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GRAPHS, ALGEBRAS, CONFORMAL FIELD THEORIES AND INTEGRABLE LATTICE MODELS

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Recent work on the construction of lattice integrable models corresponding to the $SU(N)$ coset conformal field theories is reviewed.

1. INTRODUCTION

One of the most striking features in the recent developments in two-dimensional field theory has been the convergence of concepts and methods in two *a priori* disconnected areas: conformal field theory which deals with continuous massless (critical) fields, and integrable lattice models, which are concerned with critical or non-critical discrete systems. That these two fields overlap is not a complete surprise: lattice models at criticality are described by some conformal field theory (c.f.t.). Conversely, as will be illustrated in this talk, and much more surprisingly, it seems that most, if not all, rational c.f.t. may be realized by some integrable lattice model. The convergence between the two areas is actually much deeper. Many concepts which looked proper to one approach turn out to appear in some way in the other. This is the case of the Yang-Baxter equation and quantum group ideas, developped in the framework of integrable models [1] which play also an important role in rational c.f.t. [2]. On the other hand, the Feigin-Fuchs construction and its screening operators seem to have analogues within quantum algebras [3] (see also [4]) and the characters of infinite dimensional algebras appear in integrable models [5].

The classification of rational c.f.t. must thus have much in common with the classification of integrable models. Admittedly, the former looks a better posed problem than the latter. On the one hand, one has to classify modular invariant combinations of characters, or fusion algebras, whereas the construction of integrable models relies on a series of Ansätze: solutions of the Yang-Baxter equation, with a certain group-theoretic structure, for face models of a certain type, etc... It is my belief, however, that we have much to learn about the classification of c.f.t. from the study of these lattice models.

In this talk, I would like to illustrate this by reviewing some recent developments in the construction of lattice realizations of the coset c.f.t. $SU(N)_{k-1} \times SU(N)/SU(N)_k$. The general stategy is due to Pasquier [6] who has introduced height models based on graphs, which encode which configurations of heights are admissible. Throughout this discussion, there will be two types of algebras attached to these graphs in two different senses. On the one hand, a representation of the Hecke algebra (or of its specialization, the Temperley-Lieb algebra), basic in the "Baxterization" procedure [7], is built on the space of paths on the graph. On the other hand, fusion algebras play a central role in c.f.t. These algebras are particular cases of so-called "c-algebras" [8] or "hypergroups" [9] which are associative and commutative algebras with a basis x_a and relations $x_a x_b = \sum_c N_{ab}{}^c x_c$ satisfying some additional axioms. Whenever the real structure constants are non negative integers, one may regard the matrices N_a of entries $N_{ab}{}^c$ as the adjacency matrices of a collection of graphs. Conversely given a graph, one may wonder if one may use its adjacency matrix to generate a c-algebra with non negative structure constants.

These considerations will be recurrent in the following. The simplest models for the $SU(N)$ cosets involve the graphs that represent the fusion by the fundamental representation in the Kac-Moody algebra $\widehat{SU(N)}$ at some level. This is natural since the configurations appear in the truncated Bratteli diagram, the slices of which are precisely building this graph [10], [1]. There are, however, more complicated patterns of heights, corresponding to more general graphs, for which the c-algebra may or may not have non negative structure constants.

After a short review of what is known about the classification of the coset c.f.t. (sect.2), of integrable height models (sect.3) and of the Pasquier construction in the case of $SU(2)$ (sect.4), I shall introduce a class of graphs which seem relevant for the construction of $SU(3)$ integrable models (sect.5) , discuss some of their algebraic properties (sect.6) and their associated c-algebra and what it says about the continuum limit (sect.7).

2. THE C.F.T. STANDPOINT

2.1. Classification of $SU(2)$ theories

We consider a torus of modular ratio (ratio of its two complex periods) τ, with $\mathrm{Im}\tau > 0$ and let $q = \exp 2i\pi\tau$. The partition function Z of a c.f.t. on this torus may be expressed in terms of the Virasoro generators as

$$Z = \mathrm{tr}_{\mathcal{H}}\, q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} \tag{2.1}$$

By definition the Hilbert space of a "rational c.f.t." splits into a sum of products of representation spaces of the left and right chiral algebras with multiplicities $N_{\lambda\bar{\lambda}}$

$$\mathcal{H} = \oplus N_{\lambda\bar{\lambda}} \mathcal{V}_\lambda \otimes \mathcal{V}_{\bar{\lambda}}. \tag{2.2}$$

Thus Z reads

$$A = \sum_{\lambda\bar{\lambda}} N_{\lambda\bar{\lambda}} \chi_\lambda(q) \chi_{\bar{\lambda}}(\bar{q}) \tag{2.3}$$

The characters $\chi_\lambda(q) = \mathrm{tr}_{\mathcal{V}_\lambda} q^{L_0 - \frac{c}{24}}$ are generating functions for the number of states with a given eigenvalue of L_0 in the representation space $\mathcal{V}_\lambda$ and are explicitly known functions for a variety of chiral algebras. As a function of τ, Z must be modular invariant, which expresses the independence of this physical quantity with respect to the choice of coordinates on the torus. This fundamental observation [11] opens the route to a classification of families of rational c.f.t.'s. Given a certain set of characters of some chiral algebra (Virasoro, Kac-Moody,...), what are all the modular invariant combinations with non negative integer coefficients that may be formed with them? This is a well posed problem because it turns out that (for some not very well understood reason) all these characters form finite dimensional representations of the modular group. In particular, under $\tau \to -1/\tau$, there is a unitary matrix S such that

$$\chi_\lambda(-\frac{1}{\tau}) = \sum_{\lambda'} S_{\lambda\lambda'} \chi_{\lambda'}(\tau). \tag{2.4}$$

This classification program has so far been carried out completely only for theories with a $\widehat{SU(2)}_k$ Kac-Moody algebra (of level k) and some associated coset theories. In particular, the coset theories $SU(2)_{k-1} \times SU(2)_1/SU(2)_k$ are the only unitary conformal theories with a central charge $c < 1$. In fact

$$c = 1 - \frac{6}{(k+1)(k+2)} \qquad k = 2,3,\cdots \tag{2.5}$$

The classification is simpler to describe for the Kac-Moody theories for which λ labels an integrable weight of level k, (or "altitude" $n = k+2$) i.e. corresponds to an integer or half-integer $SU(2)$ spin j, $0 \le j \le \frac{1}{2}k$ by $\lambda = 2j+1$, hence $1 \le \lambda \le k+1 = n-1$. For a given $n = k+2$, modular invariant solutions turn out to be in one-to-one correspondence with simply laced Lie algebras (ADE) of Coxeter number n [12]. The correspondence is particularly manifest on the diagonal terms ($\lambda = \bar\lambda$) of the partition function

$$Z = \sum_{1\le\lambda\le n-1} N_\lambda |\chi_\lambda|^2 + \text{off diagonal terms} \tag{2.6}$$

where N_λ is the multiplicity of λ as an exponent of the A,D,E Lie algebra.

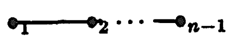
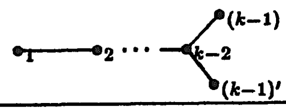
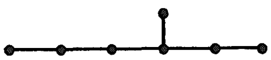
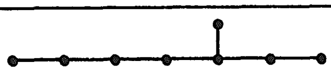
Table I: List of $SU(2)_{k-1} \times SU(2)_1/SU(2)_k$ coset modular invariants, with $n' = n \pm 1$, $k+2 = \max(n,n')$.

$n \ge 2$	$\frac{1}{2}\sum_{\lambda=1}^{n'-1}\sum_{\mu=1}^{n-1} \lvert\chi_{\lambda\,\mu}\rvert^2$	(A_{n-1})
$n = 4\rho+2 \ge 6$	$\frac{1}{2}\sum_{s=1}^{n'-1}\left\{\sum_{\mu\text{ odd}=1}^{2\rho-1} \lvert\chi_{\lambda\,\mu}+\chi_{\lambda\,4\rho+2-\mu}\rvert^2 + 2\lvert\chi_{\lambda\,2\rho+1}\rvert^2\right\}$	$(D_{2\rho+2})$
$n = 4\rho \ge 8$	$\frac{1}{2}\sum_{\lambda=1}^{n'-1}\left\{\sum_{\mu\text{ odd}=1}^{4\rho-1} \lvert\chi_{\lambda\,\mu}\rvert^2 + \lvert\chi_{\lambda\,2\rho}\rvert^2 + \sum_{\mu\text{ even}=2}^{2\rho-2}(\chi_{\lambda\,\mu}\chi^*_{\lambda\,4\rho-\mu} + \text{c.c.})\right\}$	$(D_{2\rho+1})$
$n = 12$	$\frac{1}{2}\sum_{\lambda=1}^{n'-1}\left\{\lvert\chi_{\lambda\,1}+\chi_{\lambda\,7}\rvert^2 + \lvert\chi_{\lambda\,4}+\chi_{\lambda\,8}\rvert^2 + \lvert\chi_{\lambda\,5}+\chi_{\lambda\,11}\rvert^2\right\}$	(E_6)
$n = 18$	$\frac{1}{2}\sum_{\lambda=1}^{n'-1}\left\{\lvert\chi_{\lambda\,1}+\chi_{\lambda\,17}\rvert^2 + \lvert\chi_{\lambda\,5}+\chi_{\lambda\,13}\rvert^2 + \lvert\chi_{\lambda\,7}+\chi_{\lambda\,11}\rvert^2 + \lvert\chi_{\lambda\,9}\rvert^2 + [(\chi_{\lambda\,3}+\chi_{\lambda\,15})\chi^*_{\lambda\,9} + \text{c.c.}]\right\}$	(E_7)
$n = 30$	$\frac{1}{2}\sum_{\lambda=1}^{n'-1}\left\{\lvert\chi_{\lambda\,1}+\chi_{\lambda\,11}+\chi_{\lambda\,19}+\chi_{\lambda\,29}\rvert^2 + \lvert\chi_{\lambda\,7}+\chi_{\lambda\,13}+\chi_{\lambda\,17}+\chi_{\lambda\,23}\rvert^2\right\}$	(E_8)

In Table I, is presented the parallel classification of the coset theories of central charge (2.5), which also exhibits the exponents of the ADE algebras. Characters are labelled by a pair of weights λ and μ of respective altitude $n' = n \pm 1$ and n, $k+2 = \max(n,n')$. I recall that the exponents are defined as the integers that label the eigenvalues of the Cartan matrix C, or what is more suited for our future purposes, of the adjacency matrix G of the corresponding Dynkin diagram: $C = 2\mathbb{1} - G$. The eigenvalues of G are $2\cos\pi m/n$, m taking r(= rank of the algebra) values ranging between 1 and $n-1$: the exponents. (Table II)

Table II

List of Dynkin diagrams and of their exponents

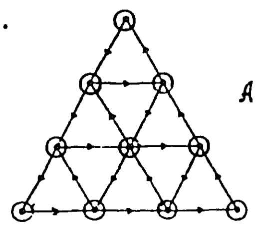
	Graph	Coxeter number n	Exponents
A_{n-1}		n	$1,2,\cdots,n-1$
D_k		$2(k-1)$	$1,3,\cdots,2k-3;k-1$
E_6		12	$1,4,5,7,8,11$
E_7		18	$1,5,7,9,11,13,17$
E_8		30	$1,7,11,13,17,19,23,29$

2.2. The case of $SU(N)$, $N>2$.

The generalization of these considerations to $SU(N)$ for arbitrary N follows a well known route. One considers conformal theories with a $\widehat{SU(N)}$ Kac-Moody algebra [13]. At level k, the only admissible representations are associated with "integrable weights", i.e. with points of the Weyl alcove:

$$\lambda=\sum_{i=1}^{N-1}\lambda_i\Lambda_i, \quad \text{with } \lambda_i\geq 1,\ \sum\lambda_i\leq n-1 \tag{2.7}$$

hence may be associated with Young tableaux with at most k columns (and $N-1$ rows). In this expression, Λ_i, $i=1,\cdots,N-1$ are the $N-1$ fundamental weights of $SU(N)$, and we have introduced the "altitude" $n=k+N$. In fig. 1, the edges are oriented along the vectors e_j, $j=1,\cdots,N$, that are the weights of the fundamental representation $\square$: $e_1=\Lambda_1$, $e_j=\Lambda_j-\Lambda_{j-1}$ for $1<j<N$, $e_N=-\Lambda_{N-1}$.



$\mathcal{A}^{(6)}$ I

FIGURE 1

The Weyl alcove of $SU(3)$ at level 3 or altitude 6.

From these theories with a $\widehat{SU(N)}$ Kac-Moody algebra, the coset construction produces more conformal theories, in particular the so-called $SU(N)$ minimal models

$$\frac{SU(N)_{k-1}\times SU(N)}{SU(N)_k} \tag{2.8}$$

of central charge

$$c=(N-1)\Big(1-\frac{N(N+1)}{n(n-1)}\Big) \tag{2.9}$$

that will be realized as critical lattice models in the following.

Table III. List of some known $SU(3)_{k-1} \times SU(3)_1/SU(3)_k$ coset modular invariants, denoted $(G^{(n)})$; $n = k+3$.

$$(A^{(n)}) \qquad Z = \frac{1}{3} \sum_{\lambda \in P_{++}^{(n-1)}} \sum_{\mu \in P_{++}^{(n)}} |\chi_{\lambda,\mu}|^2$$

$$(D^{(n)}) \qquad Z = \frac{1}{9} \sum_{\lambda \in P_{++}^{(n-1)}} \sum_{\mu \in Q \cap P_{++}^{(n)}} |\sum_{k=0}^{2} \chi_{\lambda,\sigma^k \mu}|^2 \quad \text{if 3 divides } n$$

$$(D^{(n)*}) \qquad Z = \frac{1}{9} \sum_{\lambda \in Q \cap P_{++}^{(n)}} \Big(\sum_{k=0}^{2} \chi_{\lambda,\sigma^k \mu}\Big)\Big(\sum_{k=0}^{2} \chi_{\lambda,\sigma^k \bar{\mu}}\Big)^* \quad \text{if 3 divides } n$$

$$(E^{(8)}) \qquad Z = \frac{1}{3} \sum_{\lambda \in P_{++}^{(7)}} \Big[|\chi_{\lambda,(1,1)} + \chi_{\lambda,(3,3)}|^2 + |\chi_{\lambda,(3,2)} + \chi_{\lambda,(1,6)}|^2 + |\chi_{\lambda,(2,3)} + \chi_{\lambda,(6,1)}|^2$$
$$+ |\chi_{\lambda,(4,1)} + \chi_{\lambda,(1,4)}|^2 + |\chi_{\lambda,(1,3)} + \chi_{\lambda,(4,3)}|^2 + |\chi_{\lambda,(3,1)} + \chi_{\lambda,(3,4)}|^2\Big]$$

$$(E^{(12)}) \qquad Z = \frac{1}{3} \sum_{\lambda \in P_{++}^{(11)}} \Big[|\chi_{\lambda,(1,1)} + \chi_{\lambda,(10,1)} + \chi_{\lambda,(1,10)} + \chi_{\lambda,(5,5)} + \chi_{\lambda,(5,2)} + \chi_{\lambda,(2,5)}|^2$$
$$+ 2|\chi_{\lambda,(3,3)} + \chi_{\lambda,(3,6)} + \chi_{\lambda,(6,3)}|^2\Big]$$

$$(E_{MS}^{(12)}) \qquad Z = \frac{1}{3} \sum_{\lambda \in P_{++}^{(11)}} \Big[|\chi_{\lambda,(1,1)} + \chi_{\lambda,(10,1)} + \chi_{\lambda,(1,10)}|^2 + |\chi_{\lambda,(3,3)} + \chi_{\lambda,(3,6)} + \chi_{\lambda,(6,3)}|^2$$
$$+ |\chi_{\lambda,(5,5)} + \chi_{\lambda,(5,2)} + \chi_{\lambda,(2,5)}|^2 + |\chi_{\lambda,(4,7)} + \chi_{\lambda,(7,1)} + \chi_{\lambda,(1,4)}|^2$$
$$+ |\chi_{\lambda,(7,4)} + \chi_{\lambda,(1,7)} + \chi_{\lambda,(4,1)}|^2 + 2|\chi_{\lambda,(4,4)}|^2$$
$$+ \big(\chi_{\lambda,(2,2)} + \chi_{\lambda,(8,2)} + \chi_{\lambda,(2,8)}\big)\chi^*_{\lambda,(4,4)} + \text{c.c.}\Big]$$

$$(E_{MS}^{(12)*}) \qquad Z = \frac{1}{3} \sum_{\lambda \in P_{++}^{(11)}} \Big[|\chi_{\lambda,(1,1)} + \chi_{\lambda,(10,1)} + \chi_{\lambda,(1,10)}|^2 + |\chi_{\lambda,(3,3)} + \chi_{\lambda,(3,6)} + \chi_{\lambda,(6,3)}|^2$$
$$+ |\chi_{\lambda,(5,5)} + \chi_{\lambda,(5,2)} + \chi_{\lambda,(2,5)}|^2 + 2|\chi_{\lambda,(4,4)}|^2$$
$$+ \big(\chi_{\lambda,(4,7)} + \chi_{\lambda,(7,1)} + \chi_{\lambda,(1,4)}\big)\big(\chi_{\lambda,(7,4)} + \chi_{\lambda,(1,7)} + \chi_{\lambda,(4,1)}\big)^* + \text{c.c.}$$
$$+ \big(\chi_{\lambda,(2,2)} + \chi_{\lambda,(8,2)} + \chi_{\lambda,(2,8)}\big)\chi^*_{\lambda,(4,4)} + \text{c.c.}\Big]$$

$$(E^{(24)}) \qquad Z = \frac{1}{3} \sum_{\lambda \in P_{++}^{(11)}} \Big[|\chi_{\lambda,(1,1)} + \chi_{\lambda,(22,1)} + \chi_{\lambda,(1,22)} + \chi_{\lambda,(5,5)} + \chi_{\lambda,(5,14)} + \chi_{\lambda,(14,5)}$$
$$+ \chi_{\lambda,(11,11)} + \chi_{\lambda,(11,2)} + \chi_{\lambda,(2,11)} + \chi_{\lambda,(7,7)} + \chi_{\lambda,(7,10)} + \chi_{\lambda,(10,7)}|^2$$
$$+ |\chi_{\lambda,(7,1)} + \chi_{\lambda,(16,7)} + \chi_{\lambda,(1,16)} + \chi_{\lambda,(1,7)} + \chi_{\lambda,(7,16)} + \chi_{\lambda,(16,1)}$$
$$+ \chi_{\lambda,(5,8)} + \chi_{\lambda,(11,5)} + \chi_{\lambda,(8,11)} + \chi_{\lambda,(8,5)} + \chi_{\lambda,(5,11)} + \chi_{\lambda,(11,8)}|^2\Big]$$

In spite of numerous and vigorous efforts, no complete classification has been achieved for $SU(N)$, $N \geq 3$ theories as yet. The full description of the commutant of the action of the modular group on $SU(N)$ characters, i.e. the general form of the modular invariants with arbitrary (complex or rational) coefficients has been obtained [14] but the effect of imposing the positivity of these coefficients has not yet been mastered. Infinite series of modular invariants have been constructed [15], some exceptional cases have been exhibited [16], and a theorem asserting that these cases exhaust the possibilities for $N = 3$ whenever $k+3$ is a prime number has been established [17]. Finally the case of $SU(N)_{k=1}$ has been completely analyzed in [18]. (For further comments and a different approach to this problem, see Alvarez-Gaumé's contribution at this meeting.) From the modular invariants presently known for the $SU(3)$ Kac-Moody theories, one can manufacture a subfamily of modular invariants pertaining to the cosets (2.8) which are diagonal in the weights relative to level $k-1$ and 1: this results in Table III. It involves the following notations: Q denotes the root lattice of $SU(3)$, $P_{++}^{(n)}$ is the Weyl alcove (2.7), σ is the Z_3 automorphism

$$\sigma\lambda = \lambda_2\Lambda_1 + (n - \lambda_1 - \lambda_2)\Lambda_2 \tag{2.10}$$

and all the weights are represented by their components λ_1, λ_2.

2.3. Fusion algebra.

An internal associative and commutative operation on the representations of chiral algebras (or "chiral vertex operators") has been introduced recently [19],[2]. In Kac-Moody theories, it amounts to a tensor product of the representations of the finite dimensional algebra, truncated by the altitude.

$$(\lambda) * (\mu) = \oplus_\nu N_{\lambda\mu}{}^{\nu}(\nu) \tag{2.11}$$

Most remarkably, Verlinde [19] found a close expression for the integer fusion coefficients $N_{\lambda\mu}{}^{\nu}$ in terms of the unitary matrix S implementing the modular transformation (2.4):

$$N_{\lambda\mu}{}^{\nu} = \sum_\rho \frac{S_{\lambda\rho}S_{\mu\rho}S^*_{\nu\rho}}{S_{1\rho}}. \tag{2.12}$$

Here and in what follows, "1" refers to the identity representation of $SU(N)$, i.e. to the apex of the graph of fig. 1. If N_λ denotes the matrix of entries $N^{\nu}_{\lambda\mu}$, it is readily seen that the matrices N satisfy the fusion algebra

$$N_\lambda N_\mu = \sum_\nu N_{\lambda\mu}{}^{\nu} N_\nu. \tag{2.13}$$

Hence, these matrices form the regular representation of the fusion algebra.

It is natural to encode the fusion rules in a collection of graphs: each matrix N_λ is regarded as the adjacency matrix of a graph. In particular, if we take $\lambda = \Box$, the fundamental representation of dimension N, the resulting graph is nothing else than the graph of fig. 1. Thus conversely, the formula (2.12) tells us everything about the spectrum of the adjacency matrix of that graph: its orthonormalized eigenvectors are $S_{\mu\rho}$, corresponding to the eigenvalues $S_{\Box\rho}/S_{1\rho}$.

3. HEIGHT MODELS

We now introduce a family of lattice integrable models. The degrees of freedom are attached to the sites of the lattice and interact through "interactions-round-a-face" (IRF) around each plaquette. The Boltzmann weights are thus of the form

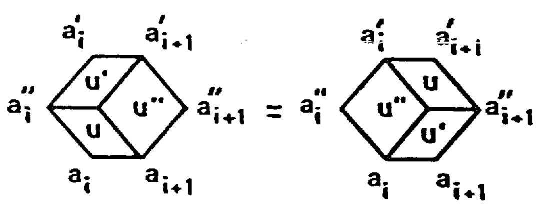
$$w(a_1, a_2, a_3, a_4) = {}^{a_4}_{a_1}\Box^{a_3}_{a_2} \tag{3.1}$$

In the simplest model of this type, the a's are integers (on a finite or infinite range) that may be regarded as describing the height of a discrete fluctuating surface, and heights at neighbouring sites must differ by ± 1. We can generalize this to a more abstract situation where the a's belong to general discrete set, and are subject to some constraint. Accordingly, these models are called "height models", or SOS (solid-on-solid), or IRF (or face) models.

In order to construct integrable models, one seeks a one-parameter family of commuting row-to-row transfer matrices, and this may be found if the Boltzmann weights satisfy the YB equation

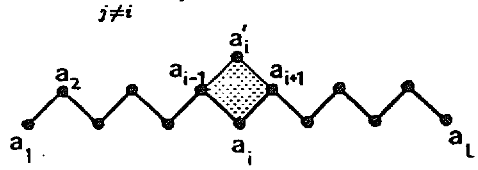
$$\begin{aligned}&\sum_{b''} w(a_i a_{i+1}; b'' a_i''|u)\, w(a_i'' b''; a_{i+1}' a_i'|u') \\ &w(b'' a_{i+1}; a_{i+1}'' a_{i+1}'|u'') = \sum_{b''} w(a_i'' a_i; b'' a_i'|u'') \\ &w(a_i a_{i+1}; a_{i+1}'' b''|u')\, w(b'' a_{i+1}''; a_{i+1}' a_i'|u) \qquad (3.2)\end{aligned}$$

with $u'' = u' - u$, which is represented diagrammatically as follows:



One also introduces the face transfer matrix acting on configurations attached to diagonals

$$\begin{aligned}&< a_1' a_2' \cdots a_L' |X_i(u)| a_1 \cdots a_L > \qquad (3.3)\\ &= \prod_{j \neq i} \delta_{a_j a_j'}\, w(a_{i-1} a_i; a_{i+1} a_i'|u)\end{aligned}$$



and eq. (3.2) amounts to

$$X_i(u) X_{i+1}(u') X_i(u'') = X_{i+1}(u'') X_i(u') X_{i+1}(u). \qquad (3.4)$$

For the diagonal-to-diagonal transfer matrix $X_i(u)$, we introduce the following Ansatz:

$$X_i(u) = \sin\left(\pi(\hat\lambda - u)\right)\mathbb{1} + \sin \pi u\, U_i. \qquad (3.5)$$

Then the model is :

i) critical, i.e. undergoes a second order phase transition, if $\hat\lambda$ is real: in the following, $\hat\lambda$ will turn out to be equal to the inverse of a positive integer $1/n$;

ii) integrable if the U_i satisfy the Hecke algebra

$$U_i U_j = U_j U_i \quad \text{for } |i-j| \geq 2 \qquad (3.6a)$$

$$U_i^2 = 2\cos(\pi\hat\lambda) U_i \qquad (3.6b)$$

$$U_i U_{i+1} U_i - U_i = U_{i+1} U_i U_{i+1} - U_{i+1} \qquad (3.6c)$$

iii) related to $SU(N)$ if the U's satisfy an additional N−dependent relation [20].

In fact, the justification of the Ansatz (3.5) comes from the discussion of a different family of lattice models: the $SU(N)$ vertex models, in which the degrees of freedom live on the links of the lattice and interact at vertices. For those models, the diagonal-to-diagonal transfer matrix $X_i(u)$ commutes with the quantum algebra $\mathcal{U}_q\, s\ell(N)$. Moreover these vertex models admit a reinterpretation in terms of height models of the previous type. If we try to construct more general height models, it is therefore natural to demand that their transfer matrix $X_i(u)$ commutes with the quantum algebra. Now the commutant of $\mathcal{U}_q\, s\ell(N)$ (in the tensor space of two fundamental representations) has been shown by Reshetikhin [20] to be described by the factor of the Hecke algebra (3.6), with $q = e^{i\pi\hat\lambda}$, by the additional condition that the q−analogue of the N−th Young antisymmetrizer vanishes.

We are thus looking for new representations of the Hecke algebra. Following an idea of Pasquier [6], it is suggested to regard the heights as attached to the vertices of a graph $\mathcal{G}$. The allowed configurations are encoded in the adjacency matrix G of this graph in the sense that two heights on two neighbouring sites of the lattice must be neighbours on the graph. In other words, we are seeking representations of the Hecke algebra on the set of paths on the graph $\mathcal{G}$.

4. THE CASE OF $SU(2)$

In the case of $SU(2)$ we only consider unoriented graphs $\mathcal{G}$ (their adjacency matrix G is symmetric). This is related to the reality of representations of $SU(2)$. As any matrix with non negative entries, the matrix G has an eigenvector $\psi^{(1)}$ of largest eigenvalue $\gamma^{(1)}$ with non negative components: $\psi_a^{(1)} \geq 0$ (Perron- Frobenius theorem). Then [6]

$$U_i = a_{i-1} \underset{a_i}{\overset{a_i'}{\Diamond}} a_{i+1} = \delta_{a_{i-1}a_{i+1}} \frac{\left(\psi_{a_i}^{(1)}\psi_{a_i'}^{(1)}\right)^{\frac{1}{2}}}{\psi_{a_{i-1}}^{(1)}} \tag{4.1}$$

satisfies the Hecke algebra with

$$\gamma^{(1)} = 2\cos\pi\hat{\lambda}. \tag{4.2}$$

(In the case of $SU(2)$, the additional condition mentioned above reduces the Hecke algebra to the Temperley-Lieb algebra, in which the two sides of eq. (3.6c) vanish separately.) Since we impose that $\hat{\lambda}$ be real, we look for unoriented graphs such that their largest eigenvalue be less or equal to 2. The case $\gamma^{(1)} = 2$ corresponds to conformal theories with $c = 1$ and will not concern us here. The classification of graphs with

$$\gamma^{(1)} < 2 \tag{4.3}$$

is a well known problem [21]: the solutions are either the A, D, E Dynkin diagrams or the quotients $A_{2\ell}/Z_2$. The latter graphs actually lead to the same lattice models as the $A_{2\ell}$ ones and are therefore discarded. It is thus quite gratifying to see the same ADE classification as in sect. 2.1 emerging in this very different approach. The identification of the continuum limit of the critical lattice models with the $SU(2)_{n-3} \times SU(2)_1/SU(2)_{n-2}$ coset theories is completed by the determination of their partition function on a torus [22] and of their operator algebra [23].

Let us summarize some of the features of the $SU(2)$ case. All the graphs have eigenvalues (of their adjacency matrix) of the form

$$\gamma^{(\rho)} = 2\cos\frac{\pi\rho}{n} \tag{4.4}$$

where ρ runs over the exponents of the Dynkin diagram, as recalled above (see Table II) and n stands for the Coxeter number.

i) The A_{n-1} case may be regarded as self-dual: the heights (configuration variables) may be taken as ranging from 1 to $n-1$ and the exponents (which label the eigenstates hence play the role of momenta) run on the same set. This model is nothing but the RSOS model introduced originally in [24].

ii) The spectrum of the exponents of the other D or E solutions is always a subset of the exponents of the A graph of same Coxeter number n.

5. FROM $SU(2)$ TO $SU(N)$

We want to attach a representation of the Hecke algebra to some graph. The graph may a priori be oriented and have multiple edges between two nodes.

There is a known solution [25],[10], that we call basic and denote $\mathcal{A}^{(n)}$, by analogy with the case of $SU(2)$. The graph is the Weyl alcove at level $k = n - N$ introduced above (2.7). The elements of matrix of U_i read:

$$U_i = \lambda \underset{\mu}{\overset{\mu'}{\Diamond}} \nu = \frac{\left([\alpha.\mu][\alpha.\mu']\right)^{\frac{1}{2}}}{[\alpha.\lambda]} \tag{5.1}$$

where the root $\alpha = 2\mu - \lambda - \nu$ is assumed to be non vanishing, otherwise the matrix element of U_i vanishes; $q = e^{\frac{i\pi}{n}}$, and $[x] = \frac{q^x - q^{-x}}{q - q^{-1}}$ is the q–analogue of the number x. This U_i satisfies the Hecke algebra for $\hat{\lambda} = 1/n$.

In addition to this basic solution, other solutions exist whenever n and N have a common divisor N'. Then one may construct the orbifold graph [26]

$$\mathcal{D}^{(n);N'} = \mathcal{A}^{(n)}/Z_{N'} \tag{5.2}$$

and determine the Boltzmann weights in accordance with the Hecke algebra [27].

Besides these infinite series of solutions, one expects some exceptional solutions. By analogy with the case of $SU(2)$, it has been suggested [28] to replace the original problem:

- find graphs that support a representation of the Hecke algebra

by a different problem:

- find graphs $\mathcal{G}$ with definite spectral properties, namely such that their eigenvalues be among those of some $\mathcal{A}^{(n)}$.

To be more precise, and taking the case of $SU(3)$ for definiteness, we postulate:

- $\mathcal{G}$ must be 3-colourable: to each node, we attach a triality $\tau(a)$, and $G_{ab} \neq 0$ only if $\tau(b) = \tau(a) + 1 \bmod 3$.
- G is normal (i.e. $[G, G^t] = 0$) hence diagonalizable, and its spectrum of eigenvalues is included in that of $A^{(n)}$, for some n. The latter is (by Verlinde formula and the explicit form of the $S-$matrix for $SU(N)$ Kac-Moody theories, see above sect. 1.3)

$$\gamma^{(\rho)} = \sum_{j=1}^{N=3} \exp \frac{2i\pi}{n} e_j.\rho \tag{5.3}$$

For the $\mathcal{A}^{(n)}$, all the values of ρ inside the Weyl alcove are reached: this is again the self-duality of this $A-$case. For the other cases, only a subset of these values, with possible multiplicities, called the "exponents" of the graph, gives the eigenvalues *by assumption*. The value of n defines the "altitude" of the graph.

- The graph $\mathcal{G}$ possesses an involution $a \to \bar{a}$, generalizing the complex conjugation of representations of the $\mathcal{A}$ case, that reverses the triality and the direction of arrows:

$$\tau(\bar{a}) = -\tau(a) \mod 3$$
$$G_{a\bar{b}} = G_{\bar{b}a} \tag{5.4}$$

On top of that, we may add some condition to discard the redundancies of the type encountered in the case of $SU(2)$ (cf. $A_{2\ell}/Z_2$).

It is an open problem to classify all graphs with these properties, and the best I can do is to present on fig. 2 some of the solutions that we found in [28].

6. PROPERTIES OF THESE GRAPHS

First one should stress again that we have proceeded so far in a very heuristic way: based on the experience with $SU(2)$, we have assumed that the relevant graphs for the construction of integrable models are to be found among those satisfying the conditions listed in the previous section. There are indications that some of the graphs satisfying these conditions are inappropriate. Only "good" graphs which seem relevant are shown on fig. 2. The fact that these graphs support a representation of the Hecke algebra has been checked in a certain number of cases by direct computation [28],[29]. These graphs are marked "H" on fig. 2.

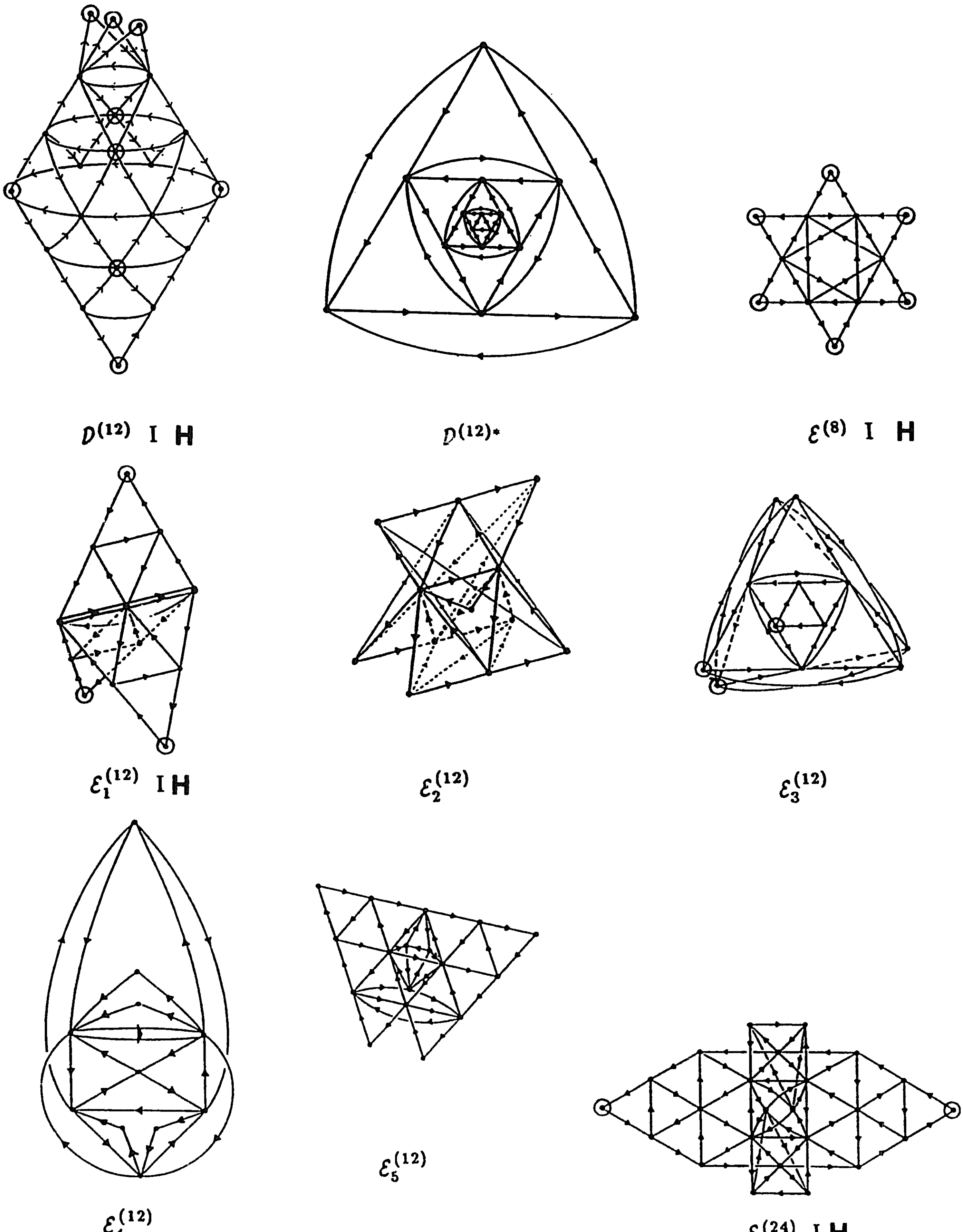
Algebraic properties and intertwiners.

Consider a general graph $\mathcal{G}^{(n)}$ and the graph $\mathcal{A}^{(n)}$ with the same altitude. Call $\psi_a^{(\rho)}$ and $\phi_\lambda^{(\rho)}$ their respective orthonormalized eigenvectors for the eigenvalue (5.3). By Verlinde formula, $\phi_\lambda^{(\rho)} = S_{\lambda\rho}$. Construct the set of numbers

$$V_{ab}^{\lambda} = \sum_{\substack{\rho \\ \text{exponents of } G}} \frac{\phi_\lambda^{(\rho)}}{\phi_1^{(\rho)}} \psi_a^{(\rho)} \psi_b^{(\rho)*} \tag{6.1}$$

Formula (6.1) is an extension of (2.12) to which it reduces for $G = A$.

The numbers V_{ab}^{λ} have the following properties:



$\mathcal{D}^{(12)}$ I H
$\mathcal{D}^{(12)*}$
$\mathcal{E}^{(8)}$ I H
$\mathcal{E}_1^{(12)}$ I H
$\mathcal{E}_2^{(12)}$
$\mathcal{E}_3^{(12)}$
$\mathcal{E}_4^{(12)}$
$\mathcal{E}_5^{(12)}$
$\mathcal{E}^{(24)}$ I H

FIGURE 2

i) $\sum_\mu A_{\lambda\mu} V_{ab}^\mu = \sum_c V_{ac}^\lambda G_{cb}$, i.e. for fixed a, the rectangular matrix V_{ab}^λ intertwines between the matrices A and G: $AV = VG$.

ii) $V_{ab}^1 = \delta_{ab}$ $\qquad V_{ab}^\Box = G_{ab}$

iii) V_{ab}^λ are non negative integers.

iv) The square matrices $V^\lambda = (V_{ab}^\lambda)$ satisfy

$$V^\lambda V^\mu = \sum_\nu N_{\lambda\mu}{}^\nu V^\nu \qquad (6.2)$$

Only property iii) is non trivial to prove: it is relatively easy to see that these numbers are integers [28], their positivity, however, requires a proof. It may be checked by inspection of all the cases relative to $SU(2)$ and of the various known cases of $SU(3)$ (the case A is already proved by (2.12)). Property iv), on the other hand, means that these matrices form a representation of the $\widehat{SU(N)}_{n-N}$ fusion algebra for which the fusion coefficients $N_{\lambda\mu}{}^\nu$ are the regular representation (see (2.13)). This suggests yet another possible reformulation of our problem: the classification of all non-negative integer valued representations of the fusion algebra of the Kac-Moody algebras $\widehat{SU(N)}$ might have much to do with the classification of the desired graphs. In the case of $SU(2)$, the problem may be shown to be equivalent to condition (4.3) and therefore leads again to ADE.

Why are the coefficients V_{ab}^λ integers and what is the algebraic interpretation of these numbers?

It has been shown some time ago [30] that in a conformal field theory, the partition function on an *annulus*, with fixed or free boundary conditions on the edges, is a *linear* form in the characters of the chiral algebra. In particular, in our height models, if we fix the height on the two sides to take fixed values 1 and λ on the $\mathcal{A}$ graph, resp. a and b on the $\mathcal{G}$ graph, there are strong indications that the corresponding partition functions are

$$Z_{1\lambda}^{(A)} = \chi_{1,\lambda}(\tau) \qquad (6.3a)$$

$$Z_{ab}^{(G)} = \sum_\lambda V_{ab}^\lambda \chi_{1,\lambda}(\tau) \qquad (6.3b)$$

$$= \sum_\lambda V_{ab}^\lambda Z_{1\lambda}^{(A)} \qquad (6.3c)$$

in terms of the coset characters labelled by two weights of levels $k-1$ and k. This has been proved in detail in the case of $SU(2)$ in [31]. Moreover Cardy [32] has proved that for the $\mathcal{A}$ models, the coefficients in (6.3b) must be the fusion coefficients. The reason why for a general graph $\mathcal{G}$ the coefficients in (6.3b) must form a representation of the ("thermal", i.e. relative to the $(1,\lambda)$ operators) fusion algebra is still mysterious. Could it be even more general and hold in an arbitrary c.f.t.? For "good" boundary conditions, the coefficients of the partition function as a linear form in the characters would correspond to a representation of some fusion algebra?

Property (6.3c) actually holds already on a discrete and finite lattice. One may show that the partition functions with fixed boundary conditions on the $\mathcal{G}$ and $\mathcal{A}$ graphs are related by (6.3c). This has been established in the case of $SU(2)$ by mapping the lattice models onto one another in [31] or by making use of algebraic properties of the representations of the Temperley-Lieb algebra attached to these graphs in [1]. It is very likely to extend to general $SU(N)$ [33],[29]. In the case of $SU(2)$, this reflects the reducibility of these representations of the Temperley-Lieb algebra. It has been argued in [3] that the only irreducible representations of this algebra are those corresponding to paths running from the end point 1 of the A_n graph to the generic point λ. The representations attached to other paths on A_n or to paths on other graphs are non irreducible and traces over such representations decompose as indicated by (6.3).

7. CONTINUUM LIMIT

To make the connection between the integrable lattice models and their continuum limit described by some c.f.t., one can appeal to statistical mechanics techniques as the Coulomb gas formalism. The lattice model is rephrased in terms of excitations with logarithmic interactions, i.e. in terms of a free boson field. Boundary conditions make the field compactified on a circle, the radius of which is determined by comparing some exact results obtained in the integrable model with the free field computation (see for example [34] for a review and further references.)

This has been successfully applied to the theories based on $SU(2)$ [22]. The case of $SU(N)$, $N > 2$, however, seems less tractable. The free field acquires several components, it is compactified on a torus of dimension $N-1$ equal to the rank of $SU(N)$, and its action may contain a "torsion" term antisymmetric in the components. This means that two parameters, compactification radius and coefficient of the antisymmetric term, have to be determined. For height models of the type discussed above, which are expected to be expressible in terms of *combinations* of free fields with different radii and couplings, the task appears almost insuperable.

It has been suggested in [33] that there may be more direct, algebraic connections between the lattice model and its continuum counterpart. This is based on two empiric observations:

⋆ For each known modular invariant Z of Table III, there exists one (or several) graph(s) of fig. 2, whose exponents label the diagonal terms of Z. This justifies the matching of notations in Table III and fig. 2. When there are several graphs with the same spectrum of exponents, there must exist several distinct c.f.t. with the same genus-one partition function but different operator algebras. Knowing the graph, hence the exponents, can one reconstruct the modular invariant? This may be easier when the modular invariant is a sum of blocks of characters squared. In that case, the problem just amounts to finding the appropriate partition of the set of exponents. This is the object of the second observation.

⋆ For each graph marked "I' on fig. 2, there is an associated c-algebra, with non negative structure constants, from which a partition of the set of exponents may be derived, corresponding to the blocks of a modular invariant.

In addition to the axioms of associativity and commutativity quoted in the Introduction, a c-algebra must have a unit denoted 1 and be endowed with an automorphism $a \to \bar{a}$: $N_{ab}{}^{c} = N_{\bar{a}\bar{b}}{}^{\bar{c}}$ such that $N_{ab}{}^{1} = \delta_{a\bar{b}}$. Fusion algebras of c.f.t. are examples of c-algebras, but more general c-algebras may be associated with the graphs of our construction. The generators N_a are attached to the vertices a of the graph and are represented by $|\mathcal{G}| \times |\mathcal{G}|$ matrices $N_{ab}{}^{c}$ ($|\mathcal{G}|$=number of vertices of $\mathcal{G}$). Suppose there exists some vertex a_1, denoted by abuse of notation 1, extremal in the sense that it is connected to only one pair of other vertices. We introduce the numbers

$$N_{ab}{}^{c} = \sum_{\lambda} \frac{\psi_a^{(\lambda)} \psi_b^{(\lambda)} \psi_c^{(\lambda)\,*}}{\psi_1^{(\lambda)}} \tag{7.1}$$

If some eigenvalues of G are degenerate, then the formula (7.1) may be ambiguous. The graph $\mathcal{G}$ is called type I if there exists a choice of eigenvectors for G yielding non negative integer N's. The forthcoming discussion will only apply to these type I graphs. There is a dual algebra with generators labelled by the exponents and with structure constants

$$M_{\lambda\mu}{}^{\nu} = \sum_{a \in U} \frac{\psi_a^{(\lambda)} \psi_a^{(\mu)} \psi_a^{(\nu)*}}{\psi_a^{(1)}} \tag{7.2}$$

These M are in general different from the N and take their values in the real numbers. However they turn out to be non negative in all the known type I cases.

What is the physical interpretation of the N's and the M's? In [6], local lattice operators have been constructed whose algebra is given by the M coefficients. To the best of my knowledge, no such interpretation has been found for the N's. This N algebra with positive structure constants turns out to contain as a subalgebra the thermal fusion algebra of the corresponding c.f.t. (Conversely, this means that the thermal fusion algebra is contained in a larger algebra, whose meaning remains elusive). The generators of this subalgebra have labels marked by circles on fig. 2. By a non-trivial theorem [8], the existence of such a subalgebra, together with the positivity of the N's and the M's implies the existence of a dual subalgebra of the M's, thus of an equivalence relation between exponents:

$$\lambda \approx \mu \iff \exists \text{ a generator } M_\alpha \text{ in the subalgebra such that } M_{\lambda\alpha}{}^{\mu} \neq 0. \tag{7.3}$$

What we have observed in [33] is that the equivalence classes for this relation are nothing else than the blocks of the partition function. In other words,

$$Z = \frac{1}{3} \sum_{\lambda \in P_{++}^{(n-1)}} \sum_{\text{class } i} \Big| \sum_{\mu \in \text{ class } i} \chi_{\lambda,\mu} \Big|^2 \tag{7.4}$$

reproduces the block diagonal invariants of Table III.

As this sketchy discussion shows, things are still very empiric and mysterious but these observations point to the existence of a novel and interesting algebraic connection between face lattice models and their continuum counterparts.

Let us summarize the points which would deserve more work:

1) the precise characterization of the graphs supporting a representation of the Hecke algebra. Are the conditions of sect.5 sufficient? What is the general form of the Boltzmann weights?

2) the classification of these graphs and/or of the representations of the fusion algebra on non negative integer valued matrices. Are the two problems equivalent or only similar?

3) the interpretation of the N and M algebras introduced in the last section, both from the lattice and from the continuous point of view, and of their relation with modular invariants.

4) the extension of these considerations to graphs that are not of "type I". In c.f.t. [2], it is known that non block diagonal modular invariants are obtained from block diagonal ones by the action of an automorphism of the fusion algebra, whose effect is to twist the right sector of the theory with respect to the left one. It is likely that there is an analogous mechanism in the c-algebraic discussion.

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