ON VARIOUS AVATARS OF THE PASQUIER ALGEBRA

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ABSTRACT

A Pasquier algebra is a commutative associative algebra of normal matrices attached to a graph. I review various appearances of such algebras in different contexts: operator product algebras and structure constants in conformal theories and lattice models, integrable N=2 supersymmetric models and their topological partners.

1. Introduction

Let $\mathcal{G}$ be a graph with n vertices $a, b, \cdots$, G_{ab} its adjacency matrix; as the edges of $\mathcal{G}$ may be oriented, G is non necessarily symmetric, but we shall assume that it is normal, *i.e.* it commutes with its transpose: $[G, G^t] = 0$, hence it is diagonalizable in an orthonormal basis $\psi_a^{(i)}$. Here i labels the eigenvectors and runs over n values. Among them, the Perron-Frobenius eigenvector plays a special role and will be denoted $\psi_a^{(0)}$. We then define the set of matrices M_i of matrix elements

$$(M_i)_{jk} = M_{ij}{}^k = \sum_a \frac{\psi_a^{(i)} \psi_a^{(j)} \psi_a^{(k)*}}{\psi_a^{(0)}} \ . \tag{1.1}$$

It includes the unit matrix $M_0 = \mathbb{I}$ and forms a commutative and associative algebra

$$M_i M_j = \sum_k M_{ij}{}^k M_k \tag{1.2}$$

that we shall call the *Pasquier algebra*[1]. This note is devoted to a review of various appearances of this algebra in different contexts, some known for quite some time, some more recent.

Before doing so, let us notice that there is a well known particular case of this algebra, obtained by taking for $\mathcal{G}$ the graph formed by the integrable weights of an affine algebra $\hat{g}$ at some level $k \geq 1$, with edges between the weights λ_1 and λ_2 if $\lambda_2 \subset \lambda_0 \otimes \lambda_1$, where λ_0 is one of the fundamental representations. Then both types of labels of ψ, a and i, refer to integrable weights λ, the ψ are in fact given by the matrix elements of the unitary matrix S of modular transformations of characters $\chi_\lambda(\tau)$, and the Pasquier algebra is nothing else than the Verlinde fusion algebra of $\hat{g}$ at level k [2]

$$N_{\lambda_0\lambda_1}{}^{\lambda_2} = \sum_\mu \frac{S_{\lambda_0\mu} S_{\lambda_1\mu} S^*_{\lambda_2\mu}}{S_{0\mu}} \ , \tag{1.3}$$

with 0 denoting the weight of the identity representation. Thus in that case, all $M_{ij}{}^k = N_{\lambda_0\lambda_1}{}^{\lambda_2}$ are non-negative integers. This is not the case in general.

In cases different from this fusion algebra, the labels i and a, although in equal number, are inequivalent. This suggests to perform the summation in (1.1) in an alternative way and to form the *dual* algebra generated by the matrices N_a of matrix elements

$$N_{ab}{}^c = \sum_i \frac{\psi_a^{(i)} \psi_b^{(i)} \psi_c^{(i)\,*}}{\psi_0^{(i)}} \ . \tag{1.4}$$

This is again an associative and commutative algebra, with the matrix $N_0 = \mathbb{I}$, the identity matrix. Eq. (1.4) assumes that there exists at least one vertex labelled 0 such that all the $\psi_0^{(i)}$ are non vanishing. If this point is chosen among the extremal vertices of the diagram, *i.e.* those connected with only one other vertex denoted f, then it is easy to see that $\psi_f^{(i)}/\psi_0^{(i)} = \lambda^{(i)}$ is the i-th eigenvalue of the adjacency matrix G, hence $N_f = G$. In the simplest cases, the dual algebra is thus generated by the identity matrix N_0 and the adjacency matrix of the Dynkin diagram $N_f = G$, all the matrices N_a turn out to have integral entries and, depending on the case (and the choice of the vertex 0), these integers are or are not all non negative (cf. [3]). In some selfdual cases, like the Verlinde algebra (1.3), the M and N algebras coincide.

2. Lattice models, conformal field theories and operator product algebras

2.1. Lattice operator algebra

In his original study of lattice integrable models attached to the ADE Dynkin diagrams, Pasquier was assigning a height a, a vertex of the diagram, to each lattice site, with the constraint that neighbouring sites were given neighbouring heights on the diagram. Such models admit a description in terms of states representing the height configurations along diagonals on the lattice. Let $P_a(\mathbf{r})$ be the projector on the subspace of states that have the height a at site $\mathbf{r}$. Its expectation value is the so-called *local height probability* and is an order parameter of the lattice theory. Pasquier was led to consider another set of operators

$$\phi^{(i)}(\mathbf{r}) = \sum_a \frac{\psi_a^{(i)}}{\psi_a^{(0)}} P_a(\mathbf{r}) \ . \tag{2.1}$$

The merit of this set is that its correlation functions at criticality have a simple power behaviour

$$\langle \phi^{(i)}(\mathbf{r})\phi^{(j)}(\mathbf{r}')\rangle = \delta_{ij} \frac{\text{const.}}{|\mathbf{r}-\mathbf{r}'|^{d_i}} \tag{2.2}$$

In fact, this critical behaviour is represented by one of the $c < 1$ conformal field theories, the labels i have to be chosen among the Coxeter exponents of the diagram $\mathcal{G}$ (shifted by -1 to agree with our convention that 0 labels the identity), namely the field $\phi^{(i)}$ corresponds to the spin zero primary field along the diagonal of the Kac table $r = s = \overline{r} = \overline{s} = i + 1$.

Pasquier then showed that a three-point correlation function of the lattice operators (2.1), $\langle \phi^{(i)}(\mathbf{r}_1)\phi^{(j)}(\mathbf{r}_2)\phi^{(k)}(\mathbf{r}_3)\rangle$, is always proportional to ${M_{ij}}^k$. (In that ADE case, the eigenvectors may be chosen real, and $M_{ijk} = {M_{ij}}^k$ is completely symmetric in i, j, k). This was the first occurrence of the algebra (1.1). See also [4] for a discussion of this lattice operator algebra.

2.2. Generalized graph models and their continuum limit

The construction of Pasquier was later extended to a larger class of graph lattice models related to higher rank algebras. In the next simplest case of $SU(3)$, the graphs are the $SU(3)$ weight lattice truncated at some level, or one of their orbifolds, or some suitable deformation, and the continuum limit of these models is described by $SU(3)$ coset conformal theories[5]. It was shown there that in the identification of this continuum limit, the study of the M algebra, of its dual algebra and of their selfdual subalgebras is playing an important role. One of these selfdual subalgebras describes indeed the *extended* fusion algebra of the conformal theory at hand.

2.3. Random lattice models

More recently Kostov[6] studied the ADE models on a random lattice. He considered the random motion of a particle on the graph and introduced interactions between three particles given by the matrix M_{ijk}; there i, j, k refer to the "momenta" of the particles dual to their positions on the graph. Similar considerations apply to strings whose target space is a Dynkin diagram[7]. Thus in this context too, the M's describe three-point couplings.

2.4. Structure constants of conformal field theories

A conformal field theory is fully specified in terms of the following data: c, the central charge, $(h_i, h_{\bar{\imath}})$, the conformal weights of the (finite or infinite) set of primary fields, and $C_{(i\bar{\imath})(j\bar{\jmath})(k\bar{k})}$, the structure constants of these fields. The latter are determined through a painstaking analysis of the locality equations of the four-point functions of the theory and have in fact been tabulated only for a handful of theories, mainly those related to $\widehat{su}(2)$, namely the $c < 1$ theories and $SU(2)$ WZW theories [8–9]. On the other hand, in view of its original introduction in connection with the (lattice) operator algebra, one may expect the Pasquier algebra to be related to these structure constants. Curiously, however, its quantitative role in that context had never been ascertained. Recently, with V. Petkova[10], a simple empirical observation was made.

Let $C^{(\mathcal{G})}_{(ii)(jj)(kk)}$ be the structure constants of the spin zero fields of one of the ADE WZW theories : i, j, k run over the Coxeter exponents (minus 1) of the $\mathcal{G}$ Dynkin diagram that label the primary spin zero fields. Then it was noticed by inspection of the existing data that

$$C^{(\mathcal{G})}_{(ii)(jj)(kk)} = C^{(A)}_{(ii)(jj)(kk)} M_{ijk} \ , \tag{2.3}$$

i.e. the M's describe the *relative* structure constants of the $\mathcal{G}$ theory with respect to the diagonal A theory of same Coxeter number. Notice that while the structure constants of each theory are known to be generically *transcendental* numbers, the ratios M are *algebraic* numbers, in fact square roots of simple fractions.

What is the rationale of such a result? The ratios $d = C^{(\mathcal{G})}/C^{(A)}$ of structure constants are known to satisfy quadratic equations of the type

$$d_{IJM} d_{KLM} = \sum_{N=(n,\bar{n})} d_{IKN} d_{JLN} X_{nm} X_{\overline{nm}} \tag{2.4}$$

with capitals standing for pairs of indices: $I = (i, \bar{\imath})$, etc, and no summation over M. Here X is the orthogonal crossing matrix from channel $i * j \to k * l$ to channel $i * k \to j * l$. Unfortunately, as fields of spin zero do not form a closed operator algebra, these equations do not yield a closed system on structure constants of spin zero fields ($I = (i, i)$ etc). Thus the alledged property

$$d_{(ii)(jj)(kk)} = M_{ijk} \tag{2.5}$$

has to be supplemented by an Ansatz on the other structure constants. It is generally believed, (but not proved, to the best of my knowledge), that in non diagonal theories that may be regarded as diagonal theories for some larger *extended* chiral algebra[11], the structure constants of non zero spin fields factorize:

$$d^2_{(i\bar{i})(j\bar{j})(k\bar{k})} = d_{(ii)(jj)(kk)} d_{(\bar{i}\bar{i})(\bar{j}\bar{j})(\bar{k}\bar{k})} \ . \tag{2.6}$$

and thus are determined up to a possible sign ambiguity. Even then, it is not clear how equation (2.4) admits (2.5) as a unique solution.

In fact, it is much easier to establish that property starting from the lattice theories. A small refinement of the discussion of Pasquier[1] leads indeed to the conclusion that the lattice three-point functions of the ϕ operators are not only proportional to M_{ijk} but in fact equal to M_{ijk} times a universal function independent on the graph $\mathcal{G}$ with a given Coxeter number. Thus the ratios of three-point functions, hence in the continuum, the $\mathcal{G}/A$ ratios of structure constants, are equal to the ratios M_{ijk}/N_{ijk}. The denominator N_{ijk} is the Verlinde fusion coefficient, and takes the value 1 whenever $M_{ijk} \neq 0$. The details of this argument will be presented elsewhere[10] [1].

Although at this time, this property has not been derived within the continuum theory starting from first principles but rather established by inspection, it seems to teach us something new on the relations between the fusion (or crossing) matrices and the ψ eigenvectors. In fact, this is the first explicit occurrence of these eigenvectors in the conformal field theory formalism: so far, only their eigenvalues, or equivalently the Coxeter exponents, had manifested themselves in the diagonal terms of the modular invariant partition function.

Could this property hold in theories other than those related to $SU(2)$? If it could be established with some generality, it would yield in a very cheap way the structure constants of other non diagonal theories relatively to the corresponding diagonal one: as recalled above there are several cases of conformal theories associated with an affine algebra of rank larger than $\widehat{su}(2)$ where there is a candidate graph[5]. Conversely, this might lead to constraints on these graphs and to a better understanding of their distinctive features.

Preliminary investigations seem to indicate that property (2.5) generalizes nicely to other theories, whenever the fusion coefficients ${N_{ij}}^k$ takes the value 1. When it is higher, which signals the occurrence of several amplitudes in the three-point functions, (2.5) is violated.

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[1] In fact, this result had been known for some time, but carefully kept secret, by I. Kostov ...

3. Integrable deformations of $N = 2$ superconformal field theories

We now turn to an *a priori* very different class of problems, that have to do with $N = 2$ superconformal field theories, their connections with topological field theories and their integrable perturbations. There is a huge literature on this subject, (a good introduction is found in [12]), and I shall content myself with a lightning review of the notations and basic concepts.

3.1. A quick review

$N = 2$ superconformal theories have a symmetry algebra generated by the energy-momentum tensor $T(z)$, a $U(1)$ current $J(z)$ and *two* supersymmetry fermionic generators $G^{\pm}(z)$, whose Laurent moments are denoted respectively L_n, J_n, $G^{\pm}_{r+\frac{1}{2}}$ (n is integer, and so is r in the Neveu-Schwarz sector). The primary fields are specified by the eigenvalues of L_0 and J_0, the conformal weight h and the $U(1)$ charge q. Among them, the *chiral* fields, annihilated by the generators $G^{+}_{-\frac{1}{2}}$, satisfy $h = \frac{1}{2}q$ and form a ring for the pointwise (non singular!) operator product expansion[12].

$$\lim_{z' \to z} \phi_i(z)\phi_j(z') = (\phi_i\phi_j)(z). \tag{3.1}$$

The simplest $N = 2$ theories are the *minimal* ones, of central charge $c < 3$. They fall in an ADE classification, as is apparent from their modular invariant partition function and also from their description by a Landau-Ginsburg potential, that is possible in that case [12].

Another fascinating property of the $N = 2$ theories is their connection with topological field theories (TFT's). By "twisting" the energy-momentum tensor $T(z)$, *i.e.* by changing it into $T_{\rm top} = T(z) + \frac{1}{2}\partial J(z)$ and by modifying slightly the rules of computation of the correlation functions $\langle \phi_{i_1}(z_1, \overline{z}_1)\phi_{i_2}(z_2, \overline{z}_2) \cdots \phi_{i_n}(z_n, \overline{z}_n)\rangle$, one may show that the latter become z-independent, *i.e.* topological quantities depending only on the genus of the Riemann surface on which one is constructing the theory and on the number and indices i of the fields. One is interested in deformations of this structure. In the $N = 2$ language, it is important to preserve the $N = 2$ supersymmetry while breaking the (super)conformal invariance. This is provided by perturbations that are "top components" of superfields

$$\langle \phi_{i_1}\phi_{i_2} \cdots \phi_{i_n} e^{-\sum_l t_l \int {\rm d}^2 z\, G^{-}_{-\frac{1}{2}} \overline{G}^{-}_{-\frac{1}{2}} \phi_l(z,\overline{z}) + {\rm h.c.}} \rangle \ . \tag{3.2}$$

The resulting theory is no longer critical, it has massive excitations, particles and/or solitons, and in some cases, is completely integrable. On the other hand, twisting it still produces a topological theory and it is an interesting question to know if there is a signal of integrability in the TFT. In the latter, all genus zero correlations $\langle\phi_{i_1}\phi_{i_2}\cdots\phi_{i_n}\rangle$ may be expressed in terms of the three-point functions

$$C_{ijk}(t_.) = \langle\phi_i\phi_j\phi_k e^{-\sum_l t_l \int \mathrm{d}^2 z\, G^-_{-\frac{1}{2}}\overline{G}^-_{-\frac{1}{2}}\phi_l(z,\overline{z})+\mathrm{h.c.}}\rangle\ . \tag{3.3}$$

that satisfy the following set of constraints:

(i) the tensor defined as $\eta_{ij} = C_{ij0}$ is independent of the t's and invertible: let η^{ij} denote its inverse;

(ii) C_{ijk} satisfies the integrability conditions that enable one to write it as $C_{ijk} = \frac{\partial^3}{\partial t_i \partial t_j \partial t_k} F(t_.)$, where $F(t_.)$ is some function, the free energy of the theory;

(iii) ${C_{ij}}^k = C_{ijl}\eta^{lk}$ satisfies the so-called factorization property expressing the consistency of the two decompositions of the four-point function

$$\langle\phi_i\phi_j\phi_k\phi_l\rangle = C_{ijm}{C_{kl}}^m = C_{ikm}{C_{jl}}^m, \qquad \text{etc}\ .$$

This means that all the information on the C's is encoded in the function F, called the free energy of the theory, that satisfies itself non linear partial differential equations expressing condition (iii). These equations are usually supplemented by an assumption of homogeneity:

(iv) $F(t_.)$ is a quasihomogeneous function of the t's.

The latter condition is natural from the $N = 2$ point of view where chiral fields and thus their couplings are graded by their $U(1)$ charge.

We shall return later to these Witten-Dijkgraaf-Verlinde-Verlinde equations[13].

3.2. Normalizable perturbations

In view of the discussion of the previous section and the similarity between the (deformed) $N = 2$ algebra encoded in the C's and the operator algebra of conformal field theories, it is natural to wonder whether the C may be diagonalized in an orthonormal basis. This turns out to be impossible generically and to happen only in very specific cases that we shall study. In general, the normalization has to be modified, *i.e.* we have to allow a redefinition of the C's by a diagonal change of basis. Let us thus *assume* that we may write

$${C_{ij}}^k = \frac{\rho_i\rho_j}{\rho_k}\,(M_i)_{jk} \tag{3.4a}$$

$$(M_i)_{jk} = \sum_a \lambda_a^{(i)}\psi_a^{(j)}\psi_a^{(k)*} \tag{3.4b}$$

for a suitable set of factors ρ_i. The symmetry $i \leftrightarrow j$ together with the condition that $M_0 = \mathbb{I}$ shows that M may in fact be written in the form (1.1). We called such a situation a *normalizable* perturbation of the TFT.

Now come three surprising empirical observations:

First, the condition that the C's may be written as in (3.4) turns out to be very restrictive, and in the cases where only one t parameter is left non vanishing, may be analysed in detail with the result that it seems to happen if and only if the corresponding $N = 2$ theory is integrable. This has been verified in all cases where the deformed algebra is known explicitly, in particular for the ADE minimal theories and for a class of $N = 2$ theories based on the $\widehat{su(N)}$ algebras. For example, for the A_n theory, the "normalizability" property (3.4) takes place only for perturbations by either t_1, or t_2 or t_{n-1} that are known to be integrable and even believed to be the only ones [14]. This has been extended to the other D or E cases: in some cases, this simple criterion has suggested perturbations that had not been identified before as integrable and in which integrability has now been established by other means [15]. This connection remains quite mysterious. Could the normalizability property be useful in the construction of conserved quantities or of the S matrix solution of the Yang-Baxter equation ?

Secondly, when we write the M algebra (3.4b), we note that in contrast with the original discussion of sect. 1, we are not given a graph to start with. We may, however, form the dual algebra following equation (1.4) and using the ψ's that diagonalize the M matrices in (3.4b). The second surprise is that these numbers ${N_{ac}}^c$ turn out to be integers, and that among the matrices N_a, $(a \neq 0)$, at least one has non negative entries. That N_a may be regarded as the adjacency matrix of a graph, and, third surprise!, this graph has the same topology as the pattern of extrema of the Landau-Ginsparg potential [2]. As the latter are interpreted as associated with the ground states of the theory, with solitons interpolating between them, the interpretation of the matrix N_a is that it displays the flow of solitons (of a certain type[14]) between these ground states. For instance, when an ADE minimal $N = 2$ theory (or its topological partner) is perturbed by the least relevant chiral field, one of the N_a turns out to be the adjacency matrix of the corresponding ADE Dynkin diagram! In other words, the normalized chiral algebra M for $t_{\text{least rel.}} \neq 0$ is nothing else than the Pasquier algebra of the ADE diagram. That observation had been originally made by Lerche and Warner[16]. All this seems to extend to other non ADE theories.

Clearly a good and systematic explanation of these curious facts is badly missing.

4. Topological field theories

4.1. Dubrovin's solutions as restrictions of the ADE ones

Recently Dubrovin [17] has found a class of solutions to the Witten-Dijkgraaf-

[2] The reader is referred to [3] for a display of these graphs

Verlinde[2] equations mentioned above, associated with finite Coxeter groups. Let's recall that Coxeter groups are linear groups generated by reflections in a real Euclidean space, and that finite Coxeter groups are classified [18]. Beside the Weyl groups of the simple Lie algebras, A_p, B_p, C_p, (the two latter Coxeter groups being identical), D_p, E_6, E_7, E_8, F_4 and G_2, there are the groups H_3 and H_4 of reflections of the regular icosaedron and of a regular 4-dimensional polytope, and the infinite series $I_2(k)$ of the reflection groups of the regular k-gones in the plane. In Dubrovin's work, the homogeneity degrees of the variables t_i and of F are respectively $1-(d_i-2)/h$ and $2+2/h$ where h is the Coxeter number of the group G and d_i are the degrees of the G invariant polynomials in the coordinates of V. This suggests to label the t parameters and their conjugate field in (3.3) by the degree -2 of these invariant polynomials $i = d_i - 2$: hence, 0 labels the identity, $h-2$ the field of higher charge or weight. The solutions of ADE type that Dubrovin finds are nothing else than the ADE TFT's discussed in sect. 3. As for the others, it has been shown in [19] they may be obtained as *restrictions* of the former, *i.e.* their C algebra is a subalgebra of some specialization of a C algebra of ADE type. Moreover, one shows that those are all the possible such restrictions, subject to the condition that the subalgebra contains the fields of lowest (0) and highest $(h-2)$ labels : the former is the identity operator and the latter its dual in the $N=2$ sense that $\eta_{0,h-2}=1$. This supports a conjecture of Dubrovin that his Coxeter solutions are the only ones satisfying the condition

$$0 < \text{degree}\,(F) - 2 \leq \text{ degree}\,(t_.) \leq 1 \ . \tag{4.1}$$

4.2. Chebishev specialization

As we have seen in sect. 3, the specialization to all $t_i = 0$ but $t_{h-2} = 1$ of the C algebra has several nice properties, in the ADE cases. It is "normalizable" and its normal form is the Pasquier algebra of ADE type: this means that the ψ's that diagonalize it are the eigenvectors of the adjacency (or of the Cartan) matrix of ADE type. In the simplest case of the A_{k+1} topological theory, this specialization reproduces the fusion algebra of $\widehat{su(2)}_k$ that admits a polynomial representation in terms of Chebishev polynomials. This specialization is thus called in general the Chebishev specialization.

Does the Chebishev specialization of the other, non ADE, solutions enjoy similar properties? One shows [19] that (i) for all the $B_n, F_4, G_2, H_3, H_4, I_2$ cases, the C matrices may again be brought to a normal form by a diagonal change of basis; (ii) the normal form M_i of C_i has only non negative entries; (iii) in contrast with the ADE cases, the matrices of the dual algebra are no longer all with integral entries; the dual algebra, however, is generated by the identity matrix and the matrix N_f

$$(N_f)_a{}^a = 0 \tag{4.2}$$

$$(N_f)_a{}^b = 2\cos\frac{\pi}{m(a,b)} \qquad a \neq b\ ,$$

where the integer $m(a,b) = m(b,a)$ takes a value different from 2 or 3 (hence $N_a{}^b \neq 0, 1$) only for one pair of vertices (a,b); see Table I below. These decorated, so-called Coxeter graphs are the unique graphs such that the associated N_f matrix of (4.2) has all its eigenvalues less than 2 [20]. It would be interesting to see directly why the dual algebra requires this property.

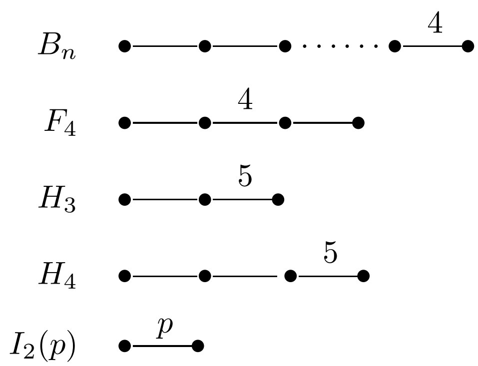

Table I : Coxeter graphs
The well known ADE diagrams are not represented here; neither is $G_2 = I_2(6)$; $m(a,b) = m(b,a) = 3$ unless otherwise specified above the edge (a,b).

Let us finally return to a point mentionned above and discussed earlier [16]. It is quite curious that this Chebishev specialization of a perturbed chiral algebra (hence relative to a *massive* theory) yields a fusion algebra (in the A case) or at least (in the D, E cases) an algebra related to the operator product expansion of a conformal (hence *massless*) field theory, as discussed in sect 2.4. Moreover, in view of Dubrovin's results, it is quite intriguing to see that (projections on spin zero fields of) consistent operator algebras of $\widehat{su}(2)$ theories (containing the operator of largest Coxeter exponent) are classified by Coxeter groups. A direct and general way of establishing this property could provide a new route to the general program of classification of conformal field theories.

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