VERLINDE NIM-REPS FOR CHARGE CONJUGATE $SL(N)$ WZW THEORY

V.B. Petkova
Institute for Nuclear Research and Nuclear Energy
72 Tsarigradsko Chaussee,
1784 Sofia, Bulgaria
petkova@inrne.bas.bg

J.-B. Zuber
SPhT, CEA Saclay, 91191 Gif-sur-Yvette, France
zuber@spht.saclay.cea.fr

Abstract

We compute the representations ("NIM-reps") of the fusion algebra of $\widehat{sl}(N)$ which determine the boundary conditions of $\widehat{sl}(N)$ WZW theories twisted by the charge conjugation. The problem is related to the classical problem of decomposition of the fundamental representations of $sl(N)$ onto representations of $B_l = so(2l+1)$ or $C_l = sp(2l)$ algebras. The relevant NIM-reps and their diagonalisation matrix are thus expressed in terms of modular data of the affine B or C algebras.

Keywords: Conformal field theories, WZW theories, boundary conditions, NIM-reps

1. Introduction

It is now well understood that the possible boundary conditions of a rational conformal field theory are determined by the set of non-negative integer valued matrix representations, or NIM-reps, of the fusion algebra of this theory [2]. In the present paper we address the problem of determining NIM-reps and related data for those theories of WZW type, that are described by a modular invariant partition function twisted by complex conjugation

$$Z = \sum_{\lambda} \chi_\lambda(\tau, z)\chi_{\lambda^*}(\tau, z)^* \,. \tag{1}$$

To be specific, we restrict here to the $\widehat{sl}(N)$ current algebra. This exercise has the double merit of illustrating the power of certain methods of general application, and of exhibiting a nice algebraic pattern: indeed, it turns out that the problem is intimately connected to the classical problem of decomposing the representations of $sl(N)$ onto representations of the $B_l = so(2l+1)$ or $C_l = sp(2l)$ algebras, with $N = 2l$ or $2l+1$. This work generalises the previous results for $N = 3$ [1],[2] and $N = 4$ [3]. [1]

2. The $A_{N-1} = sl(N)$ and the affine $\widehat{sl}(N)$ algebras

To proceed, we need to introduce notations. As we are dealing with pairs of Lie algebras, we consistently use different types of labels for their representations etc. For the $\widehat{sl}(N)$ theories under study, weights will be denoted by Greek letters. At a given level k or shifted level $h = k + N$ these weights belong to the Weyl alcove

$$\mathcal{P}_{++}^{(A_{N-1},h)} := \{\lambda = \sum_{i=1}^{N-1} \lambda_i \Lambda_i \mid \lambda_i \geq 1\,, \sum_{i=1}^{N-1} \lambda_i \leq h-1\}\ , \tag{2}$$

where Λ_i, $i = 1, \cdots, N-1$ are the $sl(N)$ fundamental weights. The Weyl vector is $\rho = \sum_{i=1} \Lambda_i$. The number of weights in $\mathcal{P}_{++}^{(A_{N-1},h)}$ equals $\binom{h-1}{N-1}$. The alcove is invariant under the action of C, the complex conjugation of representations, $C:\ \lambda = (\lambda_1, \cdots, \lambda_{N-1}) \mapsto \lambda^* = (\lambda_{N-1}, \cdots, \lambda_1)$, and of the $\mathbb{Z}_N$ automorphism σ, related to the cyclic symmetry of the affine Dynkin diagram of type A

$$\sigma(\lambda) = (h - \sum_{i=1}^{N-1} \lambda_i, \lambda_1, \cdots, \lambda_{N-2})\ . \tag{3}$$

Basic in our discussion is the symmetric, unitary matrix $S = (S_{\lambda\mu})$ of modular transformations. Under the action of C and σ,

$$S_{\lambda^*\mu} = S_{\lambda\mu^*} = (S_{\lambda\mu})^* \qquad S_{\sigma(\lambda)\mu} = e^{2\pi i \tau(\mu)/N}\, S_{\lambda\mu}\ , \tag{4}$$

where $\tau(\lambda) := \sum_{i=1}^{N-1} i(\lambda_i - 1)$ is the $\mathbb{Z}_N$ grading of weights –the "N-ality".

We want to find a set of matrices $\{n_\gamma\}_{\gamma \in \mathcal{P}_{++}^{(N;\,h)}}$ with non negative integer entries such that their matrix product reads

$$n_\lambda\, n_\mu = \sum_\nu N_{\lambda\mu}{}^\nu\, n_\nu \tag{5}$$

where $N_{\lambda\mu}{}^{\nu}$ are the fusion matrices of the $\widehat{sl}(N)$ theory at that level. The n_λ must satisfy Cardy consistency condition

$$n_{\lambda a}{}^{b} = \sum_{j\equiv j(\mu)\,,\ \mu\in\,\mathrm{Exp}^{(h)}} \frac{S_{\lambda\mu}}{S_{\rho\mu}}\,\psi_a^j\,\psi_b^{j\,*} \tag{6}$$

with ψ the unitary matrix diagonalising them; $j = j(\mu)$ labels a proper choice of basis. Their eigenvalues are thus of the form $\chi_\lambda(\mu) := S_{\lambda\,\mu}/S_{\rho\,\mu}$, and are specified by the weights μ labelling the *diagonal* terms in (1), called "exponents". In the case under study, the exponents are the real, i.e. self-conjugate, weights $\mu = \mu^*$ in the alcove. Depending on the parity of N, those have a different structure:

$$\mathrm{Exp}^{(h)} \ni \mu = \begin{cases} (m_1,\cdots,m_l,m_l,\cdots,m_1)\,, & \\ \qquad 2\sum_{i=1}^{l} m_i \le h-1 & \text{if } N=2l+1 \\ (m_1,\cdots,m_{l-1},m_l,m_{l-1},\cdots,m_1)\,, & \\ \qquad 2\sum_{i=1}^{l-1} m_i + m_l \le h-1 & \text{if } N=2l \end{cases} \tag{7}$$

and their number is

$$|\mathrm{Exp}^{(h)}| = \ \#\text{real weights} = \begin{cases} \binom{\lfloor\frac{h-1}{2}\rfloor}{l} & \text{if } N=2l+1 \\ \binom{\lfloor\frac{h}{2}\rfloor}{l} + \binom{\lfloor\frac{h-1}{2}\rfloor}{l} & \text{if } N=2l \end{cases} \tag{8}$$

In general, the NIM-rep matrices satisfy $n_\lambda^T = n_{\lambda^*}$; in the present case, because of the reality of the exponents μ, their eigenvalues $\chi_\lambda(\mu)$ are real and satisfy $\chi_\lambda(\mu) = \chi_{\lambda^*}(\mu)$ and one concludes that the matrices n_λ are symmetric. Moreover, because a real weight μ has a N-ality τ equal to $0 \mod N$, resp. 0 or $N/2 \mod N$, for N odd, resp. even, eq. (4) implies that n_λ is only a function of the orbit of λ under σ, resp. σ^2. As usual, it is sufficient to find the generators $n_{\Lambda_i+\rho} = n_{\Lambda_{N-i}+\rho}$ associated with the fundamental weights to fully determine the NIM-rep. If the matrices $n_\lambda = (n_{\lambda\,a}{}^{b})$ are regarded as adjacency matrices of graphs, it is natural to refer to the labels a, b of their entries as *vertices*. On the latter, we do not know much a priori, besides that their number equals the number of exponents (8). The set of vertices is denoted by $\mathcal{V}$.

Along with the NIM-rep matrices n_λ, we are also interested in finding a related set of matrices $\hat{N}_a = (\hat{N}_{ba}{}^{c})$, satisfying $\hat{N}_a\hat{N}_b = \sum_c \hat{N}_{ab}{}^{c}\hat{N}_c$ and

$$n_\lambda\,\hat{N}_a = \sum_{b\in\mathcal{V}} n_{\lambda a}{}^{b}\,\hat{N}_b\ . \tag{9}$$

These matrices, associated with the vertices of the graph, span the "graph algebra", which in this particular case is commutative. The set includes the unit matrix attached to a special vertex denoted $1 : \hat{N}_1 = I$. Then the previous relation evaluated for $a = 1$ gives

$$n_\lambda = \sum_{b\in\mathcal{V}} n_{\lambda 1}{}^b \, \hat{N}_b \ , \tag{10}$$

i.e. the NIM-rep matrices are ≥ 0 integer linear combinations of the $\hat{N}$. The matrix ψ in (6) diagonalises both n and $\hat{N}$ and (10) can be also rewritten as

$$\chi_\lambda(\mu) = \sum_{a\in\mathcal{V}} n_{\lambda 1}{}^a \ \hat{\chi}_a(j(\mu)) \,, \qquad \mu \in \mathrm{Exp}^{(h)} \,, \tag{11}$$

where $\hat{\chi}_a(j) = \psi_a^j/\psi_1^j$ are the eigenvalues of $\hat{N}_a$.

In the present context the equations (9, 10) have a natural group theoretic interpretation. This is clear already in the simplest case $N = 3$ [2]. The reality of the exponents (7) implies that they can be identified with an integrable weight $\mu \to j(\mu)$ of $\widehat{sl}(2)$ at a related level. Then depending on the parity of h, the coefficients $n_{\lambda 1}{}^a$ originate from different patterns of decomposition of the representations of $sl(3)$ into those of $sl(2)$. Namely the graphs are determined by the fundamental NIM-rep which is either $n_{\Lambda_1+\rho\, 1}{}^a = 1 + \delta_{a\, 2\omega}$, or $n_{\Lambda_1+\rho\, 1}{}^a = \delta_{a\, 3\omega}$, with ω the $sl(2)$ fundamental weight, thus reflecting the two ways of decomposing the 3-dimensional $sl(3)$ representation. As we shall see, this example is the first in the series for odd N, with C_l and B_l taking over the rôle of $sl(2)$ for h even or odd respectively. The "branching coefficients" interpretation of the NIM-reps and the equations (5,10) has been discussed also in the context of the discrete subgroups of $SU(2)$. See also [5] for a related recent discussion. Given the diagonalisation matrix ψ_a^j one can compute as well the structure constants of the algebra dual to the graph algebra, the Pasquier algebra, which admits important physical interpretations [2].

3. *B* and *C* algebras

We now briefly introduce relevant notations for the Lie algebras B_l and C_l and their affine extensions $B_l^{(1)}$ and $C_l^{(1)}$.

In the B_l algebra, we denote the integrable weights by Latin letters, keeping however the Greek ω_i for the fundamental weights and $\bar{\rho}$ - for the Weyl vector. As the dual Coxeter number is $h^\vee = 2l - 1$, the Weyl alcove at level k is the set

$$\mathcal{P}_{++}^{(B_l,h)} = \{m = \sum_{i=1}^{l} m_i\omega_i \mid m_i \geq 1,\ m_1 + 2\sum_{i=2}^{l-1} m_i + m_l \leq h-1\}\ , \tag{12}$$

where the notation h is again used for the shifted level $h = k + 2l - 1$. The number of integrable weights is $|\mathcal{P}_{++}^{(B_l,h)}| = \binom{\lfloor\frac{h+2}{2}\rfloor}{l} + 2\binom{\lfloor\frac{h+1}{2}\rfloor}{l} + \binom{\lfloor\frac{h}{2}\rfloor}{l}$. These weights are graded according to a $\mathbb{Z}_2$ grading $\tau(m) := m_l - 1 \bmod 2$ and the $\tau = 0$ weights label a subalgebra of the Verlinde fusion algebra. The $\mathbb{Z}_2$ automorphism of the affine B_l Dynkin diagram acts on the weights in the alcove as $\sigma(m) = (h - m_1 - 2\sum_{i=2}^{l-1} m_i - m_l\, , m_2, \ldots, m_l)$.

For the C_l algebra, we use parallel notations: fundamental weights are again denoted ω_i, $i = 1, \cdots, l$; the dual Coxeter number is $h^\vee = l+1$ whence the shifted level $h = k + l + 1$; the Weyl alcove reads

$$\mathcal{P}_{++}^{(C_l,h)} = \{m = \sum_{i=1}^{l} m_i\omega_i \mid m_i \geq 1,\ \sum_{i=1}^{l} m_i \leq h-1\}\ ; \tag{13}$$

the number of weights in the alcove is $|\mathcal{P}_{++}^{(C_l,h)}| = \binom{h-1}{l}$; the $\mathbb{Z}_2$ grading reads $\tau(m) := \sum_{i=1}^{l} i(m_i - 1) \bmod 2$. The $\mathbb{Z}_2$ automorphism of the affine C_l Dynkin diagram acts on the weights in the alcove as $\sigma(m) = (m_{l-1}, \ldots, m_1\, , h - \sum_{i=1}^{l} m_i)$.

The S matrices of B and C type are real and satisfy a $\mathbb{Z}_2$ analog of the σ symmetry property (4).

4. Results

We may summarise our results as follows. In general the eigenvalues in the r.h.s. of (11) are expressed by the modular matrices S of B_l or C_l

$$\hat{\chi}_a(j) = \frac{S_{aj}}{S_{1j}}\ , \tag{14}$$

in which the weights of B or C algebras label both the graph vertices $a \in \mathcal{V}$ and the basis $j = j(\mu)$ into (6), related to a projection of the set of exponents (7) to the B or C alcoves; the vertex $a = 1$ in (14) is identified with the B or C Weyl vector $\bar{\rho}$, i.e. the shifted weight of the identity representation.

The situation depends on the parities of N and of the shifted level h.

1) For $N=2l+1$, h even:

The set of exponents $\mathrm{Exp}^{(h)}$ (7) is identified with the C_l integrable alcove $\mathcal{P}_{+,+}^{(C_l,\frac{h}{2})}$

$$\begin{aligned} \mathcal{P}_{+,+}^{(C_l,\frac{h}{2})} \ni j(\mu) &= (m_1, m_2, \ldots, m_l) \\ \Leftrightarrow \mu &= (m_1, \ldots, m_l, m_l, \ldots, m_1) \in \mathrm{Exp}^{(h)} \end{aligned} \tag{15}$$

and the same alcove parametrises as well the set of graph vertices $\mathcal{V} \equiv \mathcal{P}_{+,+}^{(C_l,\frac{h}{2})}$. The Pasquier and graph algebras are identical and coincide with the C_l Verlinde fusion algebra, $\hat{N}_a = N_a$. Accordingly ψ_a^j in (6) is provided by the C_l modular matrix S,

$$\psi_a^j = S_{aj}\,, \qquad a, j \in \mathcal{P}_{+,+}^{(C_l,\frac{h}{2})}\,. \tag{16}$$

The decomposition formula (10) for the fundamental NIM-reps reads

$$n_{\Lambda_i+\rho} = \sum_{k=0}^{i} \hat{N}_{\omega_{i-k}+\bar{\rho}}\,, \qquad i = 1, 2, \ldots, l\,, \tag{17}$$

reproducing the sl(3) result $n_{(2,1)} = I + \hat{N}_{2\omega}$. For $h = 2l+2$ the alcove $\mathcal{P}_{+,+}^{(C_l,\frac{h}{2})}$ consists of one point, the identity, and (17) degenerates to $n_{\Lambda_i+\rho} = n_\rho = 1$ for all i.

2) For $N = 2l+1$, h odd:

The set of exponents (7) is identified with a subset of the alcove $\mathcal{P}_{+,+}^{(B_l,h)}$

$$\begin{aligned} &\mathrm{Exp}^{(B)} = \\ &\{\mathcal{P}_{+,+}^{(B_l,h)} \ni j = (m_1, m_2, \ldots, m_l) \,|\, \tau(j) = 1\,,\ m_1 < \frac{h-m_l}{2} - \sum_{i=2}^{l-1} m_i\,\} \end{aligned} \tag{18}$$

where (note $\tau(j)+1 = m_l = 0 \bmod 2$)

$$\mathrm{Exp}^{(B)} \ni j \Leftrightarrow \mu = (m_1, \ldots, \frac{m_l}{2}, \frac{m_l}{2}, \ldots, m_1) \in \mathrm{Exp}^{(h)}\,.$$

Another subset of $\mathcal{P}_{+,+}^{(B_l,h)}$ parametrises (for $l \geq 2$) the vertices

$$\mathcal{V} = \{\mathcal{P}_{+,+}^{(B_l,h)} \ni a = (m_1, m_2, \ldots, m_l) \,|\, \tau(a) = 0\,,\ m_1 < \frac{h-m_l}{2} - \sum_{i=2}^{l-1} m_i\,\} \tag{19}$$

For $l=1$ the set of exponents and the set of vertices $\mathcal{V}$ are parametrised by the subsets of $\mathcal{P}_{+,+}^{(A_1,h)}$, $1\le m\le \frac{h-1}{2}$, or $1\le m\le h-2$, m odd, respectively [2]. The eigenvector matrix in (6) is given for $l=1$ by $\psi_a^j=\sqrt{2}\,S_{aj}$, and for any $l\ge 2$ by

$$\psi_a^j = 2\,S_{aj}\,, \quad a\in\mathcal{V}\,, j\in \mathrm{Exp}^{(B)}\,. \tag{20}$$

As empirically observed, there exists a basis (i.e., a preferred correspondence of the two sets of indices $\mathcal{V}$, $\mathrm{Exp}^{(B)}$), in which the matrix ψ_a^j is symmetric and hence the Pasquier algebra is identical to the graph algebra. The matrices $\hat N_a$ are expressed (for $l\ge 2$) via the B_l Verlinde fusion matrices,

$$\hat N_{ab}{}^c = N_{ab}{}^c - N_{ab}{}^{\sigma(c)}\,, \qquad a,b,c\in\mathcal{V} \tag{21}$$

and are checked to be nonnegative. The fundamental NIM-reps are

$$n_{\Lambda_i+\rho} = \hat N_{\omega_i+\bar\rho}, \quad i=1,2,...,l-1\,, \qquad n_{\Lambda_l+\rho} = \hat N_{2\omega_l+\bar\rho}\,, \tag{22}$$

reproducing for $l=1$ the sl(3) decomposition $n_{(2,1)}=\hat N_{3\omega}$.

3) For $N=2l$, arbitrary h:

The set of exponents (7) is identified with a subset of the alcove $\mathcal{P}_{+,+}^{(C_l,h)}$

$$\mathrm{Exp}^{(C)} = \{\mathcal{P}_{+,+}^{(C_l,h)} \ni j = (m_1,m_2,\dots,m_l)|\ m_1,\dots,m_{l-1} - \text{even}\} \tag{23}$$

$$\mathrm{Exp}^{(C)} \ni j \Leftrightarrow \mu = (\frac{m_1}{2},\dots,\frac{m_{l-1}}{2}\,, m_l\,, \frac{m_{l-1}}{2},\dots,\frac{m_1}{2}) \in \mathrm{Exp}^{(h)}\,.$$

A subset of $\mathcal{P}_{+,+}^{(C_l,h)}$ parametrises the vertices

$$\mathcal{V} = \mathcal{P}_{+,+}^{(C_l,\lfloor\frac{h}{2}\rfloor+1)} \cup \sigma_1(\mathcal{P}_{+,+}^{(C_l,\lfloor\frac{h}{2}\rfloor+1)}) \subset\ \mathcal{P}_{+,+}^{(C_l,h)} \tag{24}$$

where

$$\sigma_1(m_1,\dots,m_l) := (h-m_1-2\sum_{i=2}^{l} m_i, m_2,m_3,\dots,m_l)\,. \tag{25}$$

For h odd (24) is a disjoint union of two subsets of $\mathcal{P}_{+,+}^{(C_l,h)}$ of the same cardinality.

The eigenvector matrix ψ_a^j is expressed by the C_l modular matrix S

$$\psi_a^j = (\sqrt{2})^{l-1}\,S_{aj}\,, \qquad a\in\mathcal{V}\,,\ j\in\mathrm{Exp}^{(C)}\,. \tag{26}$$

Empirical data suggest that in general ($l > 2$) ψ in (26) is not symmetrisable for h even. For h odd $\hat{N}_a$ are nonnegative, while for h even they may have signs. The same applies to the matrices of the Pasquier algebra, in which the role of the identity is played by $j = j(\rho) = (2, \ldots, 2, 1)$, with $\psi_a^{j(\rho)} > 0$. The fundamental NIM-reps are

$$n_{\Lambda_i+\rho} = \sum_{m=0}^{\lfloor i/2 \rfloor} \hat{N}_{\omega_{i-2m}+\bar{\rho}}\,, \qquad i = 1, 2, \ldots, l\,. \tag{27}$$

For $h = 2l+1$ (27) degenerates to $n_{\Lambda_i+\rho} = \hat{N}_{\bar{\rho}}$, for i even, $n_{\Lambda_i+\rho} = \hat{N}_{\omega_1}$, for i odd, and the graph algebra is isomorphic to $\mathbb{Z}_2$. In general the graph algebra matrices are expressed by the C_l Verlinde matrices

$$\hat{N}_{ab}{}^c = \sum_{p=0}^{l-1} \sum_{l \geq i_1 > i_2 > \ldots > i_p \geq 2} (-1)^{\lfloor \frac{i_1}{2} \rfloor + \ldots + \lfloor \frac{i_p}{2} \rfloor} N_{ab}{}^{\gamma_{i_1,\ldots,i_p}(c)}\,, \tag{28}$$

for $a, b, c \in \mathcal{V}$, where $\gamma_{i_1,\ldots,i_p} = \sigma_{i_1} \ldots \sigma_{i_p}\, \sigma_1^{i_1+\ldots+i_p}$. Here $\sigma_l = \sigma$, σ_1 appears in (24), and in general the maps σ_s for $s = 1, \ldots, l$ are defined recursively, along with a sequence of subsets $\mathcal{A}_s$ of the C_l alcove, s.t. $\mathcal{A}_{l+1} = \mathcal{P}_{+,+}^{(C_l,h)}$, and $\mathcal{A}_2 = \mathcal{V}$,

$$\begin{aligned} &\sigma_s(m) = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (29)\\ &(m_{s-1}, \ldots, m_1\,,\, h - \sum_{k=1}^{s} m_k - 2 \sum_{k=s+1}^{l} m_k\,,\, m_{s+1}, \ldots m_l)\,,\ m \in \mathcal{A}_{s+1}\\ &\mathcal{A}_s = \{ m \in \mathcal{A}_{s+1} |\ m_s = \langle m, \alpha_s^\vee \rangle\ <\ \langle \sigma_s(m), \alpha_s^\vee \rangle\ \}\,, \quad s = l, \ldots, 1\,. \end{aligned}$$

In the simplest example in this series, the case $\widehat{sl}(4)$, the formula (28) reduces to two terms as in (21), and (27) reproduces the graphs displayed in [3], see the Figure. In this particular case $\mathcal{V}$ is represented by the lower "half-alcove", i.e., the points $m = (m_1, m_2) \in \mathcal{P}_{+,+}^{(C_2,h)}$, $2m_2 < h - m_1$.

The formulae above ensure that the solutions of (5) we find are integer valued. Furthermore they are non-negative, however we lack a general proof of this.

Note that the relations (17),(22),(27) for $h - N > 1$ reflect three different decompositions of the A_{N-1} fundamental representations, in particular the same equalities hold for the corresponding classical dimensions. These decompositions are derived using the projections of A_{N-1} weights $P_i(\mu) = \sum_{j=1}^{N-1} \mu_j\, P_i(\Lambda_j)$, where $P_i(\Lambda_j) = P_i(\Lambda_{N-j})$, and $i = 1, 2, 3$, corresponding to the three cases above. More explicitly for

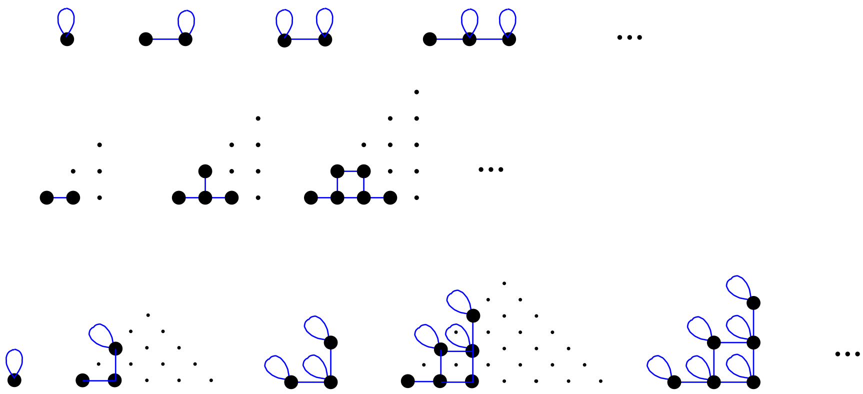

Figure 1. The graphs associated with the NIM-rep matrices $n_{\Lambda_1+\rho}$ for $sl(N)$, $N=3,4,5$ and $h=N+1,N+2,\cdots$, drawn on the alcoves of B or C type.

$i=1$, corresponding to $N=2l+1$ and $i=3$, for $N=2l$, we have $P_i(\Lambda_j)=\omega_j$, where ω_j are the fundamental weights of C_l, while for $i=2$ and $N=2l+1$, $P_2(\Lambda_j)=\omega_j^\vee$, where $\omega_j^\vee$ are the coweights of B_l, i.e., $\omega_j^\vee=\omega_j\,, i=1,\dots l-1\,,\ \omega_l^\vee=2\omega_l$.

Let us introduce the notation $W_{(i,h)}$ to apply to the three cases $i=1,2,3$, determined by the three projections P_i: $W_{(1,h)}$ is the C_l affine Weyl group W of the case 1), $W_{(2,h)}$ is the B_l extended affine Weyl group $\tilde W=\{1,\sigma\}\ltimes W$ of the case 2) and $W_{(3,h)}=\{W\,\gamma\,,\ \gamma\in\Gamma\}$, where Γ is the set of maps $\gamma_{i_1,\dots,i_p}$ in (28), with $\det(\gamma_{i_1,\dots,i_p}):=(-1)^{\lfloor\frac{i_1}{2}\rfloor+\dots+\lfloor\frac{i_p}{2}\rfloor}$. With these data at hand one derives a general representation of the NIM-reps in terms of their classical counterpart ${\bar n_{\lambda+\rho\,a}}^b$, i.e., the multiplicity of the finite dimensional representation of highest weight b in the decomposition of the product of the A_{N-1} representation λ times the representation a of C_l (or B_l). It reads

$$\begin{aligned} {n_{\lambda+\rho\,a}}^b &= \sum_{w\in W_{(i,h)}} \det(w)\,{\bar n_{\lambda+\rho\,a}}^{w(b)} \\ &= \sum_{\gamma\in\mathcal{G}_\lambda} m_\gamma^{(\lambda)} \sum_{w\in W_{(i,h)}} \det(w)\,\delta_{w(b)-a\,,\,P_i(\gamma)}\,. \end{aligned} \tag{30}$$

Here $\mathcal{G}_\lambda$ is the weight diagram of the A_{N-1} representation of highest weight λ and $m_\gamma^{(\lambda)}$ is the multiplicity of the weight μ. A formula similar to the second equality in (30) but with $W_{(i,h)}$ replaced by the horizontal Weyl group $\overline{W}$ gives ${\bar n_{\lambda+\rho\,a}}^b$, for a,b in the dominant Weyl chamber $\mathcal{P}_{+,+}$. Because of the classical nature of the multiplicities in (30), the sums in that formula are finite. E.g., for the fundamental weights the first equality for $n_{\Lambda_i+\rho\,1}{}^b$ reduces to the first term, i.e., to the classical branching coefficient $\bar{n}_{\Lambda_i+\rho\,1}{}^b$ as in (17),(22),(27), for all levels but the trivial value $h = N+1$. The formula (30) is analogous to the formula [6] for the fusion multiplicities, recovered formally by identifying the two algebras and setting $P =$Id, see also [5] for a related recent discussion.

The interested reader is invited to consult [7] for further details and a discussion of the two alternative routes which led us to these results.

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Notes

1. A different and more general approach to the problem of describing the boundary conditions associated with (1) has been developed earlier in [4], we thank J. Fuchs and C. Schweigert for informing us about this. The results presented in [4] do not look to us explicit enough to allow a direct comparison with the formulae here.

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