac{1}{4\pi\epsilon_0}\frac{e^2v^4\gamma^2}{r^2c^3} \\ =& \frac{2}{3}\frac{1}{4\pi\epsilon_0}\frac{e^2c\beta^4\gamma^2}{r^2}. \qquad (2.4) \end{aligned}$$

For questions 4 and 5,

$$\begin{aligned} Pt =& \frac{2}{3}\frac{1}{4\pi\epsilon_0}\frac{e^2c\gamma^4}{r^2}\frac{2\pi r}{c} \\ \frac{Pt}{E} =& \frac{4\pi e^2}{3}\frac{1}{4\pi\epsilon_0}\frac{1}{Er}\left(\frac{E}{m}\right)^4 \\ =& \frac{(4\pi)\alpha}{3}\frac{\hbar c}{Er}\left(\frac{E}{m}\right)^4 \qquad (2.5) \end{aligned}$$

Hence for question 4, at LEP at 100GeV with 511keV electron mass:

$$\frac{Pt}{E} \approx 0.03 \qquad (2.6)$$

so the beam lost 3 GeV per revolution – the same amount that was imparted.

For the LHC at 6.5TeV, with 0.938 GeV proton mass we have

$$\frac{Pt}{E} \approx 10^{-9}. \qquad (2.7)$$

3 Variables

1. *Show that the rapidity transforms under boosts as* $y \to y + \frac{1}{2}\log\frac{1-\beta}{1+\beta}$
2. *Show that the pseudorapidity equals the rapidity for massless particles*
3. *Heavy particles tend to have a low rapidity at the LHC. Why?*
   Answer: if $p_z \ll E$ then the ratio is roughly 1 in the definition and y is small.

For 1, if we make a boost along the beamline, we send

$$E \to \gamma(E - \beta p_z), \qquad p_z \to \gamma(p_z - \beta E) \tag{3.1}$$

and thus

$$\begin{aligned} y \to& \frac{1}{2}\log\frac{E - \beta p_z + p_z - \beta E}{E - \beta p_z - p_z + \beta E} \\ =& \frac{1}{2}\log\frac{(1-\beta)E + (1-\beta)p_z}{(1+\beta)(E - p_z)} \\ =& y + \frac{1}{2}\log\frac{1-\beta}{1+\beta}. \end{aligned} \tag{3.2}$$

For 2, for massless particles

$$\begin{aligned} y \to& \frac{1}{2}\log\frac{1+\cos\theta}{1-\cos\theta} \\ =& \frac{1}{2}\log\frac{\cos^2\theta/2}{\sin^2\theta/2} \\ =& -\log\tan\theta/2. \end{aligned} \tag{3.3}$$

4 Breit-Wigner in action: the Delta resonance

For the exercise for this part of the lectures, you are shown the total cross-section measured from scattering positively charged pions from protons in a fixed target, shown here in figure 1. In the class you are not shown that the peak cross-section is at $\frac{8\pi}{p_{CM}^2}$. You are given instead that the peak is at the pion *kinetic energy* of 190 MeV (where $E_k \equiv E_\pi - m_\pi$), and the peak cross-section is at 190 mb. The charged pion mass is 139 MeV and the mass of the proton is 938 MeV.

1. Show that the mass of the resonance is 1232 MeV
2. Calculate the spin of the resonance formed!

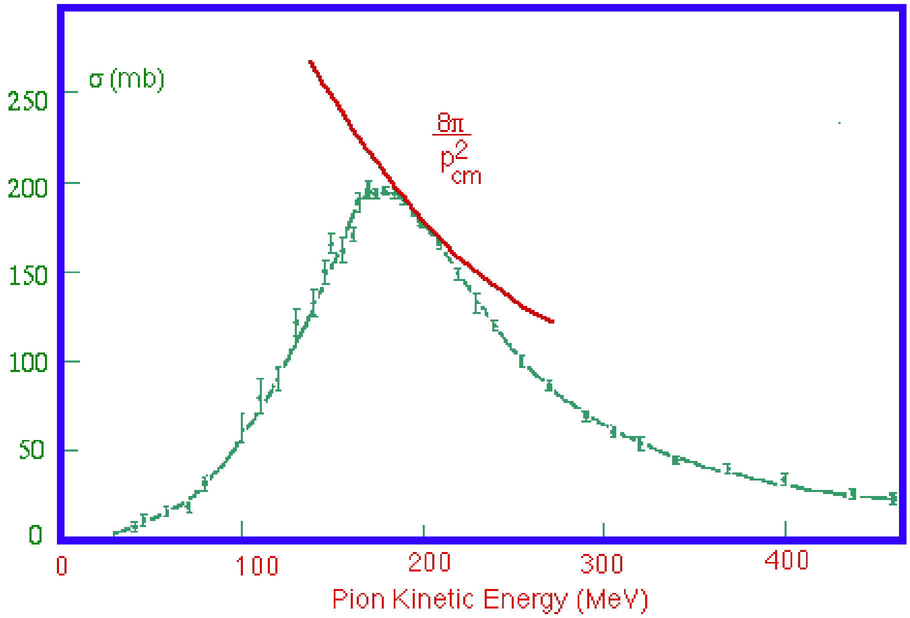

Figure 1

As can be seen from the plot, the peak is at $E_k^\pi = 190$ MeV. The proton rest four-momentum is $p_p = (m_p, 0, 0, 0)$ and the pion one is $(E_\pi, k_\pi, 0, 0) = (m_\pi + E_{kin,\pi}, k_\pi, 0, 0)$.

$$\begin{aligned}
E_\pi \equiv & m_\pi + E_k \\
E_{CM} \equiv & \sqrt{(E_\pi + m_p)^2 - p_\pi^2} = \sqrt{m_\pi^2 + m_p^2 + 2E_\pi E_p} \\
= & \sqrt{m_\pi^2 + m_p^2 + 2(m_\pi + E_k)E_p}
\end{aligned} \tag{4.1}$$

Then we need, in the CMS frame

$$\begin{aligned}
E_p =& m_\Delta - E_\pi \\
m_p^2 =& m_\Delta^2 - 2m_\Delta E_\pi + m_\pi^2 \\
E_\pi =& \frac{m_\Delta^2 + m_\pi^2 - m_p^2}{2m_\Delta} \\
p_{CM} =& \sqrt{\left(\frac{m_\Delta^2 + m_\pi^2 - m_p^2}{2m_\Delta}\right)^2 - m_\pi^2} \\
=& \sqrt{\frac{(m_\Delta^2 - m_\pi^2 - m_p^2)^2 - 4m_\pi^2 m_p^2}{2m_\Delta}}.
\end{aligned} \tag{4.2}$$

For 190MeV pion kinetic energy, $m_p = 938$ MeV, $m_\pi = 139$ MeV we get $E_{CM} = 1232$ MeV, so the mass of the resonance is 1233 MeV. Then

$$\begin{aligned}
p_{CM}^2 =& 0.051\ (\mathrm{GeV})^2 \\
\frac{8\pi(0.197)^2}{p_{CM}^2} \times 10^{-30}(m^2) =& \frac{8\pi(0.197)^2}{p_{CM}^2} \times 10\mathrm{mb} \\
=& 190\ \mathrm{mb}.
\end{aligned} \tag{4.3}$$

Since the pion has no spin, and the proton has spin 1/2, we need a factor of 4 in the numerator of the Breit-Wigner cross-section – i.e. a spin 3/2 particle! This is the $\Delta(1232)$ particle, and in this case it is *doubly charged*.