TD: practical QFT

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1 Computing amplitudes from a general lagrangian

These questions are about computing Feynman rules and amplitudes based on general Lagrangians. One way of working it out is to recall that the time-ordered correlators in an interacting theory can be worked out using ones in the free theory when we include the interaction terms as a perturbation, see e.g. equation (4.31) in Peskin and Schroeder:

$$\langle \Omega | T \left\{ \phi(x_1)\phi(x_2)\cdots\phi(x_n) \right\} |\Omega\rangle = \frac{\langle 0 | T \left\{ \phi(x_1)\phi(x_2)\cdots\phi(x_n) \exp[-i \int dt H_{\rm int}(t)] \right\} |0\rangle}{\langle 0 | T \left\{ \exp[-i \int dt H_{\rm int}(t)] |0\rangle \right\} |0\rangle}. \tag{1.1}$$

The interaction terms in the Hamiltonian density are just those of the Lagrangian with a minus sign:

$$\mathcal{L} = \mathcal{L}_{\rm free} + \mathcal{L}_{\rm int} = \mathcal{L}_{\rm free} - H_{\rm int}. \tag{1.2}$$

So if we expand $\exp[-i \int dt H_{\rm int}(t)]$ this is just

$$\exp[-i \int dt H_{\rm int}(t)] = \exp[i \int d^d x \mathcal{L}_{\rm int}(x)] = 1 + i \int d^d x \mathcal{L}_{\rm int}(x) + ... + \frac{1}{n!}\left(i \int d^d x \mathcal{L}_{\rm int}(x) \right)^n + ...$$

We can then use Wick's theorem to compute any correlator we want in the interacting theory. We can then derive the Feynman rules based on this. Since there are $n!$ ways that fields can be contracted into a vertex:

$$\phi\,\phi \left[i \int d^d x \phi(x)\,\phi(x)\,\phi(x)\,\phi(x) \right] \phi\,\phi$$

this is the same as

$$\mathrm{Vertex}[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)] = i \frac{\partial^4 \mathcal{L}_{\rm int}}{\partial\phi(x_1)\partial\phi(x_2)\partial\phi(x_3)\partial\phi(x_4)}. \tag{1.3}$$

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For computing scattering processes, incoming and outgoing states are momentum eigenstates. In position space we have

$$\underbrace{\phi(x)|p\rangle} = e^{-ip\cdot x}|0\rangle, \qquad \langle\, 0|\underbrace{\phi}\,(x) = \langle 0|e^{ip\cdot x}, \tag{1.4}$$

so we can use this to fourier-transform our amplitudes. Suppose we have an interaction that looks like $\mathcal{L}_{\text{int}} \supset -c\phi(x)\mathcal{O}(x)$ and have an incoming state of ϕ:

$$\begin{aligned} S \supset & \langle \dots | -ic \int d^d x\, \underbrace{\phi(x)\mathcal{O}(x)|p} \dots\rangle \\ = & \langle \dots | -ic \int d^d x e^{-ip\cdot x}\mathcal{O}(x)| \dots\rangle. \end{aligned} \tag{1.5}$$

We then use the Feynman propagator:

$$\begin{aligned} D_F(x-y) \equiv & \langle 0|T(\phi(x)\phi(y))|0\rangle \\ = & \int \frac{d^d p}{(2\pi)^d}\frac{i}{p^2 - m^2 + i\epsilon}e^{-ip\cdot(x-y)}, \end{aligned} \tag{1.6}$$

so for each vertex we will end up with a factor of $e^{-ip\cdot(x-y)}$. We can assign an ordering on the propagators as to whether the momenta is going *in* to a vertex or coming out: imagine 3 incoming particles:

$$S \supset \langle 0| - i\frac{a}{6}\int d^4x\phi(x)\phi(x)\phi(x)|p_1, p_2, p_3\rangle \to -ia\int d^4 x e^{-i(p_1+p_2+p_3)\cdot x} = -ia(2\pi)^4\delta(p_1+p_2+p_3). \tag{1.7}$$

Hence if we have a derivative acting on one ϕ then we have

$$S \supset \langle ...|i\int d^4x\partial_\mu \underbrace{\phi(x)\mathcal{O}(x)|p_1} ...\rangle \to \partial_\mu(e^{-ip\cdot x})\mathcal{O}|...\rangle \tag{1.8}$$

so we have a factor of $-ip_\mu$ for *incoming* momenta and ip_μ for outgoing:

$$V_\mu(q)\phi_1(p_1)\partial^\mu\phi_2(p_2) \to -ip_2^\mu V_\mu(q)\phi_1(p_1)\phi_2(p_2) \qquad p_2 \text{ incoming.} \tag{1.9}$$

When we have complex fields like Fermions or complex scalars, then we also have fermion number flow, which may be in the opposite direction to the momenta.

For fermions we have

$$\begin{aligned} \underbrace{\psi|p, \text{fermion}\rangle} &= \underbrace{\psi b^\dagger}\,|0\rangle = e^{-ip\cdot x}u(p)|0\rangle \\ \underbrace{\overline{\psi}|p, \text{antifermion}\rangle} &= \underbrace{\overline{\psi}d^\dagger}\,|0\rangle = e^{-ip\cdot x}\overline{v}(p)|0\rangle \\ \langle p, \underbrace{\text{fermion}|\overline{\psi}} &= \langle 0|\;\underbrace{b\overline{\psi}}\; = \langle 0|e^{ip\cdot x}\overline{u}(p) \\ \langle p, \underbrace{\text{antifermion}|\psi} &= \langle 0|\;\underbrace{d\psi}\; = \langle 0|e^{ip\cdot x}v(p). \end{aligned} \tag{1.10}$$

1.1 Exercises

Work out the Feynman rules for the *vertices* involving three or more fields of the following Lagrangians, where ϕ_i are real scalars, Φ_i are complex scalars and Ψ_i are (Dirac) fermions, and $F_{\mu\nu}$ is the field strength of QED:

$$\begin{aligned}
\mathcal{L}_1 =& \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{a}{6}\phi^3 - \frac{\lambda}{24}\phi^4 && (1.11)\\
\mathcal{L}_2 =& |\partial_\mu \Phi|^2 - m^2|\Phi|^2 - \frac{\lambda}{4}|\Phi|^4 && (1.12)\\
\mathcal{L}_3 =& |\partial_\mu \Phi_i|^2 - m_1^2|\Phi|^2 - m_2^2|\Phi|^2 - \frac{\lambda_{11}}{4}|\Phi_1|^4 - \frac{\lambda_{22}}{4}|\Phi_2|^4 - \lambda_{12}|\Phi_1|^2|\Phi_2|^2 && (1.13)\\
\mathcal{L}_4 \supset& c_L \Phi \overline{\Psi}_1 P_L \Psi_2 + c_R \Phi \overline{\Psi}_1 P_R \Psi_2 && (1.14)\\
\mathcal{L}_5 \supset& \overline{\Psi}_2 \gamma^\mu \gamma^\nu (c_L P_L + c_R P_R)\Psi_1 F_{\mu\nu}. && (1.15)
\end{aligned}$$

Here $P_L \equiv \frac{1}{2}(1-\gamma_5)$, $P_R \equiv \frac{1}{2}(1+\gamma_5)$.

2 Processes in QFT

In the course we will need to compute scattering *amplitudes* and also cross-sections and widths.

2.1 Amplitudes

For amplitudes, we have some important definitions:

$$\begin{aligned}
{}_{\text{out}}\langle p_1 p_2 ..|k_1 k_2 ...\rangle_{\text{in}} \equiv& \langle p_1 p_2..|S|k_1 k_2 ...\rangle\\
S \equiv& 1 + iT\\
\langle p_1 p_2 ..|iT|k_1 k_2 ...\rangle \equiv& (2\pi)^4 \delta^{(4)}(k_1 + k_2 + ... - p_1 - p_2 - ...) i\mathcal{M}. && (2.1)
\end{aligned}$$

So then

$$\begin{aligned}
i\mathcal{M} =& \text{connected, amputated Feynman diagrams}\\
& \text{computed with momentum space Feynman rules.} && (2.2)
\end{aligned}$$

NOTE however that this is not quite accurate. The Feynman rules compute correlators and we must also take into account the difference between the asymptotic states and the fields: this is the LSZ theorem. This picks up the field-renormalisation factor, i.e. we have

$$i\mathcal{M} = (\sqrt{Z})^n \times \text{connected, amputated momentum} - \text{space Feynman diagrams.} \quad (2.3)$$

2.2 Decays

Let's practice with some decay processes. To do this we will need the formulas for summing over spin states:

$$\sum_{\text{spins}} u(p)\overline{u}(p) = \not{p} + m$$
$$\sum_{\text{spins}} v(p)\overline{v}(p) = \not{p} - m$$
$$\sum_{\text{spins},\varepsilon\cdot\text{q}=0} \varepsilon_\mu(q)\varepsilon^*_\nu(q) = \begin{cases} -\eta_{\mu\nu} & \text{Massless vector} \\ -(\eta_{\mu\nu} - \frac{q_\mu q_\nu}{m_V^2}) & \text{Vector of mass } m_V \end{cases} \tag{2.4}$$

There are subtleties to do with gauge invariance on the last line but provided the amplitudes respect the Ward identities the above are correct.

We will also need some "Diracology":

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix} \tag{2.5}$$

where

$$\sigma^\mu = (1_2, \sigma)$$
$$\overline{\sigma}^\mu = (1_2, -\sigma) = \sigma_\mu \tag{2.6}$$

and thus, since $(\sigma^\mu)^\dagger = \sigma^\mu$, we have

$$(\gamma^0)^\dagger = \gamma^0, \qquad (\gamma^i)^\dagger = -\gamma^i.$$

This gives

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0.$$

Also $(\gamma^0)^* = (\gamma^0)^T = \gamma^0$, $(\gamma^i)^* = -(\gamma^i)^T$.

We define

$$\gamma_5 = \gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\epsilon^{\mu\nu\rho\kappa}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\kappa, \qquad \epsilon^{0123} = 1, \epsilon_{0123} = -1. \tag{2.7}$$

We can also derive identities like

$$\gamma^\mu\gamma^\nu\gamma_\mu = (2-d)\gamma^\nu \tag{2.8}$$

where "d' is the dimension, so 4, and

$$\text{tr}[\gamma^\mu\gamma^\nu] = 4\eta^{\mu\nu}$$
$$\text{tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\kappa] = 4\left(\eta^{\mu\nu}\eta^{\rho\kappa} - \eta^{\mu\rho}\eta^{\nu\kappa} + \eta^{\mu\kappa}\eta^{\nu\rho}\right)$$
$$\text{tr}[\gamma_5\gamma^\mu\gamma^\nu] = 0$$
$$\text{tr}[\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\kappa] = -4i\epsilon^{\mu\nu\rho\kappa}. \tag{2.9}$$

Finally we will need (from the Particles notes):

$$K(x,y,z) \equiv x^2 + y^2 + z^2 - 2xy - 2xz - 2yz = (x-y-z)^2 - 4yz$$
$$p^a_{\mathrm{CM}} = \frac{K^{1/2}(s, m_A^2, m_B^2)}{2\sqrt{s}}. \tag{2.10}$$

$$d\Pi_2 = \frac{1}{16\pi^2} d\Omega \frac{p^f_{\mathrm{CM}}}{\sqrt{s}} = \frac{1}{32\pi^2} d\Omega \frac{K^{1/2}(s, m_1^2, m_2^2)}{s} \tag{2.11}$$
$$\int d\Omega = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \,\sin\theta$$
$$= 2\pi \int_{-1}^{1} d(\cos\theta) \tag{2.12}$$

so if the integrand is independent of $\cos\theta$ we have

$$\int d\Pi_2 = \frac{1}{8\pi} \frac{K^{1/2}(s, m_1^2, m_2^2)}{s}. \tag{2.13}$$

and the width:

$$\Gamma(i \to f) = \frac{1}{2E_{\mathbf{p}}} \int d\Pi_f |\mathcal{M}_{p\to f}|^2$$
$$\underset{\text{rest frame}}{=} \frac{1}{2m_i} \int d\Pi_f |\mathcal{M}_{i\to f}|^2. \tag{2.14}$$

In the following I will label fields with uppercase letters ($\Phi, \Psi, \overline{\Psi}, Z$) and the corresponding particles with lowercase ones: ϕ, ψ, z with $\overline{\psi}$ representing the antifermion of ψ.

2.2.1 Scalar to a pair of fermions

Now consider a simple decay of a real scalar:

$$\mathcal{L} \supset y\Phi\overline{\Psi}\Psi$$

Let us pretend that the fermion is massless.

1. Derive the matrix element for $\mathcal{M}(\phi \to \overline{\psi}\psi)$.

2. Compute the spin-summed square matrix element $\sum_{\text{spins}} |\mathcal{M}|^2$.

3. Write down the decay width.

2.2.2 Fermion to fermion scalar

Let's use the example Lagrangian from earlier:

$$\mathcal{L}_4 \supset c_L \Phi \overline{\Psi}_1 P_L \Psi_2 + c_R \Phi \overline{\Psi}_1 P_R \Psi_2$$

1. Derive the matrix element for $\mathcal{M}(\psi_2 \to \psi_1 + \phi)$ where all particles are massive.
2. Compute the spin-averaged square matrix element $\frac{1}{2}\sum_{\text{spins}} |\mathcal{M}|^2$.
3. Write down the decay width.

2.2.3 Massive vector to a pair of massless fermions

Using the interaction between a massive vector and a massless fermion:

$$\mathcal{L} \supset c_L Z_\mu \overline{\Psi} \gamma^\mu P_L \Psi \tag{2.15}$$

1. Derive the matrix element for $\mathcal{M}(Z \to 2\psi)$ where Z is massive with mass m_Z.
2. Compute the spin-averaged square matrix element $\frac{1}{3}\sum_{\text{spins}} |\mathcal{M}|^2$.
3. Write down the decay width.
4. The uncertainty on the measured value of the Z boson width is less than 2.3 MeV (see `https://pdg.lbl.gov/2022/listings/rpp2022-list-z-boson.pdf`). While the uncertainty on the direct measurement of the decay of the Z to *invisible* particles is 16 MeV, the uncertainty on the SM fit value is only 1.5 MeV (technically we could have more invisible width but only if we also decrease the decays into other channels). So if we assume that the decay to our new invisible fermions has a partial width equal to the total uncertainty of 2.3 MeV, how big could c be?