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Revisiting SU(N) integrals

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Abstract

In this note, I revisit integrals over SU(N) of the form $\int DU\, U_{i_1j_1}\cdots U_{i_pj_p}U^\dagger_{k_1l_1}\cdots U^\dagger_{k_nl_n}$. While the case $p=n$ is well known, it seems that explicit expressions for $p=n+N$ had not appeared in the literature. Similarities and differences, in particular in the large N limit, between the two cases are discussed.

Keywords: matrix integrals, group integration, lattice QCD

(Some figures may appear in colour only in the online journal)

1. Introduction and results

In this note, we consider the SU(N) integrals

$$\int DU\, U_{i_1j_1}\cdots U_{i_pj_p}U^\dagger_{k_1l_1}\cdots U^\dagger_{k_nl_n}, \tag{1}$$

with DU the normalized Haar measure, or their generating functions

$$Z_{p,n}(J,K)=\int DU\,(\mathrm{tr}\,(KU))^p(\mathrm{tr}\,(JU^\dagger))^n, \tag{2}$$

where J and K are arbitrary $N\times N$ matrices.

Such integrals, mainly over the group U(N), have been the object of numerous publications in the past, in the context of lattice gauge theories [1–4], or in the large N limit [5, 6], or for their connections with combinatorics [7, 8]. Integrals over SU(N) seem to have received less attention, see, however, [2, 9]. Recent work by Rossi and Veneziano [10] has prompted this new investigation.

Let us first recall the physical motivations for studying the SU(N) integrals (1) or (2). Such integrals appear in the context of lattice calculations of baryon spectrum. Indeed, consider a SU(N) lattice gauge theory, with link variables denoted $U_\ell\in$ SU(N), Wilson action

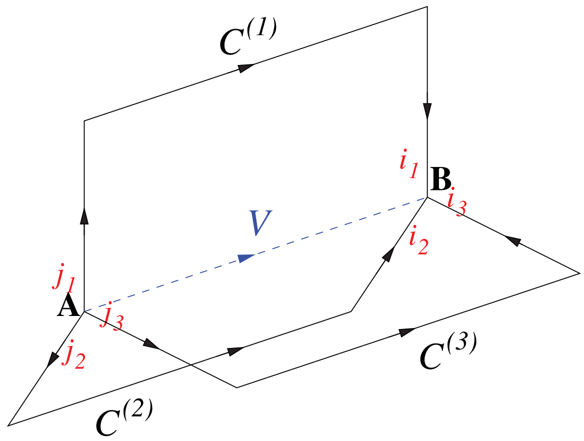

Figure 1. A book observable for $N = 3$.

$\mathcal{S} = \beta \sum_{\text{plaquettes}} \text{tr}\, U_P$, and lattice averages $\langle\cdot\rangle := \int \prod_\ell DU_\ell (\cdot) \text{e}^{\mathcal{S}} / \int \prod_\ell DU_\ell \text{e}^{\mathcal{S}}$. Following [10], introduce the baryonic Wilson loop or 'book observable'

$$\langle \mathcal{B} \rangle := \langle \epsilon_{i_1 \cdots i_N} \epsilon_{j_1 \cdots j_N} \prod_{a=1}^{N} U^{(a)}_{i_a j_a} \rangle$$

where the ordered products $U^{(a)} = \prod_{\ell \in \mathcal{C}^{(a)}} U_\ell$, $a = 1, \cdots N$, stand for N (static) quark lines joining two points A and B, a distance r apart, see figure 1. $\mathcal{B}$ represents a baryon made of N quarks, created at A and annihilated at B. To lowest order in a small β (strong coupling) expansion, $\langle \mathcal{B} \rangle = \beta^{\mathcal{A}} \epsilon_{i_1 \cdots i_N} \epsilon_{j_1 \cdots j_N} \int DV \prod_{a=1}^{N} V_{i_a j_a}$, with $\mathcal{A}$ the total area of the N 'sheets' of the book, and the last V-integration is carried out on the 'junction' of these sheets and is given by $Z_{N,0}$. Higher order corrections in β may involve some other integrals $Z_{n+N,n}$.

By $\mathbb{Z}_N$ invariance, it is clear that the above integrals vanish if

$$p - n \neq 0 \quad \text{mod}\, N. \tag{3}$$

From a representation theoretic point of view, the number of independent terms in (2), (i.e. of independent tensors with the right symmetries in (1)), is given by the number of invariants in $(N)^{\otimes p} \otimes (\bar{N})^{\otimes n}$, where (N) and $(\bar{N})$ denote the fundamental N-dimensional representation of SU(N) and its complex conjugate.

$Z_{n,n}(J,K)$ is a well known function of (traces of powers of) JK, ('Weingarten's function' [1]), at least for $n < N$, and one may collect all $Z_{n,n}$'s into

$$Z_W(J,K) = \sum_{n \geqslant 0} \frac{\kappa^{2n}}{(n!)^2} Z_{n,n}(J,K). \tag{4}$$

For the convenience of the reader, a certain number of known results on these integrals and their generating function are recalled in the appendix. This case $n = p$ in equations (1) and (2) will be referred to as 'the ordinary case'.

We now turn to the determination of the $Z_{n+N,n}$. First, for $n = 0$, $\int DU\, U_{i_1 j_1} \cdots U_{i_N j_N}$ is given by the only invariant in $\otimes^N (N)$, namely by the totally antisymmetric tensor product, (i.e. the determinant of U, equal to 1 in SU(N)). Hence

$$\int DU\; U_{i_1j_1}\cdots U_{i_Nj_N} = A\epsilon_{i_1\cdots i_N}\epsilon_{j_1\cdots j_N}$$

with a constant A determined by contraction with $\epsilon_{i_1\cdots i_N}$ and use of $\epsilon_{i_1\cdots i_N}U_{i_1j_1}\cdots U_{i_Nj_N} = \epsilon_{j_1\cdots j_N}\det U$, hence $\int DU\ \det U = \int DU = 1 = A\epsilon_{j_1\cdots j_N}\epsilon_{j_1\cdots j_N} = AN!$, and $A = 1/N!$. Thus

$$\int DU\; U_{i_1j_1}\cdots U_{i_Nj_N} = \frac{1}{N!}\epsilon_{i_1\cdots i_N}\epsilon_{j_1\cdots j_N}. \tag{5}$$

Equivalently,

$$Z_{N,0}(J,K) = \det K. \tag{6}$$

More generally, for $n < N$, $Z_{N+n,n}(J,K)$ is the product of $\det K$ by a polynomial of $M := JK$, invariant under $M \mapsto VMV^\dagger$, for any $V \in \mathrm{SU}(N)$, and homogeneous of degree n in M, hence a polynomial that may be expanded on traces of powers of M

$$Z_{N+n,n} = \int DU\ (\mathrm{tr}\,(UK))^{N+n}\,(\mathrm{tr}\,(U^\dagger J))^n = \det K \sum_{\alpha\vdash n} d_\alpha t_\alpha. \tag{7}$$

with a sum over partitions of n denoted $\alpha = [1^{\alpha_1}, 2^{\alpha_2}, \cdots, p^{\alpha_p}]$, $\sum_q q\,\alpha_q = n$, and with the notations

$$t_\alpha := \prod_q t_q^{\alpha_q}, \qquad t_q := \mathrm{tr}\,(JK)^q. \tag{8}$$

The coefficients d_α are determined through recursion formulae resulting from a contraction of (1) with a Kronecker delta:

$$\delta_{jk}\frac{\partial^2}{\partial J_{lk}\partial K_{ji}} Z_{N+n,n} = (N+n)n\ \delta_{il}\ Z_{N+n-1,n-1}. \tag{9}$$

The first coefficients are readily determined (10)

$$\begin{array}{llll}
n=1 & d_{[1]}=1 & & \\
n=2 & d_{[2]}=-\dfrac{1}{N} & d_{[1^2]}=\dfrac{N+1}{N} & \\
n=3 & d_{[3]}=\dfrac{4}{N(N-1)} & d_{[1,2]}=-\dfrac{3(N+1)}{N(N-1)} & d_{[1^3]}=\dfrac{(N+1)^2-2}{N(N-1)} \\
n=4 & d_{[4]}=-\dfrac{30}{N(N-1)(N-2)} & d_{[1,3]}=\dfrac{8(2(N+1)^2-3)}{(N+1)N(N-1)(N-2)} & d_{[2^2]}=\dfrac{3((N+1)^2+6)}{(N+1)N(N-1)(N-2)} \\
 & d_{[1^2,2]}=-\dfrac{6((N+1)^2-4)}{N(N-1)(N-2)} & d_{[1^4]}=\dfrac{(N+1)^4-8(N+1)^2+6}{(N+1)N(N-1)(N-2)} &
\end{array}$$

etc.

These coefficients are in fact simply related to the analogous coefficients z_α in the expansion of the ordinary generating function, see appendix, equation (A.3). If z_α, $\alpha \vdash n$, is written in the form

$$z_\alpha = \frac{P_\alpha(N)}{N^2(N^2-1)\cdots(N^2-(n-1)^2)} \tag{11}$$

with $P_\alpha(N)$ a polynomial of N, then

$$d_\alpha = \frac{P_\alpha(N+1)}{(N+1)\cdots(N-(n-2))}. \tag{12}$$

This follows from the comparison between the two systems of recursion formulae, see below section 2, and may be verified on the first coefficients (10) and (A.5).

Note that the coefficients d_α decrease more slowly than the z_α, for fixed n, as N grows. This behavior has consequences on the large N limit of the generating function Z_D defined by

$$Z_D = \sum_{n\geqslant 0} \frac{\kappa^n}{n!} Z_{N+n,n} = \int DU\, \mathrm{e}^{\kappa\,\mathrm{tr}\,UK\,\mathrm{tr}\,U^\dagger J}\,(\mathrm{tr}\,UK)^N =: \tilde{Z}_D \det K. \tag{13}$$

Note also the different summations in (4) and (13): here κ is a homogeneity parameter for the eigenvalues of $M = JK$ while in (4), it is κ^2 that plays that role. In both cases, however, we take κ of order N and set $\kappa = N\tilde{\kappa}$. Then while in the ordinary case, a non trivial limit is obtained by taking the traces t_n also of order N, resulting in an exponentation $Z_W = \exp N^2 W_W$, here one finds that the traces t_n have to be kept of order 1 and then $\tilde{Z}_D = \exp NW_D$, with W_D a non trivial function of the t_n's.

Indeed one finds for the first terms

$$\lim_{N\to\infty} \frac{1}{N} \log \frac{Z_D}{\det K} = \tilde{\kappa} t_1 + \frac{\tilde{\kappa}^2}{2}(t_1^2 - t_2) + \frac{\tilde{\kappa}^3}{3}(t_1^3 - 3t_1t_2 + 2t_3) + \frac{\tilde{\kappa}^4}{4}(t_1^4 - 6t_2t_1^2 + 2t_2^2 + 8t_3t_1 - 5t_4) + \cdots \tag{14}$$

which is just the beginning of a simple formula in the large N limit:

$$W_D := \lim_{N\to\infty} \frac{1}{N} \log \frac{Z_D}{\det K} = \sum_{n\geqslant 1} \tilde{\kappa}^n \sum_{\alpha\vdash n} (-1)^{n-\sum\alpha_q} \frac{(n-1)!}{(n-\sum\alpha_q+1)!} \prod_p \frac{(\mathrm{Cat}(p-1)\,t_p)^{\alpha_p}}{\alpha_p!} \tag{15}$$

in terms of the Catalan numbers $\mathrm{Cat}(m) = \frac{(2m)!}{m!\,(m+1)!}$. This will be proved below in section 3.

2. The recursion relations

Let us return to (9) and look at the action of the differential operator $\frac{\partial}{\partial K_{ji}}$ on a typical term $t_\alpha \det K$ of (7). Making use of the identities

$$\frac{\partial}{\partial K_{ji}} \det[K] = (\mathrm{Cof} K)_{ji} \tag{16}$$

$$(\mathrm{Cof} K)_{ji}(KX)_{jl} = (\det K) X_{il} \tag{17}$$

one may write for each term in the expansion of $Z_{N+n,n}$

$$\begin{aligned}\frac{\partial^2}{\partial K_{ji}\partial J_{lj}}(t_\alpha \det K) &= \frac{\partial}{\partial K_{ji}} \sum_{q=1}^{p} q\alpha_q (K(JK)^{q-1})_{jl}\, t_{\widehat{\alpha}_{(q)}} \det K \\ &= \sum_{q=1}^{p} q\alpha_q \left((N+1)(JK)_{il}^{q-1} + \sum_{s=1}^{q-1} t_s (JK)_{il}^{q-s-1} \right) t_{\widehat{\alpha}_{(q)}} \det K \\ &\quad + \sum_{q=1}^{p} q^2 \alpha_q(\alpha_q - 1)(JK)_{il}^{2q-1} t_{\widehat{\widehat{\alpha}}_{(q)}} \det K + 2 \sum_{1\leqslant q<r\leqslant p} qr\alpha_q\alpha_r (JK)_{il}^{q+r-1} t_{\widehat{\alpha}_{(q,r)}} \det K \end{aligned} \tag{18}$$

with $\widehat{\alpha}_{(q)} := (1^{\alpha_1} \cdots q^{\alpha_q - 1} \cdots p^{\alpha_p})$, $\widehat{\widehat{\alpha}}_{(q)} := (1^{\alpha_1} \cdots q^{\alpha_q - 2} \cdots p^{\alpha_p})$, and $\widehat{\alpha}_{(q,r)} := (1^{\alpha_1} \cdots q^{\alpha_q - 1} \cdots r^{\alpha_r - 1} \cdots p^{\alpha_p})$.

In this way, (9) yields an (overdetermined) system of relations between the coefficients d_α at ranks n and $n-1$, namely

$$\sum_{\alpha\vdash n} d_\alpha \Bigg\{ \sum_{q=1}^{p} q\alpha_q \Bigg(\underline{(N+1)}(JK)_{il}^{q-1} + \sum_{s=1}^{q-1} t_s (JK)_{il}^{q-s-1} \Bigg) t_{\widehat{\alpha}_{(q)}} \det(K)$$
$$+ \sum_{q=1}^{p} q^2 \alpha_q(\alpha_q-1)(JK)_{il}^{2q-1} t_{\widehat{\widehat{\alpha}}_{(q)}} \det(K) + 2 \sum_{1\leqslant q<r\leqslant p} qr\alpha_q\alpha_r (JK)_{il}^{q+r-1} t_{\widehat{\alpha}_{(q,r)}} \det(K) \Bigg\}$$
$$= n\underline{(N+n)}\delta_{il} \sum_{\alpha'\vdash n-1} d_{\alpha'} t_{\alpha'} \det(K). \tag{19}$$

Compare these equations with those satisfied by the coefficients z_α in the similar expansion of $Z_{n,n}(J, K)$ in the ordinary case, see (A.4). Their structure is the same, except for changes in the terms of (19), underlined. In the linear system on the d_α, the parameter N in the lhs of (A.4) has been changed into $N+1$ while the right hand side is multiplied by $(N+n)$. As a result the solutions of the d_α linear system are obtained from those of the z_α one by

$$\text{for a given } n \qquad d_\alpha = (N+n)(N+n-1)\cdots(N+1)\big(z_\alpha|_{N\to N+1}\big).$$

If z_α is written as in (11), it follows that d_α has the form (12), qed.

In particular, this relation implies that z_α and d_α have the same overall sign, namely $(-1)^{\#\text{cycles}(\alpha)+n}$.

3. Exponentiation and large N limit

The differential equation (9) carries over to the generating function Z_D of (13) in the form

$$\delta_{jk}\frac{\partial^2}{\partial J_{lk}\partial K_{ji}} Z_D = \delta_{il}\left((N+1)\kappa + \kappa^2\frac{\partial}{\partial\kappa}\right) Z_D. \tag{20}$$

We write $Z_D = \det K\,\tilde{Z}_D$, in which $\tilde{Z}_D$ is a function of $M := J\,K$ invariant under $M \to VMV^\dagger$ for any $V \in \mathrm{SU}(N)$, and we rewrite the differential equation as

$$\frac{\partial^2\tilde{Z}_D}{\partial M_{lk}\partial M_{ji}} M_{jk} + (N+1)\frac{\partial\tilde{Z}_D}{\partial M_{li}} = \delta_{il}\left((N+1)\kappa + \kappa^2\frac{\partial}{\partial\kappa}\right)\tilde{Z}_D. \tag{21}$$

This may be reexpressed as a differential equation wrt the eigenvalues λ_i of M (generically M is diagonalizable). This is a standard procedure [5] with the result that for any $i, 1\leqslant i\leqslant N$

$$\lambda_i\frac{\partial^2\tilde{Z}_D}{\partial\lambda_i^2} + (N+1)\frac{\partial\tilde{Z}_D}{\partial\lambda_i} + \sum_{j\neq i}\lambda_j\frac{\frac{\partial\tilde{Z}_D}{\partial\lambda_i} - \frac{\partial\tilde{Z}_D}{\partial\lambda_j}}{\lambda_i-\lambda_j} = \left((N+1)\kappa + \kappa^2\frac{\partial}{\partial\kappa}\right)\tilde{Z}_D. \tag{22}$$

Finally we write $\tilde{Z}_D = \exp NW_D$, which results in

$$\lambda_i\left(N\frac{\partial^2 W_D}{\partial\lambda_i^2} + N^2\left(\frac{\partial W_D}{\partial\lambda_i}\right)^2\right) + (N+1)N\frac{\partial W_D}{\partial\lambda_i} + N\sum_{j\neq i}\lambda_j\frac{\frac{\partial W_D}{\partial\lambda_i} - \frac{\partial W_D}{\partial\lambda_j}}{\lambda_i-\lambda_j} = (N+1)\kappa + N\kappa^2\frac{\partial W_D}{\partial\kappa}. \tag{23}$$

In the large N limit, we rescale $\kappa = N\tilde{\kappa}$, keeping all $t_n = \sum_i \lambda_i^n$ of order 1, and after dropping the subdominant terms, we get for $w_i := \frac{\partial W_D}{\partial\lambda_i}$ the equation

$$\lambda_i w_i^2 + w_i = \tilde{\kappa} + \tilde{\kappa}^2 \frac{\partial W_D}{\partial \tilde{\kappa}}. \tag{24}$$

(Note that this is in contrast with the 'ordinary case' where the term $\sum_{j \neq i} \cdots$ contributes in the large N limit [5].) Now $\tilde{\kappa}$ is just an homogeneity variable of the λ's, and we may thus substitute $\tilde{\kappa}\frac{\partial W_D}{\partial \tilde{\kappa}} = \sum_j \lambda_j \frac{\partial W_D}{\partial \lambda_j} = \sum_j \lambda_j w_j$. The equation finally reduces to a system of *algebraic* equations for the w_i's

$$\lambda_i w_i^2 + w_i = \tilde{\kappa}(1 + \sum_j \lambda_j w_j). \tag{25}$$

Assuming $w := \sum_i \lambda_i w_i$ known, one finds

$$\begin{aligned} &\lambda_i w_i^2 + w_i = \tilde{\kappa}(1 + w) \\ &w_i = \frac{-1 + \sqrt{1 + 4\lambda_i \tilde{\kappa}(1 + w)}}{2\lambda_i} \end{aligned} \tag{26}$$

and $1 + w$ is thus the root of

$$1 + w = 1 - \frac{N}{2} + \sum_i \frac{\sqrt{1 + 4\lambda_i \tilde{\kappa}(1 + w)}}{2}. \tag{27}$$

Equation (26) should be compared with that of the generating function of Catalan numbers $\mathrm{Cat}(m) = \frac{(2m)!}{m!\,(m+1)!}$, namely $C(t) = \sum_{n \geqslant 0} \mathrm{Cat}(n) t^n$, $tC^2(t) - C(t) + 1 = 0$. We find that $w_i = \tilde{\kappa}(1 + w) C(-\tilde{\kappa}(1 + w)\lambda_i)$, hence

$$w_i = \tilde{\kappa}(1 + w) \sum_{n \geqslant 0} \mathrm{Cat}(n)(-\tilde{\kappa}(1 + w)\lambda_i)^n \tag{28}$$

and

$$w = \sum_i \lambda_i w_i = -\sum_{n \geqslant 1} \mathrm{Cat}(n-1)(-\tilde{\kappa}(1 + w))^n t_n. \tag{29}$$

Let $f_0 = 1$, $f_n := (-1)^{n-1}\mathrm{Cat}(n-1) t_n$ for $n \geqslant 1$, $y := \tilde{\kappa}(1 + w)$, then (29) reads $y = \tilde{\kappa} \sum_{n \geqslant 0} f_n y^n$, whose solution is given by Lagrange formula

$$\begin{aligned} y = \tilde{\kappa}(1 + w) &= \sum_{n \geqslant 0} \frac{\tilde{\kappa}^{n+1}}{(n+1)!} \left(\frac{d}{dz}\right)^n \left(\sum_{m \geqslant 0} f_m z^m \right)^{n+1} \Bigg|_{z=0} \\ &= \tilde{\kappa} + \sum_{n \geqslant 1} \frac{\tilde{\kappa}^{n+1}}{(n+1)!} \sum_{\alpha \vdash n} \frac{(n+1)!}{(n + 1 - \sum_q \alpha_q)! \prod_q \alpha_q!} n! f_\alpha \\ w = \sum_{n \geqslant 1} m_n \tilde{\kappa}^n &= \sum_{n \geqslant 1} \tilde{\kappa}^n \sum_{\alpha \vdash n} \frac{n!}{(n + 1 - \sum_q \alpha_q)! \prod_q \alpha_q!} f_\alpha \end{aligned} \tag{30}$$

where the multinomial coefficient appears naturally in the expansion of the $n + 1$-th power, and the $n!$ results from the n-th derivative of z^n. Upon integration we get

$$W_D = \sum_{n \geqslant 1} \tilde{\kappa}^n \sum_{\alpha \vdash n} (-1)^{n - \sum \alpha_q} \frac{(n-1)!}{(n + 1 - \sum_q \alpha_q)! \prod_q \alpha_q!} \prod_p (\mathrm{Cat}(p-1) t_p)^{\alpha_p}$$

which establishes (15).

Note the similarity of this calculation with the relation between ordinary moments m_n and non crossing (or free) cumulants f_n of a given distribution. The combinatorial or diagrammatical interpretation of (15) remains to be found.

Acknowledgments

It is a pleasure to thank Gabriele Veneziano for inspiring discussions.

Appendix

In this appendix, we recall a certain number of results on the Weingarten's functions $Z_{n,n}(J,K)$ and their generating function $Z_W(J,K)$ of (4). The $Z_{n,n}$ are well known functions of (traces of powers of) JK, at least for $n<N$, see also [2–4]. For $n \geqslant N$, the traces are no longer independent, or equivalently the tensors $\prod \delta_{i\pi(l)}\delta_{j\rho(k)}$, $\pi,\rho \in S_n$, are no longer independent, and we have to deal with a 'pseudo-inverse' of the Gram matrix, see [7, 8].

Consider the integrals on U(N) (or SU(N), this is irrelevant here, assuming $n<N$) [1, 4, 7]

$$\begin{aligned}\int DU\, U_{i_1j_1}\cdots U_{i_nj_n}U^\dagger_{k_1\ell_1}\cdots U^\dagger_{k_n\ell_n} &= \sum_{\tau,\sigma\in S_n} C([\sigma])\prod_{a=1}^{n}\delta_{i_a\ell_{\tau(a)}}\delta_{j_ak_{\tau\sigma(a)}}\\ &= \sum_{\tau,\sigma\in S_n}\sum_{\substack{Y\,\text{Young diagr.}\\ |Y|=n}}\frac{(\chi^{(\lambda)}(1))^2\chi^{(\lambda)}([\sigma])}{n!^2\, s_\lambda(I)}\prod_{q=1}^{n}\delta_{i_q\ell_{\tau(q)}}\delta_{j_qk_{\tau\sigma(q)}}\end{aligned} \tag{A.1}$$

where $\chi^{(\lambda)}([\sigma])$ is the character of the symmetric group S_n associated with the Young diagram Y, (a function of the class $[\sigma]$ of σ); thus $\chi^{(\lambda)}(1)$ is the dimension of that representation; $s_\lambda(X)$ is the character of the linear group GL(N) associated with Young diagram Y, that is a Schur function when expressed in terms of the eigenvalues of X; $s_\lambda(I)$ is thus the dimension of that representation. Finally the coefficient $C([\sigma])$ will be determined below.

Alternatively, in terms of generating functions with sources J and K

$$Z_{n,n}(J,K) = \int DU(\operatorname{tr} KU)^n(\operatorname{tr} JU^\dagger)^n = \sum_{\alpha\vdash n} n!|\alpha|\, C([\alpha])\, t_\alpha \tag{A.2}$$

where $|\alpha|$ is the cardinal of class $[\alpha]$ in S_n, thus $|\alpha| = \frac{n!}{\prod_p p^{\alpha_p}\alpha_p!}$. In (A.2), the factor $n\,!$ comes from the sum over τ and the factor $|\alpha|$ from that over the elements $\sigma\in[\alpha]$. Thus

$$Z_W(J,K) := \int_{U(N)} DU \exp[\kappa \operatorname{tr}(KU+JU^\dagger)] = \sum_{n=0}^{\infty}\frac{\kappa^{2n}}{n!}\sum_{\alpha\vdash n} z_\alpha t_\alpha \tag{A.3}$$

with $z_\alpha = |\alpha|\, C([\alpha])$.

By the same argument as in section 2, the z_α satisfy the linear system of recursion relations

$$
\begin{aligned}
&\sum_{\alpha\vdash n} z_\alpha \Bigg\{ \sum_{q=1}^{p} q\alpha_q \left(N(JK)_{il}^{q-1} + \sum_{s=1}^{q-1} t_s (JK)_{il}^{q-s-1} \right) t_{\widehat{\alpha}_{(q)}} \\
&\quad + \sum_{q=1}^{p} q^2 \alpha_q(\alpha_q - 1)(JK)_{il}^{2q-1} t_{\widehat{\widehat{\alpha}}_{(q)}} + 2 \sum_{1\leqslant q<r\leqslant p} qr\alpha_q\alpha_r (JK)_{il}^{q+r-1} t_{\widetilde{\alpha}_{(q,r)}} \Bigg\} \\
&\quad = n\delta_{il} \sum_{\alpha'\vdash n-1} z_{\alpha'} t_{\alpha'}.
\end{aligned}
\tag{A.4}
$$

Explicitly, the first coefficients $z_\alpha = |\alpha| C([\alpha])$ read

$$
\begin{aligned}
&n=1 \qquad z_{[1]} = \frac{1}{N} \\
&n=2 \qquad z_{[2]} = -\frac{1}{(N^2-1)N}, \quad z_{[1^2]} = \frac{1}{(N^2-1)} \\
&n=3 \qquad z_{[3]} = \frac{4}{(N^2-4)(N^2-1)N}, \quad z_{[1,2]} = -\frac{3}{(N^2-4)(N^2-1)}, \quad z_{[1^3]} = \frac{N^2-2}{(N^2-4)(N^2-1)N} \\
&n=4 \qquad z_{[4]} = -\frac{30}{(N^2-9)(N^2-4)(N^2-1)N}, \quad z_{[1,3]} = \frac{8(2N^2-3)}{(N^2-9)(N^2-4)(N^2-1)N^2} \\
&\qquad\qquad z_{[2^2]} = \frac{3(N^2+6)}{(N^2-9)(N^2-4)(N^2-1)N^2}, \quad z_{[1^2,2]} = -\frac{6(N^2-4)}{(N^2-9)(N^2-4)(N^2-1)N} \\
&\qquad\qquad z_{[1^4]} = \frac{N^4-8N^2+6}{(N^2-9)(N^2-4)(N^2-1)N^2}
\end{aligned}
\tag{A.5}
$$

etc.

The overall sign of z_α is $(-1)^{\#\mathrm{cycles}(\alpha)-n}$ and its large N behavior $|z_\alpha| \sim N^{-2n+\#\mathrm{cycles}(\alpha)}$.

To study the large N limit, we take $\kappa = N\tilde{\kappa}$ and $t_p = N\tau_p$, with $\tilde{\kappa}$ and τ_p of order 1. Then Brézin and Gross [5] have shown that

$$
W_W(JK) := \lim_{N\to\infty} \frac{1}{N^2} \log Z_W(J, K; N\tilde{\kappa})
\tag{A.6}
$$

exists and satisfies the coupled equations

$$
W_W = \frac{2}{N} \sum_i (\tilde{\kappa}^2\lambda_i + c)^{\frac{1}{2}} - \frac{1}{2N^2} \sum_{i,j} \log[(\tilde{\kappa}^2\lambda_i + c)^{\frac{1}{2}} + (\tilde{\kappa}^2\lambda_j + c)^{\frac{1}{2}}] - c - \frac{3}{4}
\tag{A.7}
$$

$$
\text{with} \quad c = \begin{cases} \dfrac{1}{N} \displaystyle\sum_i (\tilde{\kappa}^2\lambda_i + c)^{-\frac{1}{2}} & \text{for } \dfrac{1}{N} \displaystyle\sum_i (\tilde{\kappa}^2\lambda_i)^{-\frac{1}{2}} \geqslant 2 \qquad \text{('strong coupling')} \\ 0 & \text{for } \dfrac{1}{N} \displaystyle\sum_i (\tilde{\kappa}^2\lambda_i)^{-\frac{1}{2}} \leqslant 2 \qquad \text{('weak coupling')} \end{cases}
\tag{A.8}
$$

The solution has two determinations, in a strong coupling and in a weak coupling phase. Here we are concerned with the strong coupling regime in which we may expand

$$
W_W = \sum_{n\geqslant 1} \tilde{\kappa}^{2n} \sum_{\alpha\vdash n} w_\alpha \tau_\alpha
\tag{A.9}
$$

with the first terms

$$W_W = \tilde{\kappa}^2\tau_1 + \frac{\tilde{\kappa}^4}{2}(\tau_1^2 - \tau_2) + \frac{2\tilde{\kappa}^6}{3}(2\tau_1^3 - 3\tau_1\tau_2 + \tau_3) + \frac{\tilde{\kappa}^8}{4}(24\tau_1^4 - 48\tau_1^2\tau_2 + 9\tau_2^2 + 20\tau_1\tau_3 - 5\tau_4) + \cdots \tag{A.10}$$

and the general term given in [6]

$$w_\alpha = (-1)^n \frac{(2n-3+\sum_q \alpha_q)!}{(2n)!} \prod_q \left(-\frac{(2q)!}{(q\,!)^2}\right)^{\alpha_q} \frac{1}{\alpha_q!}. \tag{A.11}$$

References

[1] Weingarten D 1978 Asymptotic behavior of group integrals in the limit of infinite rank *J. Math. Phys.* **19** 999–1001

[2] Creutz M 1978 On invariant integration over SU(*N*) *J. Math. Phys.* **19** 2043–6

[3] Bars I and Green F 1979 Complete integration of U(*N*) lattice gauge theory in the large *N* limit *Phys. Rev.* D **20** 3311–30
Samuel S 1980 U(*N*) integrals, 1/*N* and De Wit–'t Hooft anomalies *J. Math. Phys.* **21** 2695–703
Bars I 1980 U(*N*) integral for the generating functional in lattice gauge theories *J. Math. Phys.* **21** 2678–81

[4] Drouffe J-M and Zuber J-B 1983 Strong coupling and mean field methods in lattice gauge theories *Phys. Rep.* **102** 1–119

[5] Brézin É and Gross D 1980 The external field problem in the large *N* limit of QCD *Phys. Lett.* **97** B120–4

[6] O'Brien K H and Zuber J-B 1984 A note on U(*N*) integrals in the large *N* limit *Phys. Lett.* **144** B407–8

[7] Collins B 2003 Moments and cumulants of polynomial random variables on unitary groups the Itzykson-Zuber integral and free probability *Int. Math. Res. Not.* **17** 952–82
Collins B and Śniady P 2006 Integration with respect to the Haar measure of unitary, orthogonal and symplectic group *Commun. Math. Phys.* **264** 773–95

[8] Novak J 2010 Complete homogeneous symmetric polynomials in Jucys–Murphy elements and the Weingarten function *Banach Center Publ.* **89** 231–5
Zinn-Justin P 2010 Jucys–Murphy elements and Weingarten matrices *Lett. Math. Phys.* **91** 119–27

[9] Carlsson J 2008 Integrals over SU(*N*) (arXiv:0802.3409)

[10] Rossi G C and Veneziano G 2016 The string-junction picture of multiquark states: an update (arXiv:1603.05830)