Combinatorics of Bethe ansatz and fusion products

Rinat Kedem

University of Illinois Urbana-Champaign

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Binomial coefficients

- The binomial coefficient ($m \geq 0$)

$$\binom{m+p}{m} = \frac{(p+m)(p+m-1)\cdots(p+1)}{m!}$$

  is defined for any $p \in \mathbb{Z}$.
- If $p \geq 0$, the binomial coefficient counts the number of ways of choosing m distinct elements out of $p+1$ choices.
- If $p < 0$, there is an identity,

$$\binom{m+p}{m} = (-1)^m \binom{-p-1}{m}.$$

  Can be negative.

Strange identity

- For choice of non-negative integers n_i, m_i ($i > 0$) and l, define

$$p_i = \sum_i \min(i,j)(n_j - 2m_j).$$

  There is an identity

$$\sum_{m_i \geq 0} \prod_i \binom{m_i + p_i}{m_i} = \sum_{m_i \geq 0 : p_i \geq 0} \prod_i \binom{m_i + p_i}{m_i}.$$

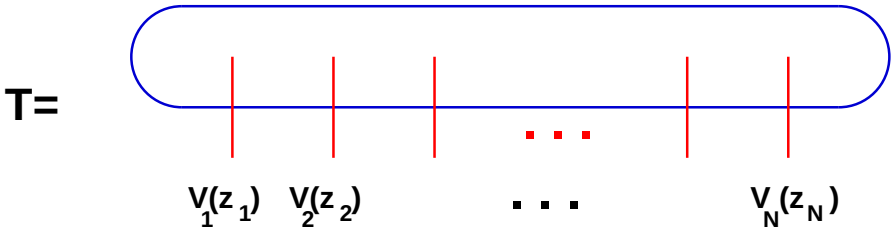
  The sum is taken over all m_i such that $0 \leq l = \sum_i i(n_i - 2m_i)$.
- Nontrivial combinatorial identity!
- There is such an identity for any simple Lie algebra $\mathfrak{g}$.
- Right hand side is a **combinatorial object**: Each term in the sum over m_i:

$$\prod_i \binom{m_i + p_i}{m_i}, \quad p_i \geq 0$$

  counts the number of ways to choose m_i distinct integers from the interval $[0, p_i]$ for each $i \geq 1$.

Completeness of Bethe ansatz states

- The inhomogeneous Heisenberg spin chain transfer matrix:



($V_i(z_i)$: irreducible reps of the Yangian $Y(\mathfrak{sl}_2)$; n_j=number of $j+1$-dimensional reps).

- The Hilbert space is

$$\mathcal{H}_{\mathbf{n}} = \bigotimes_{i=1}^{N} V_i(z_i) \underset{\mathfrak{sl}_2-\mathrm{mod}}{\simeq} \bigoplus_{l\geq 0} V(l)^{\oplus M_{l;\mathbf{n}}}$$

- **Completeness hypothesis:** There are as many Bethe vectors as the dimension of ($\mathfrak{sl}_2$-highest weight vectors in) $\mathcal{H}_{\mathbf{n}}$.

Completeness conjecture

- Hilbert space:

$$\mathcal{H}_{\mathbf{n}} \simeq \bigoplus_{l\geq 0} V(l)^{\oplus M_{l;\mathbf{n}}}$$

- Solutions to Bethe equations are parametrized by Bethe integers: Given $M = (\sum_i in_i - l)/2$ choose any partition of $M = \sum_i im_i$. Pick m_i distinct integers in the interval $[0, p_i]$.

$$p_i = \sum_j \min(i,j)(n_j - 2m_j)$$

- Completeness conjecture:

$$M_{l,\mathbf{n}} = \sum_{\substack{m_i : \sum_i i(n_i - 2m_i) = l \\ p_i \geq 0}} \prod_i \binom{m_i + p_i}{m_i}$$

- This conjecture was made by Kirillov-Reshetikhin; Proved.

- The left hand side (unrestricted sum) is a solution to a **recursion relation**. It is an alternating sum.
- Let $\mathfrak{g} = \mathfrak{sl}_2$ and $V(i)$ its $i+1$-dimensional irreducible representation.

Theorem (Kirillov)

The LHS is equal to the dimension of the space of $\mathfrak{sl}_2$-linear homomorphisms:

$$\sum_{\substack{m_i \\ \sum_i i(n_i - 2m_i) = l}} \binom{m_i + p_i}{p_i} = \mathrm{dimHom}_{\mathfrak{sl}_2}\left(\bigotimes_{i \geq 1} (V(i))^{\otimes n_i}, V(l) \right).$$

- HKOTY showed that this follows from the following interesting fact: The characters $Q_i = \mathrm{ch}\ V(i)$ of the irreducible representations of $\mathfrak{sl}_2$ satisfy the Q-system: (Kirillov-Resthikhin)

$$Q_{m+1} = \frac{Q_m^2 - 1}{Q_{m-1}}, \qquad Q_0 = 1, \qquad Q_1 = t.$$

(aka Chebyshev polynomials).

Aside: Discrete Hirota

For A_n, the characters $Q_{m,\alpha}$ of the irreducible module with highest weights $m\omega_\alpha$ satisfy

$$Q_{m+1,\alpha} Q_{m-1,\alpha} + Q_{m,i+\alpha} Q_{m,i-\alpha} = Q_{m,\alpha}^2$$

which is the combinatorial limit of the fusion relation for transfer matrices

$$T_{m+1,\alpha}(u) T_{m-1,\alpha}(u) + T_{m,\alpha+1}(u) T_{m,\alpha-1}(u) = T_{m,\alpha}(u+1) T_{m,\alpha}(u-1).$$

$T_{m,\alpha}(u)$: Transfer matrix with the auxiliary space a Yangian module with $\mathfrak{g}$-highest weight $m\omega_\alpha$ and spectral parameter u.

Define: $\tau_i(l,m) = T_{i,l+m}(l-m-i)$. Then:

$$\tau_i(l+1,m)\tau_i(l,m+1) + \tau_{i+1}(l+1,m)\tau_{i-1}(l,m+1) = \tau_i(l,m)\tau_i(l+1,m+1).$$

The various conjectures

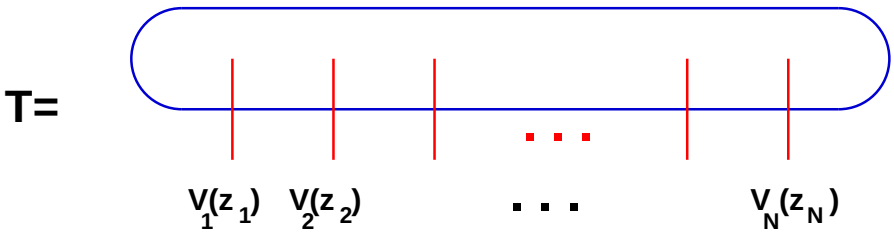
- **The Kirillov-Reshetikhin conjecture:**
  - **(KR1) Combinatorial version**: The Bethe equations (with String hypothesis) for inhomogeneous, generalized Heisenberg spin chain give a complete set of solutions for the eigenvectors of the Hamiltonian.
  - **(KR2) Representation theoretical version**: The Q-system is satisfied by the characters of KR-modules for any Lie algebra $\mathfrak{g}$. **PROVED**
- **The HKOTY conjecture**: (KR1)=(KR2).
- **(FL): The Feigin-Loktev conjecture:**
  The dimension of the fusion (graded tensor) product of localized $\mathfrak{g}[t]$-modules is independent of the localization parameters.

Theorem: [Ardonne-Kedem, 2007]: (KR1) $\Longrightarrow$ (FL)
(for the fusion product of KR-modules).

Today: How to prove HKOTY?

KR conjecture: Version 1 ($\mathfrak{g}$ simply-laced)

- Kirillov and Reshetikhin (1989): For any $\mathfrak{g}$ define generalized, inhomogeneous Heisenberg spin chain:



$V_i(z_i)$: irreducible KR-reps of the Yangian $Y(\mathfrak{g})$; Specified by $\mathbf{n} = \{n_{\alpha,i}\}_{1\leq\alpha\leq r; i\in\mathbb{N}}$

- Hilbert space

$$\mathcal{H}_{\mathbf{n}} = \bigotimes_i V_i(z_i) \underset{\mathfrak{g}}{\simeq} \bigoplus_\lambda V(\lambda)^{\oplus M_{\lambda;\mathbf{n}}}$$

- KR: The number of Bethe states is the number of ways to pick $m_{\alpha,i}$ distinct integers from the interval $[0, p_{\alpha,i}]$

$$p_{\alpha,i} = \sum_j \min(i,j) n_{\alpha,j} - \sum_{j,\beta} C_{\alpha,\beta} \min(i,j) m_{\beta,j}$$

such that for fixed $\lambda = \sum_\alpha l_\alpha \omega_\alpha$

$$q_\alpha := l_\alpha + \sum_{i,\beta} C_{\alpha,\beta} i m_{\beta,i} - \sum_i i n_{\alpha,i} = 0.$$

- **Completeness conjecture (KR1):** The Bethe ansatz gives a complete set of solutions:

$$M_{\lambda;\mathbf{n}} = \sum_{\substack{m_{\alpha,i} \\ p_{\alpha,i} \geq 0, q_\alpha = 0}} \prod_{\alpha,i} \binom{m_{\alpha,i} + p_{\alpha,i}}{m_{\alpha,i}}.$$

where

$$\otimes V_i(z_i) \underset{\mathfrak{g}}{\simeq} \underset{\lambda}{\oplus} V(\lambda)^{\oplus M_{\lambda;\mathbf{n}}}$$

The Q-system (Kirillov-Reshetikhin 89)

- Define the family $\{Q_{\alpha,i} : 1 \leq \alpha \leq \mathrm{rank}(\mathfrak{g}), i \in \mathbb{Z}_+\}$ by the recursion relation

$$Q_{\alpha,i+1} = \frac{Q_{\alpha,i}^2 - \prod_{\beta \neq \alpha} Q_{\beta,i}^{|C_{\alpha,\beta}|}}{Q_{\alpha,i-1}}, \; Q_{\alpha,0} = 1, \; Q_{\alpha,1} = t_\alpha.$$

($\mathfrak{g}$ simply-laced)

Theorem (Nakajima,Hernandez)

(Kirillov-Reshetikhin conjecture, version 2) The characters of KR-modules for **any** $\mathfrak{g}$ *satisfy the Q-system.*

- **Corollary:** $Q_{\alpha,i}$ is a polynomial in t_β!

Theorem (HKOTY)

The Q-system theorem implies

$$N_{\lambda;\mathbf{n}} := \sum_{\substack{m_{\alpha,i} \\ q_\alpha = 0}} \prod_{\alpha,i} \binom{m_{\alpha,i} + p_{\alpha,i}}{m_{\alpha,i}} = \dim \mathrm{Hom}_{\mathfrak{g}} \left(\otimes V_i, \; V(\lambda)\right)$$

HKOTY conjecture

HKOTY conjectured that

$$\sum_{m_{\alpha,i}} \prod_{\alpha,i} \binom{m_{\alpha,i}+p_{\alpha,i}}{m_{\alpha,i}} = N_{\lambda;\mathbf{n}} = M_{\lambda;\mathbf{n}} = \sum_{\substack{m_{\alpha,i} \\ p_{\alpha,i}\geq 0}} \prod_{\alpha,i} \binom{m_{\alpha,i}+p_{\alpha,i}}{m_{\alpha,i}}.$$

(both sums restricted to $q_\alpha = 0$)

- It is true in the cases where (KR1) was proved (very indirect argument, only proved in special cases).
- We need to prove this to have completeness of Bethe states.
- We also need this to prove the Feigin Loktev conjectures...

Feigin-Loktev Fusion products

- Let $\mathfrak{g}$ = Lie algebra
  $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$ the Lie algebra of polynomials with coefficients in $\mathfrak{g}$.
  The generators of $\mathfrak{g}[t]$ are $x[n] := x \otimes t^n$ with $x \in \mathfrak{g}$.
- V = a finite-dimensional $\mathfrak{g}[t]$-module.
  Define $V(z)$ ($z \in \mathbb{C}^*$): the module "localized at z":

$$x \otimes t^n v_z = x \otimes (t_z + z)^n v_z, \qquad v_z \in V(z),$$

  where $x \otimes t_z^n$ acts on v_z in the same way that $x[n]$ acts on $v \in V$.
- Example: $\mathfrak{g} = \mathfrak{sl}_2$ and V the irreducible $j+1$-dimensional rep.
  $V(z)$ = the evaluation module at z: $x \otimes t^n v = z^n x v$.
- Assume that V is a $\mathfrak{g}[t]$-module, cyclic with cyclic vector v:

$$V = U(\mathfrak{g}[t])v$$

- Any $\mathfrak{g}[t]$-module is also a $\mathfrak{g}$ module and $V(z) \simeq V$ as $\mathfrak{g}$-modules.

Lemma

Let $\{z_1, ..., z_N\}$ be distinct complex numbers and $\{V_1(z_1), ..., V_N(z_N)\}$ be $\mathfrak{g}[t]$ modules localized at z_i (finite-dimensional, cyclic with cyclic vectors v_i). Then as $\mathfrak{g}$-modules,

$$V_1 \otimes \cdots \otimes V_N \simeq U(\mathfrak{g}[t])\,(v_1 \otimes \cdots \otimes v_N)$$

This is also a finite-dimensional, cyclic $\mathfrak{g}[t]$-module, with cyclic vector $v_1 \otimes \cdots \otimes v_n$.

Grading

- The algebra $U(\mathfrak{g}[t])$ is graded by degree in t.
  $U^{\leq n}$ = polynomials in generators of $\mathfrak{g}[t]$ with total degree $\leq n$ in t.
- Let $V = U(\mathfrak{g}[t])v$
- V inherits a filtration $\mathcal{F}$ from the action $U(\mathfrak{g}[t])$:

$$\mathcal{F}[n] = U^{\leq n}v, \qquad \mathbb{C}v \subset \mathcal{F}[0] \subset \cdots \subset \mathcal{F}[n] \subset \mathcal{F}[n+1] \subset \cdots \subset V$$

- Define the graded module:

$$\overline{V} = \mathrm{Gr}\ \mathcal{F} = \bigoplus_n \mathcal{F}[n]/\mathcal{F}[n-1].$$

  The graded components are $\mathfrak{g}$-modules.

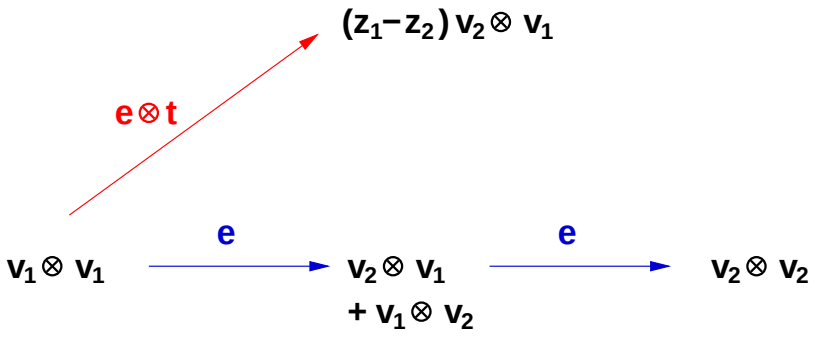
Definition (The Feigin-Loktev fusion product)

$$V_1 \star \cdots \star V_N(z_1, ..., z_N) = \overline{\bigotimes_i V_i(z_i)}$$

with cyclic vector $\otimes v_i$.

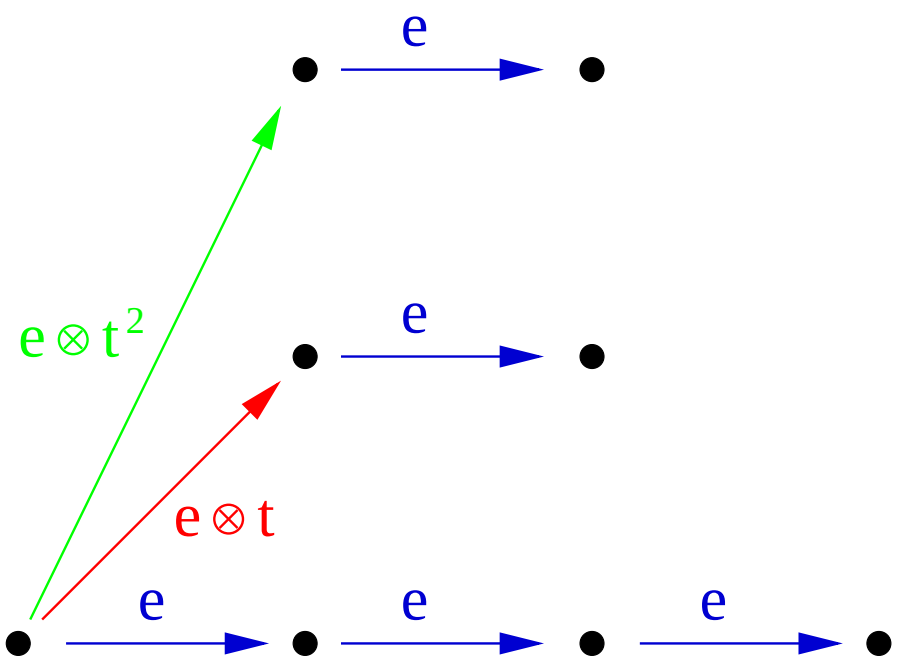
Example 1:

Let $V = 2$-dim representation of $\mathfrak{sl}_2$ with basis v_1, v_2 where v_1 is the highest weight vector. Then $V \star V(z_1, z_2)$ has this structure:



$$V \star V \simeq \underset{0}{V(2)} \oplus \underset{1}{V(0)}.$$

Example 2:



$$V \star V \star V \simeq \underset{0}{V(3)} \oplus \underset{1}{V(1)} \oplus \underset{2}{V(1)}$$

The Feigin-Loktev conjectures

- The fusion product is graded:

$$V_1 \star \cdots \star V_N(z_1, ..., z_N) = \overline{\bigotimes_i V_i(z_i)} \underset{\mathfrak{g}}{\simeq} \bigoplus_{i \geq 0} \bigoplus_{\lambda} V(\lambda)^{\oplus M_{\lambda;\mathbf{n}}[i]}$$

- Define the generating function for graded multiplicities

$$M_{\lambda;\mathbf{n}}(q) = \sum_i q^i M_{\lambda;\mathbf{n}}[i]$$

- **The conjectures:** [Feigin, Loktev 1999]
  1. $M_{\lambda;\mathbf{n}}$ are independent of the localization parameters z_i.
  2. $M_{\lambda;\mathbf{n}}$ are generalized Kostka polynomials. [Kirillov, Schilling, Shimozono, HKOTY]

The status of the conjectures:

- FJKLM 2001 (2) is true for $\mathfrak{sl}_2$-modules.
- Kedem (2004): (1) is true for tensor products of symmetric power representations of $\mathfrak{sl}_n$.
- Chari-Loktev (2005): (1) is true for tensor products of fundamental representations of $\mathfrak{sl}_n$.
- Ardonne-Kedem (2007): (1) and (2) are true for the tensor products of Kirillov-Reshetikhin modules of any $\mathfrak{g}$ if the combinatorial KR conjecture (KR1) is true.
- But (KR1) has been proven only for:
  - $\mathfrak{g} = A_n$ and $V_i(z_i)$ is the evaluation module with highest weight a rectangular Young diagram. Kirillov, Schilling, Shimozono 02
  - Special cases of modules of other non-exceptional Lie algebras Schilling, Okado, Shimozono.

We need to prove (KR1) in general to prove the FL-conjecture.

Proof that $M = N$ (Joint with Di Francesco)

- To prove the identity, relax all the restrictions and define a generating function:

$$Z_{\mathbf{n}}^{(k)}(\mathbf{u}) = \sum_{m_{\alpha,1},\ldots,m_{\alpha,k}\geq 0} \prod_{\alpha} u_{\alpha}^{q_\alpha} \prod_{i=1}^{k} u_{\alpha,i}^{q_{\alpha,i}} \binom{m_{\alpha,i}+q_{\alpha,i}}{m_{\alpha,i}}$$

$$q_{\alpha,i} = l_\alpha + \sum_{j=1}^{k-i} j \left(\sum_{\beta} C_{\alpha,\beta}\, m_{\beta,i+j} - n_{\alpha,i+j} \right).$$

- $q_{\alpha,i} = p_{\alpha,i} + q_\alpha$ with $m_{\alpha,j>k} = 0 = n_{\alpha,j>k}$.
- When $k \to \infty$ and $u_{\alpha,i>0} = 1$ the constant term in all u_α is $N_{\lambda,\mathbf{n}}$.
- Taking only non-negative powers in $u_{\alpha,i>0}$ in the same limit, the constant term gives $M_{\lambda,\mathbf{n}}$.

Theorem

The generating function factorizes:

$$Z_{\mathbf{n}}^{(k)}(\mathbf{u}) = \prod_{\alpha} \frac{\mathcal{Q}_{\alpha,1}(\mathbf{u})\mathcal{Q}_{\alpha,k}(\mathbf{u})}{\mathcal{Q}_{\alpha,k+1}(\mathbf{u})} \prod_{i=1}^{k} u_{\alpha,i}^{-1} \mathcal{Q}_{\alpha,i}(\mathbf{u})^{n_{\alpha,i}}$$

$\mathcal{Q}(\mathbf{u})$ are defined via the recursion

$$\mathcal{Q}_{\alpha,i+1}(\mathbf{u}) = \frac{\mathcal{Q}_{\alpha,i}^2 - \prod_{\beta\neq\alpha} \mathcal{Q}_{\beta,i}^{|C_{\alpha,\beta}|}}{u_{\alpha,i}\mathcal{Q}_{\alpha,i-1}}, \quad \mathcal{Q}_{\alpha,0} = 1 \text{ and } \mathcal{Q}_{\alpha,1} = 1/u_\alpha.$$

A generalized Q-system!

Theorem

$\mathcal{Q}_{\alpha,i}$ *satisfy the* $\mathcal{Q}$*-system if and only if they satisfy*

$$\mathcal{Q}_{\alpha,i+j}(\mathbf{u}) = \mathcal{Q}_{\alpha,i}(\mathbf{u}^{(j)})$$

where

$$u_\alpha^{(j)} = \frac{1}{\mathcal{Q}_{\alpha,j+1}(\mathbf{u})}, \; u_{\alpha,1}^{(j)} = \mathcal{Q}_{\alpha,j}(\mathbf{u})u_{\alpha,j+1}, \; u_{\alpha,i}^{(j)} = u_{\alpha,j+i} \; (i>1).$$

Theorem: $M = N$

- From the factorization property:

$$Z^{(k)}_{\mathbf{n}}(\mathbf{u}) = Z^{(j)}_{\mathbf{n}_1,\ldots,\mathbf{n}_j}(\mathbf{u}) Z^{(k-j)}_{\mathbf{n}_{j+1},\ldots,\mathbf{n}_k}(\mathbf{u}^{(j)}).$$

- If we evaluate this at $u_{\alpha,i} = 1 (1 \leq i < j)$, the first factor is

$$\prod_{\alpha} \frac{Q_{\alpha,1} Q_{\alpha,j}}{Q_{\alpha,j+1}} \prod_{i} Q_{\alpha,i}^{n_{\alpha,i}}$$

- The second factor is

$$\sum_{m_{j+1},\ldots,m_k} \prod_{\alpha} \left(\frac{u_{\alpha,j}}{Q_{\alpha,j+1}} \right)^{q_{\alpha,j}} Q_{\alpha,j}^{q_{\alpha,j+1}} \prod_{i=j+1}^{k} u_{\alpha,i}^{q_{\alpha,i}} \binom{q_{\alpha,i} + m_{\alpha,i}}{m_{\alpha,i}}$$

- Terms with $q_{\alpha,j+1} \geq 0$ and $q_{\alpha,j} < 0$ do not contribute to the constant term in u_α.

Theorem

There is an equality of power series in $\{t_\alpha\}$:

$$PS_{u_1,\ldots,u_r} Z^{(k)}_{\lambda;\mathbf{n}}(\mathbf{u}) \Big|_{u_{\alpha,i}=1,\forall\alpha,i} = PS_{u_1,\ldots,u_r} Z^{(k)}_{\lambda;\mathbf{n}}(\mathbf{u})^{(+)} \Big|_{u_{\alpha,i}=1,\forall\alpha,i}$$

where $Z^{(k)}_{\lambda;\mathbf{n}}(\mathbf{u})^{(+)}$ is the generating function defined with summation restricted to $q_{\alpha,i} \geq 0$.

Corollary

The constant term of this identity and $k \to \infty$:

$$M_{\lambda;\mathbf{n}} = N_{\lambda;\mathbf{n}}$$

Corollary

For all $\mathfrak{g}$ and for any set of KR-modules, the Bethe integers enumerate eigenstates of the transfer matrix.

Corollary

The FL conjectures hold for Chari's KR modules, and the graded multiplicities are the sums given by products of q-binomial coefficients of [HKOTY].

- Hernandez's theorem that the Q-system is solved by characters of KR-modules is essential: There is no other proof that solutions of the Q-systems are polynomials in $Q_{\alpha,1}$. Think generalized Chebyshev polynomials.
- This is a strong version of the Laurent phenomenon in cluster algebras [Fomin, Zelevinsky]. How general is this phenomenon?
- What about other types of modules (not of KR type)?