Mock Examination for
'Phenomenology of the Standard Model and Beyond'

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Instructions

- The expected time is 2.5 hours or less.
- You may have access to your notes.
- Write your name on every sheet of paper you hand in, number your sheets, and write your email address on the first sheet.
- Complete **three** questions from Part A. Each is worth 25%.
- If you complete more than three, the *best results only* will be taken into account.
- Complete Part B. It is worth 25%. You may prepare it in advance *and bring it with you* if you wish but do so on your own *and it must be hand-written.*

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Part A

Some units:

$$
\begin{aligned}
\hbar c =& 0.1973269602(77)\,\mathrm{GeV\,fm} \\
eV =& 1.6021766209(98)\times 10^{-19} J \\
\mathrm{kg} =& J\,\mathrm{s}^2/\mathrm{m}^2 = \mathrm{eV}/(1.6021766209(98)\times 10^{-19})\times c^2 \\
=& 5.60959\times 10^{35}\mathrm{eV} \\
e\mathrm{T} =& 1.602\times 10^{-19}\mathrm{kg.s}^{-1} \\
=& 1.602\times 10^{-19}\times 5.60959\times 10^{35}\times 6.582119569\times 10^{-16}\ (\mathrm{eV})^2 \\
=& 5.91571\times 10^{-17}(\mathrm{GeV})^2 \\
G_F =& 1.1663787\times 10^{-5}(\mathrm{GeV})^{-2}
\end{aligned}
\tag{A.1}
$$

The masses of muons, taus and (charged) pions are:

$$
m_\mu = 0.1057\ \mathrm{GeV}, m_\tau = 1.777\ \mathrm{GeV}, m_{\pi^\pm} = 0.1396\ \mathrm{GeV}, m_{K^\pm} = 0.493677\ \mathrm{GeV}.
$$

You may need the Legendre polynomials

$$
P_0(\cos\theta) = 1, \qquad P_1(\cos\theta) = \cos\theta, \qquad P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1) \tag{A.2}
$$

For massless initial states i, j and final states a, b, you may need

$$
\begin{aligned}
\Gamma(X\to a,b) =& \frac{1}{32\pi m_i}\int_{-1}^{1} d(\cos\theta)\sum_{\mathrm{spins}}\frac{1}{2s_i+1}\frac{1}{2^{\delta_{ab}}}|\mathcal{M}|^2 \\
\sigma(i,j\to a,b) =& \frac{1}{32\pi s}\int_{-1}^{1} d(\cos\theta)\sum_{\mathrm{spins}}\frac{1}{2^{\delta_{ab}}}|\mathcal{M}|^2
\end{aligned}
\tag{A.3}
$$

where $\delta_{ab} = 1$ if the states a, b are identical and zero if they are not.

A.1 Colliders and Resonances

1. (a) (i) What limits the maximum energy of a cyclotron?
   - (ii) What about a synchrotron?
   - (iii) Consider a 10 TeV *muon* collider sitting inside the LHC tunnel, which has a bending radius of 2.812km. What magnetic field would be required?
   - (iv) What fraction of the muon energy would be lost per revolution?
   - (v) What might be the advantages of a muon collider over a proton or electron one? What disadvantages are there?

(b) (i) Suppose that we built a muon collider as a Higgs factory, but we could only reach a centre of mass energy of 125 GeV. Given that the total Higgs width is about 4 MeV, and $\mathrm{BR}(h \to \mu\mu) = 2.2 \times 10^{-4}$, calculate the total cross-section for single Higgs production.

(ii) Explain why this would be a poor choice of centre of mass energy, and what would be the advantage of operating at centre of mass energies above 220 GeV.

A.2 Unitarity

Consider a theory with a complex scalar S coupled to the Standard Model Higgs doublet $H = \begin{pmatrix} G^+ \\ H^0 \end{pmatrix}$ where G^+, H^0 are complex scalars. The relevant part of the lagrangian is

$$\mathcal{L} \supset -\mu^2|H|^2 - \lambda|H|^4 - \lambda_S|S|^4 - \lambda_{HS}|S|^2|H|^2.$$

We will ignore the SM gauge bosons in this question, but assume that S transforms under a different $U(1)$ symmetry to $U(1)_Y$.

We are going to use unitarity to place bounds on these couplings in the *large scattering momentum limit* which means you can treat all particles as massless.

2. (a) Write down the possible non-zero amplitudes of scattering pairs of **scalars** and compute the zeroth partial waves $a_0 = \frac{1}{32\pi}\int_{-1}^{1} d(\cos\theta)\mathcal{M} = \frac{1}{16\pi}\mathcal{M}$ for each. NB the vertex is just

$$\text{Vertex} = i\frac{\partial^n \mathcal{L}}{\partial\phi_1 \cdots \partial\phi_n}$$

for an interaction containing **incoming** fields $\phi_1, \dots \phi_n$ (whether they are real or complex), and changing a field from incoming to outgoing is the same as complex conjugation.

(b) Considering just the amplitudes neutral under all phases (i.e. no pairs like G^+G^+ or H^0H^0 but instead $H^0\overline{H}^0$ etc), compute the corresponding 3×3 matrix of zeroth partial waves.

(c) Writing

$$\begin{pmatrix} G^+\overline{G^+} \\ H^0\overline{H^0} \end{pmatrix} \equiv \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \rho \\ \phi \end{pmatrix}$$

to transform the *pairs* of scattering fields, decompose the neutral scattering matrix into a 2×2 matrix

$$\begin{pmatrix} S\overline{S} \to S\overline{S} & S\overline{S} \to \rho \\ \rho \to S\overline{S} & \rho \to \rho \end{pmatrix}$$

and $\phi \leftrightarrow \phi$.

(d) If $\lambda_{HS} = 0$ what is the strongest bound we obtain on λ? If $m_H^2 = 2\lambda v^2$ and $v = 246$ GeV, what is the bound on the Higgs mass in this case?

(e) For $\lambda_S = 0, \lambda_{HS} > 0$ find the eigenvalues of the scattering matrices. Do the bounds on λ in general get stronger or weaker than for $\lambda_{HS} = 0$?

A.3 Drell-Yan and the splitting function

The Drell-Yan process involves quark annihilation to produce a lepton pair via an s-channel photon:

$$q(p_1) + \overline{q}(p_2) \to e^-(p_3) + e^+(p_4).$$

Since the quarks and leptons are light compared to typical scattering momenta we can treat them as massless. The squared amplitude (averaged over colours and spins) for this is

$$\sum |\mathcal{M}_{qq\to ee}|^2 = \frac{2e^4Q_q^2}{N_c}\frac{(p_1\cdot p_3)^2 + (p_1\cdot p_4)^2}{(p_1\cdot p_2)}. \tag{A.4}$$

(This is not so hard to derive).

The first correction to this invoves emission of a gluon from the *initial* state:

$$q(p_1) + \overline{q}(p_2) \to e^-(p_3) + e^+(p_4) + g(k)$$

The squared amplitude (averaged over colours and spins) for this is:

$$\sum |\mathcal{M}_{qq\to eeg}|^2 = \frac{2g_s^2e^4Q_q^2C_F}{N_c}\frac{1}{(p_1\cdot k)(p_2\cdot k)}\frac{(p_1\cdot p_3)^2 + (p_1\cdot p_4)^2 + (p_2\cdot p_3)^2 + (p_2\cdot p_4)^2}{(p_3\cdot p_4)}. \tag{A.5}$$

(This is rather tedious to derive).

This problem will show you how the splitting function comes out explicitly once we take the gluon nearly colinear to the momentum p_1.

In the lecture, we make the decomposition $k^\mu \propto p_1^\mu + k_\perp^\mu + \delta$. Since we have an explicit process, we can define instead

$$k^\mu \equiv (1-z)p_1^\mu + c_2p_2^\mu + k_\perp^\mu,$$

where $k_\perp\cdot p_1 = k_\perp\cdot p_2 = 0$, and we can take

$$k_\perp^\mu = (0, \mathbf{k}_\perp, 0).$$

3. (a) Derive an expression for c_2 in terms of $k_\perp^2, x$ and $s_0 \equiv (p_1+p_2)^2$.

   (b) Show that in the limit that the gluon is colinear with p_1, we have

   $$\sum |\mathcal{M}_{qq\to eeg}|^2 \underset{\mathbf{k}_\perp\to 0}{\longrightarrow} \propto \sum |\mathcal{M}_{qq\to ee}|^2$$

   and find the proportionality factor.

   (c) Explain why the three-body phase-space can be written as

   $$\int d\Pi_3 = \int \frac{d^3k}{(2\pi)^3 2|\mathbf{k}|}\int d\Pi_2(p_1\to p_1 - k)$$

   (d) Hence show that the partonic cross-section including radiation can be written in the colinear limit

   $$\sigma_{q\overline{q}\to eeg} \longrightarrow \int_0^1 dz \int \frac{d|\mathbf{k}_\perp|^2}{|\mathbf{k}_\perp|^2}\hat{P}_{qq}(z)\sigma_{q\overline{q}\to ee}(zp_1, p_2)$$

   and find $P_{qq}(z)$.

   (e) This process has an infra-red divergence from when the gluon is soft and/or colinear. What cancels them? Draw the relevant diagrams.

A.4 Effective Action

Suppose that we couple a new complex scalar field to the Standard Model. Let us call it $\tilde{Q}^i$. Suppose that it is a fundamental of $SU(3)$ which effectively means that there are three fields and $i \in \{1, 2, 3\}$. Then the lagrangian is

$$\mathcal{L} \supset |D_\mu H|^2 - \mu^2|H|^2 - \lambda|H|^4 - m_{\tilde{Q}}^2 \sum_i |\tilde{Q}^i|^2 - \lambda_{HQ}|H|^2 \sum_i |\tilde{Q}^i|^2. \tag{A.6}$$

4. (a) Put $H = \begin{pmatrix} G^+ \\ \frac{1}{\sqrt{2}}(v + h + iG^0) \end{pmatrix}$ and substitute into the above lagrangian. G^+, G^0 are the Goldstone bosons for electroweak symmetry breaking and we will ignore them in the following: we will keep only terms including the light higgs h and the scalar $\tilde{Q}^i$, just set G^+, G^0 to zero.

   (b) Write the *effective* mass of $\tilde{Q}^i$ as a function of v and h_{cl}.

   (c) Write down the one-loop effective potential as a function of h_{cl} for the pieces *coming only from the new scalars* $\tilde{Q}^i$, using dimensional regularisation at some scale μ. Hint: Effectively charging the field under $SU(3)$ just multiplies the result by 3.

   (d) Recall from equations (3.2) and (3.3) in part 3 of the notes that we can write $0 = v(\mu^2 + \lambda v^2) +$ loops to be at the minimum of the potential $\left.\frac{\partial V(h_{cl})}{\partial h_{cl}}\right|_{h_{cl}=0} = 0$, and then $m_h^2 = \left.\frac{\partial^2 V}{\partial h_{cl}^2}\right|_{h_{cl}=0}$ gives $2\lambda v^2 +$ loops. Derive how we modify these two equations when we add $V_{\tilde{Q}}^{(1)}(h_{cl})$ that you just calculated.

   (e) Solve the equation $0 = v(\mu^2 + \lambda v^2) + \left.\frac{\partial V_{\tilde{Q}}^{(1)}(h_{cl})}{\partial h_{cl}}\right|_{h_{cl}=0}$ for μ^2 and hence give the one-loop correction to the Higgs mass squared coming from our new set of scalars *in the effective potential limit*. We shall call this quantity $(\Delta m_h^2)_{\tilde{Q}}$.

   (f) Why don't we have a piece proportional to $m_{\tilde{Q}}^2$, which we expect from the Hierarchy problem? (Hint: consider what we would be calculating if we put $v = 0$. Or consider what we could calculate if we were "integrating out" the heavy field $\tilde{Q}$ and were matching the masses of the Higgs in the high-energy theory to the effective theory below $m_{\tilde{Q}}$.).

   (g) You should find a logarithmic correction to the Higgs mass proportional to v^2 and λ^2. This is a prototype of the correction to the tree-level Higgs mass from a top squark in Supersymmetry. Imagine that there are two degenerate top squarks, so we double the contribution. Putting $\lambda_{HQ} = y_{\text{top}}^2 = \frac{2m_{\text{top}}^2}{v^2}, v = 246\,\text{GeV}, m_t = 173\,\text{GeV}, M_W^2 = 80$ GeV, and taking the renormalisation scale $\mu = m_{\text{top}}$, find the predicted Higgs mass for $m_{\tilde{Q}} =$ 10 TeV:

$$m_h^2 = M_W^2 + 2 \times (\Delta m_h^2)_{\tilde{Q}}$$

A.5 EFTs

You may use

$$\begin{aligned}\Gamma(A \to B + \nu) =& \frac{1}{2m_A} \int \frac{d\Omega}{16\pi^2} \frac{p_{CM}}{m_A} \sum_{\text{spins}} \frac{1}{2s_A + 1} |\mathcal{M}|^2 \\ =& \frac{1}{2s_A + 1} \frac{(m_A^2 - m_B^2)}{16\pi m_A^3} \sum_{\text{spins}} |\mathcal{M}|^2\end{aligned}$$

for decay of particle A to particle B plus a (massless) neutrino, where $s_A = 0$ if particle A has spin 0, and $s_A = 1/2$ for fermions.

In this problem we will study the decays of charged Kaons to muons plus neutrinos, starting from what we know about the decay rate of the pion to muons. In part 2 section 3 of the lectures we wrote down the operator for the Fermi effective theory and later in section 4 we wrote down the effective interactions of pions. You know by now that the weak interactions mix the down-type quarks, so we can write the charged current as

$$J_\mu^+ = \frac{1}{\sqrt{2}} \left(\sum_\ell \bar{\ell} \gamma_\mu P_L \nu_\ell + \sum_{i,j} V_{ji}^* \bar{d}^j \gamma_\mu P_L u^i \right) \tag{A.7}$$

and that the effective interaction is

$$\mathcal{L} \supset 4\sqrt{2} G_F J_\mu^+ (J^{+\,\mu})^\dagger.$$

At low energies when the quarks bind into mesons, we can write

$$\langle 0 | \bar{d} \gamma^\mu \gamma^5 \overline{u}(p) | \pi^-(q) \rangle = -i\sqrt{2} p^\mu F_\pi \delta^4(p - q). \tag{A.8}$$

whereas $\langle 0 | \bar{d} \gamma^\mu \overline{u}(p) | \pi^-(q) \rangle = 0$.

5. (a) What is the equivalent expression of (A.8) for K^-?

   (b) Using the above effective lagrangian, write down the matrix element $\mathcal{M}$ for the process $K^- \to \ell^- \nu$.

   (c) Suppose you know that

   $$\sum_{\text{spins}} |p^\mu_{\pi,K} \overline{\nu}(p_\nu) \gamma_\mu P_L \ell(p_\ell)|^2 = m_\ell^2 (m_{\pi,K}^2 - m_\ell^2)$$

   is the spin sum that you need to turn these amplitudes into widths.

   Write down expressions for the decays $\Gamma(\pi^- \to \mu^- + \overline{\nu}_\mu)$ and $\Gamma(K^- \to \mu^- + \overline{\nu}_\mu)$ in terms the masses of the muon, pion and Kaon.

   (d) If you know that

   $$\frac{|V_{us}|}{|V_{ud}|} = 0.2316,$$

and

$$\Gamma(\pi^- \to \mu^- + \overline{\nu}_\mu) = 2.5 \times 10^{-17}\,\text{GeV}$$

use this to compute the value of $\Gamma(K^- \to \mu^- + \overline{\nu}_\mu)$, assuming that the Kaon and pion have the same decay constant.

(e) Compare this to the SM value

$$\Gamma(K^- \to \mu^- + \overline{\nu}_\mu) = 3.37 \times 10^{-17}\,\text{GeV}$$

We can explain the small discrepancy by using a different decay constant for the Kaon. If we take

$$\frac{F_K}{F_\pi} = 1.193$$

Why should we expect that there is a difference, and that it is small?

(f) What do you obtain using the above ratio of decay constants for $\Gamma(K^- \to \mu^- + \overline{\nu}_\mu)$?

Part B

Describe in your own words what you think are the most important and/or interesting points from one of the topics of the course. Include whatever equations, derivations or diagrams you feel necessary. Try to be brief: up to about one page (but there is no penalty if you need more or less space). Full marks will be given for clarity rather than verbosity, and demonstration that you have understood the course material. You may prepare this in advance and bring it with you but *in handwritten form only.*