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# STRONG COUPLING EXPANSION OF LARGE-$N$ QCD AND SURFACES*

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The strong coupling expansion of large-$N$ lattice QCD and its interpretation in terms of surfaces are reexamined. The relevant class of surfaces is defined and precise diagrammatic rules given.

## 1. Introduction

Since the original paper of Wilson [1], the possible interpretation of lattice gauge theories as a model of strings has been an attractive idea. By analogy with the continuum theory [2], it was suggested that the large-$N$ limit of U($N$) gauge theory [3] was most appropriate for exhibiting a string behaviour. It was first believed that in the strong coupling expansion only noninteracting planar closed surfaces survive in that limit [4]. The fixed time slices of such surfaces would then describe the evolution of free strings. This point was then questioned by Weingarten [5], who exhibited a counterexample. The point was also discussed in refs. [6, 7], but although it was generally believed that, properly interpreted, the relevant diagrams could be seen as planar (see e.g. ref. [8]), no general discussion and proof existed. In a slightly different framework, Foerster [6] showed that loop equations [9] may generate complicated topologies.

More recently, Drouffe and one of us have characterized the relevant strong coupling diagrams in the $N\to\infty$ limit (appendix of ref. [10]). We used a cumulant expansion, but the geometrical interpretation in terms of surfaces was lacking. On the other hand, Kazakov [11] discussed how group integrations in U($N$), $N\to\infty$, give a set of rules for contracting plaquettes and build up planar, connected surfaces. However, his prescription for computing the free energy and the weight attached to some contributions was not totally clear to us. Indeed, the main difficulty of the problem lies in a proper evaluation of the free energy $F$. As $F$ is of order $N^2$ for large $N$, arbitrary powers of $N$ appear in the partition function, and it is only for

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a quantity like $F$ that the concept of dominant diagrams makes sense. The diagrammatic rules for deriving the free energy in the strong coupling expansion are, however, not simple: in general, it is not correct to restrict oneself to connected diagrams because of excluded volume effects [10]. In the most commonly used character expansion, the $O(N^2)$ behavior of the free energy is not apparent and it results from cancellations within subsets of diagrams [12, 10]. Moreover that formalism, with arbitrary representation assignments to plaquettes, is not appropriate for an interpretation in terms of surfaces. In the cumulant expansion of ref. [10] the exponentiation of connected graphs takes place, but the graphs are abstract graphs expressing the binding of plaquettes and links, and do not seem easy to interpret in terms of surfaces made of plaquettes. On the other hand, Kazakov introduces a graphical notation for the group integral and claims that exponentiation of these diagrams takes place. It is the purpose of this note to bridge between the two approaches. One may summarize the results in the following way: in the large-$N$ limit of lattice QCD with Wilson action, the total free energy $F = \ln Z$ is given by

$$\frac{1}{N^2} F = \sum_{\substack{\text{surfaces } S \\ \text{closed, connected,} \\ \text{orientable, planar}}} \beta^{|S|} \frac{1}{k} \prod_{\ell} f^{(\ell)} . \tag{1.1}$$

Here, "surface" means a set of oriented plaquettes, together with a set of prescriptions to sew the plaquettes along common links: plaquettes may be joined pairwise along a link, or with new types of contractions, depicted by Kazakov as saddles (figs. 1b, c), that we shall call cyclic. The surface may be self-intersecting, and the various sheets that cross along one link may either be independent (fig. 1d) or exchange tubes (see figs. 1e,f). The whole pattern including the tubes must be closed, connected and "planar", i.e. of genus 0. Notice that a plaquette may be occupied an arbitrary number of times; therefore $|S|$ denotes the number of plaquettes counted with their multiplicity. The surface is weighted by a product of factors attached to each link; the factors depend on the pattern of contractions of plaquettes incident on that link. For example, $f = 1$ if $2n$ plaquettes are pairwise contracted, and not connected by a tube (fig. 1a, d or d'). Kazakov [11] has given the weight attached to a single cyclic contraction. In general, we shall see that these weights are given by the cumulant expansion of the one-link integral

$$\int DU \exp N \operatorname{tr}(UJ + U^{\dagger}J^{\dagger}) = \exp[N^2 W(JJ^{\dagger})] , \tag{1.2}$$

i.e. by the coefficients of the expansion of $W$ in terms of powers of the moments $\rho_n = (1/N) \operatorname{tr}(JJ^{\dagger})^n$. These coefficients have recently been computed explicitly [13] in a work inspired by Kazakov's method. Finally, in eq. (1.1), there may appear a symmetry factor $1/k$, when the whole surface is invariant by a permutation group of its links, plaquettes and contractions, of order $k$.

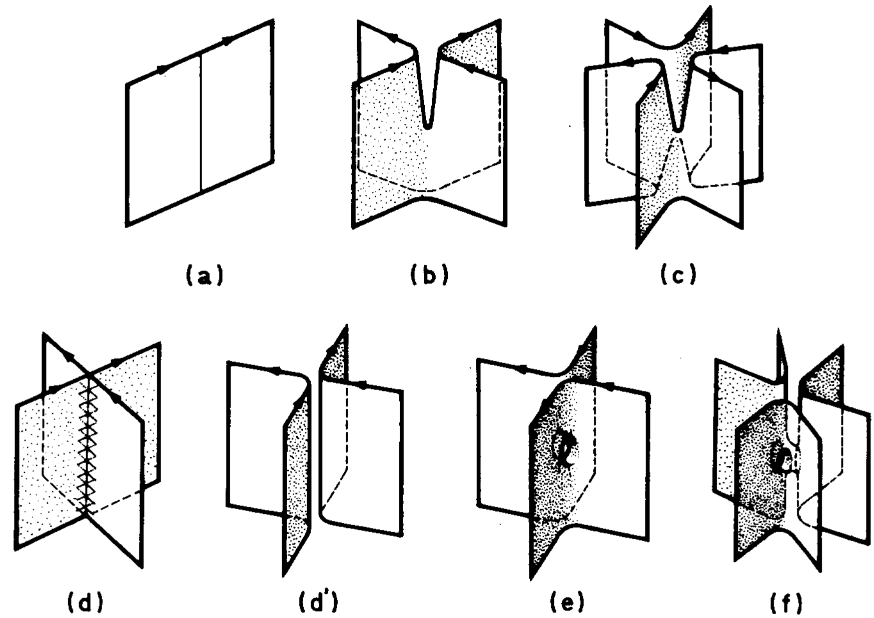


Fig. 1. Examples of contractions of plaquettes incident on a link: (a) trivial contraction; (b), (c) saddle or cyclic contraction; (d), (d') non-interacting pairwise contractions; (e), (f) tubes. Arrows indicate the relative orientation of plaquettes. The general configuration may include any superposition of all such contractions.

The logic of the derivation of eq. (1.1) is as follows:

(i) We first set up a diagrammatic expansion of the free energy: the graphs are abstract entities which encode the binding of plaquettes. This step is valid for any gauge group.

(ii) These graphs are transformed by the introduction of double lines as in ref. [2]: this is convenient for discussing the large-$N$ limit and leads to planar abstract surfaces.

(iii) The last step involves an interpretation of these abstract surfaces as real lattice "surfaces" in the sense defined above.

Before embarking on the diagrammatics, let us recall the issues at stake. First, we want to clarify a long-standing conjecture on the connection between the strong coupling expansion and surfaces, and prove eq. (1.1). Secondly, and more ambitiously, one would like to use this representation in terms of surfaces as a statistical theory of surfaces. A lot of work has been done recently on the statistics and critical properties of random surfaces [14, 15]. All these works consider free surfaces (planar or not), without self-interactions; as the universality properties of such a system are not fully understood, it would be interesting to know what are the precise classes of random surfaces and their self-interactions, if any, which reproduce the long-distance physics of large-$N$ QCD. We shall briefly return to this point in the last section of this paper and see that although the above representation is not quite appropriate, it is suggestive.

## 2. Cumulant expansion

We first recall and develop the results sketched in the appendix of ref. [10]. The U($N$) action under study is the Wilson plaquette action

$$S=\sum_{\text{plaquettes}} S(U_{\text{p}})=\sum_{\text{p}} \beta N \operatorname{tr}(U_{\text{p}}+U_{\text{p}}^{\dagger})\,. \tag{2.1}$$

The partition function is then reexpressed in terms of the external field functional:

$$\begin{aligned} Z&=\int \prod_{\ell} DU_{\ell} \prod_{\text{p}} \mathrm{e}^{S(U_{\text{p}})} \qquad (2.2)\\ &=\prod_{\text{p}} \mathrm{e}^{S((1/N^4)(\partial/\partial J_{\text{p}}))} \prod_{\ell} \int DU_{\ell} \exp[N \operatorname{tr}(U_{\ell}J_{\ell}+U_{\ell}^{\dagger}J_{\ell}^{\dagger})]|_{J=J^{\dagger}} \end{aligned}$$

where

$$\operatorname{tr}\left(\frac{1}{N^4}\frac{\partial}{\partial J_{\text{p}}}\right)\equiv\frac{1}{N^4}\sum_{\alpha\beta\gamma\delta}\frac{\partial}{\partial(J_{\ell'})_{\alpha\beta}}\frac{\partial}{\partial(J_{\ell})_{\beta\gamma}}\frac{\partial}{\partial(J_{\ell'''}^{\dagger})_{\gamma\delta}}\frac{\partial}{\partial(J_{\ell''}^{\dagger})_{\delta\alpha}}$$

if

$$\operatorname{tr} U_{\text{p}}=\operatorname{tr}(U_{\ell}U_{\ell'}U_{\ell''}^{\dagger}U_{\ell'''}^{\dagger})\,.$$

The external-field functional

$$\mathrm{e}^{N^2 W(JJ^{\dagger})}=I(JJ^{\dagger})=\int DU\, \mathrm{e}^{N\operatorname{tr}(UJ+U^{\dagger}J^{\dagger})} \tag{2.3}$$

has been extensively studied for finite $N$ and in the large-$N$ limit [16]. For our present purpose, it is sufficient to say that, for small $JJ^{\dagger}$, and $N\to\infty$,

$$W(JJ^{\dagger})=\sum_{n=1}^{\infty}\sum_{\substack{\alpha_k\geqslant 0\\ \sum k\alpha_k=n}} W_{\alpha}\left[\frac{1}{N}\operatorname{tr}(JJ^{\dagger})\right]^{\alpha_1}\left[\frac{1}{N}\operatorname{tr}(JJ^{\dagger})^2\right]^{\alpha_2}\cdots\left[\frac{1}{N}\operatorname{tr}(JJ^{\dagger})^n\right]^{\alpha_n}, \tag{2.4}$$

where the second sum runs over the partitions $\alpha$ of the integer $n$. A general expression of $W_{\alpha}$ has been derived recently [13]:

$$W_{\alpha}=(-1)^n\frac{(2n+\sum\alpha_k-3)!}{(2n)!}\prod_{k=1}^{N}\left(-\frac{(2k)!}{(k!)^2}\right)^{\alpha_k}\frac{1}{\alpha_k!}\,. \tag{2.5}$$

$W(JJ^{\dagger})$ is the generating function of the "connected integrals":

$$\begin{aligned}\left(\int DUU_{i_1j_1}\cdots U_{i_nj_n}U_{k_1l_1}^{\dagger}\cdots U_{k_nl_n}^{\dagger}\right)_{\text{conn.}}&\equiv N^{2-2n}\frac{\partial^{2n}W}{\partial J_{j_1i_1}\cdots\partial J_{l_nk_n}^{\dagger}}\\ &=\sum_{\rho,\sigma} N^{2-2n-C_{\rho}}\frac{n!\,W_{[\rho]}}{n_{[\rho]}}\delta_{i_1l_{\rho\sigma1}}\cdots\delta_{i_1l_{\rho\sigma n}}\delta_{j_1k_{\sigma1}}\cdots\delta_{j_nk_{\sigma n}}\,,\end{aligned} \tag{2.6}$$

where now the sum runs over the permutations $\rho$, $\sigma$ of the symmetric group $\Sigma_n$; $[\rho]$ denotes the class of the element $\rho$ characterized by its cyclic structure

$$[\rho]=[1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n}],$$

$n_{[\rho]}$ is the number of elements in the class $[\rho]$:

$$n_{[\rho]}=\frac{n!}{1^{\alpha_1}\alpha_1!2^{\alpha_2}\alpha_2!\cdots}. \tag{2.7}$$

$c_{[\rho]}=\Sigma_k\alpha_k$ is the total number of cycles in class $[\rho]$ and $W_{[\rho]}=W_\alpha$ is given in eq. (2.5).

The advantage of this formalism is that it enables us to compute the free energy in a well-defined way. According to the general theory of diagrammatic expansions ([10] and further references therein), the free energy is the sum of contributions of connected diagrams made of vertices coming from

$$\beta N^{-3}\,\mathrm{tr}\frac{\partial}{\partial J_\mathrm{p}} \quad \text{or} \quad \beta N^{-3}\,\mathrm{tr}\frac{\partial}{\partial J_\mathrm{p}^\dagger},$$

and of contractions (named "$\alpha$-sites" in [10]) read off eq. (2.6) (fig. 2). We recall that arbitrary powers of $S(U_\mathrm{p})$ are generated in the expansion of $\exp[S(U_\mathrm{p})]$, and hence several distinct vertices may refer to the same plaquette on the lattice. The links emanating from the plaquettes (i.e. the derivatives $\partial/\partial J_\ell$) must be contracted in all possible ways using the cumulants of (2.4).

Along a given link, the same set of contractions may come from several origins. For example, if six plaquettes incident on a link are contracted according to

$$\frac{1}{N}\mathrm{tr}\,(JJ^\dagger)\frac{1}{N}\mathrm{tr}\,(JJ^\dagger)^2,$$

such a term may come either from the product of two distinct cumulants of order 1 and 2:

$$N^2W_{[1]}\frac{1}{N}\mathrm{tr}\,(JJ^\dagger)\times N^2W_{[2]}\frac{1}{N}\mathrm{tr}\,(JJ^\dagger)^2,$$

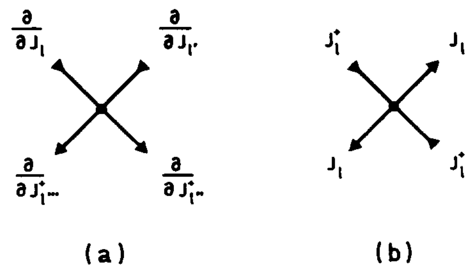


Fig. 2. (a) Graphical representation of a vertex associated with a plaquette $\mathrm{tr}\,(U_\ell U_{\ell'}U^\dagger_{\ell''}U^\dagger_{\ell'''})$. (b) Graphical representation of a link contraction associated with the term $\mathrm{tr}\,(J_\ell J_\ell^\dagger)^2$.

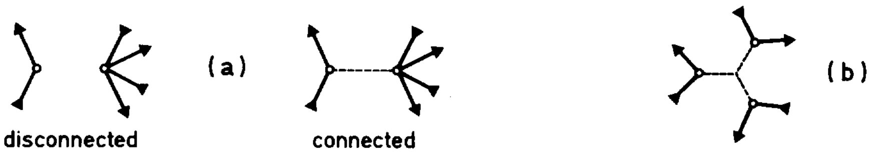


Fig. 3. (a) The two contributions to $(1/N)\,\mathrm{tr}\,(JJ^\dagger)(1/N)\,\mathrm{tr}\,(JJ^\dagger)^2$. (b) A connected contraction $(1/N)\,\mathrm{tr}\,(JJ^\dagger)^3$.

or from a single cumulant of order 3:

$$N^2 W_{[1,2]}\left(\frac{1}{N}\,\mathrm{tr}\,(JJ^\dagger)\right)\left(\frac{1}{N}\,\mathrm{tr}\,(JJ^\dagger)^2\right).$$

The former will be referred to as a disconnected contraction, the latter as a connected one, and they will be denoted graphically as in fig. 3. These two contributions come with different powers of $N$. In general, the connected contractions will be suppressed by inverse powers of $N^2$. It would seem that in the large-$N$ limit, only disconnected contractions contribute: this is not true, however, because of the connectivity requirement on diagrams contributing to $F$. The dotted lines just introduced for connected contractions contribute to the connectivity (see fig. 4 for an example).

To complete the prescriptions of this diagrammatic expansion, we still have to answer two questions:

(i) What is the weight attached to a graph?

If a certain cumulant $t_\rho = \prod_k [(1/N)\,\mathrm{tr}\,(JJ^\dagger)^k]^{\alpha_k}$ is used on a link, it gives rise to $n!\times n_{[\rho]}$ different contractions (corresponding to $\rho\in[\rho]$, any $\sigma$ in eq. (2.6)). Each

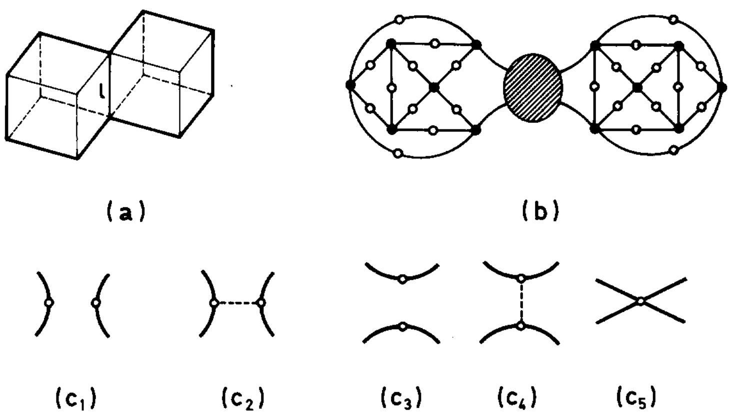


Fig. 4. (a) Diagram made of two cubes sharing one link. (b) Its representation in terms of vertices and contractions; the contractions along $\ell$ (shaded blob) have to be chosen from the set depicted in (c): $c_1$ leads to a disconnected diagram and is discarded in the computation of the free energy, $c_4$ gives a sub-dominant diagram as $N\to\infty$, $c_2$, $c_3$, $c_5$ survive in that limit.

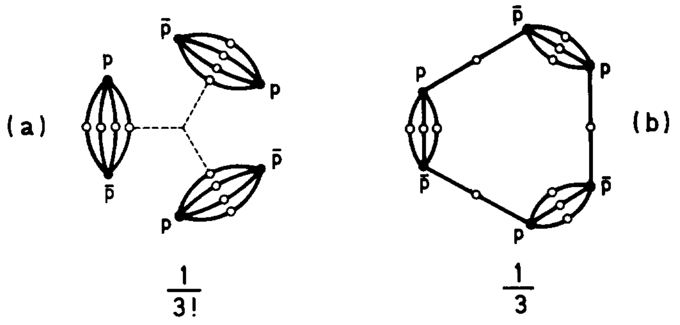


Fig. 5. Example of symmetry factors. Both graphs are contributions to $\int DU(\mathrm{tr}\, U_{\mathrm{p}}\, \mathrm{tr}\, U_{\mathrm{p}}^{\dagger})^3$.

such contraction is weighted by the integer

$$f_{[\rho]}=\frac{n!}{n_{[\rho]}}W_{[\rho]}\,. \tag{2.8}$$

Notice, however, that all these contractions do not necessarily lead to dominant diagrams in the large-$N$ limit (see below). If a disconnected contraction is used on a link, it is weighted by the corresponding product of $f_{[\rho]}$. Therefore each graph carries a factor $\prod_{\ell} f^{(\ell)}$ product over all its links of the factors $f^{(\ell)}_{[\rho]}$. These factors in turn depend on the way the plaquettes are sewed along these links. A given graph must then be embedded in all possible ways on the lattice. That is, each plaquette vertex in the graph must be associated with a lattice plaquette. Several plaquette vertices may be assigned to the same lattice plaquette. Hence, each graph may be embedded in several ways. Some of these embeddings carry an extra symmetry factor $1/k$, which is a remnant of the factorials in the expansion of $\mathrm{e}^S$: as usual, $k$ is the order of the symmetry group of the graph with its embedding assignments. A few examples are displayed in fig. 5. Notice that in contrast with the case studied in refs. [17, 18], $k$ may not necessarily be seen as the number of wrappings of a simpler surface. In the case of [17, 18], the only possible symmetry of a connected diagram is cyclic. Here, there are other types of contractions which may ensure connectivity and lead to larger symmetries: see fig. 5a for an example.

(ii) What are the relevant diagrams in the large-$N$ limit?

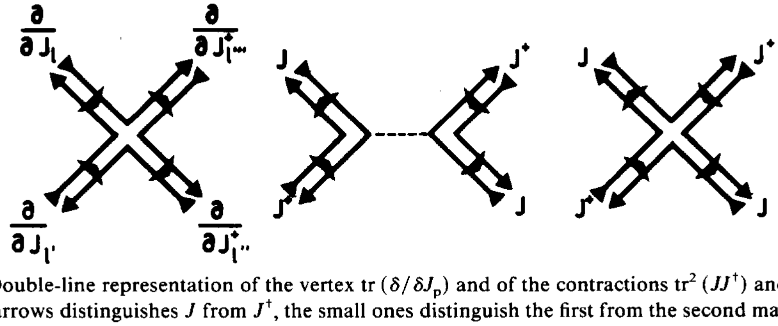


Fig. 6. Double-line representation of the vertex $\mathrm{tr}\,(\delta/\delta J_{\mathrm{p}})$ and of the contractions $\mathrm{tr}^2\,(JJ^{\dagger})$ and $\mathrm{tr}\,(JJ^{\dagger})^2$. The big arrows distinguishes $J$ from $J^{\dagger}$, the small ones distinguish the first from the second matrix index.

The counting of powers of $N$ is most easily performed if double lines are introduced to represent the conservation and contractions of matrix indices [2]: this is illustrated for a few simple vertices and cumulants in fig. 6. If the diagram under study has $P$ vertices (plaquettes) and $v_\alpha$ connected contractions of the type $N^2(\mathrm{tr}\,(JJ^\dagger/N)^{\alpha_1}\cdots(\mathrm{tr}\,(JJ^\dagger)^n/N)^{\alpha_n}$, and if there are $f$ closed index loops, the power of $N$ is

$$\# = -3P+\sum_\alpha v_\alpha \sum_k (2-\alpha_k)+f. \tag{2.9}$$

The number $l$ of double lines joining vertices to connected cumulants is

$$l=4P. \tag{2.10}$$

One then considers the closed surface spanned by the index loops; more precisely $f$ disks filling the interior of each index loop are pasted along the $l$ double lines. The resulting object is a closed surface which may have $c$ connected parts, connected only by the dotted lines introduced above, and its genus is given by Euler's formula:

$$2c-g=f-l+P+\sum_\alpha v_\alpha \sum_k \alpha_k. \tag{2.11}$$

Hence

$$\begin{aligned}\# &= 2c-g-2\sum_\alpha v_\alpha\left(\sum \alpha_k-1\right)\\ &\leqslant 2-g\\ &\leqslant 2.\end{aligned} \tag{2.12}$$

The first inequality expresses that the $c$-connectivity of the surface results form the multi-trace cumulants, more precisely that the number $c-1$ of cuts of dotted lines to separate the surface into its $c$ connected parts cannot be larger than the total number $\sum v_\alpha(\sum \alpha_k-1)$ of these dotted lines. This first inequality (2.12) is thus saturated when the original diagram is minimally connected, i.e. when erasing any dotted line makes it disconnected. In particular, connected contractions must not be used "inside" each connected part, but only along the links where two connected parts touch each other. The second inequality (2.12) tells us that each connected part of the surface must have a genus 0, i.e. have the topology of a sphere, to contribute to the leading ($\mathrm{O}(N^2)$) order of the free energy. Therefore, the surfaces that survive in the large-$N$ limit may be seen as trees of components with the topology of a sphere connected by the dotted lines: loosely speaking, each spherical component has contributed $N^2$, each dotted line $1/N^2$. We may summarize the digrammatic rules obtained so far by an equation of the form (1.1), where the sum runs over minimally connected planar abstract graphs.

### 3. Interpretation in terms of surfaces

We now want to translate this planarity of an abstract surface in terms of a geometrical surface made of plaquettes. It turns out that the previous representation with double lines is again very useful and gives a sort of dual picture of the surface. The duality between the connectivity graph and the geometrical diagram had been already noticed in ref. [17] (see also [18]). Let us consider one particular connected part of the diagram, it has the topology of a sphere; assume first that all its cumulants are the trivial contractions $(1/N)\,\mathrm{tr}\,(JJ^{\dagger})$: then this part of the diagram is nothing but the dual of the corresponding set of plaquettes: vertices correspond to plaquettes, links connecting adjacent plaquettes are dual to links along which these plaquettes join, and the previous faces (filled with disks) are dual to sites of the original lattice. The set of plaquettes may therefore be identified after excision of a small neighbourhood of each lattice site with the surface *interior* to the double lines; this is exemplified in fig. 7a. Consider now the effect of a nontrivial cumulant; as shown above, for $N\to\infty$, only single trace, i.e. cyclic cumulants $\mathrm{tr}\,(JJ^{\dagger})^n$, must be used within each connected part. The geometric interpretation of such a contraction requires the introduction of a new type of "half-bond" contraction between plaquettes [11]: suppose that the $2n$ oriented plaquettes incident on that link are labelled by an integer running between 1 and $2n$, even or odd according to their relative orientation with respect to the link. Then the contraction identifies the lower halves of the link on plaquettes $P_{2k-1}$ and $P_{2k}$, and its upper halves on plaquettes $P_{2k}$ and $P_{2k+1}$ (for $k=1,\ldots,n$, and $P_{2n+1}\equiv P_1$) (see fig. 1b, c). These configurations are reminiscent of the "topology switches" which occur in loop equations [6, 9]. With the introduction of this new type of saddle-like contraction of plaquettes, it is easy to see that the double-line picture of the diagram is nothing but a flattening of the surface, the region inside the double lines representing again the plaquettes, and a small neighbourhood of each site on the lattice being excised; this case is illustrated in fig. 7b. Finally the dotted lines that make the diagram connected may be regarded as thin tubes connecting the corresponding sheets of the surface (figs. 1e,f). It is clear that the genus of the surfaces obtained in this way is still zero and that the class of surface that emerge from this reinterpretation and their counting are those which have been presented in sect. 1 and eq. (1.1). In particular, notice that in listing all the inequivalent surfaces, the relative location of tubes and/or saddles along a link must be regarded as irrelevant. Notice also that the prescription of summing over configurations containing tubes, which had been overlooked in previous works on the subject, is essential in reproducing the correct $\beta$-expansion. For example, it ensures that the sum of all contributions to $F$ depicted in fig. 8 vanishes.

Representation (1.1) for the free energy leads by differentiation with respect to $\beta$ to a similar representation for the internal energy $\langle(1/N)\,\mathrm{tr}\,U_{\mathrm{p}}\rangle$, and generalizes to the expectation value of Wilson loops: the surfaces must have the loop as a boundary.

It is also interesting to wonder about the convergence properties of these surface sums. Clearly the sum (1.1) ordered by increasing area $|S|$ is convergent since it reproduces the ordinary strong coupling expansion [19]. Is it also absolutely conver-

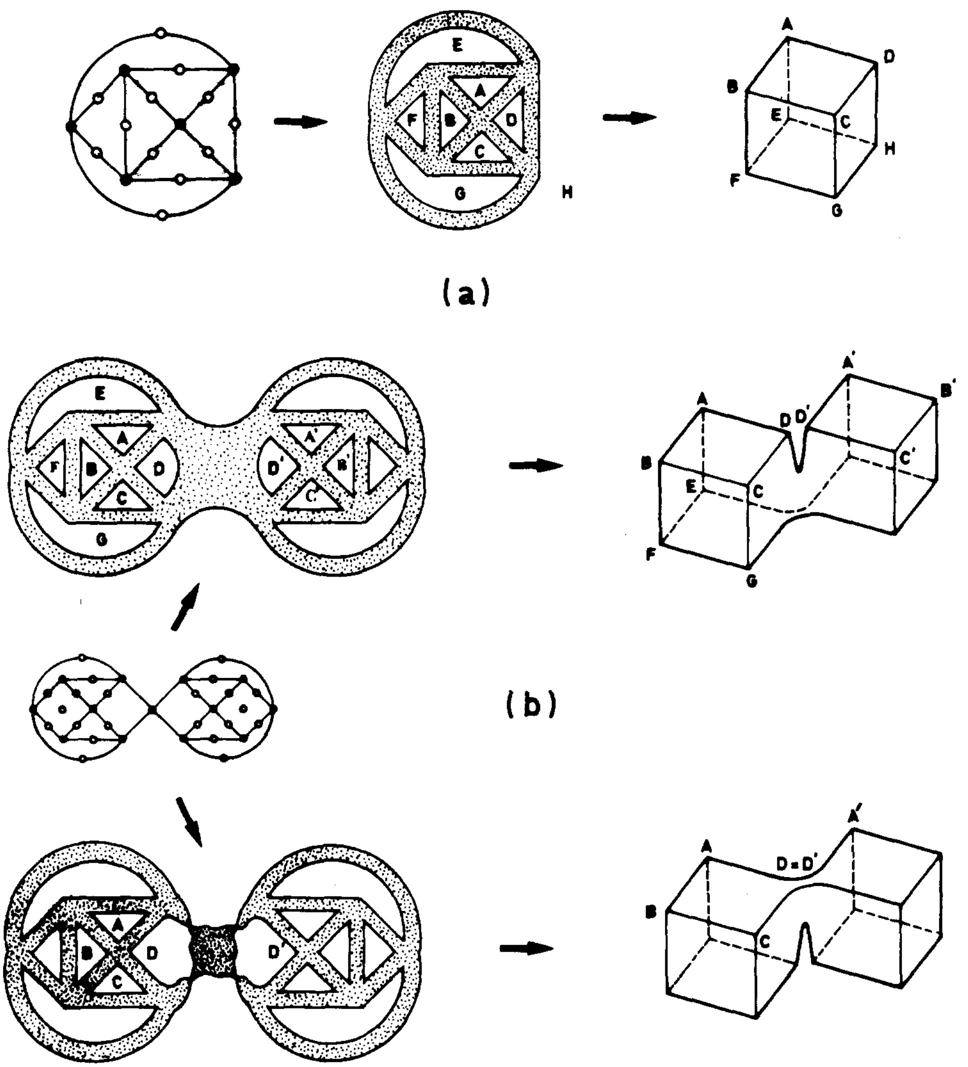


Fig. 7. From the abstract graph to its double-line representation to the geometrical diagram: the surface of the latter is in one-to-one correspondence with the interior of the double lines. (a) Simple cube, (b) double cube with a saddle contraction. In the latter case, two contractions are possible: although the second one is represented here with twists, both correspond to orientable genus-0 surfaces.

gent? This is likely, because the graphs of sect. 2 are restricted to be planar [20]; the growth of some coefficients in eq. (2.5), however, makes the argument nontrivial and would require a more detailed analysis.

So far, we have been concerned with the leading terms in the large-$N$ limit. Similar techniques may be used to study $1/N^2$ corrections. From the analysis of the sect. 2, it must be clear that corrections come from three different origins:

(i) the cumulant coefficients $W_\alpha$ have subdominant terms beyond (2.5);

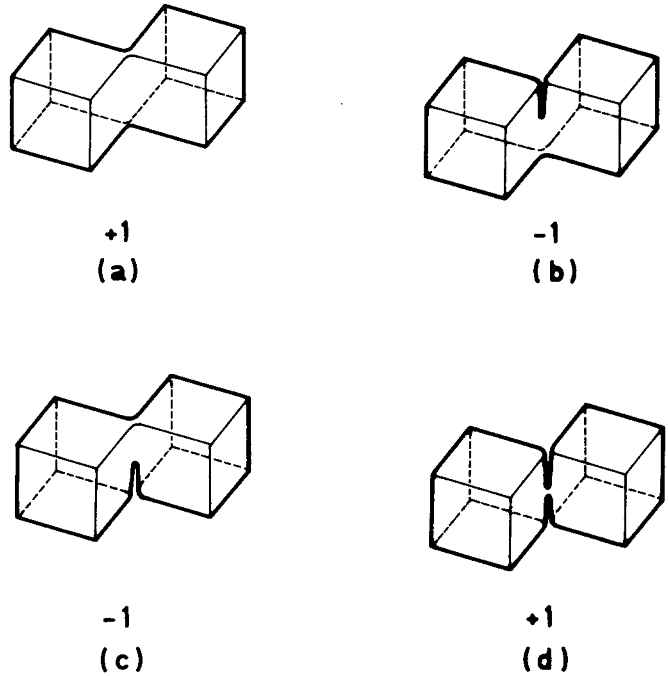


Fig. 8. The four surfaces corresponding to the diagram of fig. 4a: surface (a) corresponds to the contraction $c_3$ of fig. 4, (b) and (c) to $c_5$, and (d) to $c_2$.

(ii) nonminimal use of connected cumulants gives extra powers of $1/N^2$;

(iii) each handle in a connected component gives a $1/N^2$ factor.

Kazakov's method for generating the $W'_\alpha$ suggests that subdominant terms may also be given a geometrical representation in terms of higher-genus surfaces. For example the $1/N^2$ correction to the coefficient of the cyclic contraction tr $(J\bar{J})^n$ is represented by a handle on top of a saddle, etc. As corrections of type (ii) and (iii) may also be interpreted in terms of higher-genus surfaces, it seems that an expression similar to (1.1) holds for the $N^{-2k}$ corrections to $F/N^2$ in terms of genus-$k$ surfaces. That the U($N$) theory considered here (with Wilson's action) admits a $1/N^2$ expansion should be compared with Aizenman and Fröhlich's recent remarks [22] about the possible anomalies in the $N$-dependence and the disease of such an expansion. Notice that the procedure followed here has blindly interchanged the strong coupling expansion and the large-$N$ limit. On the other hand, it seems to us that with Wilson's action, the actual realization of Aizenman–Fröhlich anomalies necessarily involves a number of plaquettes of order $N$ (or in the character expansion language, high representations) and hence are exponentially suppressed as $\beta^N$ and invisible in the present method. At any rate, it is likely that the $1/N^2$ expansion if it exists, is asymptotic [22].

## 4. Discussion and conclusion

The reinterpretation of U($\infty$) lattice gauge theory in terms of surfaces presented in this paper is slightly awkward for several reasons:

(i) It is restricted to the strong coupling regime. Four-dimensional U($N$) theories ($N > 2$) are known to experience a first-order transition: this prevents one from using the present surface picture for the study of the continuum, zero coupling limit. See, however, refs. [11, 18] for suggestions on that problem.

(ii) It is certainly a very cumbersome way of deriving the $\beta$-expansion: for a given geometric configuration, there is in general a huge number of associated surfaces. Because of the signs in eq. (2.5), there are many cancellations. In practice, it is much easier to use the character expansion and rearrange it to work out the $N \to \infty$ limit [12, 10].

Moreover, the cumulant formalism used above, which produced the contractions of plaquettes, does not incorporate in a natural way the possibility of changes of variables and choices of gauge that simplify tremendously actual group integrals.

(iii) A typical example of such unnatural cancellations is the case of "spikes". The previous rules allowed a plaquette occupied several times to be connected with itself along (at least) one link which is not shared by any other plaquettes. However, the sum of such contributions to $F$ vanishes, as is clear if one returns to the original integral. Therefore, spikes may be forbidden in the enumeration of surfaces. But of course, this leaves the possibility that the surface "backtracks" when it reaches a link common to several distinct plaquettes.

(iv) The previous rules defined a rather unappealing theory of surfaces. First there are signs, related to the local structure of the surface, namely to the signature of the permutation of contractions along the half-bonds. These signs ruin the hope of considering this representation as a statistical theory of surfaces. On the other hand, it would be nice to abstract some natural form of interaction of a continuous surface from this study, corresponding for example to the cyclic half-bond contractions. In this direction, Kazakov [11] had proposed to reinterpret the cyclic contractions along half-bonds as reflecting the internal curvature of the surface; however, the consistency of this picture is not clear: for example, the patterns of fig. 9 which only differ by a change of scale, come with very different (opposite!) weights*.

Clearly, all these difficulties and objections make it desirable to perform some kind of resummation of this surface expansion. There has been a recent attempt by Kostov [18], using a rather different approach. The surfaces involved may again contain cyclic contractions along half-bonds, but no longer any tube: one still has a representation of the form (1.1) in terms of this new class of surfaces, but as the price to be paid, the explicit numbers $f_p$ in the weight $\beta^{|S|} \prod_\ell f_p^{(\ell)}$ are replaced by an infinite number of functions of $\beta, f_p(\beta)$, to be self-consistently determined. For

* We are indebted to J.M. Drouffe for this example.

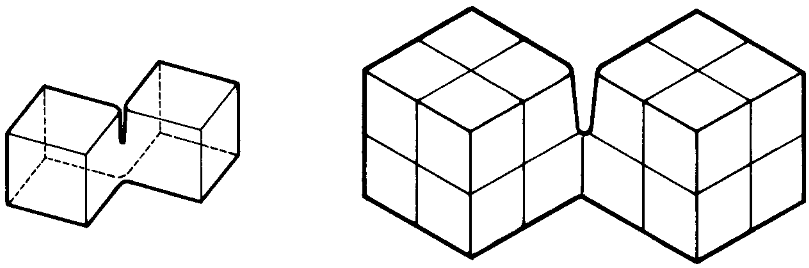

Fig. 9. Two surfaces that differ only by a change of scale: (a) is made with a cyclic contraction and comes with a factor −1; (b) is made only with trivial contractions and comes with a factor +1.

example, configuration (d) of fig. 8 does not appear in this representation, but its contribution is provided by the dressing of the link of a simple cube. However, the weights $f^{(\ell)}(\beta)$ are still not positive definite, and therefore, this representation is still not appropriate for a statistical interpretation. The problem of building a statistical theory of (planar?) surfaces from U($\infty$) lattice gauge theory still remains a challenge.

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