Lectures on Phenomenology of the Standard Model and Beyond
Part 1: Particle physics

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Abstract

These are the lecture notes for the first part of a course given at the ICFP Master 2 from January 2026.

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Contents

1 Introduction

This course on the phenomenology of the Standard Model (SM) and Beyond is about understanding how we test the Standard Model, what theories might lie beyond it, and how we might test them. The course is therefore divided into three parts; this first part concerns particle physics and the link between Quantum Field Theory and collider experiments.

You have already had courses in Quantum Mechanics and Quantum Field Theory which includes QED and several advanced topics. You should be familiar with protons, neutrons, electrons, photons and neutrinos (which were introduced to explain the continuous spectrum of beta decays, but were only actually detected in the 1960s). And you almost certainly have heard of *quarks* as the constituents of the nucleons. However, the particle content of the SM is given in fig. 1. The electroweak sector – the Z/W bosons and the Higgs – will only be introduced to you later in QFT2; so will neutrinos, and non-chiral interactions. On the other hand, you should also be being introduced to QCD in QFT2 from the beginning of the course. Since QED and QCD are the only unbroken gauge symmetries that we know about (there might be a hidden dark QCD ...) there is a lot of very important physics that can be understood about them. And since the energy frontier is dominated by a proton-proton collider (and will be for the foreseeable future) we need to know something (well, a lot) about it to understand what we see.

You have most likely already met some aspects of particle physics introduced from a Quantum Mechanical perspective, which is sufficient for dealing with early experiments involving scattering of particles in a potential, or two-particle interactions that can be approximated as such; and even to some extent (as we shall soon see) production of resonances. However, processes involving a change of the *number* of particles, and/or processes involving a *dynamical* gauge field, require the full machinery of Quantum Field Theory (QFT). We shall therefore start these notes with some topics perhaps familiar from Scattering Theory courses (production of resonances, partial waves) but showing how they appear in QFT. These are of wide use in understanding experiments from low-energy scattering of pions on protons to the production of the Higgs boson (or even hypothetical heavy bosons) but require little prior knowledge of the detail of the Standard Model. They are also used to constrain the properties of new models from purely theoretical grounds.

We will then move on to some essential QFT for understanding QCD at colliders: the infrared divergences that appear in QFT calculations arising from soft photons or gluons, how they are cancelled, and what practical consequences they have. This is crucial to understanding the production of "jets" of particles in high-energy collisions, and therefore understanding any interactions where the strong gauge coupling is involved.

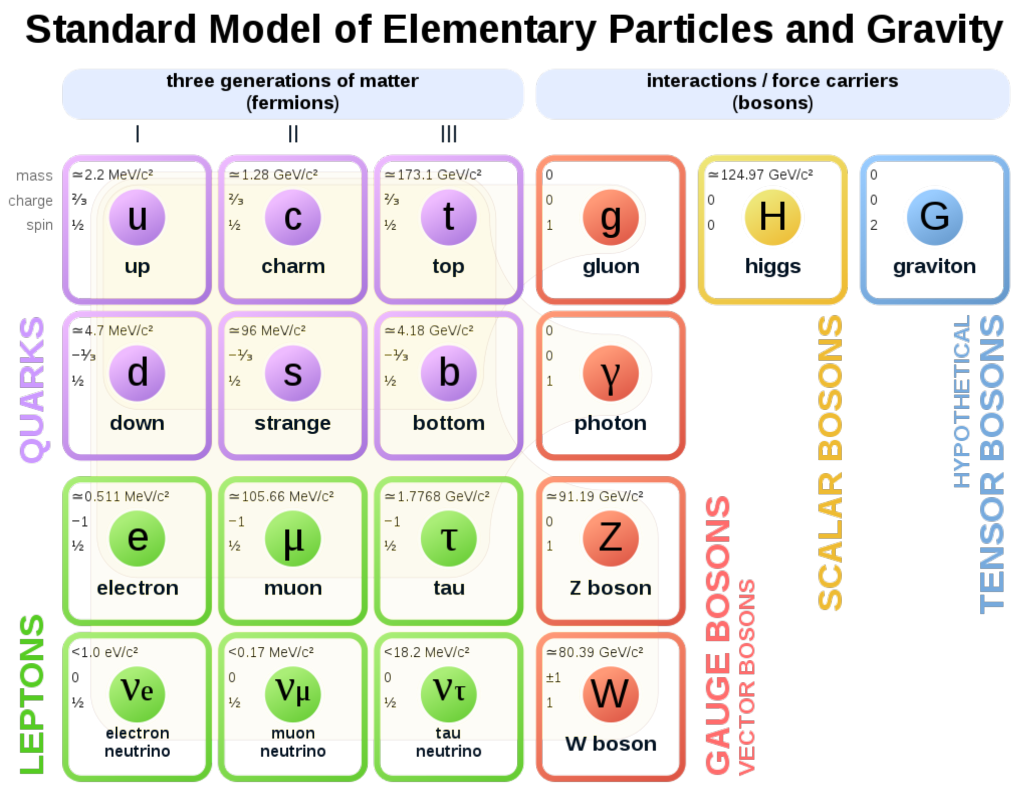


Figure 1: All of the known fundamental particles in the Standard Model (plus the graviton), along with their quantum numbers.

2 Resonances

Here we will discuss the most important processes that occur in collider experiments from a theorist's point of view.

At colliders, particles can be produced as *intermediate resonances*, *on-shell* and *off-shell*. Resonances appear in many places in physics (e.g. in electronics, in resonating cavities that may be important for looking for axion-like particles) and you can compute resonant particle production using the Breit-Wigner formula using just quantum mechanics.

2.1 Derivation from Quantum Mechanics

We assume we can decompose the system in terms of energy eigenstates as

$$\Psi(t) = \sum_m a_m(t) e^{-iE_m t}\psi_m, \tag{2.1}$$

where ψ_m are spatially-dependent orthogonal wavefunctions *of the free Hamiltonian* which we assume to be valid at infinite distance/times: we are imagining scattering of particles arriving from infinity and then passing off to infinity again. The interactions are therefore supposed to happen over some finite time (and distance) and can be treated as a time-dependent perturbation. We will denote the perturbed Hamiltonian H; then applying the Schrödinger equation $i\frac{d\Psi}{dt} = H\Psi$ and denoting $H_{mn} = \int \psi_m^* H\psi_n$ we have

$$i\dot{a}_m = \sum_{m\neq n} H_{nm} e^{-i(E_m-E_n)t} a_m. \tag{2.2}$$

Consider that the state a_0 is unstable. If we start from $a_0(0) = 1$ then it will decay, so we can approximate $a_0(t) = e^{-t/2\tau} = e^{-\Gamma t/2}$ so that $|a_0|^2 = e^{-t/\tau}$. Then for a given channel a_f we have

$$i\dot{a}_f = H_{10} e^{-i(E_0-E_f)t} e^{-t/2\tau} = H_{10} e^{-i(E_0-E_f-i\Gamma/2)t}. \tag{2.3}$$

Integrating this we have

$$\begin{aligned} a_f =& H_{f0} \frac{e^{-i(E_0-E_f-i\Gamma/2)t} - 1}{E_0 - E_f - i\Gamma/2} \\ \underset{t\to\infty}{\longrightarrow}& \frac{H_{f0}}{E_f - E_0 + i\Gamma/2}. \end{aligned} \tag{2.4}$$

Let us say that there several states with the same p^2 but different directions that the particle can decay to. Then the total probability for the particle to decay is

$$1 = \sum_{n\in f} |a_n|^2 = \int_{-\infty}^{\infty} dE\, \frac{|H_{f0}|^2}{(E-E_0)^2 + \Gamma^2/4} n_f(E) \tag{2.5}$$

where n_f is the density of states for a given E. This function is strongly peaked around E_0,

$$\begin{aligned} I =& \int_{-\infty}^{\infty} f(s) \frac{1}{(s-m)^2 + \Gamma^2/4} \approx 2f(m) \int_0^{\infty} \frac{1}{s^2 + \Gamma^2/4} \\ =& 2f(m)(2/\Gamma) \int_0^{\pi/2} du, \qquad s = \Gamma/2 \tan u \\ =& 2\pi f(m)/\Gamma \end{aligned} \tag{2.6}$$

and this leads to Fermi's golden rule:

$$\Gamma = 2\pi |H_{f0}|^2 n_i(E_0). \tag{2.7}$$

We can then turn this around and look at particle *production* through a resonance, i.e. a process where an intermediate state is formed:

$$a + b \to X^* \to f. \tag{2.8}$$

Now we can split the states into "incoming" and "outgoing" ones; we know that a_0 decays to final states $\{f\}$, so in the absence of production of a_0 we have

$$i\dot{a}_0 = \sum_{m\in\{f\}} H_{0m} e^{-i(E_m - E_0)t} a_m(t) = -i(\Gamma/2)a_0. \tag{2.9}$$

If we now add incoming states we have

$$\begin{aligned} i\dot{a}_0 =& \sum_{m\in\{i\}} H_{0m} e^{-i(E_m - E_0)t} a_m(t) + \sum_{m\in\{f\}} H_{0m} e^{-i(E_m - E_0)t} a_m(t) \\ =& - i(\Gamma/2)a_0 + \sum_{m\in\{i\}} H_{0m} e^{-i(E_m - E_0)t} a_m(t). \end{aligned} \tag{2.10}$$

Let us assume that the incoming state is not depleted but there is a steady-state flux maintaining $a_1 = 1$. We can then integrate the equation starting from $t = 0$ to t (or alternatively regard it as a first-order approximation starting from $a_1(0) = 1$):

$$a_0(t) =\frac{H_{01} e^{-i(E_1 - E_0)t}(1 - e^{-\Gamma t/2})}{E_1 - E_0 + i\Gamma/2} \to |a_0(t)|^2 \underset{t\to\infty}{=} \frac{|H_{01}|^2}{(E_1 - E_0)^2 + \Gamma^2/4}. \tag{2.11}$$

This is the probability of finding the state a_0 at some late time, assuming that it is being replenished by an influx of the incoming state. But we know that our intermediate state is constantly decaying and therefore:

$$\text{production of state } 0 = -\text{ rate of decay of state } 0. \tag{2.12}$$

For an exponentially decaying particle with $N = N_0 e^{-\Gamma t}$,

$$\frac{dN}{dt} = - \Gamma N \tag{2.13}$$

and therefore in our steady-state situation

$$\begin{aligned} \text{production of state 0 through channel } i =& \Gamma |a_0|^2 \\ =& \frac{|H_{0i}|^2 \Gamma}{(E_1 - E_0)^2 + \Gamma^2/4}. \end{aligned} \tag{2.14}$$

But we are interested in the cross-section; this is the *rate of production of the state* 0, or

$$\begin{aligned} \text{production of state 0 through channel } i =& \sigma(i \to 0) \times \text{particle flux.} \\ \Gamma \times |a_0|^2 =& \sigma(i \to 0) \times \frac{v}{V}, \end{aligned} \tag{2.15}$$

where v the relative speed of the incident particles and V is the volume of space which we can later take to infinity.

We can then use Fermi's golden rule to remove the matrix element; splitting the width into partial widths

$$\Gamma = \sum_m \Gamma_m, \qquad \Gamma_i = 2\pi n_i(E_i)|H_{0i}|^2 \tag{2.16}$$

and obtain

$$\sigma(i \to X^*) = \frac{V}{v} \times \frac{\Gamma}{(E_1 - E_0)^2 + \Gamma^2/4} \times \frac{\Gamma_i}{2\pi n_0(E_0)}. \tag{2.17}$$

The traditional computation in QM involves particles hitting a static target. Then the density of states in non-relativistic QM when we consider the kinetic energy to be dominated by one particle is just

$$n_0 = \frac{V}{(2\pi)^3} 4\pi k^2 \frac{dk}{dE} = \frac{V}{(2\pi)^3} 4\pi k^2/v, \qquad \frac{dE}{dk} = k/m = v, \tag{2.18}$$

and hence

$$\sigma(i \to X^*) = \frac{\pi}{k^2} \frac{\Gamma\,\Gamma_i}{(E_1 - E_0)^2 + \Gamma^2/4}, \tag{2.19}$$

which is the celebrated Breit-Wigner formula (note the difference between Γ_i and the total width). If we want to consider the process for a particular channel, we multiply by the branching fraction into that channel, Γ_f/Γ:

$$\sigma(i \to X^* \to f) = \frac{\pi}{k^2} \frac{\Gamma_i \Gamma_f}{(E_1 - E_0)^2 + \Gamma^2/4}. \tag{2.20}$$

It is quite remarkable that we can calculate the cross-section of such a process knowing just the widths. However, it is straightforward to extend the derivation to the relativistic case; we just have to exchange k for the three-momentum *in the centre of mass frame* and include an extra factor of two *if the particles are identical*. In appendix A.1 I show that we arrive at

$$\sigma(i \to X^* \to f) = \frac{2^{\delta_i}\pi}{k_{CMS}^2} \frac{\Gamma_i \Gamma_f}{(E_{CMS} - E_0)^2 + \Gamma^2/4}. \tag{2.21}$$

where $\delta_i = 1$ if the initial-state particles are identical and 0 otherwise.

We can go further and add in particle spins and angular momenta: suppose we have incoming states with spins s_1, s_2 in an incoherent mixture where only one combination can produce the given resonance; then we must *average* over all $(2s_1 + 1)(2s_2 + 1)$ states. However, the intermediate state has a definite spin s_X and can then also be in any of its $(2s_X + 1)$ states that we must *sum* over. We then obtain

$$\sigma(i \to X^* \to f) = \frac{2s_X + 1}{(2s_1 + 1)(2s_2 + 1)} \frac{2^{\delta_i}\pi}{k_{CMS}^2} \frac{\Gamma_i \Gamma_f}{(E_{CMS} - E_0)^2 + \Gamma^2/4}. \tag{2.22}$$

We can use these formulas to measure the spin! For elastic scattering where $\Gamma_i = \Gamma_f \simeq \Gamma$, we have a peak cross-section of

$$\sigma \simeq 2^{\delta_f} \times \frac{4\pi}{k_{\rm cm}^2} \frac{2s_X+1}{(2s_1+1)(2s_2+1)}. \tag{2.23}$$

2.2 Derivation from QFT

We can also derive the Breit-Wigner formula in the relativistic theory.

2.2.1 Decays and the Optical theorem

First we need to recall the optical theorem and some expressions for decays and cross-sections.

For incoming states $2 \to n$ scattering of incoming states a with masses m_A, m_B we have

$$\sigma = \frac{1}{4E_A E_B |v_A - v_B|} \int |\mathcal{M}_{A,B\to n}|^2 d\Pi_n \tag{2.24}$$

We can use $v_A = p_A/E_A$ so that in the CM frame $4E_A E_B|v_A - v_B| = 4p^a_{CM}(E_A + E_B) = 4p^a_{CM}\sqrt{s}$ where p^a_{CM} is the absolute value of the momentum in the CM frame (of both particles), given by

$$\begin{aligned} K(x,y,z) \equiv& x^2 + y^2 + z^2 - 2xy - 2xz - 2yz = (x-y-z)^2 - 4yz \\ p^a_{\rm CM} =& \frac{K^{1/2}(s, m_A^2, m_B^2)}{2\sqrt{s}}. \end{aligned} \tag{2.25}$$

so

$$\sigma_{CM} = \frac{1}{4p^a_{\rm CM}\sqrt{s}} \int |\mathcal{M}_{A,B\to n}|^2 d\Pi_n. \tag{2.26}$$

The phase-space integral is given by

$$d\Pi_n = (2\pi)^4 \delta^{(4)}(p_A + p_B - \sum_f p_f)\Bigg(\prod_{f=1}^n \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f}\Bigg). \tag{2.27}$$

For two-body final states f with masses m_1, m_2 we have

$$d\Pi_2 = \frac{1}{16\pi^2} d\Omega \frac{p^f_{\rm CM}}{\sqrt{s}} = \frac{1}{32\pi^2} d\Omega \frac{K^{1/2}(s, m_1^2, m_2^2)}{s}. \tag{2.28}$$

In the case of elastic scattering, so that $m_A = m_1, m_B = m_2$ we have

$$\sigma_{CM} \to \int \frac{|\mathcal{M}|^2}{64\pi^2 s} d\Omega. \tag{2.29}$$

where

$$\begin{aligned} \int d\Omega &= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, \sin\theta \\ &= 2\pi \int_{-1}^{1} d(\cos\theta) \end{aligned} \tag{2.30}$$

as the scattering process is symmetric about the collision axis in the CM frame. We must not forget that the incoming momenta define a direction and so the integral over $|\mathcal{M}|^2$ of the phase space will no longer be independent of the angle.

On the other hand, for decays, we have the formula

$$\begin{aligned}\Gamma =&\frac{1}{2E_{\mathbf{p}}}\int d\Pi_f|\mathcal{M}_{p\to f}|^2\\ \underset{\text{rest frame}}{=}&\frac{1}{2m_i}\int d\Pi_f|\mathcal{M}_{i\to f}|^2.\end{aligned} \tag{2.31}$$

This is essentially the relativistic counterpart of Fermi's golden rule.

The optical theorem is just a consequence of the statement that $S^\dagger S = 1$. Writing $S = 1 + iT$ we have $TT^\dagger + i(T - T^\dagger) = 0$; in terms of matrix elements this gives

$$\begin{aligned}{}_{\text{out}}\langle\{p\}|iT|\{k\}\rangle_{\text{in}} \equiv& i\mathcal{M}(\{k\}\to\{p\})(2\pi)^4\delta^4(\{k\}-\{p\})\\ \mathcal{M}^\dagger(\{k\}\to\{p\}) =& \mathcal{M}^*(\{p\}\to\{k\}).\end{aligned} \tag{2.32}$$

Then

$$\begin{aligned}\langle\{p\}|T^\dagger T|\{k\}\rangle =& \sum_n\left(\prod_{i=1}^n\frac{d^3q}{(2\pi)^3}\frac{1}{2E_i}\right)\langle\{p\}|T^\dagger|\{q_i\}\rangle\langle\{q_i\}|T|\{k\}\rangle\\ -i[\mathcal{M}(\{k\}\to\{p\})-\mathcal{M}^*(\{p\}\to\{k\})](2\pi)^4\delta^4(\{k\}-\{p\}) =& \sum_n\left(\prod_{i=1}^n\frac{d^3q}{(2\pi)^3}\frac{1}{2E_i}\right)\\ &\times\mathcal{M}(\{k\}\to\{q_i\})\mathcal{M}^*(\{p\}\to\{q_i\})(2\pi)^4\delta^4(\{k\}-\{q\})(2\pi)^4\delta^4(\{p\}-\{q\}).\end{aligned} \tag{2.33}$$

If we consider elastic processes where $\{k\} = \{p\}$ then

$$\begin{aligned}2\text{Im}[\mathcal{M}(\{k\}\to\{k\})](2\pi)^4\delta^4(\{k\}-\{k\}) =& \sum_n\left(\prod_{i=1}^n\frac{d^3q}{(2\pi)^3}\frac{1}{2E_i}\right)\\ &\times|\mathcal{M}(\{k\}\to\{q_i\})|^2(2\pi)^4\delta^4(\{k\}-\{q\})(2\pi)^4\delta^4(\{k\}-\{q\})\\ 2\text{Im}[\mathcal{M}(\{k\}\to\{k\})] =& \sum_n\left(\prod_{i=1}^n\frac{d^3q}{(2\pi)^3}\frac{1}{2E_i}\right)|\mathcal{M}(\{k\}\to\{q_i\})|^2(2\pi)^4\delta^4(\{k\}-\{q\})\\ =&\sum_n\int d\Pi_n|\mathcal{M}(\{k\}\to\{q_i\})|^2\end{aligned} \tag{2.34}$$

This is a completely general expression and can be applied to different processes. The classic case is $2\to 2$ scattering:

$$\begin{aligned}2\text{Im}[\mathcal{M}(k_1,k_2\to k_1,k_2)] =& 2\sqrt{(s-m_1^2-m_2^2)^2-4m_1^2m_2^2}\sigma(k_1,k_2\to\text{anything})\\ =&4E_{\text{cm}}p_{\text{cm}}\sigma(k_1,k_2\to\text{anything}),\end{aligned} \tag{2.35}$$

where $p_{cm} = |\mathbf{p}_{cm}|$ is defined in the CM frame by $\mathbf{p}_{cm} = \mathbf{k}_1 = -\mathbf{k}_2$.

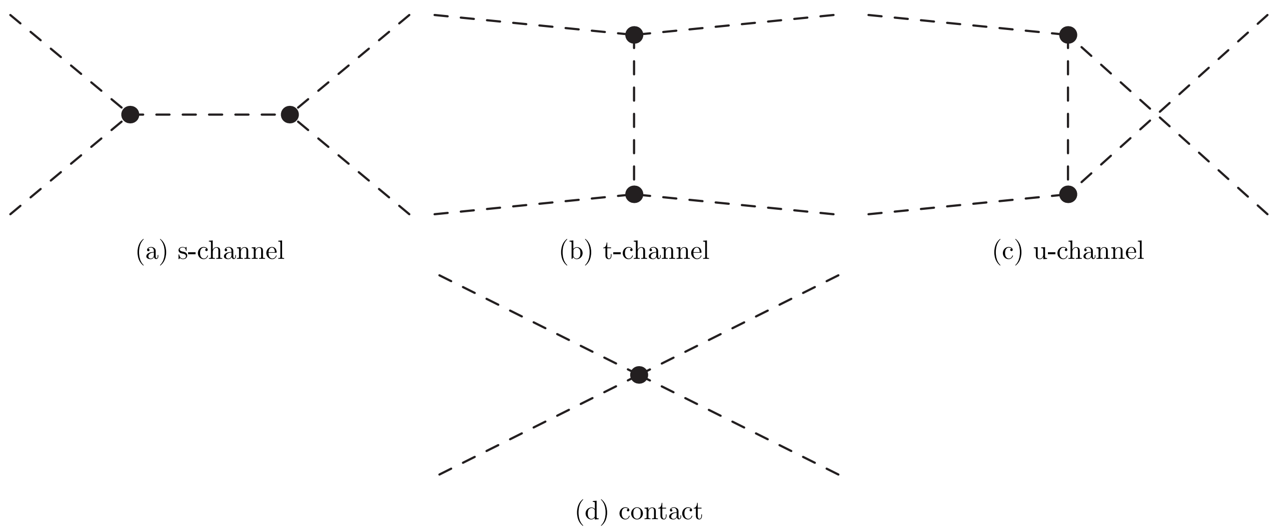


Figure 2: Tree-level $2 \to 2$ scattering diagrams involving scalars.

2.2.2 $1 \to 1$

Now let us apply the same thing to $1 \to 1$ matrix elements. We have

$$\begin{aligned} 2\mathrm{Im}[\mathcal{M}(k_1 \to k_1)] &= \sum_n \int d\Pi_n |\mathcal{M}(\{k\} \to \{q_i\})|^2 \\ &\underset{\text{on shell}}{=} 2m_1\Gamma \end{aligned} \tag{2.36}$$

i.e. this gives the total width of the particle according to the expressions we had above (that you will have seen in the QFT1 course).

2.2.3 Relativistic Breit Wigner

Now recall in any QFT amplitude what are called s-channel diagrams; recall in two-body scattering of $k_1, k_2 \to p_1, p_2$ we have Mandelstam variables:

$$\begin{aligned} s \equiv& (k_1 + k_2)^2 = (p_1 + p_2)^2 \\ t \equiv& (k_1 - p_1)^2 = (k_2 - p_2)^2 \\ u \equiv& (k_1 - p_2)^2 = (k_2 - p_1)^2. \end{aligned} \tag{2.37}$$

So an s-channel resonance would involve the two incoming particles combining into an intermediate resonance; but resonances are also possible in the t- and u-channels. I illustrate the types of interaction in figure 2.

What happens at loop level? We replace the propagators by the sum of connected diagrams, which become sums over $1PI$ diagrams as in figure 3, where we write connected diagrams as grey blobs and

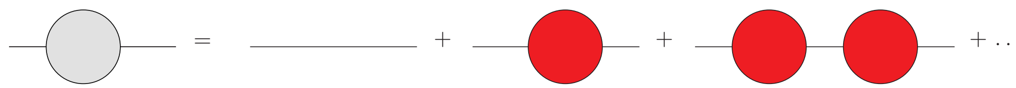


Figure 3: Sum of 1PI propagators yielding the exact propagator. 1PI diagrams are shown as red solid blobs, and the sum of all connected diagrams is shown as grey blobs.

1PI diagrams as red solid blobs. We have

$$\frac{i}{p^2-m_0^2}+\frac{i}{p^2-m_0^2}\left(-i\Pi(p^2)\right)\frac{i}{p^2-m_0^2}+\ldots=\frac{i}{p^2-m_0^2-\Pi(p^2)}. \tag{2.38}$$

Let us examine this a bit closer. This full resummed propagator can have a pole at some value of p^2, however in general it is no longer at m_0^2 (which is the value of the mass parameter in the Lagrangian). Indeed, the "pole mass" becomes the solution to the self-similar equation

$$m^2 =m_0^2+\mathrm{Re}[\Pi(m^2)]. \tag{2.39}$$

If we use a minimal-subtraction scheme then we must compute these corrections to the pole mass; alternatively it can be possible to choose our renormalisation conditions to fix $m_0 = m_{\rm pole}, \mathrm{Re}[\Pi(m_{\rm pole})] = 0$ which is known as an "on-shell scheme." In either case we then have in the vicinity of the pole

$$\Pi(p^2) = \Pi(m^2) + (p^2-m^2)\Pi'(m^2) + ... \tag{2.40}$$

and

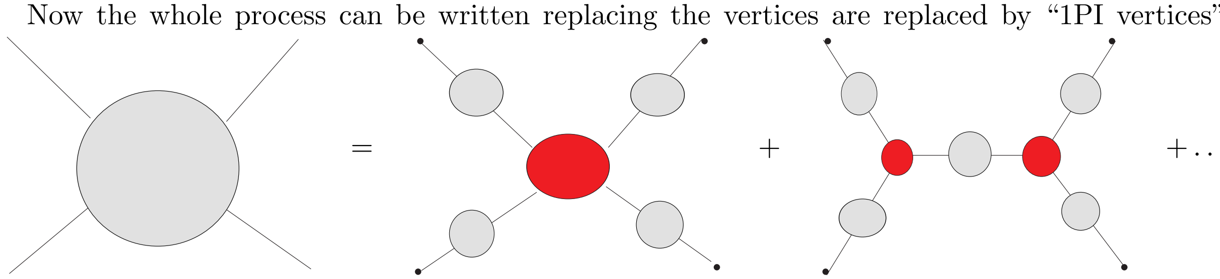
$$\frac{i}{p^2-m_0^2-\Pi(p^2)} =\frac{i}{(p^2-m^2)(1-\Pi'(m^2))-i\mathrm{Im}[\Pi(m^2)]+...}. \tag{2.41}$$

If we define $Z \equiv \frac{1}{1-\Pi'(m^2)}$ then we have

$$\frac{i}{p^2-m_0^2-\Pi(p^2)} =\frac{iZ}{p^2-m^2-iZ\mathrm{Im}[\Pi(m^2)]+\mathcal{O}((p^2-m^2)^2)}. \tag{2.42}$$

So provided that $Z\mathrm{Im}[\Pi(m^2)]$ is not large, we have a pole-like behaviour in the propagator at $p^2 = m^2$.

Now the whole process can be written replacing the vertices are replaced by "1PI vertices" as



We will look again at this later. However, for now we see that including all loop diagrams there is still this pole-like behaviour near the pole-mass. This is of course just in the amplitude; once we square the amplitude to obtain a cross-section we will have

$$\sigma \sim \left|\frac{iZ}{s-m^2-iZ\mathrm{Im}[\Pi(m^2)]}\right|^2 \propto \frac{1}{(s-m^2)^2+Z^2\mathrm{Im}[\Pi(m^2)]^2} \tag{2.43}$$

This clearly appears like a relativistic version of a Breit-Wigner propagator. To demonstrate that the term in the denominator is the width, recall that from the LSZ reduction formula:

$$\mathcal{M}(i \to f) = \prod_{j\in\{i\}} \sqrt{Z_j} \prod_{k\in\{f\}} \sqrt{Z_k} \times \text{sum of amputated diagrams in momentum space.} \tag{2.44}$$

This is because "amputated diagrams" are defined in terms of correlators (e.g. $\langle\phi(x_1)\phi(x_2)...\rangle$) and matrix elements are defined in terms of asymptotic states *on shell*. We can see from the above that the factors of $\sqrt{Z}$ take into account the difference in normalisation between the terms in the lagrangian and the on-shell states.

Then we have

$$i\mathcal{M}_{1\to 1} = -\, iZ\Pi(m^2) \tag{2.45}$$

and hence

$$\begin{aligned} Z\mathrm{Im}[\Pi(m^2)] =& -\,\mathrm{Im}[\mathcal{M}_{1\to 1}] \\ =& -\, m\Gamma \end{aligned} \tag{2.46}$$

and hence we can identify the Breit-Wigner form:

$$\begin{aligned} \sigma \sim& \frac{1}{(s-m^2)^2 + m^2\Gamma^2} \\ \underset{s\simeq m^2}{\propto}& \frac{1}{(\sqrt{s}-m)^2 + \Gamma^2/4} \end{aligned} \tag{2.47}$$

which is the same factor we found in the QM derivation. In appendix A.2 I give a reproduction of the whole Breit-Wigner formula from QFT.

3 Unitarity constraints

In this section we will show how to place constraints on the couplings of *any* theory based just on *theoretical* input, i.e. just by requiring unitarity of the S-matrix, and specifically considering just $2 \to 2$ scattering.

3.1 General derivation

Here I shall give a general derivation of unitarity constraints for interactions of scalars. First, we define the S-matrix in terms of the interaction matrix T as $S = 1 + iT$. Then in terms of matrix elements of scattering from (multiparticle) states a with a set of momentum $\{p\}$ to a set of states b with a set of momenta $\{k\}$ we have

$$T_{ba} \equiv {}_{\rm out}\langle\{k,b\}|iT|\{p,a\}\rangle_{\rm in} \equiv i\mathcal{M}(\{p,a\}\to\{k,b\})(2\pi)^4\delta^4(\{k\}-\{p\}) \equiv i\mathcal{M}_{ba}(2\pi)^4\delta^4(\{k\}-\{p\}), \tag{3.1}$$

and so

$$\mathcal{M}^\dagger(\{k,b\}\to\{p,a\}) = \mathcal{M}^*(\{p,a\}\to\{k,b\}). \tag{3.2}$$

Now S must be a unitary matrix, and so the the constraints from unitarity come from

$$SS^\dagger = 1 \longrightarrow T^\dagger T + i(T - T^\dagger) = TT^\dagger + i(T - T^\dagger) = 0. \tag{3.3}$$

Then we insert a complete set of states to evaluate $T^\dagger T$:

$$\langle\{k,b\}|T^\dagger T|\{p,a\}\rangle = \sum_n d\Pi_n \langle\{k,b\}|T^\dagger|\{q_n,c_n\}\rangle\langle\{q_n,c_n\}|T|\{p,a\}\rangle. \tag{3.4}$$

Now specialising to the case of $2\to2$ scattering, we can rewrite the equation as

$$-i(\mathcal{M}_{ba}^{2\to2} - (\mathcal{M}_{ba}^{2\to2})^\dagger) = \sum_c \frac{1}{2^{\delta_c}}\frac{|\mathbf{p}_c|}{16\pi^2\sqrt{s}}\int d\Omega \mathcal{M}_{ca}^{2\to2}\overline{\mathcal{M}}_{cb}^{2\to2} + \underbrace{\sum_{n>2} d\Pi_n \mathcal{M}_{ca}^{2\to n}\overline{\mathcal{M}}_{cb}^{2\to n}}_{\geq 0}. \tag{3.5}$$

Here $\delta_c = 0$ if the particles in c are not identical, and 1 if they are identical, to allow us to keep the same phase space region of integration (otherwise we double count), and $\mathbf{p}_c$ is the three-momentum in the centre of mass frame for the pair c.

We have cheated a little with the underbrace on the right hand side. When we diagonalise $\mathcal{M}_{ba}^{2\to2}$ (and its hermitian conjugate simultaneously, see below) – which means diagonalising on the space of pairs of incoming/outgoing particles – and then look at the diagonal terms in the new basis of states, *then* the higher-order terms must be greater than zero.

In the centre of mass frame, for incoming states $a = \{a_1, a_2\}$ we can define the momenta of both particles to lie along the z axis. For the outgoing states b they lie in the same plane; however for the intermediate states c they can have any direction. We define the three-vectors vectors $\mathbf{p}_a, \mathbf{k}_b, \mathbf{p}_c$ to lie along the unit vectors

$$\begin{aligned}\hat{p}_a =&(0,0,1)\\ \hat{k}_b =&(0,\sin\theta_b, z_b)\\ \hat{p}_c =&(\sin\theta_c\cos\phi_c, \sin\theta_c\sin\phi_c, z_c),\end{aligned} \tag{3.6}$$

where

$$z_b \equiv \cos\theta_b, \qquad z_c \equiv \cos\theta_c.$$

This means that the first incoming state has momentum along $\hat{p}_a$ and the second $-\hat{p}_a$; similarly the states b and c are back-to-back.

We next decompose the matrix elements into partial waves. Since we are dealing with spinless interactions, they can only depend on the magnitude of the momenta and the angle between them (so $\mathcal{M}_a$ cannot depend on the azimuthal angle ϕ). Then

$$\begin{aligned}\mathcal{M}_{ca} =&16\pi\sum(2J+1)P_J(z_c)\hat{a}_J^{ca}(s)\\ \mathcal{M}_{cb} =&16\pi\sum(2J+1)P_J(\hat{k}_b\cdot\hat{k}_c)\hat{a}_J^{cb}(s),\end{aligned} \tag{3.7}$$

where P_J are the Legendre polynomials, satisfying

$$\int_{-1}^{1} dz P_J(z) P_{J'}(z) = \frac{2}{2J+1}\delta_{JJ'}, \qquad P_0(z) = 1, \tag{3.8}$$

so

$$\hat{a}_J^{ca} = \frac{1}{32\pi}\int_{-1}^{1} dz P_J(z)\mathcal{M}_{ca}. \tag{3.9}$$

Equivalently we can use the Wigner-Eckart theorem that states that, since the S-matrix is a spinless operator, the the matrix elements have to be independent of the z-component of the angular momentum.

This allows us to write

$$-2\pi i(\hat{a}_J - \hat{a}_J^\dagger)^{ba} \geq \sum_c \frac{2^{-\delta_c}|\mathbf{p}_c|}{\sqrt{s}}(2J'+1)(2J''+1)\int d\phi_c dz_c dz_b P_J(z_b)P_{J'}(z_c)P_{J''}(\hat{k}_b\cdot\hat{k}_c)\hat{a}_{J'}^{ca}(\hat{a}_{J''}^{cb})^*. \tag{3.10}$$

Next we require the famous identity (which is referred to as the *addition theorem for spherical harmonics* and can be found in e.g. [1] eq. (4.28) or Weinberg's book "Quantum Mechanics" eq. (4.3.36)):

$$P_J(\hat{k}_b\cdot\hat{k}_c) = \frac{4\pi}{2J+1}\sum_{m=-J}^{J} Y_{Jm}(\theta_b,\phi_b)Y_{Jm}^*(\theta_c,\phi_c) \tag{3.11}$$

where the spherical harmonics satisfy

$$Y_{Jm} \propto e^{im\phi}P_J^m(\cos\theta), \qquad Y_{J0} = \sqrt{\frac{2J+1}{4\pi}}P_J(\cos\theta). \tag{3.12}$$

In our case we have $\phi_b = 0$ so

$$\begin{aligned}-2\pi i(\hat{a}_J - \hat{a}_J^\dagger) \geq \sum_c \frac{2^{-\delta_c}|\mathbf{p}_c|}{\sqrt{s}}(2J'+1)(2J''+1)\int d\phi_c dz_c dz_b P_J(z_b)P_{J'}(z_c)\hat{a}_{J'}^{ca}\bar{\hat{a}}_{J''}^{cb}\frac{4\pi}{2J''+1}\\ \times\sum_m e^{im\phi_c}P_{J''}^m(z_b)P_{J''}^m(z_c)\end{aligned} \tag{3.13}$$

and thus we pick out the $m=0$ value when we perform the integral over ϕ_c. Finally

$$-\frac{i}{2}(\hat{a}_J - \hat{a}_J^\dagger)^{ba} \geq \sum_c \frac{2^{-\delta_c}|2\mathbf{p}_c|}{\sqrt{s}}\hat{a}_J^{ca}\bar{\hat{a}}_J^{cb}. \tag{3.14}$$

This is true for each partial wave separately.

Now we make the definition:

$$a_J^{ba} \equiv \sqrt{\frac{4|\mathbf{p}_b||\mathbf{p}_a|}{2^{\delta_a}2^{\delta_b}s}}\hat{a}_J^{ba}. \tag{3.15}$$

Then we have

$$-\frac{i}{2}(a_J - a_J^\dagger)^{ba} \geq a_J^{ca}\overline{a}_J^{cb} = a_J^{cb}\overline{a}_J^{ca}. \tag{3.16}$$

Now, since we could have done this in either order, the matrix a^J_{ba} is normal, and thus both it and $a^\dagger$ can be diagonalised with the same unitary matrix, meaning that we can write for the eigenvalues (a^i_J):

$$\mathrm{Im}(a^i_J) \geq |a^i_J|^2. \tag{3.17}$$

This applies separately for *each* value of J, but in practice often only the $J = 0$ or first few harmonics are useful.

This is a very powerful fundamental constraint. We can write it in various ways, such as

$$0 \leq |a^i_J - \frac{i}{2}|^2 = \frac{1}{4} - \mathrm{Im}(a^i_J) + |a^i_J|^2 \leq \frac{1}{4} \tag{3.18}$$

which makes it clear that the allowed values lie in a circle of radius $\frac{1}{2}$ centred on $\frac{i}{2}$ in the complex plane, or

$$\begin{aligned} 0 \leq & \mathrm{Im}(a^i_J) \leq 1 \\ -\frac{1}{2} \leq & \mathrm{Re}(a^i_J) \leq \frac{1}{2} \\ |a^i_J|^2 \leq & 1. \end{aligned} \tag{3.19}$$

For clarity, in general we impose $|\mathrm{Re}(a^i_J)| \leq \frac{1}{2}$ and

$$a_J^{ba} = \frac{1}{32\pi}\sqrt{\frac{4|\mathbf{p}_b||\mathbf{p}_a|}{2^{\delta_a}2^{\delta_b}s}} \int_{-1}^{1} dz P_J(z)\mathcal{M}_{ba}. \tag{3.20}$$

For the zeroth partial wave ($J = 0$) and in the limit of s much larger than any mass in the amplitude, we have

$$a_0^{ba} \underset{s\to\infty}{=} \frac{1}{32\pi}\frac{1}{\sqrt{2^{\delta_a}2^{\delta_b}}} \int_{-1}^{1} dz \mathcal{M}_{ba}. \tag{3.21}$$

In the case of particles with spin, we can either use the full spherical harmonics (and sum over incoming/outgoing spins) or there is a more efficient treatment involving Wigner $d^J_{\mu_i\mu_j}$-functions, see e.g. [2].

3.2 Partial wave amplitudes

Along the way of our proof, we wrote the amplitudes in terms of partial waves. This is a technique that at least used to be very useful for experimental analyses studying resonances: we can use the partial waves as a set of basis functions and fit the experimentally observed differential cross-sections to them. To do this, we need to write the cross-sections in terms of our amplitudes:

$$\begin{aligned}\sigma_{CM}(a\to b) =& \frac{1}{2^{\delta_b}}\frac{1}{4p_a\sqrt{s}}\int d\Pi_2|\mathcal{M}|^2\\ \frac{d\sigma_{CM}}{d\Omega} =& \frac{1}{2^{\delta_b}}\frac{p^b_{CM}}{64\pi^2 p^a_{CM}s}\bigg|\sum_J(16\pi)(2J+1)\hat{a}_J P_J(z)\bigg|^2\\ =& \frac{2^{\delta_a}}{256\pi^2(p^a_{CM})^2}\bigg|\sum_J(16\pi)(2J+1)a_J P_J(z)\bigg|^2.\end{aligned} \tag{3.22}$$

Since we expect that the amplitudes should be dominated by the first few partial waves, we can use them to fit the differential distributions. This can also be useful to give information about the spins of the particles involved.

The total cross-section is also

$$\sigma_{CM}(a\to b) =\frac{4\pi}{(p^a_{CM})^2}2^{\delta_a}\sum_J(2J+1)|a_J|^2. \tag{3.23}$$

Note that this bears a certain relation to the expression that we gave for Breit-Wigner cross-sections; you can see this fleshed out in [3] section 3.8. On a resonance the amplitude will be dominated by the partial waves corresponding to the spin of the intermediate state! For this reason, experimentalists may plot $\frac{d\sigma}{d\Omega}$ for a given resonance and fit the result as a series in $\cos\theta \leftrightarrow P_J(\cos\theta)$. This allows the a_J to be extracted; for a sharp resonance they will sit on the boundary of the unitarity circle!

From a theorist's point of view, the above analysis also allows us to put bounds on each partial wave and on the total cross-section! Since we know that the coefficients a_J are bounded by unitarity, the total cross-section has a maximum. This has important implications for thermal dark matter models (and notably gives an upper bound on the mass of thermal dark matter).

You may have seen similar formulas in Quantum Mechanics under scattering theory (which typically invesigates plane waves scattering from potentials). This is the QFT derivation that directly links the coefficients a_J to Feynman diagrams. To make the connection more explicit, consider scattering of non-identical particles (i.e. one becomes the source of the potential), so that $a = b$ but $\delta_a = 0 = \delta_b$. Our constraint is

$$\mathrm{Im}(a_J) \geq |a_J|^2 \to |1+2ia_J|^2 \leq 1 \tag{3.24}$$

and therefore we can write it as:

$$1+2ia_J = \eta_J e^{2i\delta_J}, \qquad |\eta_J| \leq 1. \tag{3.25}$$

In elastic scattering we can satisfy the equality. We then have

$$a_J = e^{i\delta_J}\sin\delta_J, \qquad \hat{a}_J = \frac{1}{p_{CM}}e^{i\delta_J}\sin\delta_J. \tag{3.26}$$

which are expressions found in QM textbooks.

3.3 Applications

3.3.1 Perturbativity bounds

This seems fairly dry, so we shall first apply it to the almost trivial example of $\lambda\phi^4$ theory (let us suppose we have a symmetry $\phi \to -\phi$ to exclude a cubic term):

$$\mathcal{L} \supset -\frac{1}{2}m^2\phi^2 - \frac{1}{24}\lambda\phi^4. \tag{3.27}$$

There is only one scattering amplitude to calculate, two scalars to two scalars, and at *tree-level* it is not even dependent on the momenta:

$$i\mathcal{M}^{\text{tree}} = -\, i\lambda. \tag{3.28}$$

Obviously, since this is independent of $\cos\theta$ and since $P_0(z) = 1$ then

$$\hat{a}_0 = -\frac{\lambda}{16\pi}, \qquad \hat{a}_{J>0} = 0. \tag{3.29}$$

Now we have identical incoming and outgoing particles, so $|\mathbf{p}_a| = |\mathbf{p}_b| = |\mathbf{p}_{CM}|$

$$\begin{aligned} a_0 =& \sqrt{\frac{4|\mathbf{p}_b||\mathbf{p}_a|}{2^{\delta_a}2^{\delta_b}s}}\hat{a}_0 \\ =& \frac{|\mathbf{p}_{CM}|}{\sqrt{s}}\hat{a}_0. \end{aligned} \tag{3.30}$$

At small momenta this gives us no information, but at large scattering momenta $\sqrt{s} \simeq 2|\mathbf{p}_{CM}|$ and so

$$a_0 \underset{s\to\infty}{=} -\frac{\lambda}{32\pi} + \text{loop corrections}. \tag{3.31}$$

The bound can then be interpreted as

$$|\text{Re}(a_0)| \le \frac{1}{2} \to |\lambda| \le 16\pi. \tag{3.32}$$

This is not a firm bound; really we know that if λ exceeds this value then the loop corrections must contribute to restore unitarity. Instead we can interpret it as the *breakdown of perturbativity* of the theory: above this value loop corrections are *necessary* to correct the tree-level predictions, and once we have loop effects compensating for tree-level contributions we can no longer trust the perturbation series.

- **Question:** what happens if we were to include a $-\frac{a}{6}\phi^3$ term in the Lagrangian in the limit $s \to \infty$?
- **Question:** Imagine a very similar theory but with a complex field:

  $$\mathcal{L} \supset -m|\phi|^2 - \frac{\lambda}{4}|\phi|^2$$

  Consider two different scattering processes:

(a) $\phi\phi \to \phi\phi$

(b) $\phi\overline{\phi} \to \phi\overline{\phi}$

Compute the amplitude for each process. Then derive the unitarity bounds in the $s \to \infty$ limit on λ coming from each separately.

This last example shows that the scattering splits into two different scattering $(1,1)$ submatrices (i.e. there is only one state in them).

In general we can have matrix equations to solve, for example imagine if there were two scalars that mix like

$$\mathcal{L} \supset -\frac{1}{24}\lambda_{11}\phi_1^4 - \frac{1}{24}\lambda_{22}\phi_2^4 - \frac{1}{4}\lambda_{12}\phi_1^2\phi_2^2$$

then we could consider scattering $\phi_1\phi_1, \phi_1\phi_2$ and $\phi_2\phi_2$ and naively we would have a 3×3 matrix:

$$a_0 = \begin{pmatrix} \phi_1\phi_1 \to \phi_1\phi_1 & \phi_1\phi_1 \to \phi_1\phi_2 & \phi_1\phi_1 \to \phi_2\phi_2 \\ \phi_1\phi_2 \to \phi_1\phi_1 & \phi_1\phi_2 \to \phi_1\phi_2 & \phi_1\phi_2 \to \phi_2\phi_2 \\ \phi_2\phi_2 \to \phi_1\phi_1 & \phi_2\phi_2 \to \phi_1\phi_2 & \phi_2\phi_2 \to \phi_2\phi_2 \end{pmatrix} \underset{s\to\infty}{=} -\frac{1}{16\pi}\begin{pmatrix} \frac{\lambda_{11}}{2} & 0 & \frac{\lambda_{12}}{2} \\ 0 & \lambda_{12} & 0 \\ \frac{\lambda_{12}}{2} & 0 & \frac{\lambda_{22}}{2} \end{pmatrix}$$

In fact it is a 2×2 and a 1×1 matrix because of the $\mathbb{Z}_2$ symmetry on ϕ_1, ϕ_2. The constraints from unitarity would then come from the eigenvalues of this matrix.

3.3.2 Cutoff

Imagine now a theory with a derivative coupling:

$$\mathcal{L} \supset -\frac{1}{\Lambda}\phi(\partial_\mu\phi)^2. \tag{3.33}$$

This dimension-5 coupling comes with a mass scale Λ and leads to trilinear interactions, similar to the previous example. But now the interactions are proportional to p^2. Being schematic we expect

$$\mathcal{M}^{\text{tree}} \underset{s\to\infty}{\sim} \left(\frac{p^2}{\Lambda}\right)^2 \frac{1}{p^2 - m^2} \sim \frac{s}{\Lambda^2}. \tag{3.34}$$

Naively this leads to

$$a_0 \underset{s\to\infty}{\sim} \frac{1}{32\pi}\frac{s}{\Lambda^2} \tag{3.35}$$

and we have

$$s \lesssim 16\pi\Lambda^2. \tag{3.36}$$

BUT actually this example is a bit specious because a careful computation leads to zero for the amplitude!

Instead, consider the interaction (for massless ϕ)

$$\mathcal{L} \supset \frac{c}{\Lambda^4}(\partial_\mu\phi\partial^\mu\phi)^2. \tag{3.37}$$

If we compute the scattering amplitude we get

$$\begin{aligned}
i\mathcal{M}^{\text{tree}} =& \frac{ic}{\Lambda^4}\Big[8p_1\cdot p_2 p_3\cdot p_4 + 8p_1\cdot p_3 p_2\cdot p_4 + 8p_1\cdot p_3 p_2\cdot p_4\Big] \\
=& ic\frac{s^2}{\Lambda^4}[3+\cos^2\theta]
\end{aligned} \tag{3.38}$$

Now we compute from (3.21):

$$\begin{aligned}
a_0 \underset{s\to\infty}{=}& \frac{1}{32\pi}\frac{1}{\sqrt{2^{\delta_a}2^{\delta_b}}}\int_{-1}^{1} dz\mathcal{M}_{ba} \\
=& \frac{1}{64\pi}\frac{cs^2}{\Lambda^4}\int_{-1}^{1} dz[3+z^2] \\
=& \frac{1}{64\pi}\frac{cs^2}{\Lambda^4}\frac{20}{3}
\end{aligned} \tag{3.39}$$

This leads to

$$\frac{5}{48\pi}\frac{cs^2}{\Lambda^4} \le \frac{1}{2} \tag{3.40}$$

and we have

$$s^2 \lesssim \frac{24\pi}{5}\Lambda^4/c. \tag{3.41}$$

This is different from the constraint of the previous section; we do not expect loop corrections to save us at energies beyond this cutoff value but instead we need a contribution from *additional higher-dimensional operators*: the theory as it stands is not sufficient. This is a taste of understanding effective field theories.

As a final note on this topic, we can also have higher-dimensional operators involving fermions of the form

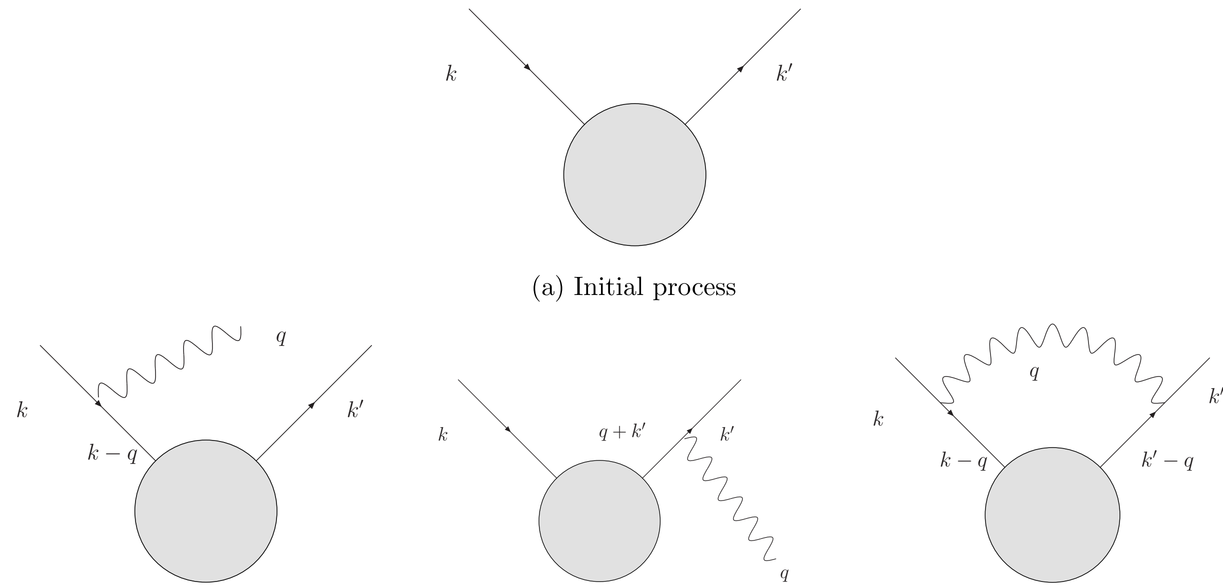
$$\mathcal{L} \supset -\frac{1}{\Lambda^2}(\overline{\Psi}\gamma^\mu P_{L/R}\Psi)(\overline{\Psi}\gamma_\mu P_{L/R}\Psi). \tag{3.42}$$

These contain no explicit derivatives, but factors of momentum will appear from the fermion wave-functions and in general it is also possible to derive a similar bound on the validity of the theory.

4 Infrared divergences in QFT

Now for a change of topic. In the build up to studying QCD, we need to discuss infra-red divergences in QFT. This section mainly follows Weinberg I chapter 13. But beware! Throughout these notes I use the metric $(+,-,-,-)$ whereas Weinberg uses $(-,+,+,+)$.

The construction of the S-matrix and the derivation of the LSZ formula all assume the existence of asymptotic states that can be arbitrarily separated at early or late times. However, even in QED this is violated: electromagnetism has a potential $V(r) \sim \frac{1}{r}$ which is non-trivial at large distances;



(a) Initial process

(b) Initial State Radiation

(c) Final State Radiation

(d) Virtual photon/gluon

Figure 4: A basic scattering process and first real and virtual corrections to it.

recall from QM scattering of a charge from a central potential that the probability of scattering is 1 (the total cross-section diverges!). Within QFT the explanation for this is that the photon is massless (so can propagate infinite distances) and it shows up as various singularities in amplitudes. However, for physical processes we still find reasonable results; the singularities cancel, as we shall see.

4.1 IR singularities for one soft photon

Consider the process in fig. 4a. We have labelled a generic process as a grey blob, and we are only considering one incoming and one outgoing leg from it; perhaps this represents deep inelastic scattering of an electron on a nucleon. In figure 4d we have the one-loop correction to this, which we can assume involves a photon for now (later it could be a gluon). Just counting the number of powers of the coupling e, when we interfere with the tree-level diagram it will contribute at order e^2 (and a contribution at order e^4 for which we also need the two-loop diagrams). On the other hand, the processes in figures 4b and 4c involve initial- and final-state radiation, and when we square them also lead to processes of order e^2. So at least perturbatively they have the same weight. But at first sight they look like they have nothing to do with each other; we have

$$\begin{aligned}\sigma(eX \to eY) =& \frac{1}{4p_{CM}\sqrt{s}} \int d\Pi_n |\mathcal{M}_{(a)} + \mathcal{M}_{(d)} + ...|^2 \\ \sigma(eX \to e\gamma Y) =& \frac{1}{4p_{CM}\sqrt{s}} \int d\Pi_{n+1} |\mathcal{M}_{(b)} + \mathcal{M}_{(c)}|^2. \end{aligned} \tag{4.1}$$

However, what if the energy of the photon is very small? Photon number is not conserved; moreover, if we have some apparatus for measuring the amount of energy arriving in photons it can only be sensitive down to some threshold energy $E_{\rm thr}$. So below a certain energy these two processes are actually indistinguishable and the total cross-section *can be added.*

Let us write the matrix element for fig. 4a as $\overline{u}_\beta(k')\mathcal{M}_{\beta\alpha}u_\alpha(k)$. Then it is actually straightforward to write down the matrix elements for the emission processes:

$$\begin{aligned}\mathcal{M}_{(b)} =& - ieQ_i\varepsilon_\mu(q)\overline{u}_\beta(k')\mathcal{M}_{\beta\alpha}\frac{i(\not{k}-\not{q}+m)}{(k-q)^2-m^2+i\epsilon}\gamma^\mu u_\alpha(k)\\ =& - ieQ_i\varepsilon_\mu(q)\overline{u}_\beta(k')\mathcal{M}_{\beta\alpha}\frac{i(\not{k}-\not{q}+m)}{-2k\cdot q+i\epsilon}\gamma^\mu u_\alpha(k)\\ \mathcal{M}_{(c)} =& - ieQ_f\varepsilon_\mu(q)\overline{u}_\beta(k')\gamma^\mu\frac{i(\not{k}'+\not{q}+m)}{2k'\cdot q+i\epsilon}\mathcal{M}_{\beta\alpha}u_\alpha(k). \end{aligned} \tag{4.2}$$

When we put these together to make a cross-section, we'll integrate over the photon momentum q; we can write[1]

$$\begin{aligned} d\Pi_{n+1} =& \int\frac{d^3q}{(2\pi^3)2E_q}\left(\prod_{f=1}^{n}\frac{d^3p_f}{(2\pi^3)2E_f}\right)\delta^4(p_{\rm tot}-q-\sum_f p_f)\\ =& \int\frac{d^3q}{(2\pi^3)2E_q}d\Pi_n(p_{\rm tot}-q). \end{aligned}\tag{4.3}$$

Now let us consider that in our experiment we can detect as an "exclusive" process the emission of a hard photon if its momentum is greater than some threshold $E_{\rm thr}$. If this is not too large then the integral over $d\Pi_n$ is roughly independent of q; and we can also neglect terms of order q in the numerator of our matrix elements. Then, using the equations of motion:

$$\begin{aligned}\overline{u}(k')\gamma^\mu(\not{k}'+m) =&\overline{u}(k')\gamma^\mu\gamma^\rho(k'_\rho+m)\\ =&\overline{u}(k')[2(k')^\mu+(-\not{k}'+m)\gamma^\mu]\\ =&\overline{u}(k')2(k')^\mu \end{aligned}\tag{4.4}$$

and similarly for $u(k)$, we have

$$\mathcal{M}_{(b)}+\mathcal{M}_{(c)} \simeq e\varepsilon_\mu(q)\overline{u}_\beta(k')\mathcal{M}_{\beta\alpha}u_\alpha(k)\left[\frac{2Q_ik^\mu}{-2k\cdot q+i\varepsilon}+\frac{2Q_f(k')^\mu}{2k'\cdot q+i\varepsilon}\right]\tag{4.5}$$

and using the naive relation for the spin sum

$$\sum_{\rm spins}\varepsilon_\mu(q)\varepsilon_\nu(q) = -\eta_{\mu\nu}\tag{4.6}$$

we have for the total matrix element

$$\sum\left|\mathcal{M}_{(b)}+\mathcal{M}_{(c)}\right|^2 \simeq - e^2\left(\sum\left|\mathcal{M}_{(a)}\right|^2\right)\left(\frac{2Q_ik^\mu}{-2k\cdot q+i\varepsilon}+\frac{2Q_f(k')^\mu}{2k'\cdot q+i\varepsilon}\right)^2.\tag{4.7}$$

[1] In fact this can be generalised/is the starting point for showing how to write phase-space integrals recursively in terms of a sequence of two-body phase space integrals etc.

The generalisation to any number of incoming and outgoing states is obvious here; we just need to compute integrals of the form

$$\int^{E_{\text{thr}}} \frac{d^3q}{(2\pi)^3 2E_q} \frac{1}{[(2k_i\cdot q)\pm i\varepsilon][(2k_j\cdot q)\pm i\varepsilon]}. \tag{4.8}$$

The signs in the denominator are negative for incoming states, and positive for outgoing ones. Simplifying the integral a little and defining $\eta_i = -1$ for incoming states, $+1$ for outgoing ones we will have

$$\int d\Pi_{n+1}\sum\left|\mathcal{M}_{(b)}+\mathcal{M}_{(c)}\right|^2 \simeq \left(\int d\Pi_n \sum\left|\mathcal{M}_{(a)}\right|^2\right)\times\left[-\frac{e^2}{(2\pi)^4}\sum_{ij}\eta_i\eta_j Q_i Q_j \mathrm{Re}J_{ij}(E_{\text{thr}})\right] \tag{4.9}$$

where (note a difference of sign w.r.t. Weinberg because of the different metric)

$$\mathrm{Re}J_{ij}(E_{\text{thr}}) \equiv \pi k_i\cdot k_j \int^{E_{\text{thr}}} \frac{d^3q}{|\mathbf{q}|^3}\frac{1}{[E_i-\hat{\mathbf{q}}\cdot\mathbf{k}_i][E_j-\hat{\mathbf{q}}\cdot\mathbf{k}_j]}. \tag{4.10}$$

Here three-momenta are in bold, and the hatted three-momenta are of unit length. This is clearly very badly behaved as $|\mathbf{q}|\to 0$; but we also see *collinear singularities* when $\mathbf{q}$ is parallel to $\mathbf{k}_{i,j}$ where the denominators become large ... but do not technically diverge unless the mass of the external states is zero. Those will be important later, but since they are finite they do not require a regulator.

There does remain the issue of what to do with the *lower* bound on the integral. There are two typical ways of dealing with this, and the choice depends partly on taste and partly on application. We can either introduce a ficitious *photon mass*, or we can use *dimensional regularisation*. For the latter, you will be (very) familiar with it for *virtual* processes, but we can also regularise the whole of spacetime in this way. The phase space integrals in $d = 4-2\epsilon$ dimensions look very similar; we just send $\frac{d^3p}{(2\pi)^3}\to\frac{d^{d-1}p}{(2\pi)^{d-1}}$. This is very practical and is most widely used in QCD calculations (and makes sure that we do not break Lorentz/gauge symmetries). However, here it will be more transparent for now to use a photon mass/IR cutoff on the integral, since we have already introduced a UV cutoff and want to see what happens to that.

Hence, introducing Feynman parameters, we can write

$$\begin{aligned}\mathrm{Re}J_{ij}(E_{\text{thr}}) =& \pi k_i\cdot k_j\int_\mu^{E_{\text{thr}}}\frac{d|\mathbf{q}|}{|\mathbf{q}|}\int d\Omega\int_0^1 dx\frac{1}{[x(E_i-\hat{\mathbf{q}}\cdot\mathbf{k}_i)+(1-x)(E_j-\hat{\mathbf{q}}\cdot\mathbf{k}_j)]^2}\\ =& 4\pi^2 k_i\cdot k_j\log\frac{E_{\text{thr}}}{\mu}\int_0^1 dx\frac{1}{(xk_i+(1-x)k_j)^2}.\end{aligned} \tag{4.11}$$

The calculation is somewhat tedious (Weinberg describes it as elementary) but I give the intermediate steps in appendix A.3. The result, after defining

$$\beta_{ij} \equiv \sqrt{1-\frac{m_i^2 m_j^2}{(k_i\cdot k_j)^2}} \tag{4.12}$$

is

$$\mathrm{Re}J_{ij}(E_{\mathrm{thr}}) = \frac{2\pi^2}{\beta_{ij}} \log \frac{1+\beta_{ij}}{1-\beta_{ij}} \log \frac{E_{\mathrm{thr}}}{\mu}. \tag{4.13}$$

The logarithmic divergence in μ is an apparent pathology; we have to find how it should be cancelled. Of course, we already gave the game away that the solution is to include the *virtual* photons. The matrix element for that process is

$$\mathcal{M}_{(d)} = ie^2 Q_i Q_f \int \frac{d^d q}{(2\pi)^4} \frac{1}{q^2 - i\varepsilon} \overline{u}_\beta(k') \gamma^\mu \frac{i(\not{k} - \not{q} + m)}{(k-q)^2 - m^2 + i\epsilon} \mathcal{M} \frac{i(\not{k}' - \not{q} + m)}{(k'-q)^2 - m^2 + i\epsilon} \gamma_\mu u_\alpha(k) \tag{4.14}$$

and again ignoring the terms in the numerators of $\not{q}$, and ignoring the dependence of $\mathcal{M}$ on q^2:

$$\mathcal{M}_{(d)} \simeq ie^2 Q_i Q_f (k \cdot k') \mathcal{M}_{(a)} \int \frac{d^d q}{(2\pi)^4} \frac{1}{q^2} \frac{1}{(-2k\cdot q)(-2k' \cdot q)}. \tag{4.15}$$

Now, however, we need to include a cutoff on the virtual photon momenta; let us call it Λ. Let us suggestively define

$$\begin{aligned} J_{ij}(\Lambda) \equiv & -i(k \cdot k') \int^{|q|<\Lambda} d^d q \frac{1}{q^2} \frac{1}{(k\cdot q - i\eta_i \varepsilon)(-k' \cdot q - i\eta_j \varepsilon)} \\ = & -i \frac{(k\cdot k')}{E_i E_j} \int^{|q|<\Lambda} d^d q \frac{1}{q^2} \frac{1}{(q^0 - \mathbf{v}\cdot \mathbf{q} - i\eta_i\varepsilon)(-q^0 + \mathbf{v}' \cdot \mathbf{q} - i\eta_j \varepsilon)} \end{aligned} \tag{4.16}$$

where η_i is $+1$ for *final* state particles and -1 for *initial* state particles, and $\mathbf{v}, \mathbf{v}'$ are the velocity vectors corresponding to $\mathbf{k}, \mathbf{k}'$. Now to recover integrals of the same form as the Bremsstrahlung ones we integrate over q^0 using Cauchy's theorem, closing the contour either in the upper or lower half plane. From the pole at $q^2 - i\varepsilon = (q^0 - |\mathbf{q}|)(q^0 + |\mathbf{q}|) - i\varepsilon$ so $q^0 = \pm(|\mathbf{q}| - i\varepsilon)$ we obtain

$$J_{ij}(\Lambda) = 2\pi (k\cdot k') \int^{|q|<\Lambda} d^{d-1} q \frac{1}{2|\mathbf{q}|^3} \frac{1}{(E_i - \hat{\mathbf{q}} \cdot \mathbf{k})(E_j - \hat{\mathbf{q}}\cdot \mathbf{k}')} + \text{other pole(s)}. \tag{4.17}$$

The other poles at $q^0 = \mathbf{v}\cdot\mathbf{q} + i\eta_i\varepsilon$ and $q^0 = \mathbf{v}'\cdot\mathbf{q} - i\eta_j\varepsilon$ can either be removed entirely (if $\eta_i = -\eta_j$) or lead to an additional piece, which only yields an irrelevant phase.

Then we can write

$$\mathcal{M}_{(d)} = \frac{e^2}{(2\pi)^4} \frac{1}{2} \sum_{ij} \eta_i \eta_j Q_i Q_j \mathcal{M}_{(a)} J_{ij}(\Lambda), \tag{4.18}$$

where we now sum over all possible states i, j and have introduced a factor $1/2$ to correctly count the virtual corrections (there is only one virtual diagram for two real emission ones). Then

$$\int d\Pi_{n+1} \sum |\mathcal{M}_{(a)} + \mathcal{M}_{(d)}|^2 - |\mathcal{M}_{(a)}|^2 = \left(\int d\Pi_n \sum |\mathcal{M}_{(a)}|^2 \right) \times \mathrm{Re}\left[\frac{e^2}{(2\pi)^4} \sum_{ij} (\eta_i \eta_j Q_i Q_j J_{ij}(\Lambda)) \right]. \tag{4.19}$$

Since the piece proportional to the photon mass μ is the same in both cases, let us define

$$A_{i\to f} \equiv -\frac{e^2}{8\pi^2}\sum_{ij}\eta_i\eta_j Q_i Q_j \frac{1}{\beta_{ij}}\log\frac{1+\beta_{ij}}{1-\beta_{ij}} \longrightarrow \frac{e^2}{(2\pi)^4}\sum_{ij}\mathrm{Re}\left(\eta_i\eta_j Q_i Q_j J_{ij}(\Lambda)\right) \equiv \quad -A_{i\to f}\log\frac{\Lambda}{\mu} \tag{4.20}$$

we find

$$\sigma(i\to f+\text{up to one soft photon with } E<E_{\text{thr}}) = \sigma_{(a)}\times\left[1+A_{i\to f}\log\frac{E_{\text{thr}}}{\Lambda}\right]. \tag{4.21}$$

There are two things to note here:

1. The initial/final states could have included *hard* photons and not affected the discussion: we could have equally written the RHS as
$$\sigma(i\to f+\text{only hard photons with } E>E_{\text{thr}})\times\left[1+A_{i\to f}\log\frac{E_{\text{thr}}}{\Lambda}\right]$$
2. The second factor in the square brackets *can be negative*: it is possible that taking soft photons into account means that the cross-section with only soft photons is (much) less than the original tree-level cross-section.

But there remains the factor dependent on the ratio of the threshold to the cutoff. What is the cutoff? And what would have happened if we used dimensional regularisation?

The answer is that the cutoff has to relate to the other scales in the calculation, especially the total momentum transfer in the process q^2. In dimensional regularisation we would find a μ_{UV} appearing not as the photon mass but as the RG scale in the problem. If we expanded our grey blobs we would find propagators and vertices with other scales in them, and the masses of the external particles.

4.2 Generalisation to many photons

On the other hand, a very important effect does remain of all this. Consider that instead of just one particle, we consider emission of N soft photons, and therefore at the same time include up to N loops. Since we are only considering the IR, we can iterate our above argument treating each loop as independent; then each successive loop contributes the same factor, and we divide by the number of ways of permuting the ends of the loop (2^N) and the loops themselves ($N!$). Thus

$$\begin{aligned}\mathcal{M}^{(N)}_{\text{virtual}} =& \mathcal{M}_{(a)}\times\frac{1}{2^N N!}\left(\frac{e^2}{(2\pi)^4}\sum_{ij}\eta_i\eta_j Q_i Q_j J_{ij}(\Lambda)\right)^N \\ \to \mathcal{M}_{\text{virtual}} =& \mathcal{M}_{(a)}\exp\left[\frac{1}{2}\frac{e^2}{(2\pi)^4}\sum_{ij}\eta_i\eta_j Q_i Q_j J_{ij}(\Lambda)\right].\end{aligned} \tag{4.22}$$

Now for the real emissions; here it is a little more complicated because we cannot integrate over arbitrary photon emission energies: if we were to sum N diagrams then eventually $N\Lambda$ would exceed the total energy of the process! Instead, however, we observe that we were always splitting our integrals into an integral over $|\mathbf{q}|$ and an integral over $d\Omega$. So if we iterate our phase space integrals then we end up with

$$\int \sum |\mathcal{M}^{(N)}_{\rm real}|^2 d\Pi_N = \sigma_{(a)} \times \frac{1}{N!} A^N \int_{E_{\rm thr} > E_r > \mu, \sum_r E_r < E} \prod_{r=1}^N \frac{dE_r}{E_r}. \tag{4.23}$$

Here we just require that the sum of all photons is less than some total E; this can be replaced by a step function[2]

$$\theta(E - \sum E_r) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin Eu}{u} \exp\left(iu \sum E_r \right) \tag{4.24}$$

which replaces $\int d|\mathbf{q}| \rightarrow \int d|\mathbf{q}| e^{-iu|\mathbf{q}|}$ in each term, to give

$$\sum_N \int \sum |\mathcal{M}^{(N)}_{\rm real}|^2 d\Pi_N = \sigma_{(a)} \frac{1}{\pi} \int du \frac{\sin Eu}{u} \exp\left(A \int_\mu^E \frac{dE}{E} e^{iEu} \right). \tag{4.25}$$

In the end, the IR regulator μ drops out (as it should) and, taking $E, E_{\rm thr}$ of the same order and much less than Λ we arrive at

$$\sigma(i \rightarrow f + \text{soft photons with total energy} < E) \simeq \sigma_{(a)} \exp\left[A_{i\rightarrow f} \log \frac{E_{\rm thr}}{\Lambda} \right]. \tag{4.26}$$

Let us unpack this. First, we note that the factor $A_{i\rightarrow f}$ is always positive when connecting two particles of the same type: the logarithm is positive because $\beta_{ij} \leq 1$, and for two incoming or two outgoing fields we need $\eta_i = \eta_j, Q_j = -Q_i$; whereas for one incoming and one outgoing we have $\eta_i = -\eta_j, Q_i = Q_j$. Then, since $E_{\rm thr} < \Lambda$ the exponent is negative. If $A_{i\rightarrow f}$ becomes large or $\Lambda \gg E_{\rm thr}$, this means that the probability of scattering only into the original hard states and soft photons vanishes ...

... The conclusion is that, as the energy transfer increases, the probability of scattering *without emitting a hard photon* dwindles to zero. This is the QFT derivation of *Bremsstrahlung* radiation (and Peskin and Schroeder present a heuristic derivation of it as such). In such a regime, we need a description for computing scattering cross-sections that goes beyond computing Feynman diagrams for $2 \rightarrow 2$ processes. Fortunately, for QED the gauge coupling is so small that this issue is mostly a technical one; but when we discuss QCD we need to take this very seriously.

We can also understand the exponents a little better if we examine the factors β_{ij} at large momentum transfer, so when

$$\begin{aligned} Q^2 \equiv -q^2 =& -(k_i - k_f)^2 \gg m_{i,f}^2 \\ k_i \cdot k_f \simeq& Q^2/2. \end{aligned} \tag{4.27}$$

we have

$$\beta_{if} \simeq 1 - \frac{1}{2} \frac{m_i^2 m_f^2}{Q^4/4} \tag{4.28}$$

[2]The identity here just follows from using $\frac{e^{iEu}}{u+i\varepsilon}$ in the integrand and Cauchy's theorem.

and

$$\sigma \simeq \sigma_{(a)} \exp\left[-\frac{e^2}{8\pi^2} \sum_{ij} \eta_i \eta_j Q_i Q_j \log \frac{Q^4}{m_i^2 m_f^2} \log \frac{E_{\text{thr}}}{\Lambda} \right]. \tag{4.29}$$

The appearance of the double logarithm in the exponent is called a *Sudakov double logarithm.* For example, to compare with Peskin and Schroeder, section 6.5 we must not forget to sum over both J_{ij} and J_{ji}, so for one incoming and one outgoing electron we have

$$\begin{aligned}\sigma(eX \to eY + \text{soft photons with total energy} < E) \simeq & \sigma_{(a)} \exp\left[\frac{2e^2}{8\pi^2} \log \frac{Q^4}{m_i^2 m_f^2} \log \frac{E_{\text{thr}}}{\Lambda} \right] \\ = & \sigma_{(a)} \exp\left[\frac{\alpha}{\pi} \log \frac{Q^2}{m^2} \log \frac{E_{\text{thr}}^2}{\Lambda^2} \right] \\ = & \sigma_{(a)} \left| \exp\left[\frac{\alpha}{2\pi} \log \frac{Q^2}{m^2} \log \frac{E_{\text{thr}}^2}{\Lambda^2} \right] \right|^2 . \end{aligned} \tag{4.30}$$

To understand the appearance of the double logarithm, let us take another look at equation (4.10). There is clearly a logarithmic divergence coming from the $\frac{1}{|\mathbf{q}|^3}$ part of the integral. However, we can get an additional divergence from the integral over $d\Omega$ which will multiply this, from the regions where $\hat{\mathbf{q}} \propto \mathbf{k}_i, \hat{\mathbf{q}} \propto \mathbf{k}_j$ – i.e. when they are collinear; if the energy of the incoming/outgoing states is much bigger than their masses (which is usually the case) then we can expand $E_i \simeq |\mathbf{k}_i| + \frac{m^2}{2|\mathbf{k}|} + ...$:

$$\text{Re} J_{ij}(E_{\text{thr}}) \propto \int^{E_{\text{thr}}} \frac{d|\mathbf{q}|}{|\mathbf{q}|} \times \int d\Omega \frac{1}{[E_i - \hat{\mathbf{q}} \cdot \mathbf{k}_i][E_j - \hat{\mathbf{q}} \cdot \mathbf{k}_j]} \tag{4.31}$$

and examining the angular integrals we can split it into two regions around $\cos\theta_i$ being the angle of $\mathbf{k}_i$ and $\cos\theta_j$ being the angle of $\mathbf{k}_j$:

$$\int d\cos\theta \frac{1}{[E_i - \hat{\mathbf{q}} \cdot \mathbf{k}_i][E_j - \hat{\mathbf{q}} \cdot \mathbf{k}_j]} \simeq \int d\cos\theta \frac{1}{E_i - \hat{\mathbf{q}} \cdot \mathbf{k}_i} \frac{1}{E_j - \mathbf{k}_i \cdot \mathbf{k}_j/|\mathbf{k}_i|} + \frac{1}{E_i - \mathbf{k}_i \cdot \mathbf{k}_j/|\mathbf{k}_j|} \frac{1}{E_j - \hat{\mathbf{q}} \cdot \mathbf{k}_j}. \tag{4.32}$$

Each of these gives a singularity; just looking at the one around $\mathbf{k}_i$:

$$\begin{aligned}\int d\cos\theta \frac{1}{E_i - \hat{\mathbf{q}} \cdot \mathbf{k}_i} \sim & \frac{1}{|\mathbf{k}_i|} \int d\cos\theta \frac{1}{1 - \cos\theta + \frac{m^2}{2|\mathbf{k}|^2}} \\ \sim & \frac{1}{|\mathbf{k}_i|} \log \frac{m^2}{2|\mathbf{k}|^2}. \end{aligned} \tag{4.33}$$

Since there are two regions each contributes a similar logarithm, but each multiplies the $\log \frac{E_{\text{thr}}}{\mu}$ which becomes $\log \frac{E_{\text{thr}}}{\Lambda}$. Intuitively then the fact that we have a double logarithm is because the photons can be *both* soft *and* collinear.

4.3 Summary and inclusive processes

For QED (and any interacting theory with massless mediators), there are infra-red divergences that arise because the interactions do not really disappear at infinity so the states at infinity are not really well defined: they are still interacting via arbitrarily soft photons. But we showed that, once we include these soft photons *in the final state*, the IR divergences cancel. However, there also exist colinear divergences which, in QED, are cut off by the electron mass; in a theory of massless electrons – or when we can treat the charge particles as massless, such as in QCD – then final states are not enough. To cancel these *general* IR divergences, as proved by Lee and Nauenberg, *we need to sum over all indistinguishable* **initial and final** *states*: this is necessary in QCD.
We also showed that in theories where the energy transfer and/or the coupling is large then the probability of producing a final state with only soft photons – so without *hard* photons – is essentially zero. So what if we include *hard* radiation in our definition of the final state? In other words, we ask a different question: what is the cross-section for $i \to f + \text{any radiation}$? This would be equivalent (roughly) to taking $E_{\text{thr}} \simeq \Lambda$. We would then expect our double logarithm to cancel out, and instead can safely compute perturbative cross-sections. In other words,

$$\sigma(eX \to eY + \text{only soft photons with total energy} < E) \to 0$$

$$\sigma(eX \to eY + \text{any radiation}) \approx \sigma(eX \to eY) + \mathcal{O}(\alpha).$$

5 QCD

Now we will discuss everything that you need to know about QCD from a phenomenological perspective. In QFT-II you will have (just) learnt that the strong interactions are described by an $SU(3)$ gauge theory. The quarks are in the fundamental representation and gluons, which mediate the interaction, are in the adjoint representation. The gauge coupling g_3 is usually quoted in terms of $\alpha_s \equiv \frac{g_3^2}{4\pi}$ in analogy to the fine-structure constant. In QFT the couplings are not constant but depend upon the energy scale (inverse to the distance scale), and for the strong coupling at low energies (long distances) this becomes ... strong. This happens around about 100 MeV – 1 GeV depending on the application; below this scale perturbation is no longer useful. On the other hand, the coupling becomes *weaker* at *higher energies* (shorter distances), meaning that for high-energy interactions we can treat QCD as a perturbative theory and compute interactions using Feynman diagrams.

5.1 Mesons and baryons

The quark model coupled via QCD was settled on as the correct theory partly because of its explanation of the pattern of properties and masses of mesons and baryons. Baryons are fermions and the total number of baryons in experiments is always observed to be conserved, so Baryon number B is a useful quantum number – although we know that it must be violated in nature. However, spin 3/2 baryons were observed and it is not possible to write down a local renormalisable QFT describing fundamental 3/2 (or higher spin) particles with non-trivial couplings (known as the Velo-Zwanziger problem, see e.g. [4–6]). This points already to an underlying theory of spin 1/2 quarks where baryons are made of an odd number, so three is the most conservative choice. On the other hand, mesons are bosonic and must therefore be built from a pair of quarks. If we start from the idea that the quarks are light and are treated the same by the strong interactions, then we can explain why there are baryons/mesons with almost identical masses but different charges.

Settling on the charges that you now know – three families of up-type quarks with charge 2/3 and three families of down-type quarks with charge $-1/3$, where the up/down quarks are very light, and the strange is heavier but still lighter than the QCD scale, then there is a quite good $SU(2)$ symmetry respected by the strong interactions from rotating the up/down quarks into each other, and an approximate $SU(3)$ symmetry when we include the strange quark. This explains the pattern of properties of the mesons and baryons that we observe.

But what about QCD itself? If we endow the quarks with a new quantum number or charge under the new force which must be the same for all quarks, then for quark-antiquark pairs this is fine, but how do we explain the existence of bound states of three quarks, which must be neutral under this new force? Since you now know QCD is a gauge theory, let us write the quarks with and index $i = 1...N_c$ where N_c is the number of colours. Then mesons are like

$$\pi^0 = \frac{1}{\sqrt{2}}(\overline{u}^i u_i - \overline{d}^i d_i), \qquad \pi^+ = \overline{d}^i u_i.$$

But for baryons? Especially, since we know that there exists the particle $\Delta^{++} \propto u_i u_j u_k$, we know that the Pauli exclusion principle tells us this must be perfectly antisymmetric on the colour indices. This essentially fixes us with $SU(3)$, because there exists the completely antisymmetric tensor ϵ^{ijk} which *is invariant under $SU(3)$ rotations*:

$$\epsilon^{ijk} \to U^i_{i'} U^j_{j'} U^k_{k'} \epsilon^{i'j'k'} = (\det U)\epsilon^{ijk} = \epsilon^{ijk}. \tag{5.1}$$

So our baryons are antisymmetric combinations of quarks such as $\epsilon^{ijk} u_i u_j u_k$ for Δ^{++} and $\epsilon^{ijk} u_i u_j d_k$ for the proton.

Nonetheless, there are other ways of checking that $SU(3)$ is indeed the theory of the strong interactions and plenty of consequences.

5.2 R-ratio

One of the measurements that convinced people that QCD was the correct theory of the strong interactions was the measurement in lepton colliders of the ratio

$$R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}. \tag{5.2}$$

In QED at energies $s \gg m_\mu$ the production of muons takes place via an s-channel photon:

$$\sigma(e^+e^- \to \mu^+\mu^-) = \frac{4\pi\alpha^2}{3s}. \tag{5.3}$$

In QCD, if we examine this at scales above the mass of the proton, then at leading order we can treat quarks as free particles when calculating the cross-section (and worry later about what happens subsequently to them). So then the production of all hadrons should be dominated by the same diagram for quark-antiquark pairs, but the coupling is now modified to their charge Q_i. Therefore

$$\sigma(e^+e^- \to q_i\overline{q}_i) = \frac{4\pi\alpha^2}{3s} \times Q_i^2 \times N_C. \tag{5.4}$$

Here we have a factor N_C of the number of colours, because the photon is colourless and so the vertex has to be a delta-function on the colour indices. So we should find

$$R \simeq N_C \times \sum_i Q_i^2. \tag{5.5}$$

If we look at energies above (twice) the mass of the b-quark (charge $-1/3$) then

$$\begin{aligned} R \simeq& N_C \times \left[3 \times \left(\frac{-1}{3}\right)^2 + 2 \times \left(\frac{2}{3}\right)^2\right] \\ =& \frac{11}{3}. \end{aligned} \tag{5.6}$$

Below (twice) the mass of the b-quark we should have

$$\begin{aligned} R \simeq& N_C \times \left[2 \times \left(\frac{-1}{3}\right)^2 + 2 \times \left(\frac{2}{3}\right)^2\right] \\ =& \frac{10}{3}. \end{aligned} \tag{5.7}$$

This is exactly what can be seen from figure 5.

Let us now return to the comments I made at the end of section 4. We know that the coupling of the strong interactions is strong, so how can we make sense of the above cross-section, will there not be a Sudakov double logarithm? The answer to this is no, if we look for

$$\sigma(e^+e^- \to \text{hadrons}) = \sigma(e^+e^- \to q_i\overline{q}_i) + \sigma(e^+e^- \to q_i\overline{q}_i g) + ... \tag{5.8}$$

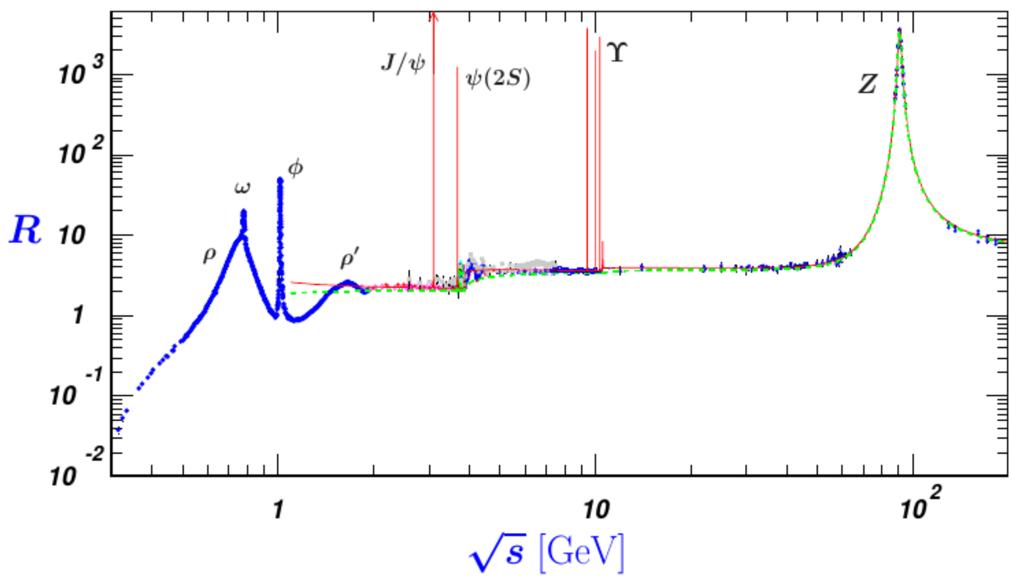


Figure 5: R-ratio from low energies up to the Z resonance. Taken from the RPP under "cross-sections and plots". The green line is naive quark model prediction given in the text; the solid red one is the three-loop perturbative QCD prediction.

and include all *hard* emissions too. In fact this calculation is very useful. We show the lowest order diagrams in figure 6. If we ignore the quark masses, then the collinear divergences can be regulated either through dimensional regularisation or through a gluon mass (instead of the electron/quark mass like we used before). Since we used a photon mass earlier, let us give the results using a gluon mass[3]:

$$\begin{aligned}\sigma_{\text{Real emission}}(e^+e^- \to q_i\overline{q}_i g) &= \frac{g_s^2}{8\pi^2}C_F\sigma_{\text{Tree}} \times \left[\log^2\frac{m_g^2}{Q^2} + 3\log\frac{m_g^2}{Q^2} + 5 - \frac{\pi^2}{3}\right],\\ \sigma_{\text{Virtual}}(e^+e^- \to q_i\overline{q}_i) &= \frac{g_s^2}{8\pi^2}C_F\sigma_{\text{Tree}} \times \left[-\log^2\frac{m_g^2}{Q^2} - 3\log\frac{m_g^2}{Q^2} - \frac{7}{2} + \frac{\pi^2}{3}\right].\end{aligned} \tag{5.9}$$

Here

$$\begin{aligned}(T^aT^a)_i^j \equiv& C_F\delta_i^j,\\ C_F =& \frac{N_c^2-1}{2N_c} \underset{N_c=3}{\rightarrow} \frac{4}{3},\end{aligned} \tag{5.10}$$

is the group-theory factor associated to the quark couplings. Note that the virtual diagram contains no UV-cutoff because the result given is the *renormalised* one; instead the factors in the logarithms involving Q^2 remain, as I stated in section 4. The total result is then

$$R = \left(N_C \times \sum_i Q_i^2\right) \times \left[1 + \frac{3\alpha_s C_F}{4\pi} + ...\right] \tag{5.11}$$

[3]They are rather long to derive; you can find them for $e^+e^- \to \mu^+\mu^-(+\gamma)$ in Schwarz, section 20.1.

where $\alpha_s \equiv \frac{g_s^2}{4\pi}$ is the equivalent of the fine structure constant for the strong coupling. This is a very well-behaved perturbation series, because α_s at high energies is not too large. In fact, the R-ratio is so accurately measured that we can even use it to extract α_s (from the PDG):

$$\alpha_s(\mathrm{R}-\mathrm{ratio}) = 0.1171 \pm 0.0031 \tag{5.12}$$

compared to the overall average value:

$$\alpha_s(\text{pdg average}) = 0.1179 \pm 0.0009. \tag{5.13}$$

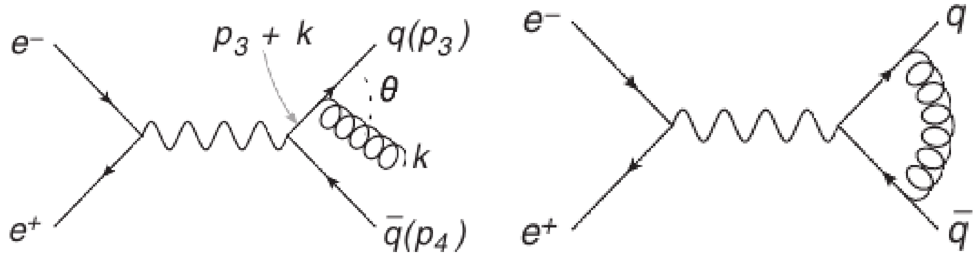


Figure 6: Lowest order diagrams correcting the prediction for the R-ratio.

In the above, the Sudakov double logarithms cancelled exactly and we had a simple final form. However, while it should be true that by including all possible final states we should not cause the overall cross-section to diverge, there are still cases where Sudakov logarithms remain in the cross-sections when we include several mass scales: then we expect an incomplete cancellation between (double) logorithms of ratios of the different masses, and they can lead to important corrections.

5.3 Parton showers

5.3.1 Hard colinear emission

Now that we are interested in computing QCD processes, where we know that the coupling becomes strong at low energies/Q^2, we would like to know more than the total cross-sections. Suppose we are interested in studying the hadronic stuff that comes out of e^+e^- collisions, or pp collisions. We need to know the differential distributions and we need to be able to identify the objects that come out. We know that we can calculate the 'hard' process using perturbation theory, and we know that in the end we have to end up with hadrons that are formed in some non-perturbative process, but what about in between?

The analysis from section 4 tells us that, once strongly coupled states are produced, they will begin to radiate because they feel the effect of the other EM/colour charges. This radiation will be dominated by soft (indistinguishable) and ***colinear*** emission. Now soft radiation carries off relatively little energy by definition, but colinear emission could be very significant, especially if it is *hard*.

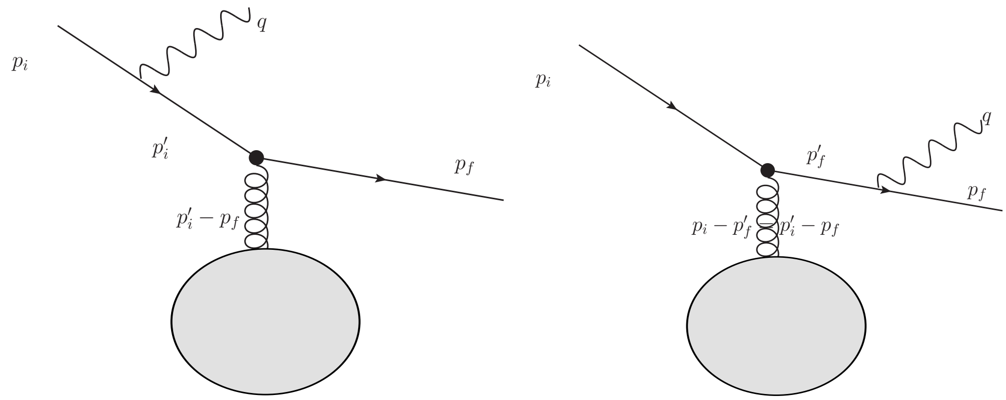


Figure 7: Diagrams showing photon emission from an incoming and outgoing electron relevant for the splitting function derivation

Imagine that a quark emits a gluon that is nearly colinear but with a large fraction of its momentum: the gluon will not hadronise straight away but may emit quarks or more gluons as it continues to radiate before it can bind together with co-travelling particles to form a colourless state.

Let us go back to electrons and photons for a minute (the generalisation to quarks and gluons is straightforward). In our previous analysis of electrons and photons we considered that q of the photon was small; now let us instead consider that its momentum *perpendicular to the electron* $k_\perp$ is small. Let us just consider the emission of one photon to start with. We can also reuse some of the analysis from section 4, but now we are interested in the case when the photon can carry a significant fraction of the electron momentum.

To be concrete, let us consider very similar diagrams in section 4 for real emission, so we consider incoming and outgoing states that can both emit a photon. Now, however, we will relabel the initial and final momenta as p_i, p_f and we neglect the electron masses, but keep all the factors of q in the numerator. We show the new processes in figure 7. We are also considering now an electron *pair* that are either produced at one vertex or the scattering of one electron against a bunch of other stuff (why will be apparent soon). On shell we have $p_i^2 = p_f^2 = q^2 = 0$. In both diagrams there is a virtual electron with momenta that we will label p_i', p_f', so

$$\begin{aligned} p_i' \equiv& p_i - q \\ p_f' \equiv& p_f + q. \end{aligned} \tag{5.14}$$

These lead to the on-shell conditions

$$\begin{aligned} (p_i' - p_i)^2 =& q^2 = 0 = (p_i')^2 - 2 p_i \cdot p_i' \\ (p_f')^2 =& 2 p_f \cdot p_f'. \end{aligned} \tag{5.15}$$

The quantities $(p_i')^2, (p_f')^2$ are non-zero and Lorentz-invariant; we can use them as a measure of "virtuality." Now to be concrete we want the photon to be nearly colinear with the *incoming* electron and carry some fraction z of its momentum (we should consider each external leg in turn). So we have $q = zp_i + k_\perp + ...$ where we need to add something so that $q^2 = 0$ without $k_\perp^2 = 0$. So aligning our choice of frame with p_i we have

$$\begin{aligned} p_i =&(p, 0, 0, p) \\ q =&(zp, \mathbf{k}_\perp, zp + \delta) \\ p_i' =&((1-z)p, -\mathbf{k}_\perp, (1-z)p - \delta). \end{aligned} \tag{5.16}$$

Since we are taking our emitted photon on-shell, we can put

$$\begin{aligned} 0 = q^2 =& - k_\perp^2 - 2\delta z p + \mathcal{O}(\delta^2) \\ \longrightarrow \delta =& - \frac{k_\perp^2}{2zp} + \mathcal{O}(k_\perp^2). \end{aligned} \tag{5.17}$$

This means we can also write

$$\begin{aligned} (p_i')^2 =& - k_\perp^2 + 2(1-z)\delta p \\ =& - \frac{k_\perp^2}{z}, \\ p_i \cdot p_i' =& \frac{1}{2}(p_i')^2 = - \frac{k_\perp^2}{2z}. \end{aligned} \tag{5.18}$$

These later definitions are Lorentz invariant so we can compute some amplitudes; the relevant matrix elements are essentially the same as before:

$$\begin{aligned} \mathcal{M}_{(b)} =& - ieQ_i \varepsilon_\mu(q) \overline{u}_\beta(p_f) \mathcal{M}_{\beta\alpha} \frac{i(\not{p}_i')}{(p_i')^2 + i\epsilon} \gamma^\mu u_\alpha(p_i) \\ \mathcal{M}_{(c)} =& - ieQ_f \varepsilon_\mu(q) \overline{u}_\beta(p_f) \gamma^\mu \frac{i(\not{p}_f')}{(p_f')^2 + i\epsilon} \mathcal{M}_{\beta\alpha} u_\alpha(p_i). \end{aligned} \tag{5.19}$$

Now consider what happens when q is nearly colinear to the initial-state electron and carries a fraction z of its momentum away, so $q \sim zp_i + k_\perp + \mathcal{O}(k_\perp^2)$. We see that $(p_i')^2$ is of order $k_\perp^2$ so naively when we square the matrix elements $|\mathcal{M}_{(b)}|^2$ will be the most important. However, the numerator for that case also vanishes as $k_\perp^2$ so we must also consider the cross-terms! This is why I included them: we know from gauge invariance that they must be there, and it turns out that they are *just as important as the diagonal term*! On the other hand, we can indeed safely ignore $\mathcal{M}_{(c)}|^2$ in this limit.

The most efficient way to do this calculation is to use explicit expressions for the electron helicities and insert them in the amplitudes. That way, we can sum the total amplitudes and then square them, rather than having to do a spin sum for each product of $|\mathcal{M}_{(b)}|^2, \mathcal{M}_{(b)}\mathcal{M}_{(c)}^\dagger$ etc. This is the approach used in Peskin and Schroeder section 17.5, but it is not clear why they do it and it is not transparent (plus there is a detour discussing the *equivalent photon approximation* that we do not want to go into here). So instead I present the full traditional calculation of the spin-summed squared matrix element.

We have three spin sums to compute. For reference, let us define the no-emission amplitude as

$$\sum |\mathcal{M}_{(a)}(p_i,p_f)|^2 \equiv \sum \mathrm{tr}\Big[\not{p}_f\mathcal{M}(p_i,p_f)\not{p}_i\mathcal{M}^\dagger(p_i,p_f)\Big]. \tag{5.20}$$

Note here that I specify the momenta involved explicitly, because in our real emission amplitudes the matrix elements will be functions of (p_i',p_f) and (p_i,p_f'). In general $\mathcal{M}(p_i,p_f)\neq\mathcal{M}(p_i',p_f)$. *However* there is a case when $\mathcal{M}(p_i',p_f)=\mathcal{M}(p_i,p_f')$: that is when the incoming/outcoming electron interact with $\mathcal{M}$ at one vertex, which is precisely the case that we are interested in! With this proviso, the form of $\mathcal{M}$ could be $\propto\gamma^\mu$ for exchange of a photon (or gluon in QCD) or it could be a scalar coupling for Higgs exchange: the actual form does not matter, and that is why I keep it general to not clutter the expressions.

We replace everywhere $\varepsilon_\mu\varepsilon_\nu\to-\eta_{\mu\nu}$:

$$\sum |\mathcal{M}_{(b)}(p_i',p_f)|^2 = -\frac{e^2Q_i^2}{(p_i')^4}\sum \mathrm{tr}\Big[\not{p}_f\mathcal{M}\not{p}_i'\gamma^\mu\not{p}_i\gamma_\mu\not{p}_i'\mathcal{M}^\dagger\Big], \tag{5.21}$$

$$\sum \mathcal{M}_{(b)}(p_i',p_f)\mathcal{M}^\dagger_{(c)}(p_i,p_f') = -\frac{e^2Q_iQ_f}{(p_i')^2(p_f')^2}\sum \mathrm{tr}\Big[\not{p}_f\mathcal{M}\not{p}_i'\gamma^\mu\not{p}_i\mathcal{M}^\dagger\not{p}_f'\gamma_\mu\Big], \tag{5.22}$$

$$\sum \mathcal{M}_{(c)}(p_i,p_f')\mathcal{M}^\dagger_{(b)}(p_i',p_f) = -\frac{e^2Q_iQ_f}{(p_i')^2(p_f')^2}\sum \mathrm{tr}\Big[\not{p}_f\gamma^\mu\not{p}_f'\mathcal{M}\not{p}_i\gamma_\mu\not{p}_i'\mathcal{M}^\dagger\Big]. \tag{5.23}$$

As promised, the first term looks naively the most important as $k_\perp^2\to0$ but this is not the case.

The full calculation is messy and involves a fair amount of gamma algebra, so I relegate it to appendix A.4. The result for the diagonal term is

$$\begin{aligned}\sum |\mathcal{M}_{(b)}|^2 =& 2\frac{e^2Q_i^2}{(p_i')^4}\sum \mathrm{tr}\Big[\not{p}_f\mathcal{M}\not{p}_i'\not{p}_i\not{p}_i'\mathcal{M}^\dagger\Big]\\ &\longrightarrow -\frac{2z}{1-z}\frac{e^2Q_i^2}{(p_i')^2}\sum|\mathcal{M}_{(a)}(p_i',p_f)|^2.\end{aligned} \tag{5.24}$$

For the other two we can directly make the replacement in the numerator of $p_f'\to p_f+zp_i$, $p_i'\to(1-z)p_i$ to get

$$\begin{aligned}\sum \mathcal{M}_{(b)}\mathcal{M}^\dagger_{(c)}+\mathcal{M}_{(c)}\mathcal{M}^\dagger_{(b)} =& -(1-z)\frac{e^2Q_iQ_f}{(p_i')^2(p_f')^2}\sum\mathrm{tr}\Big[(\not{p}_f\gamma^\mu\not{p}_f'+\not{p}_f'\gamma^\mu\not{p}_f)\mathcal{M}\not{p}_i\gamma_\mu\not{p}_i\mathcal{M}^\dagger\Big]\\ &\longrightarrow -\frac{4(1-z)}{z}\frac{e^2Q_iQ_f}{(p_i')^2}\sum\frac{|\mathcal{M}_{(a)}(p_i',p_f)|^2}{1-z}.\end{aligned} \tag{5.25}$$

So adding them together, and noting that here we must have $Q_f=Q_i$ we get

$$\begin{aligned}\sum|\mathcal{M}_{ee\gamma}|^2 \to& -2\frac{2-2z+z^2}{z(1-z)}\frac{e^2Q_i^2}{(p_i')^2}\sum|\mathcal{M}_{(a)}(p_i',p_f)|^2\\ =&2\left[\frac{1+(1-z)^2}{1-z}\right]\frac{e^2Q_i^2}{k_T^2}\sum|\mathcal{M}_{(a)}((1-z)p_i,p_f)|^2.\end{aligned} \tag{5.26}$$

So we have shown again that the squared amplitude *factorises* in this limit! And moreover, we have shown that a process of a momentum p_i emitting a photon of momentum $zp_i + ...$ splits into an amplitude squared with an incoming electron of momentum $(1-z)p_i$ plus a universal factor! Clearly we can sum over all external legs and obtain the same behaviour. This expression is also useful, because it shows that we are not only dominated by colinear emission, but also by the $1-z$ part, or the region when $z \sim 1$ – the photon carries away most of the electron's momentum!

Adding now the phase space factors we have

$$\int d\Pi_{n+1} \sum |\mathcal{M}_{ee\gamma}|^2 \to \int \frac{d^3q}{(2\pi)^3 2E_q} d\Pi_n(p_{\text{tot}} - q) \sum |\mathcal{M}_{ee\gamma}|^2. \tag{5.27}$$

For cross-sections, we need that

$$\begin{aligned}\sigma_{CM} =& \frac{1}{4p_{CM}\sqrt{s}} \int d\Pi_{n+1} \sum |\mathcal{M}_{ee\gamma}|^2 \\ =& \frac{1}{2s} \int d\Pi_{n+1} \sum |\mathcal{M}_{ee\gamma}|^2\end{aligned} \tag{5.28}$$

for massless particles, and $s = (p_1 + p_2)^2 = 2p_1 \cdot p_2$, so if we scale the momentum of one particle we scale s accordingly and thus $\sigma(p_1, p_2)$. Hence

$$d\sigma_n(xp_i, p_f) = \frac{1}{x} \times \left(\frac{1}{2s} \int \int d\Pi_n \sum |\mathcal{M}_{ee}(xp_i, p_f)|^2 \right) \tag{5.29}$$

For the integral over d^3q, we can use $q = zp_i + \mathbf{k}_\perp + ...$ and split

$$\begin{aligned}\int d^3q =& \int_0^1 dzp_i \int d^2\mathbf{k}_\perp \\ =& 2\pi \int_0^1 dzp_i \int |\mathbf{k}_\perp| d|\mathbf{k}_\perp| \\ =& \pi \int_0^1 dzp_i \int d|\mathbf{k}_\perp|^2,\end{aligned} \tag{5.30}$$

to obtain

$$\begin{aligned}d\sigma_{n+1} =& \pi \int_0^1 dzp_i \int d|\mathbf{k}_\perp|^2 \frac{1}{(2\pi)^3 2(zp)} d\Pi_n(p_{\text{tot}} - q) \sum |\mathcal{M}_{ee\gamma}|^2 \\ =& \frac{1}{8\pi^2} \sum_i \int_0^1 \frac{dz}{z} \int \frac{d\mathbf{k}_\perp^2}{|\mathbf{k}_\perp|^2} \left[\frac{1 + (1-z)^2}{1-z} \right] e^2 Q_i^2 [(1-z) d\sigma_n((1-z)p_i, p_f)] \\ \underset{x=1-z}{\equiv}& \sum_i \int_0^1 dx \int \frac{d\mathbf{k}_\perp^2}{|\mathbf{k}_\perp|^2} \hat{P}_{ee}(x) d\sigma_n(xp_i, p_f).\end{aligned} \tag{5.31}$$

The above universal function is called a *splitting function* and we have defined it as

$$\hat{P}_{ee}(x) \equiv \frac{\alpha}{2\pi} \frac{1 + x^2}{1 - x}, \tag{5.32}$$

where now x denotes the fraction of the fraction of the electron momentum that *remains*. To be clear, we now talk about a process where the incoming momentum is $p_i = (p_i, 0, 0, p_i)$ and we integrate over $d\sigma_n(xp_i, p_f)$ where the incoming momentum is $(xp_i, -\mathbf{k}_\perp, xp_i - \delta)$ and the emitted photon has momentum $q = ((1-x)p_i, \mathbf{k}_\perp, zp + \delta)$, so the momentum split to $(x, 1-x)$ pieces.

For QCD, we have

$$\hat{P}_{qq}(x) = \frac{\alpha_s C_F}{2\pi} \frac{1+x^2}{1-x}, \tag{5.33}$$

i.e. we just change the coupling from e to g_s and have a factor of C_F from the group theory factors.

If we had considered final state radiation, so emission nearly colinear with p_f, then the analysis would have followed through the same, except we would not have the additional $(1-z) \to x$ factor from the difference in initial kinematics from the partons (5.29). So we would have

$$d\sigma_{n+1} = \sum_f \int_0^1 \frac{dx}{x} \int \frac{d\mathbf{k}_\perp^2}{|\mathbf{k}_\perp|^2} \hat{P}_{ee}(x) d\sigma_n(p_i, xp_f). \tag{5.34}$$

We will need both of these versions later. In this case, to be clear again, we now must introduce the virtuality into the "final" momentum from $d\sigma_n$, so we have $p'_f = (p_f, -\mathbf{k}_\perp, p_f + \delta)$ splitting to a photon with $q = ((1-x)p_f, \mathbf{k}_\perp, (1-x)p_f + \delta)$ and a final state electron with momentum $xp_f, 0, 0, xp_f$: the virtuality of the intermediate electron is now $\frac{x\mathbf{k}_\perp^2}{1-x}$.

Note that formally the above is still divergent. In the QED case, for the colinear divergence we can cut the above integral off by the electron mass at the lower end, and at the upper range of the $|\mathbf{k}_\perp|$ integral we will just have s. The divergence in the x integral at $x = 1$ is just the soft limit, and can be dealt with as we did before by including a photon mass or using dimensional regularisation to regularise it, and then including virtual corrections to eliminate it. In practice, however, we will actually want to put some finite ranges on the above integrals.

All of this formalism is necessary for showing what happens at colliders and how we make sense of QCD calculations. First we will examine parton showers, and then parton distribution functions.

5.3.2 Cascade

The fact that the squared amplitude factorises in the colinear limit is very useful, because the generalisation to multiple emissions is now (almost) obvious. Imagine that we are considering radiation from a final-state electron and consider first *two* photons being emitted, as shown in figure 8. Our expectation is that the electron with momentum p''_f must be the most off-shell, so if the transverse momenta are $p'_\perp, p''_\perp$ for the two electrons/photons, we should have $|p''_\perp| \gg |p'_\perp|$ and this is indeed what we need. To make sure of this, note that we now have three matrix elements (two initial photons, two

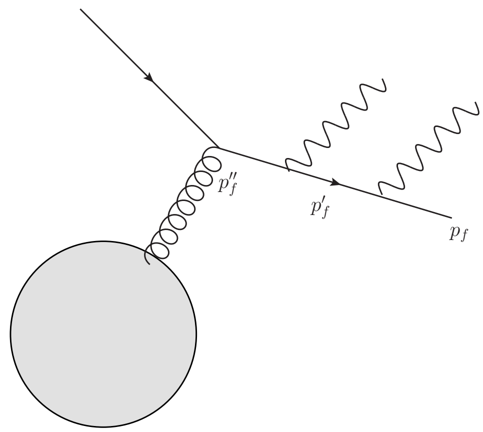


Figure 8: Shower of emission from final state particle

final photons, and one each):

$$\mathcal{M}^{\text{two ISR photons}} \sim \mathcal{M}\gamma^\mu \frac{\not{p}'_i}{(p'_i)^2}\gamma^\nu \frac{\not{p}''_i}{(p''_i)^2} \tag{5.35}$$

$$\mathcal{M}^{\text{two FSR photons}} \sim \frac{\not{p}'_f}{(p'_f)^2}\gamma^\mu \frac{\not{p}''_f}{(p''_f)^2}\gamma^\nu \mathcal{M} \tag{5.36}$$

$$\mathcal{M}^{\text{one ISR,one FSR}} \sim \frac{\not{p}'_f}{(p'_f)^2}\gamma^\mu \mathcal{M}\gamma^\nu \frac{\not{p}'_i}{(p'_i)^2}. \tag{5.37}$$

We should consider the square of this whole thing, so now 9 terms. Notably we will have the interference between (5.37) and the other two diagrams. On the other hand, if $(p'_f)^2 \ll (p''_f)$ then clearly when considering the singularity in p'_f we can ignore p''_f, and treat one photon at a time (whereas in the other limit the interference becomes important):

$$\begin{aligned} d\sigma_{n+2} \propto & \left(\frac{\alpha}{2\pi}\right)^2 \int_{m^2}^{s} \frac{d(p''_\perp)^2}{(p''_\perp)^2}) \int_{m^2}^{(p''_\perp)^2} \frac{d(p'_\perp)^2}{(p'_\perp)^2} \\ = & \frac{1}{2}\left(\frac{\alpha}{2\pi}\right)^2 \log^2 \frac{s}{m^2}. \end{aligned} \tag{5.38}$$

Clearly for n photons, with the hierarchy $p'_\perp \ll p''_\perp ...$ we have

$$\frac{1}{n!}\left(\frac{\alpha}{2\pi}\right)^n \log^n \frac{s}{m^2}$$

... in other words, we see an exponential appearing, our Sudakov factor again. If we choose any other hierarchy, then the interference terms will lead to smaller singularities and a lower power of the logarithm, so – especially when n is large – we can neglect them.

Indeed we can intepret the process as an electron starting with virtuality $t_0(= p_0^2 - m^2)$ and ending on-shell at $t = 0$. Since $p''_f \gg p'_f$ in the above example, the transverse momenta of the photons are

$$\begin{aligned} \mathbf{q}''_\perp = & \mathbf{p}''_\perp - \mathbf{p}'_\perp \approx \mathbf{p}''_\perp \\ \mathbf{q}'_\perp = & \mathbf{p}'_\perp \end{aligned} \tag{5.39}$$

and so $|\mathbf{q}''_\perp| \gg |\mathbf{q}'_\perp|$: in other words *the dominant process involves the emission of photons with the largest transverse momentum (but still "nearly" colinear) closest to the hard process.* This is actually very important. In simulations we can compute the hard process and then match it with a cascade of emissions simulated using universal splitting functions.

5.3.3 Parton shower algorithm

To see the appearance of the shower, let us start from equation (5.34) and allow σ_n to have one final-state leg off-shell with virtuality t_0 (by which we mean $t = q^2 - m^2$ for the external leg). Recall that

$$t = (p')^2 \propto k_\perp^2 \longrightarrow \frac{d\mathbf{k}_\perp^2}{|\mathbf{k}_\perp|^2} = \frac{dt}{t}. \tag{5.40}$$

Next we will say that we integrate t to some minimum value t_{min}: for a photon cascade this would be the mass of the particles, but for a QCD cascade this should be around the scale at which QCD becomes strong and we need to introduce hadronisation effects. We get:

$$d\sigma_{n+1} = \sum_f \int_{x_{\text{min}}}^{x_{\text{max}}} \frac{dx}{x} \int_{t_{\text{min}}}^{t_0} \frac{dt}{t} \hat{P}_{ji}(x) d\sigma_n(p_i, xp_f). \tag{5.41}$$

Note that we need to take into account the resolution when we integrate over x, since we cannot resolve photons/gluons below the threshold and recall that in final-state radiation we have also picked up a virtuality of $\frac{x\mathbf{k}_\perp^2}{1-x}$ in our integral $d\sigma_n(p_i, xp_f)$.

We can now iterate this process. We can view it as before as ordering the integrations by virtuality:

$$d\sigma_{n+k+1} = \sum_f \left(\int_{x_{\text{min}}}^{x_{\text{max}}} \frac{dx_0}{x_0} \int_{t_{\text{min}}}^{t_0} \frac{dt}{t} \hat{P}_{ji}(x_0) \right) \left(\int_{x_{\text{min}}}^{x_{\text{max}}} \frac{dx_1}{x_1} \int_{t_{\text{min}}}^{t_1} \frac{dt}{t} \hat{P}_{ji}(x_1) \right) \cdots \left(\int_{x_{\text{min}}}^{x_{\text{max}}} \frac{dx_k}{x_k} \int_{t_{\text{min}}}^{t_k} \frac{dt}{t} \hat{P}_{ji}(x_k) \right) d\sigma_n \tag{5.42}$$

with $t_0 > t_1 \cdots > t_k$; but this gives the same result as integrating over all possible orders of branchings and then dividing by symmetry factors to remove duplicates:

$$d\sigma_{m>n} = \sum_{k=1}^{\infty} \sum_f \frac{1}{k!} \left(\int_{x_{\text{min}}}^{x_{\text{max}}} \frac{dx}{x} \int_{t_{\text{min}}}^{t_0} \frac{dt}{t} \hat{P}_{ji}(x) \right)^k d\sigma_n \tag{5.43}$$

This gives us an exponential factor:

$$\begin{aligned} d\sigma_{\text{total}} =& d\sigma_n + d\sigma_{m>n} \\ =& d\sigma_n \exp\left[\int_{x_{\text{min}}}^{x_{\text{max}}} \frac{dx}{x} \int_{t_{\text{min}}}^{t_0} \frac{dt}{t} \hat{P}_{ji}(x) \right]. \end{aligned} \tag{5.44}$$

Then we can use

$$\frac{\sigma_n}{\sigma_{\text{total}}} = \text{probability of no soft emmission.} \tag{5.45}$$

Viewing the integrals again as ordered by virtuality again, and recall that the above integral takes us from a maximum virtuality of t_0 down to $t_{\min}$, we can define the function $\Delta(t_0, t)$, called the **Sudakov factor**, which is the probability of finding *no (resolvable) radiation between* t_0 *and* t. We just determined this probability in $[t_0, t_{\min}]$ but we expect that $\Delta(t_0, t)\Delta(t, t_{\min}) = \Delta(t_0, t_{\min})$ by the usual laws of probability. So we can use this to determine the Sudakov factor in general as follows:

$$\begin{aligned}\Delta(t_0, t + \delta t) =&\Delta(t_0, t) \times \Big(1 - \text{prob of emission between } t \text{ and } \delta t\Big)\\ \equiv&\Delta(t_0, t)\Big(1 - R(t)\delta t\Big)\\ =&\Delta(t_0, t) + \delta t\frac{d\Delta}{dt}.\end{aligned} \tag{5.46}$$

We determine $R(t)$ therefore by differentiating:

$$R(t_0) = -\frac{d\log\Delta(t_{\min}, t_0)}{dt} = \int \frac{dx}{x}\hat{P}_{ji}(x). \tag{5.47}$$

We can integrate the above equation from $\Delta(t_0, t_0) = 1$ to give

$$\Delta(t_0, t) = \exp\left[-\int_{t_0}^{t} R(t)\right] = \exp\left[-\sum_j \int_{x_{\min}}^{x_{\max}} \frac{dx}{x}\int_{t_0}^{t}\frac{dt}{t}\hat{P}_{ji}(x)\right]. \tag{5.48}$$

Having noted that our shower proceeds as an evolution in virtuality (for final-state radiation towards lower virtuality; for QCD production involving parton distribution functions it is more complicated) we can use this as an algorithm: we start with a state of some virtuality t_0 and want to generate branchings, so we need the probability of emitting radiation at some value of t. But that is simply

$$dP(\text{emission at t}) = \Delta(t_0, t)R(t) = -\frac{d\Delta}{dt}. \tag{5.49}$$

If we want to simulate events drawn from some probability distribution distributed with some density $p(t)$ so that $dP = p(t)dt$ then we can use the *inverse transform method* (see e.g. [7]). Since P is the cumulative probability we want to draw events from uniform slices of $P \in [P_{\min}, 1]$, so we can use a random variable $\rho \in [0, 1]$ with a flat distribution (that we can generate easily). We then take the inverse $P^{-1}(\rho) = t$: $d\rho = \frac{dP}{dt}dt = p(t)dt$ as required. So we proceed by:

- Start from virtuality t_0 and generate a random number $\rho \in [0, 1]$.
- Solve for $\Delta(t_0, t) = \rho$. If $t < t_{\min}$ then there is no emission and the chain stops (this corresponds to finding $\rho < P_{\min}$, i.e. $\rho < \Delta(t_0, t_{\min})$.
- Similarly generate a random momentum fraction x; and angle for the emitted radiation (distributed evenly in ϕ).
- Repeat for all (new) branches until all have terminated.

This forms the starting point for the Monte-Carlo algorithms used in *parton shower* programs such as `Pythia8` and `Herwig++` (although they now use newer, slightly more complicated, algorithms).

These programs do the same job for QCD showers and for electromagnetic showers. The end of the shower is a whole bunch of leptons and partons, which must then be *hadronised* into colourless states. This job is also performed by the same programs, but the algorithms rely on different models of QCD in the strong coupling regime (e.g. "Lund string model") and we will not discuss that here.

5.4 Jets

We have discussed what happens in QCD processes: a hard scattering processes emits initial- and final-state radiation leading to a parton shower, but what do detectors actually see? Most of the radiation is nearly colinear with the hard partons, and the hadronic calorimeters detect energy deposits and momenta with a good angular resolution, so we can try to group together those deposits and reconstruct what the underlying hard process might be. Such a grouping is a 'Jet.' There are a number of algorithms proposed for doing this. They all take hadronic four-momenta as inputs (which may be inputs from our hadron shower program, from a detector simulation and its inferring of energy deposits into four-mometna – or indeed from actual detector data).

To do this, we need to have some measure d_{ij} between two four-vectors. Since they are optimised for colliders where the interaction point is roughly known, this should be an angular measure, is typically:

$$d_{ij} = \min(k_{Ti}^{2p}, k_{Tj}^{2p}) \frac{\Delta R_{ij}^2}{R^2} \tag{5.50}$$

where p is an algorithm-dependent number, k_{Ti} is the transverse momenta of particle i and we introduced ΔR in lecture 1:

$$\Delta R_{ij}^2 = (\eta_i - \eta_j)^2 + (\phi_i - \phi_j)^2 \tag{5.51}$$

η_i is the pseudorapidity, and ϕ_i the azimuthal angle; R is then some reference "Jet size," typically 0.4 or 0.5 (or 1 for "fat jets"). Sometimes ΔR_{ij} is defined differently in terms of the rapidity, rather than the pseudorapidity:

$$\Delta R_{ij}^2(\text{rapidity}) = (y_i - y_j)^2 + (\phi_i - \phi_j)^2. \tag{5.52}$$

For reference,

$$y \equiv \frac{1}{2} \log \frac{E + p_z}{E - p_z}, \qquad \eta \equiv -\log \tan \frac{\theta}{2}, \tag{5.53}$$

which are equivalent for massless particles.

Now we have a distance measure, we can decide whether two objects are "close" and should be grouped together, subject to conditions. For this, we need an algorithm that is not (too) dependent

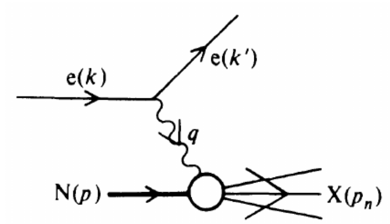


Figure 9: Deep inelastic scattering of an electron from a nucleon.

on IR effects: if we add arbitrary soft or split vectors into nearly colinear ones it should not change the number of jets or their properties. Fortunately a lot of work has gone into this.

A typical algorithm then works along the following lines:

1. Calculate d_{ij} and $d_{iB} \equiv (k_{Ti})^{2p}$ (which is a measure of the distance of i to the beamline) for all (pairs of) momenta, which become *pseudojets*.
2. If $\min(\{d_{ij}\}, \{d_{iB}\})$ is $d_{ij} \in \{d_{ij}\}$, combine i and j into a single pseudojet (this might be complicated).
3. Otherwise the minimum is d_{iB} for some i, so declare i is a final-state jet and remove from the list.
4. Continue until no pseudojets remain.

The most popular jet algorithm is called the **anti-$\mathbf{k_T}$** algorithm, and has $p = -1$ in the above (other choices are $p = 1$ **$\mathbf{k_T}$** algorithm, $p = 0$ **Cambridge-Aachen algorithm**). It produces particularly stable boundaries with respect to soft radiation.

From a phenomenologist's point of view, a given hard process should produce jets corresponding to the final-state partons, accompanied potentially by a small number of hard initial- and final-state radiation jets (but maybe none, depending on chance and whatever our definition of "hard" is – i.e. a *cut*), and then various soft jets from the underlying process. These can then be studied to compare to predictions in the Standard Model – and beyond.

5.5 Parton distribution functions

So far we have avoided talking about the structure of the proton and how this affects interactions, mainly by discussing QED as an analogy. You know now that at large (interaction) energies quarks behave like (screened) point particles, and at low energies they bind together to make mesons and baryons. The parton model treats the proton as a collection of point-like particles that are struck by the hard interaction. For a process such as $e^- p \to e^- X$ shown in figure 9 we need to sum over all the constituents of the proton. The problem is that we do not how what the momenta of the individual constituents are! So we parametrise our ignorance by saying that parton p_i has momentum $p_i^\mu = \xi P^\mu$, where P^μ is the momentum of the proton. Then

$$\sigma(e^- p \to e^- X) = \sum_i \int_0^1 d\xi f_i^0(\xi) \hat{\sigma}(e^- p_i \to e^- X) \tag{5.54}$$

where the function $f_i(\xi)$ is to be interpreted as a probability distribution of finding a parton i with (longitudinal) momentum fraction ξ. Here we have presumed that this function is somehow universal and does not depend on the momentum transferred Q; this is essentially proved by the *factorisation theorems* for our f_i^0 which we can call "bare pdfs" but there are some important subtleties that we will discuss shortly.

In the quark model, we require that the quantum numbers of the proton are explained by two up quarks and one down quark. These are referred to as "valence quarks" and, from their charges, we require (dropping the superscript for now):

$$\int d\xi [f_u(\xi) - f_{\overline{u}}(\xi)] = 2, \qquad \int d\xi [f_d(\xi) - f_{\overline{d}}(\xi)] = 1. \tag{5.55}$$

Similarly, the remaining quarks are known as "sea quarks" and must obey:

$$\int d\xi [f_s(\xi) - f_{\overline{s}}(\xi)] = 0 \tag{5.56}$$

and so on. There is also a function for gluons, f_g, but since they do not carry charge and have no conserved number, we cannot bound them by a similar equation. On the other hand, we require that the total momentum carried by all constituents adds up to the proton momentum:

$$\sum_j \int d\xi [\xi f_j(\xi)] = 1. \tag{5.57}$$

Clearly we need parton distribution functions (pdfs) to compute any process involving protons, which includes HERA, the Tevatron and the LHC. To compute them, information must be put together from many sources (including those experiments). One of these is *deep inelastic scattering* of electrons on protons.

5.5.1 Deep Inelastic Scattering

Consider an electron interacting with a parton in the nucleus via a photon exchange. We can write the matrix element as

$$i\mathcal{M} = -i\frac{Q_f e^2}{q^2} \overline{u}_e(k')\gamma^\mu u_e(k) \overline{u}_q(p_q)\gamma_\mu u_q(p_q') \tag{5.58}$$

where $q = k - k'$ is the momentum transfer. Neglecting the lepton and quark masses, we have the mandelstam invariants

$$\begin{aligned}
\hat{s} =& (p_q + k)^2 = 2p_q \cdot k = 2\xi P \cdot k = \xi s \\
\hat{t} =& (k - p_q)^2 \\
\hat{u} =& (k' - p)^2 \\
Q^2 \equiv& -q^2.
\end{aligned} \tag{5.59}$$

We note that

$$m_q^2 = (p+q)^2 \simeq 0 = 2p \cdot q + q^2 = 2\xi P \cdot q - Q^2 \tag{5.60}$$

so we can define

$$x \equiv \frac{Q^2}{2P \cdot q} = \xi. \tag{5.61}$$

Another convenient variable is

$$y \equiv \frac{2P \cdot q}{2P \cdot k} = \frac{2P \cdot q}{s}. \tag{5.62}$$

In the lab frame where $P_\mu = (m_p, 0, 0, 0), k_\mu \simeq (E, 0, 0, E), k'_\mu \simeq (E', 0, E' \sin\theta, E' \cos\theta)$ we have

$$y = \frac{2(E - E')}{2E} = \frac{E - E'}{E}, \tag{5.63}$$

so $0 \leq y \leq 1$, and

$$x \simeq \frac{-2k \cdot k'}{2m_p(E - E')} = -\frac{EE'(1 - \cos\theta)}{m_p(E - E')}. \tag{5.64}$$

So we can trade $d\Omega$ for dx and also differentiate with respect to the scattered momentum E'; we can write the differential cross-section as

$$\frac{d^2\sigma}{dxdy}(e^- p \to e^- X) = \left(\sum_i x f_i(x) Q_j^2\right) \frac{2\pi\alpha^2 s}{Q^4}[1 + (1-y)^2]. \tag{5.65}$$

From this we can extract information about the sum of the pdfs. This characteristic dependence on Q^2 and y can be seen in experimental data.

5.5.2 IR divergences and scaling

However, there is an important subtlety that we put to one side, related to several of the things we have discussed already. In the above process, imagine adding QCD radiation to the partons inside the proton. Consider just the contribution from one quark q; then we have

$$d\sigma(e^- p \to e^- X + \text{up to one gluon}) \supset \int_0^1 d\xi f_q^0(\xi) \times \left(\hat{\sigma}_0(p_e, \xi p_p) + \int_0^1 dx \int \frac{d|k_\perp|^2}{|\mathbf{k}_\perp|^2} \hat{P}_{qq}(x) \hat{\sigma}_0(p_e, \xi x p_p)\right) \tag{5.66}$$

where we have put $\hat{\sigma}_0$ for the partonic process and p_e, p_p for the electron and proton momenta. This is suggestive: we can rescale them by $\xi \to \xi/x$ in the second term (which means that now $x \in [\xi, 1]$ so that we do not have $\xi > 1$) to give

$$d\sigma(e^- p \to e^- X + \text{up to one gluon}) \supset \int_0^1 d\xi \hat{\sigma}_0(p_e, \xi p_p) \times \left(f_q^0(\xi) + \int_\xi^1 \frac{dx}{x} f_q^0(\xi/x) \int \frac{d|k_\perp|^2}{|\mathbf{k}_\perp|^2} \hat{P}_{qq}(x)\right). \tag{5.67}$$

This looks like a different kind of pdf, but we have now added infra-red divergences. If we now define

$$f_q(x,\mu^2) \equiv f_q^0(\xi) + \int_\xi^1 \frac{dx}{x} f_q(\xi/x,\mu^2) \int_0^{\mu^2} \frac{d|k_\perp|^2}{|\mathbf{k}_\perp|^2} \hat{P}_{qq}(x) \tag{5.68}$$

then our amplitude becomes

$$d\sigma(e^- p \to e^- X + \text{up to one gluon}) \supset \int_0^1 d\xi \hat{\sigma}_0(p_e, \xi p_p) \times \left(f_q(\xi,\mu^2) + \int_\xi^1 \frac{dx}{x} f_q(\xi/x,\mu^2) \int_{\mu^2}^s \frac{d|k_\perp|^2}{|\mathbf{k}_\perp|^2} \hat{P}_{qq}(x) \right). \tag{5.69}$$

In other words we have removed the colinear singularity into the pdfs! This is very practical, in contrast to the formal elimination of IR divergences as in section 4 (we would not want a gluon mass above the QCD scale ... and if we chose it below the QCD scale, then it would not help us). Therefore the pdfs perform the job of summing over the indistinguishable initial states for us! The price we must pay is a new *scale dependence* of our pdfs. The scale μ is called the *factorisation scale* and it turns out it is a price worth paying; it is often useful to take it to be equal to be Q^2, the scale of momentum transfer of the problem.

The dependence of the pdfs on the scale μ can be worked out simply by insisting that the "bare" pdfs have no dependence on it. Starting from (5.68) we have

$$\begin{aligned} \mu^2 \frac{\partial}{\partial \mu^2} f_q(x,\mu^2) =& \mu^2 \frac{\partial}{\partial \mu^2} \left[f_q^0(\xi) + \int_\xi^1 \frac{dx}{x} f_q(\xi/x,\mu^2) \int_0^{\mu^2} \frac{d|k_\perp|^2}{|\mathbf{k}_\perp|^2} \hat{P}_{qq}(x) \right] \\ =& \mu^2 \int_\xi^1 \frac{dx}{x} f_q(\xi/x,\mu^2) \frac{1}{\mu^2} \hat{P}_{qq}(x) \\ \frac{\partial}{\partial \log \mu^2} f_q(x,\mu^2) =& \int_\xi^1 \frac{dx}{x} f_q(\xi/x,\mu^2) \hat{P}_{qq}(x). \end{aligned} \tag{5.70}$$

This looks something like a renormalisation group equation! Here we just considered the effects of one flavour of quark without gluons in the initial state, but in general quarks split into gluons and quarks, and gluons split into pairs of quarks: the evolution equations mix and involve all the splitting functions. The full set are called the **DGLAP** equations, named after Dokshitzer, Gribov, Lipatov, Altarelli and Parisi. Nowadays they are implemented in computer codes such as `lhapdf`, so that they can be evaluated very simply at any desired interaction scale.

5.5.3 Generalised DIS

This expression for DIS *assumes* free quarks and a parton distribution function; there is also a more general parametrisation where we work only to leading order in the electromagnetic charge and use the fact that the photon coupling to the proton and final states is given by the electromagnetic coupling:

$$\mathcal{L} \supset -ie \int d^4x J^\mu A_\mu(x) \tag{5.71}$$

to write

$$i\mathcal{M}(ep \to ef) =(-ie)\overline{u}_e(k')\gamma^\mu u_e(k)\frac{-i}{q^2}(ie)\int d^4x e^{iq\cdot x}\langle f|J^\mu(x)|p\rangle. \tag{5.72}$$

Squaring this and summing over the electron spinors (while ignoring the electron mass) using

$$\begin{aligned} L^{\mu\nu} \equiv&\frac{1}{2}\mathrm{tr}[\not{k}'\gamma^\mu\not{k}\gamma^\nu] \\ =&2(k'^\mu k^\nu + k'^\nu k^\mu - k\cdot k'\eta^{\mu\nu}) \end{aligned} \tag{5.73}$$

gives

$$\begin{aligned} \sigma =&\frac{1}{2s}\int\frac{d^3k'}{(2\pi)^3}\frac{1}{2k'}e^4 L_{\mu\nu}\frac{1}{Q^4}\sum_f\int d^4zd^4z'e^{iq\cdot(z-z')}\langle p|J^\nu(z')|f\rangle\langle f|J^\mu(z)|p\rangle \\ =&\frac{1}{2s}\int\frac{d^3k'}{(2\pi)^3}\frac{1}{2k'}e^4 L_{\mu\nu}\frac{1}{Q^4}\int d^4ze^{iq\cdot z}\langle p|J^\mu(z)J^\nu(0)|p\rangle \\ \equiv&\frac{1}{2s}\int\frac{d^3k'}{(2\pi)^3}\frac{1}{2k'}e^4 L_{\mu\nu}\frac{1}{Q^4}W^{\mu\nu}. \end{aligned} \tag{5.74}$$

After some algebra this leads to

$$\frac{d\sigma}{dxdy} =\frac{\alpha^2 y}{2Q^4}L_{\mu\nu}W^{\mu\nu}. \tag{5.75}$$

In general then the structure function can be written in terms of Lorentz structures depending on q_μ and P_μ such that the Ward identity $q_\mu W^{\mu\nu} = 0$ is respected:

$$W^{\mu\nu} = \left(-\eta_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right)\frac{F_1(x,Q^2)}{2\pi} + \left(P^\mu - q^\mu\frac{P\cdot q}{q^2}\right)\left(P^\nu - q^\nu\frac{P\cdot q}{q^2}\right)\frac{F_2(x,Q^2)}{2\pi} \tag{5.76}$$

(and there is an F_3 for neutrino scattering which contracts with an antisymmetric part in the corresponding $L_{\mu\nu}$). The behaviour of $F_2(x, Q^2)$ is shown in figure 10 demonstrating the independence of Q^2 *at large* Q^2, which is known as ***Bjorken scaling***.

The advantage of this generalised approach is that the structure functions can be more rigorously defined in terms of observables. There is then practically a whole industry in calculating them from data. Some recent results are shown in figure 11; it is clear that at larger μ^2 there is a large increase in the momentum fraction carried by soft gluons and quark-antiquark pairs ("sea quarks").

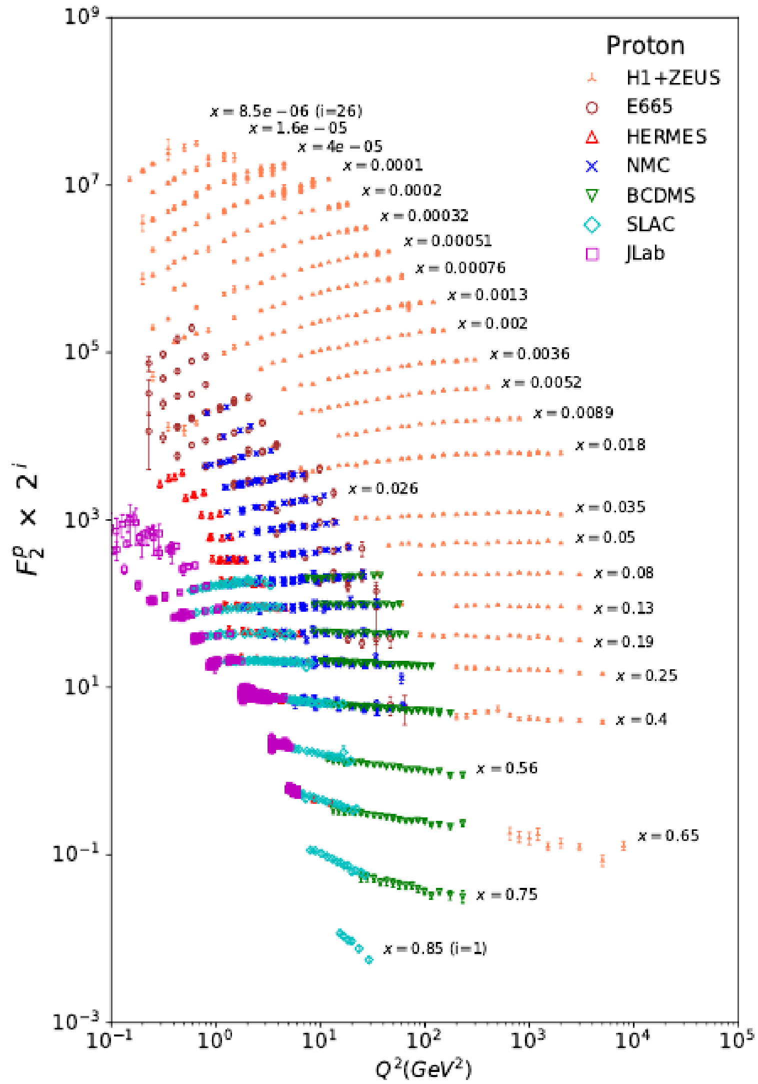


Figure 10: Structure function F_2 against Q^2 for electron-proton interactions.

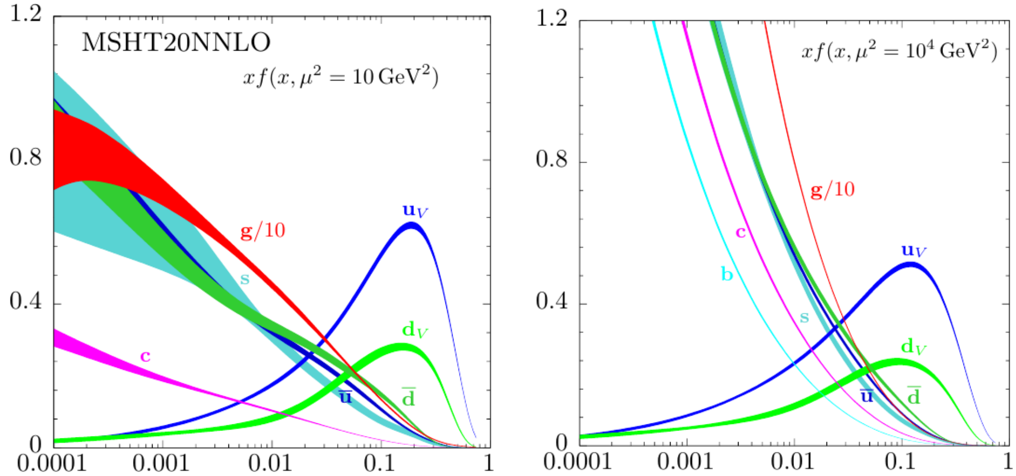


Figure 11: Parton distribution functions for two different values of Q^2, taken from the pdg review of structure functions.

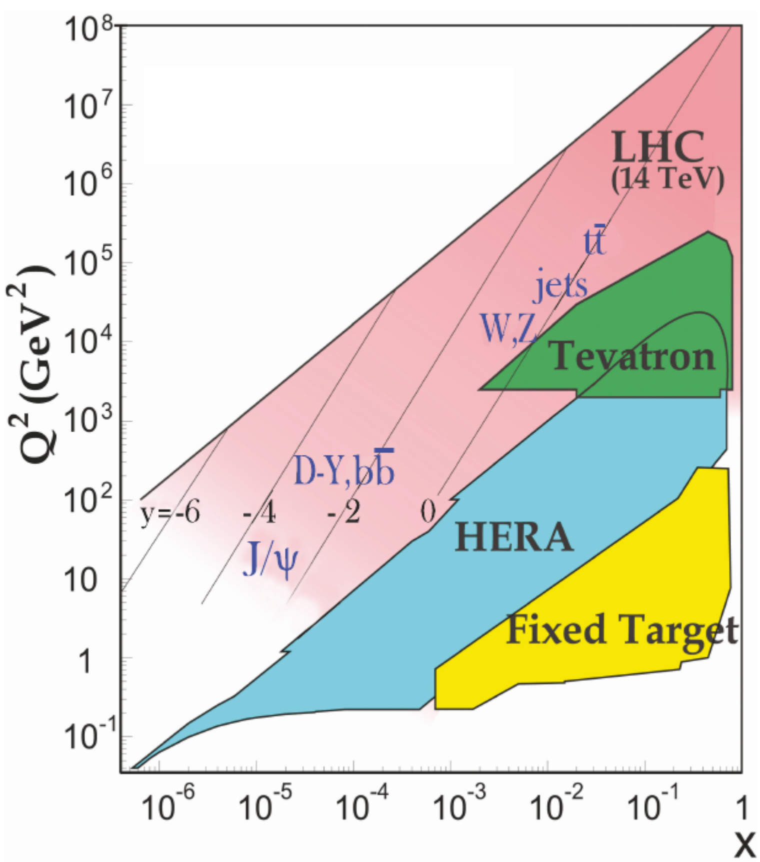


Figure 12: Regions of x against Q^2 probed by various experiments.

5.5.4 Proton-proton collisions

Finally, for proton-proton collisions, the obvious generalisation from the parton model is that cross-sections should look like

$$\sigma = \sum_{i,j \in q,\overline{q},g} \int_0^1 dx_1 \int_0^1 dx_2 f_i(x_1, Q^2) f_j(x_2, Q^2) \hat{\sigma}_{ij}(x_1 p_1, x_2 p_2). \tag{5.77}$$

This means that not all of the energy in the protons is available for collisions. Indeed from observing the pdfs in figure 11 we can see that at large energies there will be a large proportion of momentum carried by soft gluons and sea quarks at low x, and indeed a "typical" value for $x_{1,2}$ in a collision would very rarely be bigger than 0.1.

5.6 Summary: what you need to know to study physics at colliders

We have covered a lot of theory to introduce you to physics at colliders. I have gone into a lot of details that are not necessarily well explained elsewhere, but of course we were only able to scratch the surface of all the theory behind everything. The good news is that in practice you don't need to know too much to just *use* the tools and get results: there are now codes (especially `Pythia` and `Herwig`) that put it into practice. Typically a simulation requires:

1. Squared matrix elements summed over spins (or not, if you want polarised initial states).
2. Integration over parton distribution functions in case at least one incoming state is a proton/antiproton.
3. Integration over the final state phase space. Typically this is done by instead simulating *events* with the particular values of momenta chosen from a distribution weighted by the phase-space factors.
4. Decay of any short-lived states.
5. Parton showering, generating soft/colinear radiation from the initial and final states for QCD and QED.
6. A recipe for matching the parton shower onto the hard process in case there are any hard quarks/gluons in the hard process that could otherwise be confused with colinear/soft radiation.
7. Hadronisation of the final states into colourless particles.
8. Simulation of the response of the detector (if you are considering collider processes – for some applications e.g. dark matter annihilations – this is not necessary).
9. A recipe for identifying the final states as electrons, photons, muons and jets.
10. A calculation of the missing energy and other observables.
11. Analysis of the event.

Fortunately all of the theory required for this is packaged nowadays into computer programs. On the other hand, some of the elements included are as much art as science, involving various parameters that can/must be set by the user, so you need to understand something about what is going on so that it is not just a magic box and you can make sure that the results make sense. These were the two main purposes of this part of the lectures.

The codes you might use are:

- Steps 1-3 are taken care of by an *event generator*, typically `MadGraph` (or maybe `CalcHEP`), or sometimes directly in `Pythia` for $2 \to 2$ processes.
- The parton distribution function integration is taken care of in the same codes when you are considering proton-proton collisions. You have a wide variety of pdfs to choose from put together by different teams, with different tunes, and only experienced users will be able to understand the advantages/disadvantages of each.
- The parton showering/hadronisation and particle decays are typically handled by `Pythia` or `Herwig++`.
- Steps 9 and 10 are performed in a detector simulator such as `Delphes`. CMS and ATLAS use a very sophisticated simulation of their detector in the code `GEANT` which is not publicly available; on the other hand they provide various analyses of the efficiency of their particle and jet reconstruction that can be applied to the particle-level information (with or without using `Delphes`).
- Step 11 is either performed yourself (using e.g. `root` from CERN) or some analysis code such as `MadAnalysis` that makes the objects available for manipulation and inspection by the user.

A good way to get a feel for these is to download them and have a play, or follow a tutorial, many of which are available online (or you can ask me).

There are many other tools available for specialised purposes – most notably there are other event/matrix element generators that are useful for NLO calculations. Of course, I have not discussed what to do with the data collected and how to analyse it. But that depends on what you want to study and the model you are interested in, which is the subject of the rest of the course.

A Long derivations

A.1 Breit Wigner in the CMS frame

If we consider two-particle processes then we can write the equivalent of (2.5) in the centre of mass frame and over the CMS momentum $\mathbf{k}$. There is then a subtlety because if the two outgoing states are identical the two-particle state with $\mathbf{k}$ and $-\mathbf{k}$ are the same, so we have to include a factor of $2^{-\delta_f}$, where $\delta_f = 1$ if the two final states are identical:

$$\begin{aligned}
1 =& \frac{1}{2^{\delta_f}} \frac{V}{(2\pi)^3} \int d^3\mathbf{k} \frac{|H_{f0}|^2}{(E_{CMS} - E_0)^2 + \Gamma^2/4} \\
\simeq& \frac{1}{2^{\delta_f}} \frac{V}{(2\pi)^3} \int d^3\mathbf{k} |H_{f0}|^2 \frac{2\pi}{\Gamma} \delta(E_{CMS} - E_0) \\
=& \frac{1}{2^{\delta_f}} \frac{V}{(2\pi)^3} 4\pi |H_{f0}|^2 \int k^2 \frac{dk}{dE_{CMS}} dE_{CMS} \frac{2\pi}{\Gamma} \delta(E_{CMS} - E_0) \\
=& \frac{1}{2^{\delta_f}} \frac{V}{(2\pi)^3} 4\pi |H_{f0}|^2 k^2 \frac{dk}{dE_{CMS}} \frac{2\pi}{\Gamma} \bigg|_{k=k_{CMS}} \\
=& \frac{V}{2^{\delta_f}} \frac{\pi}{\Gamma} |H_{f0}|^2 k_{CMS}^2 \frac{dk}{dE_{CMS}}.
\end{aligned} \tag{A.1}$$

The same is true for the decay to the initial-state particles:

$$1 = \frac{V}{2^{\delta_i}} \frac{\pi}{\Gamma} |H_{i0}|^2 k_{CMS}^2 \frac{dk}{dE_{CMS}}. \tag{A.2}$$

Now

$$\begin{aligned}
E_{CMS} =& E_1 + E_2 = \sqrt{m_1^2 + k_{CMS}^2} + \sqrt{m_2^2 + k_{CMS}^2} \\
\frac{dE_{CMS}}{dk_{CMS}} =& \frac{k_{CMS}}{E_1} + \frac{k_{CMS}}{E_2} \qquad \left(= k_{CMS} \frac{E_{CMS}}{E_1 E_2} \right).
\end{aligned} \tag{A.3}$$

Next we need that the relative speed of the two particles in the centre of mass frame is

$$v = v_1 + v_2 = \frac{k_{CMS}}{E_1} + \frac{k_{CMS}}{E_2} = \frac{dE_{CMS}}{dk_{CMS}}. \tag{A.4}$$

Hence we arrive at

$$\begin{aligned}
\sigma(i \to X^* \to f) =& \sigma(i \to X^*) \times \frac{\Gamma_f}{\Gamma} \\
=& \frac{V}{v} \frac{\Gamma |H_{0i}|^2 \Gamma_f}{(E_{CMS} - m_X)^2 + \Gamma^4/4} \\
=& \frac{2^{\delta_i}\pi}{k_{CMS}^2} \frac{\Gamma_i \Gamma_f}{(E_{CMS} - m_X)^2 + \Gamma^4/4}
\end{aligned} \tag{A.5}$$

as quoted in the text.

A.2 Breit Wigner from QFT

Here we will complete the derivation of the full Breit-Wigner formula from QFT. In the text we had argued that the full amplitude of $i \to X \to f$ scattering at centre of mass energies near a pole m_X should be dominated by the s-channel propagator around that pole. Let us look at the matrix element more closely. We will write a generic wavefunction ψ_s^X for the particle X with spin s, and we can write $\psi_{s_i}^i, \psi_{s_f}^f$ for the incoming and outgoing wavefunctions. Then if we denote the amputated amplitudes without wavefunction factors as $\mathcal{A}$ then we have

$$\begin{aligned}\mathcal{M}(i \to X) &= \sqrt{Z_X} \prod_{j \in \{i\}} \sqrt{Z_j} \psi_{s_i}^i \mathcal{A}(i, s_i \to X, s_X) \psi_{s_X}^X \\ \mathcal{M}(X \to f) &= \sqrt{Z_X} \prod_{k \in \{f\}} \sqrt{Z_m} \psi_{s_X}^X \mathcal{A}(X, s_X \to f, s_f) \psi_{s_f}^f. \end{aligned} \tag{A.6}$$

Next we need that the propagator for X with a general spin state must look like

$$\int d^4x e^{ip\cdot x} \langle\Omega|\Psi(x)\overline{\Psi}(0)|\Omega\rangle = \frac{iZ_X \sum_{s_X} \psi_{s_X}^X(p)\overline{\psi}_{s_X}^X(p)}{p^2 - m_X^2 - iZ_X \mathrm{Im}(\Pi(m_X^2))} + \text{multiparticle states.} \tag{A.7}$$

This can be seen from inserting the representation in terms of creation and annihilation operators, or by inserting a complete set of states between $\Psi(x)$ and $\overline{\Psi}(0)$ (you can infer this from e.g. Peskin & Schroeder section 7.1). This crucially means that we can write

$$\begin{aligned}\mathcal{M}(i \to X \to f) &= \prod_{j \in \{i\}} \sqrt{Z_j} \prod_{k \in \{f\}} \sqrt{Z_m} \sum_{s_X} \psi_{s_i}^i \mathcal{A}(i, s_i \to X, s_X) \frac{iZ_X \sum_{s_X} \psi_{s_X}^X(p)\overline{\psi}_{s_X}^X(p)}{p^2 - m_X^2 - iZ_X \mathrm{Im}(\Pi(m_X^2))} \mathcal{A}(X, s_X \to f, s_f)\psi_{s_f}^f \\ &= \sum_{s_X} \mathcal{M}(i \to X, s_X) \frac{i}{p^2 - m_X^2 + im\Gamma(X)} \mathcal{M}(X, s_X \to f). \end{aligned} \tag{A.8}$$

Now when we transform this to a cross-section let us specialise to the case that the initial state contains two particles and average over their spins:

$$\begin{aligned}\sigma(i \to X \to f) =& \frac{1}{(2s_{i_1}+1)(2s_{i_2}+1)} \frac{1}{4p_{CM}^i\sqrt{s}} \frac{1}{2^{\delta f}} \sum_{s_i}\sum_{s_f}\sum_{s_X} \left|\mathcal{M}(i \to X, s_X)\right|^2 \left|\frac{1}{s - m_X^2 + im\Gamma(X)}\right|^2 \\ &\times \int d\Pi_f \left|\mathcal{M}(X, s_X \to f)\right|^2 \\ =& \frac{1}{(2s_{i_1}+1)(2s_{i_2}+1)} \sum_{s_X} \frac{2m_X\Gamma(X \to f)}{(s - m_X^2)^2 + m_X^2\Gamma^2(X)} \times \frac{1}{4p_{CM}^i\sqrt{s}} \sum_{s_i} \left|\mathcal{M}(i \to X, s_X)\right|^2. \end{aligned} \tag{A.9}$$

Then we use

$$\Gamma(X \to i) = \frac{1}{2^{\delta_i}} \frac{1}{2s_X+1} \sum_{s_i}\sum_{s_X} \frac{1}{8\pi} \frac{p_{CM}^i}{m_X\sqrt{s}} \left|\mathcal{M}(i \to X, s_X)\right|^2 \tag{A.10}$$

to write $s = m^2 + 2m(\sqrt{s} - m) + ...$:

$$\sigma(i \to X \to f) = \frac{2s_X + 1}{(2s_{i_1}+1)(2s_{i_2}+1)} \frac{4\pi 2^{\delta_i}}{(p^i_{CM})^2} \frac{m_X^2 \Gamma(X \to i)\Gamma(X \to f)}{(s - m_X^2)^2 + m_X^2 \Gamma^2(X)}$$
$$\underset{s \simeq m_X^2}{\simeq} \frac{2s_X + 1}{(2s_{i_1}+1)(2s_{i_2}+1)} \frac{2^{\delta_i}\pi}{(p^i_{CM})^2} \frac{\Gamma(X \to i)\Gamma(X \to f)}{(\sqrt{s} - m_X)^2 + \Gamma^2(X)/4}, \tag{A.11}$$

which is exactly the same Breit-Wigner formula we found from Quantum Mechanics.

A.3 J_{ij}

In this appendix we derive expression (4.10) starting from (4.13).

Introducing Feynman parameters we can write

$$\begin{aligned} \mathrm{Re}J_{ij}(\Lambda) =& \pi k_i \cdot k_j \int_\mu^\Lambda \frac{d|\mathbf{q}|}{|\mathbf{q}|} \int d\Omega \int_0^1 dx \frac{1}{[x(E_i - \hat{\mathbf{q}} \cdot \mathbf{k}_i) + (1-x)(E_j - \hat{\mathbf{q}} \cdot \mathbf{k}_j)]^2} \\ =& 2\pi^2 k_i \cdot k_j \log\frac{\Lambda}{\mu} \int_0^1 dx \int_{-1}^1 d\cos\theta \frac{1}{[xE_i + (1-x)E_j - \cos\theta |x\mathbf{k}_i + (1-x)\mathbf{k}_j|]^2} \\ =& 4\pi^2 k_i \cdot k_j \log\frac{\Lambda}{\mu} \int_0^1 dx \frac{1}{[xE_i + (1-x)E_j]^2 - |x\mathbf{k}_i + (1-x)\mathbf{k}_j|^2} \\ =& 4\pi^2 k_i \cdot k_j \log\frac{\Lambda}{\mu} \int_0^1 dx \frac{1}{(xk_i + (1-x)k_j)^2}. \end{aligned} \tag{A.12}$$

The denominator has

$$(xk_i + (1-x)k_j)^2 = m_j^2 + 2xk_j \cdot (k_i - k_j) + x^2(k_i - k_j)^2 \tag{A.13}$$

and therefore has

$$\begin{aligned} \Delta^2 =& 4(k_i \cdot k_j - m_j^2)^2 - 4m_j^2(m_i^2 + m_j^2 - 2k_i \cdot k_j) \\ =& 4(k_i \cdot k_j)^2 - 4m_i^2 m_j^2. \end{aligned} \tag{A.14}$$

Using

$$\begin{aligned} \int_0^1 dx \frac{1}{ax^2 + bx + c} =& \frac{1}{a} \int_0^1 dx \frac{1}{(x - x_1)(x - x_2)} \\ =& \frac{1}{a} \frac{1}{x_1 - x_2} \log\frac{(1 - x_1)x_2}{(1 - x_2)x_1} \\ =& \frac{1}{\Delta} \log\frac{(1 - x_1)(-x_2)}{(1 - x_2)(-x_1)} \end{aligned} \tag{A.15}$$

for roots of the polynomial x_1, x_2. Now

$$x_1 x_2 = \frac{b^2}{4a^2} - \frac{\Delta^2}{4a^2} = \frac{c}{a} \tag{A.16}$$

and therefore

$$\begin{aligned} x_1x_2 - x_2 =& \frac{2c+b}{2a} + \frac{\Delta}{2a} \\ x_1x_2 - x_1 =& \frac{2c+b}{2a} - \frac{\Delta}{2a} \end{aligned} \tag{A.17}$$

and in this case

$$\begin{aligned} 2c+b =& 2m_j^2 + 2k_j \cdot (k_i - k_j) \\ =& 2k_j \cdot k_i \end{aligned} \tag{A.18}$$

Hence

$$\text{Re}J_{ij} = -\frac{4\pi^2 k_i \cdot k_j}{\Delta} \log \frac{2k_i \cdot k_j + \Delta}{2k_i \cdot k_j - \Delta}. \tag{A.19}$$

Defining

$$\begin{aligned} \beta_{ij} \equiv& \sqrt{1 - \frac{m_i^2 m_j^2}{(k_i \cdot k_j)^2}} \\ =& \frac{\Delta}{2|k_i \cdot k_j|} \end{aligned} \tag{A.20}$$

and note that with the metric $(+,-,-,-)$ that

$$k_i \cdot k_j = E_i E_j - |\mathbf{p}_i||\mathbf{p}_j| \cos\theta > 0 \tag{A.21}$$

(whereas for the oppisite sign metric we would need to put $k_i \cdot k_j \to -|k_i \cdot k_j|$) we have

$$\text{Re}J_{ij}(\Lambda) = \frac{2\pi^2}{\beta_{ij}} \log \frac{\Lambda}{\mu} \log \frac{1+\beta_{ij}}{1-\beta_{ij}}. \tag{A.22}$$

A.4 Colinear photon emission

Here we give the derivations of the spin-summed squared matrix element for photon emission from section 5.3.

For reference, we define the no-emmision amplitude as

$$\sum |\mathcal{M}_{(a)}(p_i, p_f)|^2 \equiv \sum \text{tr}\Big[\not{p}_f \mathcal{M}(p_i, p_f) \not{p}_i \mathcal{M}^\dagger(p_i, p_f)\Big]. \tag{A.23}$$

Note here that I specify the momenta involved explicitly.

We replace everywhere $\varepsilon_\mu \varepsilon_\nu \to -\eta_{\mu\nu}$ and square the matrix elements in (5.19):

$$\sum |\mathcal{M}_{(b)}|^2 = -\frac{e^2 Q_i^2}{(p_i')^4} \sum \text{tr}\Big[\not{p}_f \mathcal{M}(p_i', p_f) \not{p}_i' \gamma^\mu \not{p}_i \gamma_\mu \not{p}_i' \mathcal{M}^\dagger(p_i', p_f)\Big], \tag{A.24}$$

$$\sum \mathcal{M}_{(b)} \mathcal{M}_{(c)}^\dagger = -\frac{e^2 Q_i Q_f}{(p_i')^2 (p_f')^2} \sum \text{tr}\Big[\not{p}_f \mathcal{M}(p_i', p_f) \not{p}_i' \gamma^\mu \not{p}_i \mathcal{M}^\dagger(p_i, p_f') \not{p}_f' \gamma_\mu\Big], \tag{A.25}$$

$$\sum \mathcal{M}_{(c)} \mathcal{M}_{(b)}^\dagger = -\frac{e^2 Q_i Q_f}{(p_i')^2 (p_f')^2} \sum \text{tr}\Big[\not{p}_f \gamma^\mu \not{p}_f' \mathcal{M}(p_i, p_f') \not{p}_i \gamma_\mu \not{p}_i' \mathcal{M}^\dagger(p_i', p_f)\Big]. \tag{A.26}$$

Here I have given the momentum dependence of the matrix element. As stated in the text, we are concerned with the case that our incoming and ougoing electrons meet at a single vertex (or equivalently we could consider pair production of an electron/positron pair, or annihilation) so momentum conservation tells us that, for fixed q, I can take $\mathcal{M}(p_i', p_f) = \mathcal{M}(p_i, p_f') \neq \mathcal{M}(p_i, p_f)$. The form of $\mathcal{M}$ could be $\propto \gamma^\mu$ for exchange of a photon (or gluon in QCD) or it could be a scalar coupling for Higgs exchange: the actual form does not matter. Hence I can work everywhere with $\mathcal{M}(p_i', p_f)$.

Let us treat the first equation a little, using

$$\gamma^\mu \gamma_\nu \gamma_\mu = (2-d)\gamma_\nu \tag{A.27}$$

as well as

$$\begin{aligned} \not{p}_1 \not{p}_2 \not{p}_1 =& p_1^\mu p_2^\nu p_1^\rho \gamma_\mu \gamma_\nu \gamma_\rho \\ =& p_1^\mu p_2^\nu p_1^\rho [2\gamma_\mu \eta^{\nu\rho} - \gamma_\mu \gamma_\rho \gamma_\nu] \\ =& 2p_1 \cdot p_2 \not{p}_1 - p_1^2 \not{p}_2 \end{aligned} \tag{A.28}$$

$$\begin{aligned} \not{p}_i' \not{p}_i \not{p}_i' =& 2p_i' \cdot p_i \not{p}_i' - (p_i')^2 \not{p}_i \\ =& -(p_i')^2 \not{q} \end{aligned} \tag{A.29}$$

to give

$$\begin{aligned} \sum |\mathcal{M}_{(b)}|^2 =& 2\frac{e^2 Q_i^2}{(p_i')^4} \sum \mathrm{tr}\Big[\not{p}_f \mathcal{M} \not{p}_i' \not{p}_i \not{p}_i' \mathcal{M}^\dagger\Big] \\ =& -2\frac{e^2 Q_i^2}{(p_i')^2} \sum \mathrm{tr}\Big[\not{p}_f \mathcal{M} \not{q} \mathcal{M}^\dagger\Big] \\ =& \frac{-2z}{1-z}\frac{e^2 Q_i^2}{(p_i')^2} \sum |\mathcal{M}_{(a)}(p_i', p_f)|^2. \end{aligned} \tag{A.30}$$

For the other two we can directly make the replacement in the numerator of $p_f' \to p_f + z p_i, p_i' \to (1-z)p_i, q \to z p_i \to \frac{z p_i'}{(1-z)}$ to get

$$\sum \mathcal{M}_{(b)} \mathcal{M}_{(c)}^\dagger + \mathcal{M}_{(c)} \mathcal{M}_{(b)}^\dagger = -(1-z)\frac{e^2 Q_i Q_f}{(p_i')^2 (p_f')^2} \sum \mathrm{tr}\Big[(\not{p}_f \gamma^\mu \not{p}_f' + \not{p}_f' \gamma^\mu \not{p}_f) \mathcal{M} \not{p}_i \gamma_\mu \not{p}_i \mathcal{M}^\dagger\Big]. \tag{A.31}$$

Using

$$\begin{aligned} [\not{p}_1 \not{p}_2 \not{p}_3 + \not{p}_3 \not{p}_2 \not{p}_1] =& 2\not{p}_1 p_2 \cdot p_3 - 2\not{p}_2 p_1 \cdot p_3 + 2\not{p}_3 p_1 \cdot p_2. \\ \not{p}_1 \gamma^\mu \not{p}_2 + \not{p}_2 \gamma^\mu \not{p}_2 =& 2\not{p}_1 p_2^\mu + 2\not{p}_2 p_1^\mu - 2p_1 \cdot p_2 \gamma^\mu \end{aligned} \tag{A.32}$$

we obtain

$$\begin{aligned}
\sum \mathcal{M}_{(b)}\mathcal{M}_{(c)}^\dagger + \mathcal{M}_{(c)}\mathcal{M}_{(b)}^\dagger =& -(1-z)\frac{e^2Q_iQ_f}{(p_i')^2(p_f')^2}\sum \mathrm{tr}\left[(2\not{p}_f(p_f')^\mu + 2\not{p}_f'(p_f)^\mu - 2p_f\cdot p_f'\gamma^\mu)\mathcal{M}2\not{p}_i(p_i)_\mu\mathcal{M}^\dagger\right] \\
=& -4(1-z)\frac{e^2Q_iQ_f}{(p_i')^2(p_f')^2}\sum \mathrm{tr}\left[(\not{p}_f p_f'\cdot p_i + \not{p}_f' p_f\cdot p_i - p_f\cdot p_f'\not{p}_i)\mathcal{M}\not{p}_i\mathcal{M}^\dagger\right] \\
=& -4(1-z)\frac{e^2Q_iQ_f}{(p_i')^2(p_f')^2}\sum \mathrm{tr}\left[(\not{p}_f(p_f'\cdot p_i + p_f\cdot p_i) + z\not{p}_i p_f\cdot p_i - zp_f\cdot p_i\not{p}_i)\mathcal{M}\not{p}_i\mathcal{M}^\dagger\right] \\
=& -8(1-z)p_i\cdot p_f\frac{e^2Q_iQ_f}{(p_i')^2 2zp_i\cdot p_f}\sum \frac{|\mathcal{M}_{(a)}(p_i',p_f)|^2}{(1-z)} \\
=& -\frac{4(1-z)}{z(1-z)}\frac{e^2Q_iQ_f}{(p_i')^2}\sum |\mathcal{M}_{(a)}(p_i',p_f)|^2 \qquad \text{(A.33)}
\end{aligned}$$

So adding them together, and noting that here we must have $Q_f = Q_i$ we get

$$\begin{aligned}
\sum |\mathcal{M}_{ee\gamma}|^2 \to& -2\frac{2-2z+z^2}{z(1-z)}\frac{e^2Q_i^2}{(p_i')^2}\sum |\mathcal{M}_{(a)}(p_i',p_f)|^2 \\
=& -2\left[\frac{1+(1-z)^2}{z(1-z)}\right]\frac{e^2Q_i^2}{(p_i')^2}\sum |\mathcal{M}_{(a)}(p_i',p_f)|^2 \\
=& 2\left[\frac{1+(1-z)^2}{(1-z)}\right]\frac{e^2Q_i^2}{(k_\perp)^2}\sum |\mathcal{M}_{(a)}(p_i',p_f)|^2 \qquad \text{(A.34)}
\end{aligned}$$

where we put $(p_i')^2 \to -k_\perp^2/z$ in the denominator.

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