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On structure constants of sl(2) theories

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Abstract

Structure constants of minimal conformal theories are reconsidered. It is shown that *ratios* of structure constants of spin zero fields of a non-diagonal theory over the same evaluated in the diagonal theory are given by a simple expression in terms of the components of the eigenvectors of the adjacency matrix of the corresponding Dynkin diagram. This is proved by inspection, which leads us to carefully determine the *signs* of the structure constants that had not all appeared in the former works on the subject. We also present a proof relying on the consideration of lattice correlation functions and speculate on the extension of these identities to more complicated theories.

1. Introduction

The computation of the structure constants of the operator product algebra is the most delicate and tedious step in the determination of all the parameters of a conformal field theory. In fact, this determination has been completed only for relatively few theories, mainly minimal $c < 1$ theories and sl(2) WZW theories. In their pioneering work, Dotsenko and Fateev [1] for minimal theories and Zamolodchikov and Fateev [2] for WZW theories computed the structure constants of what are now recognized as the diagonal or "A" theories. A few years later, starting with some work by Christe and Flume [3] on the determination of OP subalgebras, much work was accomplished to extend these calculations to the non-diagonal ("D" or "E") theories [4–7]. The analysis was done case by case, and

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even though some general rules and symmetries of the structure constants were found, no universal formula was available.

In parallel, in his ADE lattice models, Pasquier [8] studied the algebra of spinless order parameters and showed that their product was proportional to the following numbers:

$$M_{ab}{}^{c} = \sum_{\alpha} \frac{\psi_\alpha^{(a)}\psi_\alpha^{(b)}\psi_\alpha^{(c)*}}{\psi_\alpha^{(1)}}. \tag{1.1}$$

Here and in the following, $\psi_\alpha^{(a)}$ refers to the αth component of the ath orthonormalized eigenvector of the Cartan matrix C (or of the adjacency matrix $G = 2\mathbb{1} - C$) of the A, D or E Dynkin diagram under consideration:

$$\begin{aligned} &G_{\alpha\beta}\psi_\beta^{(a)} = \gamma_a \psi_\alpha^{(a)}, \\ &\sum_{\alpha} \psi_\alpha^{(a)}\psi_\alpha^{(b)*} = \delta_{ab}, \\ &\sum_{a} \psi_\alpha^{(a)}\psi_\beta^{(a)*} = \delta_{\alpha\beta}; \end{aligned} \tag{1.2}$$

a runs over the exponents, $\gamma_a = 2\cos(\pi a/h)$, h is the Coxeter number and α is some labelling of the vertices of the diagram. (In the case $D_{h/2+1}$, $h = 2 \bmod 4$, the label a should be replaced by $(a; \varepsilon_a)$, where $\varepsilon_{b/2} = \pm 1$, and $\varepsilon_a = 1$ otherwise, to account for the double degeneracy of the exponent $a = h/2$.) For the Dynkin diagrams, the ψ's may be taken real (see, however, Appendix A) and the resulting M's are fully symmetric in a, b, c: we shall then write them as M_{abc}. In the particular case of the A Dynkin diagram, $\psi_\alpha^{(a)}$ turns out to be a symmetric matrix, equal to the modular S matrix of sl(2) characters, and Eq. (1.1) was then recognized as yielding the integer fusion coefficients N_{abc} (= 0 or 1) [9,10] (in this case $G_{ab} \equiv N_{2ab}, a, b = 1,2,\ldots,h-1$). The role of this matrix M_{abc} in the operator product algebra of lattice theories was reemphasized again in [11,12]. Also, together with its "dual algebra", it was utilized later in the identification of the continuous, conformal limit of a larger class of lattice integrable models attached to graphs [13], and more recently in connection with the integrability of perturbed $N = 2$ superconformal field theories [14]. Strangely enough, its *quantitative* role in the OPE was never ascertained.

In this paper we want to point out a curious fact. The numbers M_{abc} (= $N_{abc}M_{abc}$) yield the ratios of the structure constants of the *spinless* (or "scalar") fields of the D or E theories over the corresponding structure constants of the A theory with the same Coxeter number. Loosely stated (we shall be more precise below)

$$M_{abc} = \frac{D_{(a,a)(b,b)(c,c)}}{D^{(A)}_{(a,a)(b,b)(c,c)}}. \tag{1.3}$$

That these ratios should be simpler than the individual structure constants had been recognized since long [4–7]. Recall that the structure constants are typically

ratios of products of Euler Γ functions of rational arguments, hence generically transcendental numbers. In contrast the ratios (1.3) are square roots of rationals!

Although simple to express, this relation does not seem easy to derive directly from the crossing (or locality) equations, and our observation remains at this stage somehow phenomenological.... On the other hand, from the lattice point of view, a simple extension of Pasquier's discussion yields the desired result.

Structure constants involving fields with a non-zero spin (or "spin fields" in short) turn out to satisfy in many cases factorization properties that enable one to express them in terms of the M's: see Eq. (2.10) below.

In the next section, we define more carefully our notations and conventions, and present the evidence that we have. Section 3 is devoted to a derivation of this relation starting from the lattice formulation:it may be read (or skipped) independently of the former section. Our observation leaves some unanswered questions that we shall list at the end of this paper, whereas a certain number of tables and additional data are gathered in three appendices.

2. The conformal field theory approach

2.1. Conventions and normalizations

The minimal unitary representations of the Virasoro algebra are labelled by a value of the central charge $c_h = 1 - 6/h(h-1)$ and a scaling dimension $\Delta_{s,s'} = [1/4h(h-1)][(s(h-1) - s'h)^2 - 1]$, where $s = 2j+1$, $s' = 2j'+1$ and $h-1$ are positive integers, $1 \leqslant s < h$, $1 \leqslant s' < h-1$. To describe the (A, D), (A, E) non-diagonal theories we will assume that h is even (for h odd the cases (D, A), (E, A) appear instead). Furthermore for the purposes of this paper it will be enough to consider the subalgebra of the OPA for which all $s' = 1$, and accordingly, we denote $\Delta_s = \Delta_{s,1}$.

The primary fields in the subalgebra with $s' = 1$, $\Phi_A(z, \bar{z})$, are labelled by a pair of values $(a, \bar{a})$ of the s index, possibly supplemented by an index $\varepsilon = \pm$ whenever two different fields have the same scaling dimensions Δ_a and $\Delta_{\bar{a}}$. This happens only in the D_{even} case (i.e., $h = 2 \bmod 4$) for $a = h/2 = \bar{a}$. Thus the label A stands for $(a, \bar{a})$ or, if need require, for $(a, \bar{a}; \varepsilon)$.

We will consider fields with integer spin $s(A) := \Delta_a - \Delta_{\bar{a}}$ (in general $\Delta_{a,a'} - \Delta_{\bar{a},a'}$). The normalization of the 2-point (euclidean) functions will be chosen to be

$$\langle \Phi_A(1)\Phi_A(0)\rangle = g_{AA} = (-1)^{s(A)}. \tag{2.1}$$

With this choice the corresponding 2-point Wightman function is positive definite [4] and all the structure constants of the primary fields OPE expansions are real in a proper basis.

We denote these structure constants by the letter D:

$$\Phi_A(x_1)\Phi_C(x_2)|0\rangle = D^F_{AC}(z_1 - z_2)^{\Delta_f - \Delta_a - \Delta_c}(\bar{z}_1 - \bar{z}_2)^{\Delta_f - \Delta_{\bar{a}} - \Delta_c}\Phi_F(x_2)|0\rangle + \ldots, \tag{2.2}$$

reserving the notation C to those of the diagonal case $C^f_{ac} = D^{(A)(f,f)}_{(a,a)(c,c)}$. These constants are determined from the leading singularities at coinciding arguments of the 4-point functions

$$\langle \Phi_A(x_1)\Phi_C(x_2)\Phi_B(x_3)\Phi_D(x_4)\rangle = \sum_F (-1)^{s(F)} d^F_{AC} d^F_{BD} \mathscr{B}_f(z_1, a; z_2, c; z_3, b; z_4, d) \times \mathscr{B}_{\bar f}\left(\bar z_1, \bar a; \bar z_2, \bar c; \bar z_3, \bar b; \bar z_4, \bar d\right), \tag{2.3}$$

Here $\mathscr{B}_f$ are the chiral conformal blocks (in the s-channel), normalized in such a way that at coinciding arguments they reproduce the products of the Dotsenko–Fateev (DF) diagonal OPE coefficients, i.e.,

$$\lim_{\substack{z_1\to 1\\ z_3\to 0}} (z_1-1)^{\Delta_a+\Delta_c-\Delta_f} z_3^{\Delta_b+\Delta_d-\Delta_f} \mathscr{B}_{\bar f}(z_1, a; 1, c; z_3, b; 0, d) = \sqrt{C^f_{ac} C^f_{bd}}\,.$$

Taking into account the 2-point function normalization the general OPE coefficients D^F_{AC} are expressed as

$$D^F_{AC} = d^F_{AC}\sqrt{C^f_{ac} C^{\bar f}_{\bar a \bar c}}\,. \tag{2.4}$$

Thus to determine the OPE coefficients one has to find the relative structure constants d^F_{AC} entering the non-diagonal kernel in (2.3). In the diagonal A-type theory the summation in (2.3) runs over $f = \bar f$ and the constants d^F_{AC} coincide with the fusion rule coefficients N^f_{ac}, i.e., for the minimal sl(2) case under consideration, they can take the values 0, 1. The DF diagonal constants C^f_{ac} can be chosen positive, fully symmetric with respect to all indices, and normalized according to $C^1_{aa} = 1$.

2.2. *The locality requirement and the associativity equations*

The relative structure constants d^F_{AC} (to which we will often refer in what follows as to the structure constants) are determined imposing the requirement of locality, i.e. the symmetry of the euclidean correlator (2.3) under exchange of any pair of fields. The locality applied to the 3-point functions leads to relations for the 3-point normalization coefficients $D_{ACF} = D^F_{AC} g_{FF}$, implying that $D_{ACF} = (-1)^{s(A)+s(C)+s(F)} D_{CAF}$ is cyclically symmetric in A,C,F. Written in terms of the relative structure constants d^F_{AC} they read

$$d^F_{AC} = (-1)^{s(A)+s(C)+s(F)} d^F_{CA} = (-1)^{s(A)} d^C_{AF} = (-1)^{s(C)} d^A_{FC}. \tag{2.5}$$

The relations (2.5) imply in particular that all constants of type d^F_{AA} are identically zero if $s(F) = 1 \bmod 2$. Note also $d^{(1,1)}_{AB} = \delta_{AB}(-1)^{s(A)} = (-1)^{s(A)} d^A_{(1,1)B}$. Furthermore the locality condition which arises exchanging the two middle fields in the

4-point function implies taking into account the braiding properties of the chiral conformal blocks,

$$\sum_F d^F_{AC} d^F_{BD} \begin{Bmatrix} c & a \\ b & d \end{Bmatrix}_{ft} \begin{Bmatrix} \bar{c} & \bar{a} \\ \bar{b} & \bar{d} \end{Bmatrix}_{\bar{f}\bar{t}} = (-1)^{s(A)+s(D)} d^T_{AB} d^T_{CD}, \tag{2.6}$$

where $\{\begin{smallmatrix} \bullet & \bullet \\ \bullet & \bullet \end{smallmatrix}\}$ are the fusion matrices first introduced in [1] (see Appendix B for more explicit formulae). Similarly, exchanging the first pair of fields, we recover the relations in (2.5). Combining the two moves, i.e., exchanging the first and the third fields, reproduces the crossing relation of [1]

$$(-1)^{s(A)+s(B)+s(C)-s(D)} \sum_F (-1)^{s(F)} d^F_{AC} d^F_{BD} \begin{Bmatrix} a & c \\ b & d \end{Bmatrix}_{ft} \begin{Bmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{Bmatrix}_{\bar{f}\bar{t}}$$

$$= (-1)^{s(T)} d^T_{BC} d^T_{AD}. \tag{2.7}$$

With the normalization conventions adopted in this paper the fusion matrix satisfies the orthogonality relation

$$\sum_t \begin{Bmatrix} c & a \\ b & d \end{Bmatrix}_{ft} \begin{Bmatrix} c & a \\ b & d \end{Bmatrix}_{f't} = \delta_{ff'}, \tag{2.8}$$

which implies in particular the validity of Eq. (2.6) in the diagonal case.

Now consider scalar correlation functions (i.e., $\bar{a} = a$, $\bar{c} = c$, $\bar{b} = b$, $\bar{d} = d$ and hence $s(A) = 0$, etc.). Take $t = \bar{t}$ in (2.6) and sum over t. Since (2.8) enforces $f = \bar{f}$ we obtain in both sides a summation over scalars $F = (f, f; \varepsilon_f)$, $T = (t, t; \varepsilon_t)$ only, or,

$$\sum_F d^F_{AC} d_{FBD} = \sum_T d^T_{AB} d_{TCD}. \tag{2.9}$$

The summation in (2.9) runs over f (or t) such that the triplets (a, c, f) and (b, d, f) (or (a, b, t), (c, d, t), respectively) are consistent with the fusion rules, i.e., $d^F_{AC} = N^f_{ac} d^F_{AC}$, etc..

The associativity equations (2.9), the symmetry of the scalar structure constants and the normalization $d^{(1,1)}_{AA} = 1$ imply that the scalar structure constants admit a representation of the type satisfied by the M matrices in (1.1) with some variables ψ subject to the last two conditions in (1.2). Further restrictions on these unknown variables arise from the symmetry $d^{(h-c,h-c)}_{(a,a)(h-b,h-b)} = \pm d^{(c,c)}_{(a,a)(b,b)}$ implied by a corresponding symmetry of the fusion matrices (see Appendix B).

However, these data alone are not sufficient to identify these ψ's with the eigenvectors of the Cartan matrices and thus to determine the scalar structure constants, and one has to solve the full set of Eqs. (2.6).

On the other hand analyzing the explicit solutions of (2.6) found in [3–7] one observes that in all non-diagonal cases the squares of the structure constants involving only scalar fields coincide with the squares of the corresponding M matrix elements. The determination of the signs of all these constants (previously known in the D and partially in the E_6 cases [4]), shows that not only the squares

but the scalar constants themselves coincide with the M matrix elements, i.e., with notations now settled, we can rephrase our main result (1.3) in the form

$$d_{(aa)(bb)}{}^{(cc)} = M_{ab}{}^{c}. \tag{1.3'}$$

In fact, as we shall see, there is a certain freedom in the choice of signs of both the d's and the M's. The precise statement is thus that one can find a determination of these two sets of numbers satisfying (1.3').

Note that while in the E_6, E_8 cases and (in a particular basis) in the D_{even} case all M matrix elements can be chosen non-negative, in the remaining D_{odd} (i.e., $h = 0 \mod 4$) and E_7 cases some of these matrix elements are negative (see Appendix A for explicit formulae).

The former three cases are also selected by the property of factorization of their structure constants involving also spin fields – namely, whenever d_{AB}^{C} is non-zero,

$$|d_{AB}^{C}|^2 = M_{ab}^{c} M_{\bar{a}\bar{b}}^{\bar{c}}, \tag{2.10}$$

and furthermore in the D_4, E_6, E_8 cases, d_{AB}^{C} vanishes iff the product $M_{ab}^{c} M_{\bar{a}\bar{b}}^{\bar{c}}$ (for $(a, \bar{a})$, etc., in the OPA) is zero. The property (2.10) holds in the D_{even} series in the bases in which all scalar constants are non-negative – at the price of complex spin field constants appearing for some h; as in (1.3) any $a = h/2$ has to be replaced by a double index (see below for more details).

Thus up to signs all the relative structure constants in the cases E_6, E_8 and D_{even} are completely described by the corresponding M matrices.

These positivity and factorization properties are most likely a consequence of the fact that these theories may be interpreted as the "diagonal" theories for some extended chiral algebra [15]. In Appendix C we present some evidence in support (see also the second reference of [3], and [5]).

Unlike (1.3) the formula (2.10) is not universal. It fails in the D_{odd} and the E_7 models, although partial factorizations still take place.

We recall that apart from some trivial subalgebras of the diagonal OP algebra in the D cases (and the subalgebras $\{(1, 1), (h-1, h-1)\}$, present in all series) there are no closed OP subalgebras involving only scalar fields in the non-diagonal minimal theories. On the contrary the M matrices in any of the ADE cases can be interpreted as the structure constants of a closed associative algebra $x_a * x_b = M_{ab}{}^{c} x_c$.

In [16], it was noticed (in connection with some work of Dubrovin on topological field theories [17]) that the M algebras of the ADE cases admit subalgebras containing the generators x_a of smallest and largest labels ($a = 1$ and $h - 1$ in our present notations), and that the labels of these subalgebras are the exponents of finite Coxeter groups. Accordingly, we shall show below that some of the OPAs of the ADE models admit subalgebras whose spin zero fields are labelled by the exponents of the finite Coxeter groups.

In what follows we shall summarize the existing data on the general structure constants, providing in addition also the full information about their signs. Apart from some partial results this information was not present in the literature so we

have rechecked numerically the exceptional cases. The results for E_6 and E_7 are presented in detail below, while the signs of the constants involving spin fields in the rather lengthy case E_8 are not included.

2.3. ADE relative structure constants – explicit formulae

The set of fields that concern us in any ADE theory is described by the subset of fields in the corresponding modular invariant [18] for which all $s' = 2j' + 1 \equiv 1$.

The derivation of the solutions of Eqs. (2.6) is simplified by taking into account the symmetries of the structure constants [3–7],

$$\left(d^{C}_{AB}\right)^2 = \left(d^{\sigma(C)}_{A\sigma(B)}\right)^2 = \left(d^{\sigma_l(C)}_{A\sigma_l(B)}\right)^2 = \left(d^{\sigma_r(C)}_{A\sigma_r(B)}\right)^2, \tag{2.11}$$

where $\sigma = \sigma_l\sigma_r$ and

$$\sigma_r((a,\bar{a})) = (a, h-\bar{a}), \quad \sigma_l((a,\bar{a})) = (h-a,\bar{a}), \quad \text{for } a, \bar{a} \neq \tfrac{1}{2}h,$$
$$\sigma_r\left(\left(\tfrac{1}{2}h, \tfrac{1}{2}h; \varepsilon\right)\right) = \left(\tfrac{1}{2}h, \tfrac{1}{2}h; -\varepsilon\right) = \sigma_l\left(\left(\tfrac{1}{2}h, \tfrac{1}{2}h; \varepsilon\right)\right). \tag{2.12}$$

In (2.11) it is assumed that the transformations (2.12) are consistent with the content of the given non-diagonal series. Thus the first equality (2.11) with the transformation σ holds in all cases, while the rest make sense only in the cases when the transformations σ_r and σ_l keep invariant the specific set of indices. (Alternatively these transformations can be used to relate the constants in different types of theories, say A_{h-1} and $D_{h/2+1}$, etc., see below.)

Actually there are stronger restrictions than (2.11), to be described in detail below, which determine also the relative signs of the constants. They are based on the explicit symmetries [4] of the fusion matrices recalled in Appendix B. Furthermore Eqs. (2.6) are consistent with the choice

$$d^{(c,\bar{c})}_{(a,\bar{a})(b,\bar{b})} = d^{(\bar{c},c)}_{(\bar{a},a)(\bar{b},b)}. \tag{2.13}$$

Note that a change by a sign μ_A, $\mu_A^2 = 1$, $\mu_{(1,1)} = 1$, of all fields is possible, since it preserves the normalization of the 2-point function. Since we fix the signs of the diagonal structure constants C these sign factors affect the relative constants d. The sign renormalization is obviously consistent with the locality Eqs. (2.6) – in what follows we shall fix it imposing various conditions.

2.3.1. Case $D_{h/2+1}$

Let us start with the two infinite series $D_{h/2+1}$, $h = 2 \bmod 4$, or $h = 0 \bmod 4$. Each contains a subalgebra of scalar fields described by (a, a), a odd, which is also a subalgebra of the corresponding diagonal A_{h-1} series. Here the scalar $(h/2, h/2; +)$ in the D_{even} case is simply denoted $(h/2, h/2)$. Furthermore both contain a scalar – to be denoted for convenience in both cases by $(h/2, h/2; -)$, which in the D_{even} case represents the second scalar of scale dimension $\Delta_{(h-2)/4}$. Finally both possess a set of non-zero spin fields labelled by $(c, h-c)$ where c is

odd in the D_{even} case and even for D_{odd}. Now using the notation $\hat{C}$, $\hat{F}$, etc., for the non-zero spin fields as well as for the scalar $(h/2, h/2; -)$, the values of the relative structure constants read

$$\begin{aligned} d^{(t,t)}_{(a,a)(b,b)} &= N^t_{ab}, \\ d^{\hat{F}}_{(a,a)\hat{C}} &= (-1)^{(a-1)/2} N^f_{ac} N^{h-f}_{a\ h-c} = (-1)^{(a-1)/2} N^f_{ac}, \end{aligned} \tag{2.14}$$

all the others being zero. In what follows we shall often omit the fusion rule structure constants N^f_{ac}, assuming that the left and right triplets of indices are consistent with the fusion rules. Note that when $\hat{C}$ and $\hat{F}$ coincide, the sign of $d^{\hat{C}}_{(a,a)\hat{C}}$ as given by (2.14) is uniquely determined from Eqs. (2.6) [4]. (We require that the sign factors $\mu_{(a,a)}$ are trivial for the fields of the diagonal subalgebra of the D series, $\mu_{(a,a)} = 1$.) The general solution for $d^{\hat{F}}_{(a,a)\hat{C}}$ is given by the expression in (2.14) multiplied by the sign factors $\mu_{\hat{C}}\ \mu_{\hat{F}}$, e.g., one can choose $\mu_{\hat{C}} = (-1)^{s(\hat{C})}$. With the choice $\mu_{\hat{C}} = 1$ made in (2.14), these constants can be rewritten in the D_{even} case as

$$d^{\hat{F}}_{(a,a)\hat{C}} = d^{\sigma_r((f,f))}_{(a,a)\sigma_r((c,c))} = (-1)^{(a-1)/2} d^{(f,f)}_{(a,a)(c,c)},$$

where according to (2.12) $\sigma_r((h/2, h/2)) = \sigma_r((h/2, h/2; +)) = (h/2, h/2; -)$. In the D_{odd} case, (2.14) is a manifestation of the automorphism of the diagonal fusion rules used to construct the D_{odd} series [19].

The formula (2.14) describes in an unified way the constants of both D series. It also makes explicit the $\mathbb{Z}_2$ grading of both OPAs that assigns a grade 0 to the subalgebra of fields without hats and 1 to those with hats [4]. On the other hand in the D_{even} case there exists an alternative description, changing the basis of fields – namely replacing the two scalars of identical dimension with two independent linear combinations. Using (2.14) one can rewrite the structure constants for the new basis. We shall illustrate this on the case $h = 2$ mod 8, and in more detail for $h = 18$, i.e. for D_{10} since the explicit formulae will be relevant also for the case E_7 below.

Denote by ϕ and $\hat{\phi}$ the fields labelled by $(h/2, h/2)$, and $(h/2, h/2; -)$, respectively, and consider the linear combinations

$$\Psi^{\pm} = \frac{1}{\sqrt{2}}\left(\phi \pm \hat{\phi}\right). \tag{2.15}$$

(In the other subseries $h = 6$ mod 8 of the D_{even} series, the second field in the RHS of (2.15) appears multiplied with $\sqrt{-1}$, i.e., $\Psi^- = (\Psi^+)^*$.) Restricting to the case $h = 18$, denote furthermore the fields in the LHS of (2.15) by $9^{\pm}$ respectively. Then one obtains from (2.14) the following expressions for the non-zero scalar fields constants:

$$B^{9^{\pm}}_{9^{\pm}9^{\pm}} := d^{9^{\pm}}_{9^{\pm}9^{\pm}} = \sqrt{2} \tag{2.16a}$$

$$B^{a}_{9^{\pm}9^{\pm}} := d^{(a,a)}_{9^{\pm}9^{\pm}} = \tfrac{1}{2}\left[1 + (-1)^{j_a}\right], \quad a \neq 9, \quad j_a = \tfrac{1}{2}(a-1), \tag{2.16b}$$

$$B^{a}_{9^{\pm}9^{\mp}} := d^{(a,a)}_{9^{\pm}9^{\mp}} = \tfrac{1}{2}\left[1-(-1)^{j_a}\right], \quad a \neq 9, \tag{2.16c}$$

$$B^{b}_{9^{\pm}a} := d^{(b,b)}_{9^{\pm}(a,a)} = \sqrt{\tfrac{1}{2}}, \quad a, b \neq 9, \tag{2.16d}$$

$$B^{c}_{ab} := d^{(c,c)}_{(a,a)(b,b)} = 1, \quad a, b, c \neq 9. \tag{2.16e}$$

(In this basis the squares of the above constants appear in [6].) Unlike the solution for the scalar constants in the initial basis (2.14) (i.e., $\hat{F} = \hat{C} = (h/2, h/2; -)$ in the second equality) all constants in (2.16) are non-negative. Furthermore for the remaining constants in the new basis we get

$$d^{\hat{A}}_{9^{\pm}9^{\pm}} = \pm\tfrac{1}{2}\left[1+(-1)^{s(\hat{A})}\right], \quad d^{\hat{A}}_{9^{\pm}9^{\mp}} = \mp\tfrac{1}{2}\left[1-(-1)^{s(\hat{A})}\right], \quad a \neq 9 \tag{2.17a}$$

$$d^{\hat{B}}_{(a,a)9^{\pm}} = \pm\sqrt{\tfrac{1}{2}}(-1)^{j_a}, \qquad d^{\hat{B}}_{9^{\pm}\hat{A}} = \sqrt{\tfrac{1}{2}}, \quad a, b \neq 9, \tag{2.17b}$$

$$d^{\hat{C}}_{(a,a)\hat{B}} = (-1)^{j_a}, \quad a, b, c \neq 9, \tag{2.17c}$$

which in particular implies (since $s(\hat{A}) = j_a \bmod 2$ for $h = 2 \bmod 8$) the factorizability (2.10) of the squares of the constants in (2.17), if in the RHS the M matrices are also converted in the basis corresponding to (2.16).

Formulae similar to (2.16) hold in the case $h = 6 \bmod 8$, where some of the non-zero spin constants become complex in the new basis. Also the symmetry properties of the scalar constants (and the M matrices, see Appendix A) get modified since (2.1) is replaced for $A = (h/2, h/2; \pm)$ with $\langle \Psi(1)\Psi^{*}(0)\rangle = 1$. Note that the factorization property (2.10) holds in that basis.

For $h = 10$ one selects using the basis (2.15) two isomorphic subalgebras of the D_6 series which differ by some of the signs of the structure constants. They consist of the fields $\{\Phi^{+}, (1,1), (9,9), (1,9), (9,1)\}$ and $\{\Phi^{-}, (1,1), (9,9), (1,9), (9,1)\}$, respectively. The scalar fields in any of these subalgebras are labelled by the exponents $\{1, 5, 9\}$ of the Coxeter group H_3. The fields in the grade zero subalgebra of the general series $D_{h/2+1}$ correspond to the exponents of $B_{h/2}$.

2.3.2. Case E_7

The exceptional case E_7 which appears for $h = 18$ contains scalars $A = (a, a)$ labelled by the E_7 exponents $a = 1, 5, 7, 9, 11, 13, 17$ and spin fields of the type $(a, h-a)$, with the same values of a, $a \neq 9$, and the spin fields $(3, 9)$, $(9, 3)$, $(15, 9)$, $(9, 15)$.

Since the spins of the fields labelled by $(7, 11)$ or $(11, 7)$ are odd, all constants of the type $d^{(7,11)}_{AA}$ or $d^{(11,7)}_{AA}$, and those related to them using (2.11) vanish. This in particular implies that the factorization (2.10) cannot take place, e.g., for $A = (7, 7)$ since $M_{777} = 1 = -M_{77\,11}$. Furthermore because of the symmetry (2.11) this leads to the vanishing of the scalar constants $d^{(7,7)}_{(9,9)(9,9)}$ and $d^{(11,11)}_{(9,9)(9,9)}$. Similarly the constant $d^{(15,9)}_{(9,9)(9,9)}$ vanishes since $s((15, 9)) = 1 \bmod 2$, which in turn implies the vanishing of $d^{(3,9)}_{(9,9)(9,9)}$.

The results of [5–7] concerning the squares of the remaining structure constants can be furthermore summarized in the following way.

(i) The squares of all scalar structure constants coincide with the squares of the corresponding M matrices (see Appendix A). According to (2.11) the latter gives

as well all the constants obtained from the scalar ones by the σ-transformations (2.12). Hence because of the symmetry of the M matrices, $M_{h-a\,h-b\,c} = M_{abc}$, the factorization formula (2.10) holds in these particular cases.

(ii) We have

$$\left(d^{(5,13)}_{(7,7)(7,7)}\right)^2 = \tfrac{3}{4}. \tag{2.18}$$

(Compare with $M_{775} = -\frac{1}{2} = -M_{7713}$.) Hence for this as well as for those related by the σ-transformations the factorization (2.10) fails.

(iii) If the triplet $\{A, B, C\}$ consists of scalars and (or) some of the fields (3, 9), (9, 3), the square of the constants factorizes into the corresponding scalar constants (2.16) of the D_{10} case – in the basis, in which these constants are positive, i.e.,

$$\left(d^{C}_{AB}\right)^2 = B^{c}_{ab} B^{\bar{c}}_{\bar{a}\bar{b}}, \tag{2.19}$$

where 9 from (3, 9) goes to 9^-, while 9 from (9, 9) goes to 9^+ in the RHS – e.g., $(d^{(3,9)}_{(3,9)(3,9)})^2 = B^{3}_{33}\, B^{9^-}_{9^-9^-} = \sqrt{2}$, $(d^{(9,9)}_{(3,9)(9,3)})^2 = B^{9^+}_{3\,9^-}\, B^{9^+}_{9^-3} = 1$, etc.. The RHS of (2.19) provides as well the expressions for all σ-related constants.

To describe the signs of the structure constants, first note that the symmetries of the fusion matrices (see Appendix B) can be used together with Eqs. (2.6) to derive restrictions on the relative signs. Namely

$$d^{C}_{AB} = (-1)^{(a-1)/2} \epsilon_{\mathrm{l}}(B)\epsilon_{\mathrm{l}}(C) d^{\sigma_{\mathrm{l}}(C)}_{A\sigma_{\mathrm{l}}(B)} = (-1)^{(a-1)/2} \epsilon_{\mathrm{r}}(B)\epsilon_{\mathrm{r}}(C) d^{\sigma_{\mathrm{r}}(C)}_{A\sigma_{\mathrm{r}}(B)}. \tag{2.20}$$

Here $\epsilon_{\mathrm{l}}((9,3)) = \epsilon_{\mathrm{l}}((9,15)) = -1 = \epsilon_{\mathrm{r}}((3,9)) = \epsilon_{\mathrm{r}}((15,9))$, while for all the other B this factor is one. In particular $\epsilon_{\mathrm{l}}((9,9)) = 1 = \epsilon_{\mathrm{r}}((9,9))$.

Taking also into account (2.13) it remains to choose the signs of a minimal subset of constants. The choice

$$d^{(5,5)}_{(9,9)(9,9)},\quad d^{(9,9)}_{(9,9)(9,9)},\quad d^{(9,9)}_{(5,5)(7,7)},\quad d^{(7,7)}_{(7,7)(7,7)} > 0 \tag{2.21}$$

is consistent with the locality equations (2.6). (Note that (2.6) restricts only the sign of the product of the four constants in (2.21).) With this choice one obtains (1.3), i.e., the signs of the scalar constants coincide with the signs of the corresponding E_7 M matrix elements, and furthermore

$$\begin{aligned} &d^{(5,13)}_{(7,7)(7,7)},\quad d^{(5,5)}_{(3,9)(3,9)} < 0; \qquad d^{(7,7)}_{(9,3)(3,9)},\quad d^{(9,9)}_{(9,3)(3,9)} > 0, \\ &\operatorname{sign}\left(d^{(3,9)}_{(9,9)(7,7)}\right) = \operatorname{sign}\left(d^{(3,9)}_{(7,7)(7,7)}\right) = -\operatorname{sign}\left(d^{(3,9)}_{(5,5)(7,7)}\right) = -\operatorname{sign}\left(d^{(3,9)}_{(5,5)(5,5)}\right) \\ &\qquad = -\operatorname{sign}\left(d^{(3,9)}_{(3,9)(3,9)}\right). \end{aligned} \tag{2.22}$$

Clearly the couplings of the E_7 scalar field (9, 9) and those of the D_{10} field Ψ^+ with the fields belonging to the common subset completely coincide. This follows from the identity of the constants $M_{9ab} = B^{b}_{9^+a}$, checked by comparing (2.16) with the formulae in Appendix A, and furthermore from the comparison of (2.17) with the consequences of (2.20). This fact together with (2.19) can be interpreted as a

manifestation of the automorphism of the fusion rules of the extended D_{10} field algebra which upon twisting takes it into the E_7 theory [19].

2.3.3. Case E_6

This exceptional case appears for $h = 12$ and contains the scalars labelled by the E_6 exponents $a = 1, 4, 5, 7, 8, 11$ and spin fields of the type $(a, h-a)$, $a = 4, 8$, and (1, 7), (7, 1), (5, 11), (11, 5). The odd spin fields are those labelled by (4, 8), (8, 4), (5, 11), (11, 5). The results in [4,5] imply that the squares of the general structure constants are determined by the factorization formula (2.10).

Let us now add the signs of the remaining constants. The symmetry of the fusion matrix together with Eqs. (2.6) gives the following relations for the relative signs.

(i) Let all a, b, c be odd (hence $\bar{a}$, $\bar{b}$, $\bar{c}$ are also odd):

$$d^{\sigma(C)}_{A\sigma(B)} = (-1)^{(a-\bar{a})/2} d^{C}_{AB}. \tag{2.23}$$

(ii) Let a be odd and b, c even:

$$\begin{aligned} &d^{(4,8)}_{(5,5)(4,8)} = -d^{(4,4)}_{(5,5)(4,4)} = -d^{(8,8)}_{(5,5)(8,8)}; \\ &d^{(b,\bar{b})}_{A(b,\bar{b})} = (-1)^{(a-1)/2} d^{\sigma_l((b,\bar{b}))}_{A\sigma_l((b,\bar{b}))} = (-1)^{(\bar{a}-1)/2} d^{\sigma_r((b,\bar{b}))}_{A\sigma_r((b,\bar{b}))} \\ &= (-1)^{(a-\bar{a})/2} d^{\sigma((b,\bar{b}))}_{A\sigma((b,\bar{b}))}, \end{aligned} \tag{2.24a}$$

for $A = (1, 1), (1, 7), (7, 1), (7, 7)$ and $\bar{b} = h - b$;

$$\begin{aligned} &d^{(8,4)}_{(7,7)(4,8)} = d^{(4,4)}_{(7,7)(8,8)}; \\ &d^{(\bar{b},b)}_{A(\bar{b},b)} = (-1)^{(a-1)/2} d^{(b,\bar{b})}_{A(\bar{b},b)} = (-1)^{(a-1)/2} d^{(\bar{b},\bar{b})}_{A(b,\bar{b})} = (-1)^{(a-\bar{a})/2} d^{(b,\bar{b})}_{A(\bar{b},b)} \end{aligned} \tag{2.24b}$$

for $A = (11, 11), (5, 5), (11, 5), (5, 11)$ and $\bar{b} = h - b$.

(iii) Let b be odd and a, c even:

$$d^{\sigma(C)}_{A\sigma(B)} = (-1)^{s(C)+(b-\bar{b})/2} d^{C}_{AB}. \tag{2.25}$$

Note that (2.24b) follows given (2.24a) and (2.25).

Using the above relations as well as (2.13) it is sufficient to give the signs for a minimal set of constants, e.g., $d^{(7,7)}_{(7,7)(7,7)}$, $d^{(7,7)}_{(4,4)(4,4)}$, $d^{(5,5)}_{(4,4)(4,4)}$ and $d^{(1,7)}_{(1,7)(1,7)}$, $d^{(7,7)}_{(7,1)(1,7)}$, $d^{(1,7)}_{(7,7)(7,7)}$, $d^{(1,7)}_{(4,4)(4,4)}$, $d^{(7,7)}_{(4,4)(4,8)}$, $d^{(1,7)}_{(4,4)(4,8)}$. Actually given $d^{F}_{(7,7)(7,7)}$, for $F = (7, 7), (1, 7)$, Eqs. (2.6) restrict the signs of the remaining constants in this subset. Thus the choice of the sign of one of the scalar constants, e.g.,

$$d^{(7,7)}_{(7,7)(7,7)} > 0$$

ensures that all scalar constants are positive and hence they coincide with the E_6 M matrices. Furthermore choosing

$$d^{(1,7)}_{(7,7)(7,7)} > 0, \tag{2.26}$$

it follows that

$$d^{(1,7)}_{(1,7)(1,7)} \quad d^{(7,7)}_{(7,1)(1,7)}, \quad d^{(1,7)}_{(4,4)(4,4)} > 0,$$
$$\operatorname{sign}\left(d^{(7,7)}_{(4,4)(4,8)}\right) = \operatorname{sign}\left(d^{(1,7)}_{(4,4)(4,8)}\right). \tag{2.27}$$

As an example of the application of (2.24) one obtains, e.g.,

$$d^{(8,8)}_{(1,7)(8,8)} < 0,$$

while

$$d^{(8,4)}_{(1,7)(8,4)} = -d^{(1,7)}_{(8,4)(8,4)} > 0,$$
$$d^{(8,4)}_{(7,1)(8,4)} = -d^{(7,1)}_{(8,4)(8,4)} < 0.$$

In the last two equalities we have used also the relations (2.5).

The E_6 operator product algebra has a subalgebra consisting of the fields {(1, 1), (5, 5), (7, 7), (11, 11), (5, 11), (11, 5), (1, 7), (7, 1)}. The labels {1, 5, 7, 11} of the scalars in this set correspond to the exponents of the Coxeter group F_4. Furthermore this subalgebra has the smaller subalgebra {(1, 1), (7, 7), (1, 7), (7, 1)} generated by the purely chiral subalgebras {(1, 1), (1, 7)} and {(1, 1), (7, 1)}.

2.3.4. Case E_8

We shall be very brief on this last case, already studied in part in the last reference of [5], as it is fairly cumbersome, and we shall not display explicitly all the formulae (they may be obtained on request from the authors). Suffice it to say that a determination of signs in the expressions of $d^C_{AB} = \pm\sqrt{M_{abc}M_{\bar{a}\bar{b}\bar{c}}}$ (cf. (2.10)) has been completed. Imposing Eqs. (2.1), (2.5), (2.13) and (2.23) as constraints leaves a set of 423 signs (!) that are determined so as to satisfy (2.6). There are solutions such that the sign for A, B and C scalars is $+$, thus in agreement with (1.3).

Finally one observes that the exponents {1, 11, 19, 29} of H_4 appear as labels of the scalars in a subalgebra in this E_8 case. The latter subalgebra consists of the fields labelled by {$(a, \bar{a})$; $a, \bar{a} = 1, 11, 19, 29$}, and it contains furthermore the chiral subalgebras {$(a, 1)$} and {$(1, a)$}, $a = 1, 11, 19, 29$.

We conclude this section with a remark on the general case $j'_i \neq 0$ of the minimal (unitary) theories and the corresponding non-diagonal solutions. Notice that for $s' = 2j' + 1$ even, in the exceptional cases with $\bar{s} \neq h - s$, s, the spins $\Delta_{s,s'} - \Delta_{\bar{s},s'}$ (mod 1) depend on the second index s' [3]. Furthermore there are additional sign factors in the general fusion matrix, mixing both types of indices, i.e., it factorizes only up to signs. This does not change the solutions for the squares of the relative structure constants d, i.e., they are the same as the ones described above, whenever the triplets of primed indices are consistent with the

[3] Choosing as in [8] the $(h-1)(h-2)/2$ independent left (right) labels to be represented by {(s, s'); $s' = s$ mod 2} avoids this dependance.

fusion rules, but the signs of some of them will depend on the primed indices. However, the signs of the relative scalar constants are not affected, so that the property (1.3) holds true in the general case, with the proper substitution of the labels, i.e., $a \to (a, a')$, etc..

3. The lattice approach

According to Pasquier [20], an integrable SU(2) lattice model may be attached to a graph by constructing a representation of the Temperley–Lieb algebra on the space of paths on the graph. We recall hereafter the basic steps in that construction and then expose some universality properties in the calculation of the matrix elements that enter the expression of the correlation functions.

3.1. The Temperley–Lieb algebra

By definition, the Temperley–Lieb algebra is the associative algebra generated by $U_1, \ldots, U_{L-1}$ subject to the conditions

$$U_i^2 = \beta U_i, \quad \beta = 2\cos(\pi/h), \tag{3.1a}$$

$$U_i U_j = U_j U_i, \quad \text{if } |i-j| \geqslant 2, \tag{3.1b}$$

$$U_i = U_i U_{i\pm 1} U_i. \tag{3.1c}$$

There, h is an integer, to be chosen as the Coxeter number of a Dynkin diagram $\mathscr{G}$ of ADE type.

One then introduces the space $\mathscr{H}$ of paths on the graph $\mathscr{G}$, i.e., the space spanned by the states $\{|\alpha_0, \ldots, \alpha_L\rangle\}$,

$$|\alpha_0 \ldots \alpha_L\rangle = G_{\alpha_0\alpha_1} G_{\alpha_1\alpha_2} \ldots G_{\alpha_{L-1}\alpha_L} |\alpha_0\rangle \otimes |\alpha_1\rangle \otimes \cdots |\alpha_L\rangle, \tag{3.2}$$

where $\langle \alpha | \beta \rangle = \delta_{\alpha\beta}$ and the matrix elements of the adjacency matrix G of the graph $\mathscr{G}$ ensure that consecutive vertices α_i, α_{i+1} along the path are adjacent on the graph. The space $\mathscr{H}$ also supports a representation of the Temperley–Lieb algebra, provided by the formulae

$$U_i |\alpha_0 \alpha_1 \ldots \alpha_L\rangle = \sum_{\alpha_1'} \alpha_i \begin{array}{c} \alpha_{i+1} \\ \diamond \\ \alpha_{i-1} \end{array} \alpha_i' \, |\alpha_0 \alpha_1 \ldots \alpha_i' \ldots \alpha_L\rangle, \tag{3.3}$$

$$\text{with} \quad \alpha_i \begin{array}{c} \alpha_{i+1} \\ \diamond \\ \alpha_{i-1} \end{array} \alpha_i' = \delta_{\alpha_{i-1}\alpha_{i+1}} G_{\alpha_{i-1}\alpha_i} G_{\alpha_{i-1}\alpha_i'} \frac{\left(\psi^{(1)}_{\alpha_i} \psi^{(1)}_{\alpha_i'}\right)^{1/2}}{\psi^{(1)}_{\alpha_{i-1}}}.$$

This is easily seen to verify (3.1) with $\beta = \gamma_1$, the eigenvalue of the Perron–Frobenius eigenvector $\psi^{(1)}$. Note that none of these operators affects the values of α_0 and α_L.

For our purposes, it will be useful to enlarge the algebra by the operators $\phi_i^{(a)}$,

$$\phi_i^{(a)} | \alpha_0 \dots \alpha_L \rangle = \frac{\psi^{(a)}_{\alpha_i}}{\psi^{(1)}_{\alpha_i}} | \alpha_0 \dots \alpha_L \rangle. \tag{3.4}$$

For any operator of this enlarged algebra, define now the (modified) trace

$$\mathrm{Tr}\, X = \sum_{\{\alpha_0, \dots, \alpha_L\}} \psi^{(1)*}_{\alpha_0} \langle \alpha_1 \dots \alpha_L | X | \alpha_0 \dots \alpha_L \rangle \psi^{(1)}_{\alpha_L}, \quad \mathrm{Tr}\, \mathbb{1} = (\gamma_1)^L, \tag{3.5}$$

which has the properties of a Markov trace [21]. The main property that we shall use below is its cyclicity: for any two operators X and Y belonging to the algebra

$$\begin{aligned}
\mathrm{Tr}\, XY &= \sum_{\alpha_0, \alpha_1, \dots, \alpha_{L-1}, \alpha_L} \psi^{(1)*}_{\alpha_0} \psi^{(1)}_{\alpha_L} \langle \alpha_0 \dots \alpha_L | XY | \alpha_0 \dots \alpha_L \rangle \\
&= \sum_{\substack{\alpha_0, \alpha_1, \dots, \alpha_L \\ \beta_1, \dots, \beta_{L-1}}} \psi^{(1)*}_{\alpha_0} \psi^{(1)}_{\alpha_L} \langle \alpha_0 \alpha_1 \dots \alpha_{L-1} \alpha_L | X | \alpha_0 \beta_1 \dots \beta_{L-1} \alpha_L \rangle \\
&\quad \times \langle \alpha_0 \beta_1 \dots \beta_{L-1} \alpha_L | Y | \alpha_0 \alpha_1 \dots \alpha_{L-1} \alpha_L \rangle \\
&= \sum_{\alpha_0, \beta_1, \dots, \beta_{L-1}, \alpha_L} \psi^{(1)*}_{\alpha_0} \psi^{(1)}_{\alpha_L} \langle \alpha_0 \beta_1 \dots \beta_{L-1} \alpha_L | YX | \alpha_0 \beta_1 \dots \beta_{L-1} \alpha_L \rangle \\
&= \mathrm{Tr}\, YX.
\end{aligned} \tag{3.6}$$

3.2. *Height lattice models and their correlation functions*

We now consider a square lattice of finite size. To each lattice site is assigned a "height" that is a vertex of $\mathcal{G}$ with the constraint that neighbouring sites are assigned neighbouring heights on the graph. It is convenient to regard the "equal time" configurations of heights $\alpha_0 \alpha_1 \dots \alpha_L$ attached to a diagonal zigzag line across the lattice and to describe it by a state $| \alpha_0 \alpha_1 \dots \alpha_L \rangle$ in the Hilbert space $\mathscr{H}$ of the theory.

The transfer matrix between these configurations is constructed in terms of the representation U_i of the Temperley–Lieb algebra (3.3):

$$\begin{aligned}
\mathcal{T} &= \prod_{\substack{i=1 \\ i \text{ odd}}}^{L-1} X_i(u) \prod_{\substack{i=2 \\ i \text{ even}}}^{L-2} X_i(u), \\
X_i(u) &= \mathbf{1} + \frac{\sin(\pi u)}{\sin[\pi(1/h - u)]} U_i,
\end{aligned} \tag{3.7}$$

with u a spectral parameter. The commutation of row-to-row transfer matrices for two different values u and v of this spectral parameter follows from the Yang–Baxter relation satisfied by the X's:

$$X_i(u) X_{i+1}(u+v) X_i(v) = X_{i+1}(v) X_i(u+v) X_{i+1}(u), \tag{3.8}$$

which is itself a consequence of (3.1) and of simple trigonometric identities.

If the lattice has a "time" extent of M, it is appropriate to define the partition function as the modified trace of the Mth power of that transfer matrix,

$$Z_{\rm mod} = \sum_{\alpha_0,\alpha_L} \psi^{(1)*}_{\alpha_0}\psi^{(1)}_{\alpha_L} Z_{\alpha_0\alpha_L},$$
$$Z_{\alpha_0\alpha_L} = \sum_{\alpha_1,\dots,\alpha_{L-1}} \langle \alpha_0 \dots \alpha_L | \mathcal{T}^M | \alpha_0 \dots \alpha_L \rangle. \tag{3.9}$$

In such a lattice model, it is natural to consider the operator $P_\alpha(r)$ that projects on the state of height α at a certain site r. Its expectation value is the so-called *local height probability* and is an order parameter of the lattice theory. Pasquier suggested to consider another set of order parameters,

$$\Phi^{(a)}(r) = \sum_\alpha \frac{\psi^{(a)}_\alpha}{\psi^{(1)}_\alpha} P_\alpha(r). \tag{3.10}$$

The merit of this set is that its correlation functions are diagonal in the labels a and b:

$$\langle \Phi^{(a)}(r)\Phi^{(b)}(r')\rangle = \delta_{ab}\left(\frac{\text{const.}}{|r-r'|^{d_a}} + \text{subdominant terms}\right) \tag{3.11}$$

(see below). In fact, this critical behaviour is represented by one of the minimal unitary conformal field theories of central charge $c = 1 - 6/h(h-1)$, namely the one labelled $(A_{h-2}, \mathcal{G})$ in the classification of [18]. The labels a have to be chosen among the Coxeter exponents of the diagram $\mathcal{G}$ (which agrees with our convention that 1 labels the identity), namely the field $\Phi^{(a)}$ is a linear combination of the zero spin fields labelled in the Kac formula by $s = a = s'$ mod 2 [18]. For $a < h-1$, the leading term in (3.11) is given by the spin zero primary field along the diagonal of the Kac table $s = s' = a$, and only that term survives in the continuum limit.

In the transfer matrix formalism, correlation functions of these operators may be computed through the insertion of $\phi^{(a)}_i$ defined in (3.4). If the fields Φ are located at sites $r_l = (t_l, i_l)$ with, say, $t_1 \leqslant t_2 \leqslant \dots \leqslant t_q$, their correlator reads

$$\langle \Phi^{(a_1)}(r_1)\dots\Phi^{(a_q)}(r_q)\rangle$$
$$= Z^{-1}_{\rm mod} \sum_{\alpha_0,\alpha_1,\dots,\alpha_L} \psi^{(1)*}_{\alpha_0}\psi^{(1)}_{\alpha_L}$$
$$\times \langle \alpha_0 \dots \alpha_L | \mathcal{T}^{M-t_q}\phi^{(a_q)}_{i_q}\dots\phi^{(a_2)}_{i_2}\mathcal{T}^{t_2-t_1}\phi^{(a_1)}_{i_1}\mathcal{T}^{t_1} | \alpha_0 \dots \alpha_L\rangle. \tag{3.12}$$

Thus expanding the expression of $\mathcal{T}$ and of each X_i as given in (3.7), we see that the calculation of $Z_{\rm mod}$ or of any of these correlation functions is a *universal* linear combination of expressions of the form

$$\sum_{\alpha_0,\alpha_1,\dots,\alpha_L} \psi^{(1)*}_{\alpha_0}\psi^{(1)}_{\alpha_L}\langle \alpha_0 \dots \alpha_L | \mathcal{M} | \alpha_0 \dots \alpha_L\rangle,$$

where $\mathcal{M}$ is a monomial in the U_i and $\phi^{(\cdot)}_j$. Here and in the following, *universal* means independent of the explicit representation of the Temperley–Lieb algebra attached to a graph with a given h. In contrast, M_{abc} is *not* a universal number as it depends on the graph $\mathcal{G}$.

We shall now prove that:
(a) the modified partition function and the two-point functions are universal,
(b) the three-point function is of the form $\langle \phi^{(a)}\phi^{(b)}\phi^{(c)}\rangle = M_{abc} \times$ (a universal function).

In the latter, the universal function may (and will in general) depend on the labels a, b, c, but in a universal way. It also clearly depends on the locations of the three operators. Note that (a) is a particular case of (b), when one or three of the operators are chosen to be the identity and using that $M_{ab1} = \delta_{ab}$. Note finally that this universality of the three-point function is what is needed to prove the assertion on structure constants. In the ratio of two three-point functions of operators with the same labels but pertaining to the graph $\mathcal{G}$ and to the graph A of same Coxeter numbers, the universal function disappears and we find

$$\frac{\langle \phi^{(a)}(r_1)\phi^{(b)}(r_2)\phi^{(c)}(r_3)\rangle_{\mathcal{G}}}{\langle \phi^{(a)}(r_1)\phi^{(b)}(r_2)\phi^{(c)}(r_3)\rangle_{\mathrm{A}}} = \frac{M_{abc}}{N_{abc}}. \tag{3.13}$$

The fusion coefficient N_{abc} takes the value 1 whenever the three-point functions for the graph A are non-vanishing. This is a peculiarity of SU(2) that makes this discussion simpler. On the other hand, in the continuum limit, this ratio of three-point functions is nothing else than the ratio of structure constants. According to the discussion at the end of Section 2, this ratio is the same for the conformal fields on the diagonal of the Kac table that appear in this lattice approach as for those of the $s' = 1$ subalgebras considered in Section 2.

We now turn to the proof of the asserted universality. (This may also be proved using the cluster expansion techniques developed by Pasquier in [8], see also [12].) The technique that we use here is more powerful and extends to a large part to the case of more general models based on Hecke algebras relative to sl(N) algebras of higher rank [22].

Let us first establish a few simple lemmas.

Consider the operators $\phi_i^{(a)}$ defined in (3.4). At a given site i, they form an algebra

$$\phi_i^{(a)}\phi_i^{(b)} = M_{ab}{}^c\phi_i^{(c)}, \tag{3.14}$$

or more generally

$$\phi_i^{(a_1)}\dots\phi_i^{(a_l)} = M_{a_1\dots a_l}{}^c\phi_i^{(c)}, \tag{3.14}'$$

where

$$M_{a_1\dots a_l}{}^c = \sum_b \left(\prod_j \frac{\psi_b^{(a_j)}}{\psi_b^{(a)}}\right)\psi_b^{(1)}\psi_b^{(c)*} \tag{3.15}$$

satisfy $M_{a_1\dots a_l}{}^c M_{cb_1\dots b_m} = M_{a_1\dots b_m}$. On the other hand, the ϕ's relative to different sites commute among themselves but do not commute with the U's. They satisfy, however, the following identities:

$$U_i\phi_i^{(a)}U_i = \gamma_a\phi_{i-1}^{(a)}U_i = \gamma_a U_i\phi_{i-1}^{(a)}, \tag{3.16a}$$

$$U_i\phi_i^{(a)}U_{i-1}U_i = \phi_{i-2}^{(a)}U_i = U_i\phi_{i-2}^{(a)}, \tag{3.16b}$$

that are readily established using the expressions (3.3) and (3.4).

Now consider the trace of any monomial in the generators of the Temperley–Lieb algebra, U_i, $i = 1, 2, \ldots, I \leqslant L-1$, and in operators $\phi_j^{(\cdot)}$, $j = 0, \ldots, J \leqslant L$, contributing to a three-point function of the ϕ's. A second lemma asserts that the trace of such a monomial may be written as a linear combination, with universal coefficients, of traces of products of U's and ϕ's at most linear in U_I, the one of largest label. This is easily established by induction on I and the degree in U_I, using the relations (3.3), (3.14) and (3.16), and the cyclicity of the modified trace.

Then, we may always assume that $J < L$ at the possible price of replacing in the modified trace $\psi_{\alpha_L}^{(1)}$ by some more general $\psi_{\alpha_L}^{(b)}$. Moreover, if the monomial is of degree more than one in ϕ_J and $J \geqslant I$, we may use the commutativity of the ϕ's and the cyclicity of the trace to bring the ϕ_J next to one another and then use (3.14) to reduce their degree to one. Ultimately, we are dealing with a combination of monomials

$$\operatorname{tr} \mathscr{M}\big(U_1, \ldots, U_I, \phi_1^{(i_1)} \ldots \phi_J^{(i_J)}\big)$$

at most linear in U_I and if $J \geqslant I$ at most linear in ϕ_J.

The universality property will then be proved by induction on the length L. If $L = 2$, it is trivial. Let us assume it is true for all lengths up to $L-1$. For a length L, by the lemma above, $\mathscr{M}$ may be taken to be at most linear in U_{L-1}.

- If it is independent of U_{L-1}, then the summation over α_L may be carried out, with the result

$$\sum_{\alpha_L} \langle \alpha_0 \ldots \alpha_L | \mathscr{M}\big(U_1, \ldots, U_I, \phi_1^{(a_1)} \ldots \phi_J^{(a_J)}\big) | \alpha_0 \ldots \alpha_L \rangle \psi_{\alpha_L}^{(b)}$$

$$= \gamma_b \langle \alpha_0 \ldots \alpha_{L-1} | \mathscr{M}(\ldots) | \alpha_0 \ldots \alpha_{L-1} \rangle \psi_{\alpha_{L-1}}^{(b)},$$

  and we are now dealing with a chain of length $L-1$ on which the induction hypothesis applies.

- If $\mathscr{M}$ is linear in U_{L-1}, one may sum again over α_L,

$$\sum_{\alpha_L} \langle \alpha_0 \ldots \alpha_L | U_{L-1} \mathscr{M}'\big(U_1, \ldots, U_{L-2}, \phi_1^{(a_1)} \ldots \phi_{L-1}^{(a_{L-1})}\big) | \alpha_0 \ldots \alpha_L \rangle \psi_{\alpha_L}^{(b)}$$

$$= \langle \alpha_0 \ldots \alpha_{L-1} | \mathscr{M}'\big(U_1, \ldots, U_{L-2}, \phi_1^{(a_1)} \ldots \phi_{L-1}^{(a_{L-1})}\big) | \alpha_0 \ldots \alpha_{L-1} \rangle \frac{\psi_{\alpha_{L-1}}^{(1)}}{\psi_{\alpha_{L-2}}^{(1)}} \psi_{\alpha_{L-2}}^{(b)}$$

$$= \langle \alpha_0 \ldots \alpha_{L-1} | \mathscr{M}'\big(U_1, \ldots, U_{L-2}, \phi_1^{(a_1)} \ldots \phi_{L-1}^{(a_{L-1})}\big) \phi_{L-2}^{(b)} | \alpha_0 \ldots \alpha_{L-1} \rangle \psi_{\alpha_{L-1}}^{(1)},$$

  to which we may apply again the recursion hypothesis. QED.

Ultimately, we collect only one M factor times a universal combination of γ's and this proves the desired property.

4. Questions, conclusions

Although all structure constants including their relative signs are determined from the locality equations (2.6) and thereby we have been able to prove our

assertion (1.3), it seems desirable to find a more transparent and global argument to that effect.

The same applies to the factorization property (2.10). It is also not unlikely that a general procedure yields the d's of the twisted cases (like E_7) from those of the corresponding untwisted case (see, e.g., Section 4 of [7]).

The fact that a certain class of subalgebras is in one-to-one correspondence with Coxeter groups is also quite intriguing.

It is natural to wonder whether the property (1.3) connecting the relative structure constants to the matrix elements of the Pasquier algebra extends to a larger class of non-minimal theories, in particular to cosets based on $\widehat{sl(N)}$ affine algebras, $N \geqslant 3$. For some of those, graphs have been identified which allow the construction of integrable lattice models with a continuum limit described by the appropriate conformal theory [13], and it is a simple matter to find the eigenvectors ψ and to construct the M's. The latter, as well as spin zero fields, are now labelled by generalized "exponents" a taking their values among integrable weights of $\widehat{sl(N)}$ at some level k. Since essentially nothing is known about the structure constants of the non-diagonal solutions in these cases, it is difficult to assert the validity of (1.3). One may try instead to repeat the lattice approach following the steps of Section 3. One encounters, however, some difficulty due to the absence of a simple cluster expansion in those higher rank cases, or alternatively, the lack of the Kronecker delta function like in the RHS of (3.3) makes it difficult to generalize Eqs. (3.16). Preliminary results based on the consideration of lattice configurations of small size seem to point to the following conjecture:

- whenever the fusion coefficient $N_{ab}{}^{c}$ is equal to one, the property (1.3) remains true;
- on the contrary, if $N_{ab}{}^{c} > 1$, the universality property crucial in Section 3 fails. This seems to fit with the qualitative idea that $N_{ab}{}^{c} > 1$ means that there is more than one independent amplitude in the $\langle \Phi_A \Phi_B \Phi_C \rangle$ correlation function, thus some more work has to be done to recover the "universal" quantity.

On the other hand in the block diagonal cases one can exploit the existence and the locality properties of the underlying extended chiral algebras, extending the approach outlined in Appendix C. A preliminary computation suggests in particular that the M matrices in the level $k = 5$ exceptional example in the $\widehat{sl(3)}$ case [13] can be reproduced and an extension of (1.3) obtained. We hope to return to these problems.

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Appendix A

Tables of M matrices

The M matrices introduced in Section 1 are in fact not fully determined, due to a remaining arbitrariness in the ψ. The latter are assumed to be orthonormalized, which leaves a sign ambiguity for each, and slightly more in the D_{even} case.

(i) As for the sign ambiguity, it may be removed by imposing for example that the component $\psi_\alpha^{(a)}$ of each eigenvector $\psi^{(a)}$ for the vertex α at the end of the longest leg of the Dynkin diagram is non-negative. This disposes of all the cases but D_{even}.

(ii) In the latter case, the eigenspace for the exponent $a = h/2$ is of dimension 2. We choose (as in [20]) $\psi^{(h/2;-)} = 1/\sqrt{2}\,(0, 0, \ldots, 1, -1)$, with all components vanishing but on the end points of the fork, and $\psi^{(h/2,+)}$ orthogonal to it, with the sign fixed as above in (i). The reader may find explicit formulae for the (unnormalized) eigenvectors e.g. in [20].

The following M have been computed using these prescriptions. Note that they satisfy the symmetry property $M_{h-a\ h-b\ c} = M_{abc}$ in all cases but the D_{odd} one, where it is true only up to a sign.

A.1. Case $D_{h/2+1}$

For $a, b, c = 1, 3, \ldots, h-1$ but $\neq *$, where $*$ denotes $h/2$ in the D_{odd} case and $(h/2;-)$ in the D_{even} one, we have

$$M_{abc} = N_{abc} = \begin{cases} = 1, & \text{if } |b-c|+1 \leqslant a \leqslant \inf(b+c, 2h-b-c)-1, \\ = 0, & \text{otherwise,} \end{cases} \tag{A.1a}$$

$$M_{a**} = M_{*a*} = M_{**a} = (-1)^{(a-1)/2}. \tag{A.1b}$$

All the other M's vanish. Comparing with (2.14) for $C = F = (h/2, h/2;-)$ we see that Eqs. (A.1) coincide with the expressions for the scalar constants d. Alternatively in the D_{even} case, the M matrices can be rewritten in the second basis corresponding for $h = 2 \bmod 8$ to (2.16); the formulae for $h = 6 \bmod 8$ read

$$\begin{aligned} &M_{\pm\pm}^{\mp} = \sqrt{2}, \quad M_{\pm\pm}^{\pm} = 0 = M_{+-}^{\pm}, \\ &M_{\pm a}^{\pm} = M_{a\pm}^{\pm} = \tfrac{1}{2}\left[1 + (-1)^{(a-1)/2}\right] = M_{\pm\mp}^{a}, \quad a \neq \tfrac{1}{2}h, \\ &M_{\pm a}^{\mp} = M_{a\pm}^{\mp} = \tfrac{1}{2}\left[1 - (-1)^{(a-1)/2}\right] = M_{\pm\pm}^{a}, \quad a \neq \tfrac{1}{2}h, \\ &M_{\pm a}^{f} = M_{a0}^{f} = \tfrac{1}{2}N_{af\,h/2} = M_{af}^{\pm}, \quad a, f \neq \tfrac{1}{2}h, \end{aligned} \tag{A.2}$$

and for $a, b, c, \neq h/2$, the constant is N_{abc} as in the old basis. The labels $\pm$ stay for the two linear combinations $\Psi_\alpha^{(+)} = (1/\sqrt{2})(\psi_\alpha^{(h/2)} + i\psi_\alpha^{(h/2;)})$, $\Psi_\alpha^{(-)} = (\Psi_\alpha^{(+)})^*$. In the case D_4 ($h = 6$) the last line in (A.2) does not appear since it is excluded by the fusion rules, i.e., the constants take only the values $0, 1, \sqrt{2}$.

A.2. Case E_6

To make it shorter, we make use of the symmetry $M_{abc} = M_{h-a\,h-b\,c}$ to display only the values for $a \leqslant h/2$ and omit M_1, equal to the unit matrix:

$$M_4 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & \sqrt{\frac{3}{2}} & \sqrt{\frac{1}{2}} & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}} & 0 & 0 & \sqrt{\frac{1}{2}} & 0 \\ 0 & \sqrt{\frac{1}{2}} & 0 & 0 & \sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & \sqrt{\frac{1}{2}} & \sqrt{\frac{3}{2}} & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$M_5 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}} & 0 & 0 & \sqrt{\frac{1}{2}} & 0 \\ 1 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 1 \\ 0 & \sqrt{\frac{1}{2}} & 0 & 0 & \sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

A.3. Case E_7

Same remark as for E_6. The rows and columns of the matrices below correspond to the exponents 1, 5, 7, 11, 13, 17 and 9:

$$M_5 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & \sqrt{\frac{1}{2}} \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \sqrt{\frac{1}{2}} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 1 & 0 & \sqrt{\frac{1}{2}} \\ 0 & 0 & 0 & 1 & 1 & 1 & \sqrt{\frac{1}{2}} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & 0 & 1 \end{pmatrix},$$

$$M_7 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \sqrt{\frac{1}{2}} \\ 1 & -\frac{1}{2} & 1 & -1 & \frac{1}{2} & 0 & \sqrt{\frac{1}{2}} \\ 0 & \frac{1}{2} & -1 & 1 & -\frac{1}{2} & 1 & \sqrt{\frac{1}{2}} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 1 & 0 & \sqrt{\frac{1}{2}} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & 0 & 0 \end{pmatrix}$$

$$M_9 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & 0 & 1 \\ 0 & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & 0 & 0 \\ 0 & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & 0 & 0 \\ 0 & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & \sqrt{2} \end{pmatrix}.$$

A.4. Case E_8

Same remark as for E_6.

$$M_7 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \sqrt{\frac{5}{3}} & 1 & \sqrt{\frac{1}{3}} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2}\sqrt{5} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \sqrt{\frac{1}{3}} & \frac{1}{2}\sqrt{5} & \frac{1}{2}\sqrt{\frac{5}{3}} & \frac{1}{2}\sqrt{3} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2}\sqrt{3} & \frac{1}{2}\sqrt{\frac{5}{3}} & \frac{1}{2}\sqrt{5} & \sqrt{\frac{1}{3}} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2}\sqrt{5} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{1}{3}} & 1 & \sqrt{\frac{5}{3}} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$M_{11} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2}\sqrt{5} & \frac{1}{2} & 0 & 0 & 0 \\ 1 & 0 & \frac{3}{2} & 0 & 0 & \frac{1}{2}\sqrt{5} & 0 & 0 \\ 0 & \frac{1}{2}\sqrt{5} & 0 & \frac{1}{2} & \frac{1}{2}\sqrt{5} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2}\sqrt{5} & \frac{1}{2} & 0 & \frac{1}{2}\sqrt{5} & 0 \\ 0 & 0 & \frac{1}{2}\sqrt{5} & 0 & 0 & \frac{3}{2} & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2}\sqrt{5} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$M_{13} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{1}{3}} & \frac{1}{2}\sqrt{5} & \frac{1}{2}\sqrt{\frac{5}{3}} & \frac{1}{2}\sqrt{3} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2}\sqrt{5} & 0 & \frac{1}{2} & \frac{1}{2}\sqrt{5} & 0 & \frac{1}{2} & 0 \\ 1 & \frac{1}{2}\sqrt{\frac{5}{3}} & \frac{1}{2} & 2\sqrt{\frac{1}{3}} & 0 & \frac{1}{2}\sqrt{5} & \frac{1}{2}\sqrt{3} & 0 \\ 0 & \frac{1}{2}\sqrt{3} & \frac{1}{2}\sqrt{5} & 0 & 2\sqrt{\frac{1}{3}} & \frac{1}{2} & \frac{1}{2}\sqrt{\frac{5}{3}} & 1 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2}\sqrt{5} & \frac{1}{2} & 0 & \frac{1}{2}\sqrt{5} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2}\sqrt{3} & \frac{1}{2}\sqrt{\frac{5}{3}} & \frac{1}{2}\sqrt{5} & \sqrt{\frac{1}{3}} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Appendix B

Fusion matrices, quantum 6j-symbols, symmetries

Denote

$$[a]=\frac{\sin(\pi a\rho)}{\sin(\pi\rho)},\quad [b]!=\prod_{a=1}^{b}[a],$$
$$\Delta[abc]=\left(\frac{[a+b-c]![a+c-b]![b+c-a]!}{[a+b+c+1]!}\right)^{1/2}. \tag{B.1}$$

Recall the explicit expression for the quantum $U_q(\mathrm{sl}(2))$ $6j$-symbols [23],

$$\begin{Bmatrix} j_1 & j_2 & j_5 \\ j_3 & j_4 & j_6 \end{Bmatrix}_\rho$$
$$=\sqrt{[2j_5+1][2j_6+1]}\,\Delta[j_1j_2j_5]\Delta[j_3j_4j_5]\Delta[j_1j_4j_6]\Delta[j_2j_3j_6]$$
$$\times\sum_z \frac{(-1)^{z+\sum_{n=1}^4 j_n}[z+1]!}{[z-j_2-j_3-j_6]![z-j_1-j_2-j_5]![z-j_1-j_4-j_6]![z-j_3-j_4-j_5]!}$$
$$\times\frac{1}{[j_2+j_4+j_5+j_6-z]![j_1+j_3+j_5+j_6-z]![\Sigma_1^4 j_a-z]!}. \tag{B.2}$$

The thermal (i.e., for all $j'_n=0$) fusion matrix is defined as ($s_i=2j_i+1$)

$$\begin{Bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{Bmatrix}_{s_5s_6}=(-1)^{(1+j_1+j_3-j_2-j_4)(j_1+j_3-j_5-j_6)}\begin{Bmatrix} j_1 & j_2 & j_5 \\ j_3 & j_4 & j_6 \end{Bmatrix}_{1/h}. \tag{B.3}$$

One can assume that the parameters j_n, $n=1,2,\ldots,6$, in (B.3) take values in a subrange consistent with the conformal fusion rules for the given h, i.e., any of the triplets (j_1, j_2, j_5), (j_3, j_4, j_5), or (j_1, j_4, j_6), (j_2, j_3, j_6), is admissible. Accordingly the summation in the crossing equations (2.6), as well as in (2.3), accounts for these restrictions. The consistency of this truncated summation in the physical correlation functions can be established by quantum group arguments [24,25].

The signs in the RHS of (B.3) come from the transition $\rho=(h-1)/h\to\rho=1/h$ in the original expression and furthermore from the choice of normalization of the chiral blocks in (2.3). The latter differs by a sign from that in [1,4] and is adopted here to ensure the positivity of the constants C_{ab}^c in (2.4) [4].

Note that

$$\begin{Bmatrix} c & a \\ a & c \end{Bmatrix}_{f1}=\sqrt{\frac{[f]}{[a][c]}}. \tag{B.4}$$

[4] Here we correct the analogous formula (3.1) in the third reference of [4]. The sign missing in (3.1), if compared (for all $j'=0$) with (B.3) above, is due to the erroneous formula (2.2). This does not change the main results in the third reference of [4] but affects, say, some of the signs of the non-zero spin field constants in the E_6 case.

The fusion matrix satisfies a set of symmetry relations derived in the first reference of [4]. Denoting $s = h - s$, they read

$$\left\{\begin{matrix} \underline{s_1} & s_2 \\ s_3 & \underline{s_4} \end{matrix}\right\}_{\underline{s_5}s_6} = (-1)^{\Delta^4_{16}+(\Delta^5_{12}+\Delta^6_{23})(s_6-1)+\Delta^4_{123}s_2} \left\{\begin{matrix} s_1 & s_2 \\ s_3 & s_4 \end{matrix}\right\}_{s_5 s_6}, \tag{B.5a}$$

$$\left\{\begin{matrix} s_1 & s_2 \\ \underline{s_3} & \underline{s_4} \end{matrix}\right\}_{s_5\underline{s_6}} = (-1)^{\Delta^4_{35}+(\Delta^5_{12}+\Delta^6_{23})(s_5-1)+\Delta^4_{123}s_2} \left\{\begin{matrix} s_1 & s_2 \\ s_3 & s_4 \end{matrix}\right\}_{s_5 s_6}, \tag{B.5b}$$

$$\left\{\begin{matrix} s_1 & \underline{s_2} \\ s_3 & \underline{s_4} \end{matrix}\right\}_{\underline{s_5}s_6} = (-1)^{(j_2+j_4-j_5-j_6)(s_1+s_3+1)} \left\{\begin{matrix} s_1 & s_2 \\ s_3 & s_4 \end{matrix}\right\}_{s_5 s_6}, \tag{B.5c}$$

where $\Delta^5_{12} = j_1 + j_2 - j_5$, $\Delta^4_{123} = j_1 + j_2 + j_3 - j_4$.

When inserted in the general equation (2.6) these relations imply restrictions on the signs of the relative structure constants. Finally, given these relations the derivation of the structure constants (2.14) in the D series is straightforward.

Appendix C

Relation to the extended theories

In this appendix we shall sketch the implications of the factorization property (2.10) for the cases E_6 and E_8. All the arguments work for the simpler case D_4 as well.

The idea is to use (2.10) to block-diagonalize the locality equations (2.6) [26]. Indeed, whenever the overall sign of the constants in it is positive, we can attach the ratio $(M^f_{ac}\, M^f_{ac}/M^b_{at}\, M^c_{td})^{1/2}$ to the fusion matrix. Take, e.g., $A = B$, $C = D$ and $T = (1, 1)$ and denote by $\{f\}$ the equivalence class of f, i.e., $f' \sim f$ iff $\Delta_f = \Delta_{f'}$ modulo an integer. A direct check shows that the quantity

$$F^{\{f\}}_{\{a\}\{c\}} = F^{\{f\}}_{\{c\}\{a\}} = \sum_{f\in\{f\}} M^f_{ac} \left\{\begin{matrix} c & a \\ a & c \end{matrix}\right\}_{f1}, \tag{C.1a}$$

$$F^{\{a\}}_{\{a\}\{1\}} = 1, \tag{C.1b}$$

depends only on the classes $\{a\}$, $\{c\}$, as indicated by the notation (cf. also (B.4)).

This fact (an assumption in [26]) allows to split the sum over f (or $\bar{f}$) in the locality equation to a sum over the classes followed by a summation within the classes. Hence Eqs. (2.6) for the particular choice $A = B$, $C = D$, $T = (1, 1)$ admit a diagonal form, i.e.

$$\sum_{\{f\}} \left(F^{\{f\}}_{\{a\}\{c\}}\right)^2 = 1 \Leftrightarrow \sum_{f,\bar{f}} \left(d^F_{AC}\right)^2 \left\{\begin{matrix} c & a \\ a & c \end{matrix}\right\}_{f1} \left\{\begin{matrix} \bar{c} & \bar{a} \\ \bar{a} & \bar{c} \end{matrix}\right\}_{\bar{f}1} = 1. \tag{C.2}$$

Now let us look at the concrete expressions for $F^{\{f\}}_{\{a\}\{c\}}$ apart from the values already given in (C.1b).

Case E_6:

$$F^{\{1\}}_{\{5\}\{5\}} = F^{\{4\}}_{\{5\}\{4\}} = 1, \quad F^{\{1\}}_{\{4\}\{4\}} = F^{\{5\}}_{\{4\}\{4\}} = \sqrt{\tfrac{1}{2}}\,; \tag{C.3}$$

the rest zero.

Case E_8:

$$F^{\{1\}}_{\{7\}\{7\}} = \tfrac{1}{2}(\sqrt{5}-1), \quad F^{\{7\}}_{\{7\}\{7\}} = \left[\tfrac{1}{2}(\sqrt{5}-1)\right]^{1/2}; \tag{C.4}$$

the rest zero.

Given the class matrices in (C.3) and (C.4), one recovers the corresponding fusion rule coefficients $N^{\{f\}}_{\{a\}\{c\}} = 0, 1$ [26],

$$N^{\{f\}}_{\{a\}\{a\}} = \frac{F^{\{f\}}_{\{a\}\{c\}} F^{\{c\}}_{\{a\}\{f\}}}{F^{\{1\}}_{\{a\}\{a\}}}. \tag{C.5}$$

The matrix in (C.1) can be represented by

$$F^{\{f\}}_{\{a\}\{c\}} = \left(\frac{D_{\{f\}}}{D_{\{a\}} D_{\{c\}}}\right)^{1/2}, \quad D_{\{a\}} = \frac{S_{\{1\}\{a\}}}{S_{\{1\}\{1\}}}. \tag{C.6}$$

Here S is the modular matrix which according to the Verlinde formula diagonalizes the fusion rule coefficients $N^{\{f\}}_{\{a\}\{c\}}$. The fusion algebra implied by (C.3) coincides with the Ising model fusion algebra. In agreement with the analysis initiated in the second reference of [3] (see also [27]) the numbers in (C.3) are alternatively reproduced using the modular matrix elements $S_{\bar{1}\lambda}/S_{\bar{1}\bar{1}}$ (quantum dimensions) for the set of integrable representations of the level 1 affine algebra $\widehat{B_2}$. Namely, identifying $\lambda = (0, 1)$, $(1, 0)$ and $(0, 0)$ (classical B_2 dimensions 4, 5 and 1) with the classes $\{4\}$, $\{5\}$ and $\{1\}$, respectively, one has

$$D_{(0,1)} = \sqrt{2}\,, \quad D_{(1,0)} = 1 = D_{(0,0)}. \tag{C.7}$$

The second fusion algebra is the one of the level 1 $\widehat{G_2}$ WZW model or of the corresponding coset theory. Indeed the classes $\{7\}$ and $\{1\}$ in the case E_8 can be identified with the representations $\lambda = (0, 1)$ and $(0, 0)$ of $\widehat{G_2}$ since

$$D_{(0,1)} = \frac{2}{\sqrt{5}-1}, \quad D_{(0,0)} = 1. \tag{C.8}$$

One can slightly extend the construction in (C.1). Namely, taking instead of $T = (1, 1)$, arbitrary $T = (t, \bar{t})$, with $t, \bar{t} \in \{1\}$, one can define, whenever $M^t_{aa} M^t_{cc} \neq 0$,

$$F^{\{f\}}_{\{a\}\{c\};\{1\}} = F^{\{f\}}_{\{c\}\{a\};\{1\}} = \epsilon_{a,c;t} \sum_{f \in \{f\}} \frac{M^f_{ac}}{\sqrt{M^t_{aa} M^t_{cc}}} \begin{Bmatrix} c & a \\ a & c \end{Bmatrix}_{ft}, \tag{C.9a}$$

$$F^{\{a\}}_{\{a\}\{1\};\{1\}} = 1. \tag{C.9b}$$

Here $\epsilon_{a,c;t} = \epsilon_{c,a;t}$ is a sign (we suppress the dependence on $\{f\}$), such that $\epsilon_{a,c;1} = \epsilon_{a,a;t} = 1$ and $\epsilon_{a,c;t}\epsilon_{\bar{a},\bar{c};\bar{t}} = \text{sign}(d^A_{AT} d^C_{TC})$ (Since the signs are overall they are easily found by direct computation – we omit the explicit values.)

The LHS of (C.9a) takes the same values as the corresponding elements in (C.1a). Hence it admits the representation (C.6), with the values given in (C.7) and (C.8). Similarly to (C.2), the quantity (C.9a) allows to block-diagonalize the locality equations (2.6) for $A = B$, $C = D$ and T of the kind described above [5].

Vice versa, if the extended fusion matrix elements in (C.9a) are known, they can be decomposed for given $\{a\}$ and $\{c\}$ in several different ways into fusion matrix elements of the minimal model. The resulting set of relations – a linear system of equations for the decomposition coefficients M, can be solved (together with the signs), assuming the symmetry $M^f_{ac} = M^c_{af}$. The coefficients then provide a factorized solution for the relative structure constants of the minimal model locality equations. Note that in the E_6 case it is sufficient to use (C.1a), i.e., to vary $c \in \{c\}$, $a \in \{a\}$, choosing $t = 1$.

Finally let us remark that the existence of an extended theory behind some modular invariant implies a set of symmetry relations for the fusion matrices at the given h (equivalent to the class property above), which generalize (B.5).

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[5] In fact (C.9a) extends to arbitrary T ($\epsilon_{a,a;t} = \pm 1$), leading to the extended fusion matrix $\left\{\begin{smallmatrix}\{c\}\{a\}\\ \{a\}\ \{c\}\end{smallmatrix}\right\}_{\{f\},\{t\}}$, and furthermore one can reproduce in a similar way the general matrix elements. The representation (C.6) is a standard normalization condition for $\left\{\begin{smallmatrix}\{c\}\{a\}\\ \{a\}\ \{c\}\end{smallmatrix}\right\}_{\{f\},\{1\}} \equiv F^{\{f\}}_{\{a\}\{c\};\{1\}}$.

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