

Universal Vertex-IRF Transformation and Quantum Whittaker Vectors

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- “Quantum Dynamical coBoundary Equation for finite dimensional simple Lie algebras” **E.B., Ph.Roche, V.Terras**, math.QA/0512500, Adv.Math.214 (2007) 181–229
- “Universal Vertex-IRF Transformation for Quantum Affine Algebras” **E.B., Ph.Roche, V.Terras**, arXiv:0707.0955 submit. Adv.Math.
- “Quantum Whittaker vectors and dynamical coboundary equation”

8-vertex model:

2-d square lattice model

link $\rightarrow \varepsilon_j = \pm$

vertex \rightarrow Boltzmann weight

z = spectral parameter,

p = elliptic parameter

$$\mathbf{R}^{8V}(z_1/z_2)_{\varepsilon'_1, \varepsilon'_2}^{\varepsilon_1, \varepsilon_2} = z_2 \begin{array}{c} \varepsilon_1 \\ \swarrow \quad \searrow \\ \varepsilon'_2 & \varepsilon_2 \\ \downarrow & \downarrow \\ \varepsilon'_1 & z_1 \end{array}$$

$$\mathbf{R}^{8V}(z) = \begin{pmatrix} a(z; p) & 0 & 0 & d(z; p) \\ 0 & b(z; p) & c(z; p) & 0 \\ 0 & c(z; p) & b(z; p) & 0 \\ d(z; p) & 0 & 0 & a(z; p) \end{pmatrix}$$

$$\begin{aligned} a(z; p) &= \frac{\Theta_{p^4}(p^2 z) \Theta_{p^4}(p^2 q^2)}{\Theta_{p^4}(p^2) \Theta_{p^4}(p^2 q^{-2} z)} \\ b(z; p) &= q^{-1} \frac{\Theta_{p^4}(z) \Theta_{p^4}(p^2 q^2)}{\Theta_{p^4}(p^2) \Theta_{p^4}(q^{-2} z)} \\ c(z; p) &= \frac{\Theta_{p^4}(p^2 z) \Theta_{p^4}(q^{-2})}{\Theta_{p^4}(p^2) \Theta_{p^4}(q^{-2} z)} \\ d(z; p) &= pq^{-1} \frac{\Theta_{p^4}(z) \Theta_{p^4}(q^2)}{\Theta_{p^4}(p^2) \Theta_{p^4}(q^{-2} z)} \\ \Theta_p(x) &:= (x, px^{-1}, p; p)_\infty \end{aligned}$$

satisfying the Quantum Yang-Baxter Equation (QYBE) with spectral param.
 No charge conservation through a vertex \rightarrow no direct Bethe Ansatz solution
 Baxter's solution (Ann.Phys.1973) \rightarrow map onto an IRF model (SOS model)

SOS model (Interaction-Round-a-Face model):

2-d square lattice model

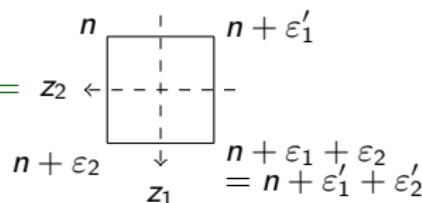
vertex \rightarrow local height n_j

$$n_j - n_k = \pm 1 \text{ (adjacent)}$$

face \rightarrow Boltzmann weight

w dynamical parameter

$$\mathbf{R}^{SOS}\left(\frac{z_1}{z_2}; w_0 q^n\right)_{\varepsilon'_1, \varepsilon'_2}^{\varepsilon_1, \varepsilon_2} =$$



$$\mathbf{R}^{SOS}(z; w) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(z; w, p) & c(z; w, p) & 0 \\ 0 & zc(z; \frac{p}{w}, p) & b(z; \frac{p}{w}, p) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$b(z; w, p) = q^{-1} \frac{\Theta_{p^2}(z)(q^2 p^2 w^{-2}, q^{-2} p^2 w^{-2}; p^2)_\infty}{\Theta_{p^2}(q^{-2} z)(p^2 w^{-2}; p^2)_\infty^2}$$

$$c(z; w, p) = \frac{\Theta_{p^2}(q^{-2}) \Theta_{p^2}(w^2 z)}{\Theta_{p^2}(w^2) \Theta_{p^2}(q^{-2} z)}$$

satisfying the **Dynamical** Quantum Yang-Baxter Equation (DQYBE):

$$\mathbf{R}_{12}^{SOS}(z_1/z_2; w q^{h_3}) \mathbf{R}_{13}^{SOS}(z_1; w) \mathbf{R}_{23}^{SOS}(z_2; w q^{h_1})$$

$$= \mathbf{R}_{23}^{SOS}(z_2; w) \mathbf{R}_{13}^{SOS}(z_1; w q^{h_2}) \mathbf{R}_{12}^{SOS}(z_1/z_2; w) \quad \text{with} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Charge conservation, solvable by Bethe Ansatz

Baxter's transformation:

It is equivalent to the following Dynamical Gauge Equivalence:

$$\mathbf{R}_{12}^{SOS}(z_1/z_2; w) = \mathbf{M}_1(z_1; wq^{h_2})\mathbf{M}_2(z_2; w)\mathbf{R}_{12}^{8V}(z_1/z_2)\mathbf{M}_1(z_1; w)^{-1}\mathbf{M}_2(z_2; wq^{h_1})^{-1}$$

with

$$\mathbf{M}(z, w, p)^{-1} = \begin{pmatrix} \vartheta_{p^4}(-p^3 w^{-2} z) & pz^{-1} \vartheta_{p^4}(-p^{-1} w^2 z) \\ w \vartheta_{p^4}(-pw^{-2} z) & w \vartheta_{p^4}(-pw^2 z) \end{pmatrix} \Lambda(z; w, p)$$

where $\Lambda(z; w, p) = \text{diagonal matrix.}$

- Eigenvalues and eigenvectors of the 8-Vertex Transfer Matrix (Baxter, 1973)
- Correlation Functions of the 8-Vertex Model from the SOS ones (Lashkevich and Pugai, 1997)

Gervais-Neveu-Felder R -matrix :

In their study of \mathfrak{sl}_{r+1} -Toda Field theory, Gervais-Neveu ($r = 1$ case, 1984), and then Cremmer-Gervais (1989), introduced the "Face-type" standard trigonometric solution

$$\begin{aligned}
 \mathbf{R}^{GN}(x) = q^{-\frac{1}{r+1}} & \left\{ q \sum_i E_{i,i} \otimes E_{i,i} + \sum_{i \neq j} \left(q - q^{-1} \frac{\nu_i}{\nu_j} \right) \left(1 - \frac{\nu_i}{\nu_j} \right)^{-1} E_{i,i} \otimes E_{j,j} \right. \\
 & \left. + (q - q^{-1}) \sum_{i \neq j} \left(1 - \frac{\nu_i}{\nu_j} \right)^{-1} E_{i,j} \otimes E_{j,i} \right\}
 \end{aligned}$$

($\prod_{i=1}^{r+1} \nu_i = 1$ and $x_i^2 = \frac{\nu_i}{\nu_{i+1}}$) of the QDYBE:

$$\mathbf{R}_{12}^{GN}(x) \mathbf{R}_{13}^{GN}(xq^{h_2}) \mathbf{R}_{23}^{GN}(x) = \mathbf{R}_{23}^{GN}(xq^{h_1}) \mathbf{R}_{13}^{GN}(x) \mathbf{R}_{12}^{GN}(xq^{h_3})$$

where $xq^h = (x_1 q^{h_{\alpha_1}}, \dots, x_r q^{h_{\alpha_r}})$.

Cremmer-Gervais-Bilal R -matrix :

It exists, in the fundamental representation of $U_q(\mathfrak{sl}_{r+1})$,

- an invertible element $\mathbf{M}(x)$
- a certain non-standard solution \mathbf{R}^J of the (non-dynamical) QYBE
(Cremmer-Gervais's R-matrix)

such that

$$\mathbf{R}^{GN}(x)\mathbf{M}_1(xq^{h_2})\mathbf{M}_2(x) = \mathbf{M}_2(xq^{h_1})\mathbf{M}_1(x)\mathbf{R}^{CG}$$

Explicitly,

$$\begin{aligned} \mathbf{R}^{CG} = q^{-\frac{1}{r+1}} & \left\{ q \sum_{t,s} q^{-\frac{2(s-t)}{r+1}} E_{tt} \otimes E_{s,s} \right. \\ & \left. + (q - q^{-1}) \sum_{i,j,k} \eta(i,j,k) q^{-\frac{2(i-k)}{r+1}} E_{i,j+i-k} \otimes E_{j,k} \right\}, \end{aligned}$$

(with $\eta(i,j,k) = 1$ if $i \leq k < j$, -1 if $j \leq k < i$, 0 otherwise) and $\mathbf{M}(x)$ is given by a Van der Monde matrix (up to a diagonal matrix $\mathcal{U}(x)$):

$$\mathbf{M}(x)^{-1} = \left(\sum_j \nu_j^{i-1} E_{i,j} \right) \mathcal{U}(x).$$

\mathfrak{g} fin. dim. or affine Lie algebra, \mathfrak{h} its Cartan subalg., let $\overset{\circ}{I} = \{\alpha_1, \dots, \alpha_r\}$,

and if $\mathfrak{g} = \mathfrak{sl}_{r+1}$, $I := \overset{\circ}{I}$, if $\mathfrak{g} = A_r^{(1)}$, $I = \{\alpha_0\} \cup \overset{\circ}{I}$

$$(\zeta^i, h_{\alpha_j}) = \delta_{i,j}, \quad \forall i, j \in I, \quad (\zeta^d, h_{\alpha_j}) = 0, \quad \forall j \in I, \quad (\zeta^d, d) = 1.$$

$U_q(\mathfrak{g})$ is the algebra generated by $e_{\pm\alpha_i}$, $i \in I$ and $q^h, h \in \mathfrak{h}$ with relations:

$$q^h e_{\pm\alpha_i} q^{-h} = q^{\pm\alpha_i(h)} e_{\pm\alpha_i}, \quad q^h q^{h'} = q^{h+h'}, \quad [e_{\alpha_i}, e_{-\alpha_j}] = \delta_{ij} \frac{q^{h_{\alpha_i}} - q^{-h_{\alpha_i}}}{q - q^{-1}},$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_q e_{\pm\alpha_i}^{1-a_{ij}-k} e_{\pm\alpha_j} e_{\pm\alpha_i}^k = 0, \quad i \neq j$$

$U_q(\mathfrak{b}^\pm)$ (resp. $U_q(\mathfrak{n}^\pm)$): sub Hopf algebra generated by $q^h, e_{\pm\alpha_i}$ (resp. $e_{\pm\alpha_i}, i \in I$). $U_q^\pm(\mathfrak{g}) = \text{Ker}(\iota^\pm)$ with $\iota^\pm : U_q(\mathfrak{b}^\pm) \rightarrow U_q(\mathfrak{h})$ projections

$U_q(\mathfrak{g})$ is a quasitriangular Hopf algebra with $\Delta(e_{\alpha_i}) = e_{\alpha_i} \otimes q_i^{h_{\alpha_i}} + 1 \otimes e_{\alpha_i}$, $\Delta(e_{-\alpha_i}) = e_{-\alpha_i} \otimes 1 + q_i^{-h_{\alpha_i}} \otimes e_{-\alpha_i}$, and $\Delta(h) = h \otimes 1 + 1 \otimes h$ with universal R-matrix:

$$R = K \widehat{R} \quad \text{with} \quad K = q^{\sum_I h_I \otimes \zeta^I} \in U_q(\mathfrak{h})^{\otimes 2}, \quad \widehat{R} \in 1^{\otimes 2} \oplus U_q^+(\mathfrak{g}) \otimes U_q^-(\mathfrak{g})$$

Universal solutions $R(x)$ of the QDYBE can be build in terms of the standard solution R of the QYBE and **Quantum Dynamical Cocycles** as

$$R(x) = J_{21}(x)^{-1} R_{12} J_{12}(x),$$

Definition : Quantum Dynamical Cocycle

$J : (\mathbb{C}^\times)^{\dim \mathfrak{l}} \rightarrow U_q(\mathfrak{g})^{\otimes 2}$ where $J(x)$ is invertible, is of zero \mathfrak{l} -weight, and satisfies the **Quantum Dynamical Cocycle Equation (QDCE)**:

$$(\Delta \otimes id)(J(x)) J_{12}(xq^{h_3}) = (id \otimes \Delta)(J(x)) J_{23}(x).$$

Quasi-Hopf interpretation, Dynamical Vertex-Operator approaches of 8–vertex correlation functions.

Explicit construction by means of an **auxiliary linear equation**:

- Standard solution $R(x)$ in the finite dimensional case (Arnaudon, Buffenoir, Ragoucy, Roche, 1998)
- Standard IRF solution and Belavin-Baxter's Vertex solution in the $A_r^{(1)}$ case (Jimbo, Konno, Odake, Shiraishi, 1999)
- Extension to other solutions (in the formalism of generalized Belavin-Drinfeld Triple) (Etingof, Schedler, Schiffmann, 2000)

Verma modules

$$\begin{aligned} \mathcal{V}_\eta &= U_q(\mathfrak{g}).|\emptyset\rangle_\eta \quad x.|\emptyset\rangle_\eta = 0, \quad \forall x \in U_q^+(\mathfrak{g}). \\ u.|\emptyset\rangle_\eta &= \Upsilon_\eta(u)|\emptyset\rangle_\eta, \quad \forall u \in U_q(\mathfrak{h}), \end{aligned}$$

Shapovalov's form

$$\mathcal{S}_\eta : \mathcal{V}_{-\eta^*} \times \mathcal{V}_\eta \rightarrow \mathbb{C}$$

unique hermitian form such that

$$\mathcal{S}_\eta(|\emptyset\rangle_{-\eta^*}, |\emptyset\rangle_\eta) = 1 \quad \mathcal{S}_\eta(v, x.v') = \mathcal{S}_\eta(x^*.v, v')$$

with $q^* = q$ $e_{\alpha_i}^* = e_{-\alpha_i} q^{h_{\alpha_i}}$ $e_{-\alpha_i}^* = q^{-h_{\alpha_i}} e_{\alpha_i}$ $h_{\alpha_i}^* = h_{\alpha_i}$
 non-degenerate if and only if \mathcal{V}_η is irreducible.

Fusion Operators of Verma Modules

Φ element of $\text{Hom}_{U_q(\mathfrak{g})}(\mathcal{V}_{\eta'}, \mathfrak{V} \otimes \mathcal{V}_\eta)$ (where \mathfrak{V} is a finite dimensional $U_q(\mathfrak{g})$ -module). We define $\langle \Phi \rangle$ by

$$\langle \Phi \rangle \in \mathfrak{V}[\eta' - \eta] / \Phi(|\emptyset\rangle_{\eta'}) = \langle \Phi \rangle \otimes |\emptyset\rangle_\eta + \sum_{(\gamma) \neq \emptyset} a_{(\gamma)} \otimes f_{(\gamma)} |\emptyset\rangle_\eta.$$

If $\mathcal{V}_{\eta'}$ is irreducible then $\langle \rangle : \text{Hom}_{U_q(\mathfrak{g})}(\mathcal{V}_{\eta'}, \mathfrak{V} \otimes \mathcal{V}_\eta) \rightarrow \mathfrak{V}[\eta' - \eta]$ is an isomorphism. For any homogeneous element v_μ of weight μ , we denote $\Phi_\eta^{v_\mu} \in \text{Hom}_{U_q(\mathfrak{g})}(\mathcal{V}_{\eta+\mu}, \mathfrak{V} \otimes \mathcal{V}_\eta)$ the unique element such that $\langle \Phi_\eta^{v_\mu} \rangle = v_\mu$.

Fusion Operators and "Face Type" Quantum Dynamical Cocycles

For any homogeneous vector $v_{\mu_i} \in \mathfrak{V}$ of weight μ_i and any $\eta \in \mathfrak{h}^*$, we have

$$\begin{aligned} \Phi_\eta^{v_\mu} |\emptyset\rangle_{\eta+\mu} &= (\pi \otimes id)(J_F(\hat{x})) \cdot (v_\mu \otimes |\emptyset\rangle_\eta) \quad \hat{x}(h_{\alpha_i}) = q, \forall i \\ ((1 \otimes \Phi_\eta^{v_{\mu_2}}) \Phi_{\eta+\mu_2}^{v_{\mu_1}}) |\emptyset\rangle_{\eta+\mu_1+\mu_2} &= . \Phi_\eta^{J_F(\hat{x}q^\eta)(v_{\mu_1} \otimes v_{\mu_2})} |\emptyset\rangle_{\eta+\mu_1+\mu_2} \end{aligned}$$

J_F verifies quantum Dynamical Cocycle equation.

Relation of matrix elements of J_F and Shapovalov's coefficients...



Definition : Generalized Translation Quadruple $(\theta_{[x]}^+, \theta_{[x]}^-, \varphi, S)$

$\theta_{[x]}^\pm \in \text{End}(U_q(\mathfrak{b}^\pm))$, φ and S are invertible elements of $U_q(\mathfrak{h})^{\otimes 2}$ such that

$$\theta_{[x]q^{h_2}}^\pm = Ad_\varphi^{\mp 1} \circ \theta_{[x]1}^\pm \quad \theta_{[x]1}^+(\widehat{R}) = (Ad_\varphi \circ \theta_{[x]2}^-)(\widehat{R})$$

$$\theta_{[x]1}^\pm(\varphi_{12}) = \theta_{[x]2}^\pm(\varphi_{12}) = \varphi_{12} \quad \theta_{[x]1}^+(S_{12}) = \theta_{[x]2}^-(S_{12})$$

$$\forall v \in U_q(\mathfrak{b}^-), \quad [\varphi_{12} K_{12} S_{21}^{-1} S_{12} \quad \theta_{[x]1}^-(K_{12} S_{21} S_{12}^{-1}) \ , \ \theta_{[x]1}^-(v)] = 0.$$

Theorem : Auxilliary Linear Problem (ABRR equation)

Let $(\theta_{[x]}^+, \theta_{[x]}^-, \varphi, S)$ be a generalized translation quadruple. The linear equation

$$\widehat{J}(x) = \theta_{[x]2}^- \left(Ad_{S^{-1}}(\widehat{R}) \widehat{J}(x) \right)$$

admits a unique solution $\widehat{J}(x) \in 1 \otimes 1 + U_q^+(\mathfrak{g}) \otimes U_q^-(\mathfrak{g})$ and $J(x) = S\widehat{J}(x)$ satisfies the Quantum Dynamical Cocycle Equation.

→ explicit formula for quantum dynamical cocycles as infinite product:

$$J_{12}(x) = S_{12} \prod_{k=1}^{+\infty} (\theta_{[x]2}^-)^k \left(S_{12}^{-1} \widehat{R} S_{12} \right)$$

Standard IRF solutions

Standard IRF Solution $\mathfrak{g} = \mathfrak{sl}_{r+1}$ or $\mathfrak{g} = A_r^{(1)}$, $\mathfrak{l} = \mathfrak{h}$

$$\theta_{[x]}^{\pm} = Ad_{B(x)^{\pm 1}}, \quad \varphi = K^{-2}, \quad S = 1,$$

with $B(x) \in U_q(\mathfrak{h})$ such that $\Delta(B(x)) = B_1(x) B_2(x) K^2$, $B_1(xq^{h_2}) = B_1(x) K^2$.
 This solution leads to the fusion matrix J_F .

Arnaudon, Buffenoir, Ragoucy, Roche, 1998 for the finite case, Jimbo, Konno, Odake, Shiraishi, 1999, for the affine case

→ "Face-type" solution $R^{GN}(x)$ in the $\mathfrak{g} = A_n$ case in the fundamental evaluation representation

→ "Face-type" solution $R^{SOS}(z; w, p)$ in the $\mathfrak{g} = A_1^{(1)}$ case (and higher rank generalizations) in the fundamental evaluation representation

Extremal Vertex Solutions

Cremmer-Gervais-Bilal's Vertex Solutions ($\mathfrak{g} = A_r$, $\mathfrak{l} = 0$)

Obtained with the following choice of Generalized Translation Quadruple:

$$\begin{aligned}\theta_{[x]}^\pm(e_{\pm\alpha_i}) &= e_{\pm\alpha_{i\mp 1}}, & \theta_{[x]}^+(e_{\alpha_1}) &= 0, & \theta_{[x]}^-(e_{-\alpha_r}) &= 0, & \varphi &= 1, \\ \theta_{[x]}^\pm(\zeta^{\alpha_i}) &= \zeta^{\alpha_{i\mp 1}}, & \theta_{[x]}^+(\zeta^{\alpha_1}) &= 0, & \theta_{[x]}^-(\zeta^{\alpha_r}) &= 0, & S &= q^{\sum_{i=1}^{r-1} \zeta^{\alpha_i} \otimes \zeta^{\alpha_{i-1}}}.\end{aligned}$$

→ Cremmer-Gervais's vertex solution R^{CG} (and higher rank generalizations) in the fundamental representation

Belavin-Baxter Elliptic Vertex solutions ($\mathfrak{g} = A_r^{(1)}$, $\mathfrak{l} = c\mathbb{C}$)

$$\begin{aligned}\theta_{[x]}^\pm &= Ad_{D^\pm(p)} \circ \sigma^\pm, & D^+(p) &= p^{\frac{2}{r+1}\varpi}, & D^-(p) &= p^{-\frac{2}{r+1}\varpi} q^{-\frac{2}{r+1}c\varpi}, \\ \sigma^\pm(e_{\pm\alpha_i}) &= e_{\pm\alpha_{[i\mp 1]}}, & \sigma^\pm(h_{\alpha_i}) &= h_{\alpha_{[i\mp 1]}}, & \sigma^\pm(\varpi) &= \varpi,\end{aligned}$$

where $\varpi = \sum_{i=0}^r \zeta^{\alpha_i}$, $[i] = i \bmod r+1$, and p is the component of x along q^c .

→ eight-vertex solution $R^{8V}(z; p)$ (and higher rank generalizations)

Vertex-IRF problem and Dynamical Coboundary Equation

In terms of **Quantum Dynamical Cocycles**, the Vertex-IRF is reformulated as

Generalized Quantum Dynamical Coboundary Problem:

Let $J_F(x)$ be the standard "Face-type" solution of the Quantum Dynamical Cocycle Equation, and $J_V(x)$ a "Vertex-type" solution

Does it exists an invertible element $M(x) \in U_q(\mathfrak{g})$ such that

$$J_F(x) = \Delta(M(x)) J_V(x) M_2(x)^{-1} M_1(xq^{h_2})^{-1} \quad ?$$

→ Such a **generalized Quantum Dynamical Coboundary** is a Vertex-IRF transformation:

$R_F(x) = J_F(x)_{21}^{-1} R J_F(x)_{12}$, and $R_V(x) = J_V(x)_{21}^{-1} R J_V(x)_{12}$ satisfy

$$R_F(x)_{12} = M(xq^{h_1})_2 M(x)_1 R_V(x)_{12} M(x)_2^{-1} M(xq^{h_2})_1^{-1}.$$

Theorem : Auxiliary Linear Problem (Buffenoir, Roche, Terras)

If $M(x)$ is expressed as

$$M(x) = M^{(0)}(x) M^{(-)}(x)^{-1} M^{(+)}(x), \quad M^{(\pm)}(x) = \prod_{k=1}^{+\infty} \mathfrak{C}^{[\pm k]}(x)^{\pm 1}$$

with $\mathfrak{C}^{[+k]}(x) = (\theta_{[x]}^+)^{k-1}(\mathfrak{C}^{[+]}(x))$, $\mathfrak{C}^{[-k]}(x) = B(x)^{-k} \mathfrak{C}^{[-]}(x) B(x)^k$. and
 if $M^{(0)}(x) \in U_q(\mathfrak{h})$, $\mathfrak{C}^{[\pm]}(x) \in 1 \oplus U_q^\pm(\mathfrak{g})$ satisfy

Fusion $\Delta(M^{(0)}(x)) = S_{12}^{-1} M_1^{(0)}(xq^{h_2}) M_2^{(0)}(x),$

$$\Delta(\mathfrak{C}^{[\pm]}(x)) = K_{12} \operatorname{Ad}_{[S_{21}]}(\mathfrak{C}_1^{[\pm]}(x)) \operatorname{Ad}_{[K_{12}^\mp S_{12}]}(\mathfrak{C}_2^{[\pm]}(x)) K_{12}^{-1}$$

Shift $\mathfrak{C}_1^{[\pm]}(xq^{h_2}) = \operatorname{Ad}_{[S_{12}^{-1} S_{21} K_{12}]}(\mathfrak{C}_1^{[\pm]}(x)),$

Hexagonal $\mathfrak{C}_2^{[-]}(x) \mathfrak{C}_1^{[+]}(xq^{h_2}) [\operatorname{Ad}_{B_2(x)} \circ \theta_{[x]2}^-] (S_{12}^{-1} \widehat{R}_{12} S_{12})$
 $= (S_{12}^{-1} \widehat{R}_{12} S_{12}) \mathfrak{C}_1^{[+]}(xq^{h_2}) \mathfrak{C}_2^{[-]}(x).$

then it satisfies quantum dynamical coboundary equation.

Explicit Universal Vertex-IRF transform

- trivial to find $M^{(0)}(x)$ solution of fusion + adequate shift in terms of S
- easy to build explicit solutions $\mathfrak{C}_{\chi^\mp}^{[\pm]}(x)$ of **fusion** and **shift**:
they are generically associated to couple $\chi = (\chi^+, \chi^-)$ of some **non singular characters** χ_\mp of the subalgebras $U_q(\mathfrak{n}^\mp)^{\omega_\mp}$ of $U_q(\mathfrak{b}^\mp)$ generated respectively by

$$\overset{\omega^+}{e}_{\alpha_i} = q^{-(\alpha_i \otimes id)(\omega^+)} e_{\alpha_i} \quad \overset{\omega^-}{e}_{-\alpha_i} = e_{-\alpha_i} q^{(\alpha_i \otimes id)(\omega^-)} \quad q^{\omega^+} := S_{21}, \quad q^{\omega^-} := KS_{21}$$

$$\overset{\omega}{\mathfrak{C}}_{\chi^\mp}^{[\pm]}(x) = Ad_{M^{(0)}(x)^{-2}}(C_{\chi^\mp}^{[\pm]}) \quad C_{\chi^\mp}^{[\pm]} = (id \otimes \chi_\mp)(S_{21} R_{12}^{(\pm)} K^{\mp 1} S_{21}^{-1})$$

- **hexagonal relation** is satisfied only for specific Lie algebras and only for Standard IRF-type dynamical cocycles and Extremal Vertex-type cocycles (verified for the Cremmer-Gervais and Belavin-Baxter case for any rank) as soon as

$$-q^{-1}(q - q^{-1})^2 \chi^+(\overset{\omega^+}{e}_{\alpha_i}) \chi^-(\overset{\omega^-}{e}_{-\alpha_i}) = 1.$$

Quantum Whittaker Vectors

Let \mathcal{V} be a $U_q(\mathfrak{g})$ -module and let χ^+ be a non-singular character of $U_q(\mathfrak{n}^+)^{-\omega^+}$, a vector $\overset{\omega}{W}_{\chi^+} \in \mathcal{V}$ is said to be a χ^+ -quantum Whittaker vector if

$$\forall x \in U_q(\mathfrak{n}^+)^{-\omega^+}, \quad x \cdot \overset{\omega}{W}_{\chi^+} = \chi^+(x) \overset{\omega}{W}_{\chi^+}.$$

Quantum Whittaker Functions

A quantum Whittaker function denoted $\overset{\omega}{W}_{\varrho^-, \varrho^+}^\xi$ associated to a central character ξ and non-singular characters ϱ^\pm of $U_q(\mathfrak{n}^\pm)^{-\omega^\pm}$, is an element of $(U_q(\mathfrak{g}))^*$ defined such that

$\forall x \in U_q(\mathfrak{n}^-)^{-\omega^-}, z \in U_q(\mathfrak{n}^+)^{-\omega^+}, y \in U_q(\mathfrak{g}), a \in \mathcal{Z}_q(\mathfrak{g})$ we have

$$\overset{\omega}{W}_{\varrho^-, \varrho^+}^\xi(x \cdot y \cdot z) = \varrho^-(x) \varrho^+(z) \overset{\omega}{W}_{\varrho^-, \varrho^+}^\xi(y).$$

$$\overset{\omega}{W}_{\varrho^-, \varrho^+}^\xi(a \cdot y) = \xi(a) \overset{\omega}{W}_{\varrho^-, \varrho^+}^\xi(y).$$



The basic motivation is the Toda Quantum Mechanics (ex. $q = 1, \mathfrak{g} = A_1$)

$$\begin{aligned}\xi(c) \mathcal{W}_{\varrho^-, \varrho^+}^\xi(e^{\phi h}) &= \mathcal{W}_{\varrho^-, \varrho^+}^\xi \left(e^{\phi h} (e_- e_+ + \frac{1}{2} h^2 + h) \right) \\ &= \left(\varrho^-(e_-) \varrho^+(e_+) e^{-2\phi} + \frac{1}{2} \partial_\phi^2 + \partial_\phi \right) \mathcal{W}_{\varrho^-, \varrho^+}^\xi(e^{\phi h})\end{aligned}$$

Then, $f(\phi) = e^\phi \mathcal{W}_{\varrho^-, \varrho^+}^\xi(e^{\phi h})$ obeys

$$(\partial_\phi^2 + 2\varrho^-(e_-) \varrho^+(e_+) e^{-2\phi} - 2\xi(c) - 1) \cdot f = 0$$

$(\phi^i)_i \mapsto \mathcal{W}_{\varrho^-, \varrho^+}^\xi(e^{\phi^i h_{\alpha_i}})$ is a wave function of q -deformed quantum Toda hamiltonian derived from the action of the casimir ($\wp^{-1} := m(S)B(\hat{x})$)

$$c_q^{\mathfrak{V}} := (\text{tr}_q^{\mathfrak{V}} \otimes id)(Ad_{q-\omega^-}(\widehat{R}_{12}^{-1}) K_{12}^{-2} Ad_{q-\omega^+}(\widehat{R}_{21}^{-1})) \rightarrow$$

$$0 = \left(\text{tr}_q^{\mathfrak{V}} \left(Ad_{\wp e^{\phi^i h_{\alpha_i}}} (C_{\varrho^-}^{[+] -1}) q^{-2\zeta^i \partial_{\phi^i}} C_{\varrho^+}^{[-]} \right) - \xi(c_q^{\mathfrak{V}}) \right) \mathcal{W}_{\varrho^-, \varrho^+}^\xi(e^{\phi^i h_{\alpha_i}})$$

Quantum Whittaker vectors can be identified as elements of a certain completion of a Verma module:

Explicit realization of quantum Whittaker vectors in Verma modules

An explicit realization of quantum Whittaker vectors is given by

$$\overset{\omega}{W}_{\chi^+}^{\xi_\eta} = M(\hat{x})^{-1} |\emptyset \rangle_\eta$$

The quantum Whittaker function is given by

$$\begin{aligned}\overset{\omega}{W}_{\chi^+, \chi^-}^{\xi_\eta}(u) &= S_\eta(\tilde{W}_{\tilde{\chi}^+}^{\xi_{-\eta}^*}, \wp u \overset{\omega}{W}_{\chi^+}^{\xi_\eta}) \\ \tilde{\omega}_{12}^\pm &= (\omega_{21}^\pm)^* \quad \tilde{\chi}^+(x) = \chi^-(Ad_\wp(x^*))^*\end{aligned}$$

Using the dynamical Coboundary equation as well as expression of Fusion operators, we deduce

$$\begin{aligned}
 \Phi_{\eta}^{\mathfrak{v} \mu} \overset{\omega}{W}_{\chi^+}^{\xi_{\eta+\mu}} &= \Phi_{\eta}^{\mathfrak{v} \mu} M(\hat{x})^{-1} |\emptyset\rangle_{\eta+\mu} \\
 &= (\overset{\mathfrak{V}}{\pi} \otimes id)(\Delta(M(\hat{x})^{-1})) \Phi_{\eta}^{\mathfrak{v} \mu} |\emptyset\rangle_{\eta+\mu} \\
 &= (\overset{\mathfrak{V}}{\pi} \otimes id)(\Delta(M(\hat{x})^{-1})) (\overset{\mathfrak{V}}{\pi} \otimes id)(J_F(\hat{x})) . (\mathfrak{v}_\mu \otimes |\emptyset\rangle_\eta) \\
 &= (\overset{\mathfrak{V}}{\pi} \otimes id)(J_V(\hat{x}) M(\hat{x})_2^{-1} M(\hat{x} q^{h_2})_1^{-1}) . (\mathfrak{v}_\mu \otimes |\emptyset\rangle_\eta) \\
 &= (\overset{\mathfrak{V}}{\pi} \otimes id)(J_V(\hat{x}) M(\hat{x} q^{\eta(h)})_1^{-1}) . \left(\mathfrak{v}_\mu \otimes \overset{\omega}{W}_{\chi^+}^{\xi_\eta} \right).
 \end{aligned}$$

These fusion formulas implies the basic finite difference equations describing the shift on η for Whittaker functions in terms of finite difference operators acting on the ϕ_i variables.

- Explicit universal solutions of quantum dynamical cocycle and coboundary equation in the finite dimensional as well as affine Lie algebra case: → universality is fundamental to build correlation functions of the model as matrix elements in highest weight reps. of quantum algebras
- Relation between “Face type” quantum dynamical cocycles and the Fusion theory of Verma modules
- Relation between quantum dynamical coboundaries and Whittaker vectors
- Relation between “Extremal Vertex type” quantum dynamical cocycles and the Fusion theory of Whittaker modules

Main Open Problems

- The explicit expression of quantum dynamical coboundary (in the quantum affine case) in the highest weight modules gives the “tail operator” used in Lashkevich/Pugai and Konno/Kojima/Weston’s works
→ New approach to compute correlation functions of the 8–vertex model
- Analysis of the quantum dynamical coboundary equation → new results on Whittaker functions

