

Renormalization of non-Abelian gauge theories in a background-field gauge.

II. Gauge-invariant operators

H. Kluberg-Stern* and J. B. Zuber†

Service de Physique Théorique, Centre d'Etudes Nucléaires de Saclay, BP No.2-91190 Gif-sur-Yvette, France

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The Ward-Slavnov identities satisfied by the Green's functions with one insertion of a gauge-invariant operator are studied in the background-field gauge. As a consequence, the counterterms for a given gauge-invariant operator must satisfy a system of equations, whose general solution is found in the simplest cases of operators of low dimension ($d \leq 6$) or low twist ($\tau \leq 3$) and conjectured in the general case. It then follows that the renormalized Green's functions satisfy the same Ward identities as the bare, regularized ones. We deduce a definite prescription for the practical calculation of the anomalous dimensions of gauge-invariant operators which do not vanish in the classical limit: this prescription is formulated in the background gauge or in the usual Fermi-type gauge.

I. INTRODUCTION

The quantization procedure for Yang-Mills theories breaks explicitly gauge invariance and thus one must check that the computed value of any measurable quantity satisfies the requirement of gauge independence: It must be independent of the breaking term introduced for the quantization. This problem is particularly tough in the case of a nonspontaneously broken gauge theory where the S matrix probably does not exist in a perturbative sense. In this case, anomalous dimensions of gauge-invariant operators appear as observables: For example, they determine, according to Ref. 1, the deep-inelastic behavior of the structure functions relevant to leptonproduction. The computation in Ref. 1 of these anomalous dimensions to first order gave rise to various problems.^{2,3} The study of an explicit example, the operator $\tilde{F}_{\mu\nu}^2$, in Ref. 4 showed that the renormalization of an operator invariant under classical gauge transformations involves a coupling to noninvariant operators and that this mixing cannot be ignored for the computation of the gauge-independent anomalous dimension, even in the one-loop approximation. This result was obtained by algebraic manipulations which did not lead to a geometrical interpretation of the spurious non-gauge-invariant operators. Also, the necessity of a blind computation of the full renormalization matrix associated with the original gauge-invariant operator clearly seems redundant. Therefore, we have taken up the problem again with the aim of a better understanding and characterization of the spurious operators and of reducing through a simple prescription the computation of gauge-independent anomalous dimensions.

A previous study of the Green's functions in the background gauge⁵ suggests that this gauge is well suited for the study of gauge-independent quanti-

ties. Let us recall that this gauge is a generalized Fermi gauge where the gauge-fixing term depends on a parameter α and on an external vector field, the background field, and is covariant under classical gauge transformations (type-1 transformations) associated with the classical field. The quantization, as well as the renormalization,⁵ preserves the invariance under these type-1 transformations, while breaking the gauge invariance related to the quantized field. Also, the variation of the background field is simply a change of gauge for the quantized field. As a consequence, the wave-function renormalization of the classical field was found to be the coupling-constant renormalization g/g_0 . This result indicated that the background gauge is convenient for a direct computation of gauge-independent quantities such as $\beta(g)$. The results of the present study, which are summarized below, indeed confirm this idea.

The renormalization of a classical gauge-invariant operator (class-I operator) which depends on gauge fields and on matter fields and which does not vanish by virtue of the classical equations of motion in the absence of sources involves either (a) class-I operators, (b) class-IIa operators which are gauge invariant, but vanish in the classical limit because of the equations of motion in the absence of sources, or (c) class-IIb operators which are not gauge invariant and which depend both on gauge and matter fields and on Faddeev-Popov fields: A precise generic expression is given for these operators. Class-II operators are renormalized only among themselves and this block-triangular structure of the renormalization matrix leads to a natural prescription for the computation of the α -independent anomalous dimensions relative to class-I operators. This prescription is stated both for the background gauge and for the Fermi gauge. In the background gauge, the

graphs with only external legs of matter and background fields need only gauge-invariant counterterms, namely of class I and class IIa. This automatic elimination of class-IIb counterterms is the main practical advantage of the background gauge over the Fermi gauge, where, however, fewer vertices appear in the effective computation. The selection of class-I counterterms at the exclusion of class-IIa counterterms in the background gauge or of class-II counterterms in the Fermi gauge involves inspection of the structure of the spurious operators which are characterized with precision. Finally, our prescription fails for gauge-invariant operators of class-IIa for which the mixing problem seems entire: No triangular structure emerges for the renormalization matrix between class-IIa and class-IIb operators. No serious argument exists concerning this question; however, the general "mythology" says that class-IIa operators are unphysical, despite their gauge invariance.

Section II is devoted to notations and a brief reminder of results of Ref. 5: The Ward identities and the renormalization of Green's functions in the background gauge. Section III gives the derivation of our results which is based on a systematic exploitation of Ward-Slavnov identities (WI). The WI for a single insertion of a class-I operator yield a constraint equation for the possible counterterms (Sec. III A). The equation involves a linear differential operator with a vanishing square. The solution of this equation is a nontrivial cohomology problem. This algebraic problem is reduced and solved exactly in the Appendix only for a few cases (operators of dimension $d \leq 6$ or of low twist $\tau \leq 3$ or $\tau \leq 5$ depending on the symmetry of Lorentz indices). In the general case the solution is conjectured: The only possible counterterms are class-I counterterms and class-II counterterms defined by the action of the linear differential operator mentioned above on a polynomial with definite dimension, ghost number, and Lorentz covariance. These class-II operators obey WI (Sec. III B) which prevent any coupling to class-I operators by renormalization. The insertion of operators leads to contributions of diagrams with self-contractions of the variables appearing in the expression of the operators at the one-loop level; these contributions are subtracted in Sec. III C without modification of the Ward identities which hold for regularized quantities to all orders of perturbation theory.

Section III D completes the proof of WI for renormalized quantities by recursion on the number of loops. In Sec. III E the generating functional for renormalized Green's functions with a single insertion of a class-I operator or a class-II operator is shown to satisfy a WI analogous to those for a bare insertion of a class-I operator, which ex-

presses the invariance under generalized supergauge transformations (type-2 transformations); this generalized type-2 invariance is the remaining symmetry associated with the initial gauge symmetry for the quantized field after renormalization. The latter preserves trivially the initial type-1 symmetry connected with the gauge transformations of the background field. Section III F establishes the α independence of the block of the renormalization matrix relative to class-I operators. We have no proof for the α dependence of the rest of this matrix; this point is particularly puzzling in view of the existence of the gauge-invariant class-IIa operators. All previous results are extended briefly to a theory including fermions in Sec. III G. The considerations of Sec. III are illustrated at various points by the operator $\vec{F}_{\mu\nu}^2$ introduced in Ref. 4.

The reader interested only in the effective computation of α -independent anomalous dimensions may skip Sec. III, since the results of this section are summarized in Sec. IV A. The computation in the background gauge is described in Sec. IV B and involves only graphs with external legs of background and matter fields. This result was claimed by Sarkar and Strubbe³ and by Crewther,⁶ for twist-two symmetric operators, in the one-loop approximation. The counterterms of the mentioned graphs involve, however, also class-IIa operators which must be eliminated by hand; we give some clues for this operation and illustrate them by some examples. A similar work is performed for the Fermi gauge in Sec. IV C. The Fermi gauge is just a special case of the background gauge where the background field is set equal to zero. The counterterms for graphs with external legs of Yang-Mills and matter fields involve also class-IIb operators, and therefore the selection of the class-I operators is, in general, more difficult in the Fermi gauge. The case of symmetric traceless operators of twist two appears particularly simple in both gauges and this study confirms computations of Ref. 1 and of Ref. 3 and 6. Higher-order perturbations discussed in Sec. IV D reintroduce some complexity for practical calculations: The superficial divergence of a diagram can be extracted only after renormalization of all subdiagrams which need, in fact, all class-I and -II counterterms. Section IV E ends with some comments relative to the contribution of class-II operators to Wilson expansion and to the renormalization of several insertions of class-I operators.

The method of investigation was developed by Zinn-Justin.⁷ It consists of a thorough use of WI derived from supergauge transformations which were introduced by Becchi, Rouet, and Stora⁸ and extended in Ref. 4. The ingredients are functional

methods, gauge-invariant regularization, and minimal renormalization.^{9,4,5} A gauge-invariant regularization⁹ is supposed to associate an unambiguous finite value to each graph in a way which preserves the equations of motion and Ward-Slavnov identities independently of the regulating parameter. No proof was given for the existence at all orders of perturbation theory of such a regularization,⁹ especially for a massless theory including composite operators. We do not attempt to prove this assumption. The functional methods are legitimate algebraic operations,⁷ once a gauge-invariant regularization has been applied.

II. PRELIMINARY

The classical Yang-Mills Lagrangian

$$\mathcal{L}(\vec{Q}_\mu) = -\frac{1}{4}(\partial_\mu \vec{Q}_\nu - \partial_\nu \vec{Q}_\mu + g \vec{Q}_\mu \times \vec{Q}_\nu)^2(x) \quad (2.1)$$

is invariant under the following local gauge transformations:

$$\begin{aligned} \delta Q_\mu^i(x) &= D_\mu^{ij}(Q) \delta \omega^j(x) \\ &= (\partial_\mu \delta^{ij} + g f^{ijk} Q_\mu^k) \delta \omega^j(x), \end{aligned} \quad (2.2)$$

where f^{ijk} denote the structure constants of a compact semisimple Lie algebra. The \times symbol is a shortcut for

$$(\vec{A} \times \vec{B})^i = f^{ijk} A^j B^k.$$

Summation over repeated indices is assumed. The quantization of this classical Lagrangian requires the insertion of a term which breaks the gauge invariance. In the background gauge of Ref. 5, the gauge term, after a translation of the gauge field $\vec{Q}'_\mu = \vec{Q}_\mu + \vec{A}_\mu$, takes the expression $-(1/2\alpha)[D_\mu^{ij}(A) \times Q_\mu^j]^2$, where the external field \vec{A}_μ is the background field. A corresponding Faddeev-Popov term must be introduced and the full action reads

$$\begin{aligned} S' = \int d^4x & \left(\mathcal{L}(\vec{Q} + \vec{A}) - \frac{1}{2\alpha} [D_\mu^{ij}(A) Q_\mu^j]^2 \right. \\ & \left. + \bar{C}_i D_\mu^{ij}(A) D_\mu^{jk}(A + Q) C_k \right). \end{aligned} \quad (2.3)$$

The generating functional of Feynman graphs, denoted by $Z(\vec{A}_\mu, \vec{\eta}_\mu, \xi_i, \bar{\xi}_i, \vec{J}_\mu, \vec{K}, \vec{L}_\mu)$ depends, of course, on the variables $\vec{\eta}_\mu$, ξ_i , and $\bar{\xi}_i$, which are the sources of \vec{Q}_μ and of the ghosts \bar{C}_i and C_i :

$$Z(\vec{A}_\mu, \vec{\eta}_\mu, \xi_i, \bar{\xi}_i, \vec{J}_\mu, \vec{K}, \vec{L}_\mu) = \int dQ dC d\bar{C} \exp \left[i \left(S + \int d^4x (\vec{\eta}_\mu \cdot \vec{Q}_\mu + \bar{C}_i \xi_i + \bar{\xi}_i C_i) \right) \right]. \quad (2.4)$$

The modified action S contains sources \vec{J}_μ , \vec{K} , \vec{L}_μ of composite operators (see Refs. 4 and 5) introduced only for a linearization of Ward identities satisfied by the generating functional of one-particle-irreducible (1PI) graphs:

$$S = \int d^4x \left(\mathcal{L}(\vec{A}_\mu + \vec{Q}_\mu) - \frac{1}{2\alpha} [D_\mu^{ij}(A) Q_\mu^j]^2 + [J_\mu^i + \bar{C}_i D_\mu^{ij}(A)] [D_\mu^{jk}(A + Q) C^k + L_\mu^j] + g \bar{C}_i (\vec{Q}_\mu \times \vec{L}_\mu)_i + \frac{g}{2} K_i (\vec{C} \times \vec{C})_i \right). \quad (2.5)$$

The auxiliary sources \vec{J}_μ , \vec{K} , and \vec{L}_μ which depend on space-time x should be set to zero for practical calculations of counterterms. The action S is invariant under the following transformations.

(a) type-1 transformations.

$$\delta A_\mu^i(x) = D_\mu^{ij}(A) \delta \omega^j(x). \quad (2.6)$$

\vec{Q}_μ , C_i , \bar{C}_i , \vec{J}_μ , \vec{K} , \vec{L}_μ transform according to the adjoint representation.

(b) type-2 transformations^{3,4,5}.

$$\begin{aligned} \delta Q_\mu^i(x) &= [D_\mu^{ij}(A + Q) C^j + L_\mu^j](x) \delta \lambda = \frac{\delta S}{\delta J_\mu^i(x)} \delta \lambda, \\ \delta A_\mu^i(x) &= -L_\mu^i(x) \delta \lambda, \\ \delta C^i(x) &= \frac{g}{2} (C \times C)^i(x) \delta \lambda = \frac{\delta S}{\delta K^i(x)} \delta \lambda, \\ \delta \bar{C}_i(x) &= -\frac{1}{\alpha} [D_\mu^{ij}(A) Q_\mu^j](x) \delta \lambda, \end{aligned} \quad (2.7)$$

where $\delta \lambda$ is the anticommuting x -independent parameter.

The generating functional of 1PI regularized graphs, $\Gamma(\vec{A}_\mu, \vec{Q}_\mu, C_i, \bar{C}_i, \vec{J}_\mu, \vec{K}, \vec{L}_\mu)$, is defined as the Legendre transform with respect to $\vec{\eta}_\mu$, ξ_i , and $\bar{\xi}_i$ of the functional W :

$$\begin{aligned} W(\vec{A}_\mu, \vec{\eta}_\mu, \xi_i, \bar{\xi}_i, \vec{J}_\mu, \vec{K}, \vec{L}_\mu) &= i \ln Z(\vec{A}_\mu, \vec{\eta}_\mu, \xi_i, \bar{\xi}_i, \vec{J}_\mu, \vec{K}, \vec{L}_\mu), \\ \Gamma(\vec{A}_\mu, \vec{Q}_\mu, C_i, \bar{C}_i, \vec{J}_\mu, \vec{K}, \vec{L}_\mu) &+ W(\vec{A}_\mu, \vec{\eta}_\mu, \xi_i, \bar{\xi}_i, \vec{J}_\mu, \vec{K}, \vec{L}_\mu) + \int d^4x (\vec{\eta}_\mu \cdot \vec{Q}_\mu + \bar{C}_i \xi_i + \bar{\xi}_i C_i)(x) = 0, \end{aligned} \quad (2.8)$$

where

$$Q_\mu^i = -\frac{\partial W}{\partial \eta_\mu^i}, \quad C_i = -\frac{\partial W}{\partial \xi_i}, \quad \bar{C}_i = +\frac{\partial W}{\partial \bar{\xi}_i}.$$

The functional $\hat{\Gamma}$ ($\Gamma = \hat{\Gamma} - (1/2\alpha) \int d^4x [D_\mu^{ij}(A)Q_\mu^j]^2$) satisfies identities (a) and (b) which follow from the invariance of the action under type-1 and -2 transformations and identity (c), which is the equation of motion for the ghost¹⁰:

(a)

$$\mathfrak{D}_x^i \hat{\Gamma} = \left[D_\mu^{ij}(A) \frac{\partial}{\partial A_\mu^j(x)} + g f^{ijk} \left(Q_\mu^j(x) \frac{\partial}{\partial Q_\mu^k(x)} + C^j(x) \frac{\partial}{\partial C^k(x)} + \bar{C}^j(x) \frac{\partial}{\partial \bar{C}^k(x)} \right. \right. \\ \left. \left. + K^j(x) \frac{\partial}{\partial K^k(x)} + J_\mu^j(x) \frac{\partial}{\partial J_\mu^k(x)} + L_\mu^j(x) \frac{\partial}{\partial L_\mu^k(x)} \right) \right] \hat{\Gamma} = 0, \quad (2.9a)$$

(b)

$$\int d^4x \left(\frac{\partial \hat{\Gamma}}{\partial Q_\mu^i(x)} \frac{\partial \hat{\Gamma}}{\partial J_\mu^i(x)} - \frac{\partial \hat{\Gamma}}{\partial C^i(x)} \frac{\partial \hat{\Gamma}}{\partial K^i(x)} - L_\mu^i(x) \frac{\partial \hat{\Gamma}}{\partial A_\mu^i(x)} \right) = 0, \quad (2.9b)$$

(c)

$$D_\mu^{ij}(A) \frac{\partial \hat{\Gamma}}{\partial J_\mu^j(x)} - \frac{\partial \hat{\Gamma}}{\partial \bar{C}^i(x)} + g [\bar{Q}_\mu(x) \times \bar{L}_\mu(x)]^i = 0. \quad (2.9c)$$

For identity (b) we have made use of the invariance of the integration measure $dQ dC d\bar{C}$ under type-2 transformations

$$\frac{\partial^2 S}{\partial Q_\mu^i(x) \partial J_\mu^i(y)} = \frac{\partial^2 S}{\partial K_\mu^i(x) \partial C_\mu^i(y)} = 0. \quad (2.10)$$

Obviously, \hat{S} ($\hat{S} = S - (1/2\alpha) \int [D_\mu^{ij}(A)Q_\mu^j]^2 d^4x$) satisfies identities (2.9) ($\hat{\Gamma} \rightarrow \hat{S}$). Identity (2.9a) is specific of the background gauge; in a Fermi gauge ($\bar{A}_\mu = \bar{L}_\mu = 0$) only an integrated version remains, corresponding to invariance under global transformations. According to Ref. 5, the renormalization procedure ("minimal renormalization") can be performed in a way that ensures the validity of identities (2.9) for the renormalized action S^R and the renormalized 1PI functional Γ^R ; the action S^R has the same functional form as the bare functional S :

$$S^R(\bar{A}_\mu, \bar{Q}_\mu, C_i, \bar{C}_i, \bar{J}_\mu, \bar{K}, \bar{L}_\mu, g, \alpha) = S(\bar{A}_\mu^0, \bar{Q}_\mu^0, C_i^0, \bar{C}_i^0, \bar{J}_\mu^0, \bar{K}^0, \bar{L}_\mu^0, g^0, \alpha^0), \quad (2.11)$$

where the bare variables denote $\bar{A}_\mu^0 = (g/g^0) \bar{A}_\mu$, $\bar{Q}_\mu^0 = Z_3^{1/2} \bar{Q}_\mu$, $C_i^0 = \bar{Z}_3^{1/2} C_i$, $\bar{C}_i^0 = \bar{Z}_3^{-1/2} \bar{C}_i$, $\bar{J}_\mu^0 = \bar{Z}_3^{1/2} \bar{J}_\mu$, $\bar{K}^0 = Z_3^{1/2} \bar{K}$, $\bar{L}_\mu^0 = (g/g^0) Z_3^{-1/2} \bar{Z}_3^{-1/2} \bar{L}_\mu$, $\alpha^0 = \alpha Z_3$, and where g^0 , Z_3 , and \bar{Z}_3 are, respectively, coupling-constant and wave-function renormalizations. In the following, when we consider insertions of composite operators $O(\bar{A}_\mu + \bar{Q}_\mu, g)$, it will be understood that we have performed an intermediate renormalization on the generating functional $Z^{\text{IR}}(\bar{A}_\mu, \bar{\eta}_\mu, \xi_i, \bar{\xi}_i, \bar{J}_\mu, \bar{K}, \bar{L}_\mu, X, g, \alpha)$:

$$Z^{\text{IR}} = \int dQ dC d\bar{C} \exp \left[i \left(S(\bar{A}_\mu^0, \bar{Q}_\mu^0, C_i^0, \bar{C}_i^0, \bar{J}_\mu^0, \bar{K}^0, \bar{L}_\mu^0, g^0, \alpha^0) \right. \right. \\ \left. \left. + \int [XO(\bar{A}_\mu^0 + \bar{Q}_\mu^0, g^0) + \bar{Z}_3^{1/2} \bar{\eta}_\mu \cdot \bar{Q}_\mu + \bar{Z}_3^{-1/2} (\bar{\xi}_i C_i^0 + \bar{C}_i^0 \bar{\xi}_i)] \right) \right]. \quad (2.12)$$

However, for simplicity of notation we shall drop in the following 0 indices and the wave-function renormalization factors $Z_i^{-1/2}$; one can readily check that the Ward identities are not altered by this simplification.

III. INSERTION OF A GAUGE-INVARIANT OPERATOR

The proof of the block-triangular structure of the renormalization matrix is given in Secs. IIIA–IIIC. We derive the Ward identities in the tree approximation for class-I and class-II operators, respectively in Secs. IIIA and IIIB. Next we introduce a prescription (Sec. IIIC) for the insertion of operators which eliminates some of the most singular self-contractions of operators and which allows us to extend the validity of our WI to all orders for the *bare regularized generating functional*. Finally, the proof of these WI for the *renormalized generating functional* to all orders is given in Sec. IIID: The WI for class-II operators

yield the block-triangular structure of the renormalization matrix; the proof of WI for class-I operators rests on our conjecture. Section III E is devoted to comments concerning the geometrical interpretation of the WI after renormalization, the role of the equations of motion in the presence of non-supergauge-invariant operators, and the peculiarities of class-IIa operators. In Sec. III F we derive new WI which imply the α independence of the block of the renormalization matrix relative to class-I operators. All previous results are extended to a theory including fermions in Sec. III G.

A. Ward identity in the tree approximation for a class-I operator

Let us add a new term to the action S of Eq. (2.5), namely

$$\mathcal{S} = S + \int d^4x X(x) O(\tilde{A}_\mu + \tilde{Q}_\mu)(x), \quad (3.1)$$

where O denotes a local classical gauge-invariant operator. Clearly, the WI for the 1PI functional Γ are not affected by this insertion. Thus, the 1PI generating functional of graphs with one insertion of O , denoted by Γ_O , $\Gamma_O = \partial\Gamma/\partial X(x)|_{X=0}$, obeys the following identities:

(a)

$$\mathfrak{D}_x \Gamma_O = 0, \quad (3.2a)$$

(b)

$$\partial_r \Gamma_O = \int d^4x \left(\frac{\partial \hat{\Gamma}}{\partial Q_\mu^i(x)} \frac{\partial}{\partial J_\mu^i(x)} + \frac{\partial \Gamma}{\partial J_\mu^i(x)} \frac{\partial}{\partial Q_\mu^i(x)} - \frac{\partial \Gamma}{\partial C^i(x)} \frac{\partial}{\partial K^i(x)} - \frac{\partial \Gamma}{\partial \bar{C}^i(x)} \frac{\partial}{\partial \bar{K}^i(x)} - L_\mu^i(x) \frac{\partial}{\partial A_\mu^i(x)} \right) \Gamma_O = 0, \quad (3.2b)$$

(c)

$$\left(D_\mu^{ij}(A) \frac{\partial}{\partial J_\mu^j(x)} - \frac{\partial}{\partial \bar{C}^i(x)} \right) \Gamma_O = 0. \quad (3.2c)$$

Let us assume that the same identities hold in the one-loop approximation for the bare regularized functional, denoted also by Γ_O (this property is realized by the dimensional regularization). Then, to the same order, the singular part $\Gamma_{O,\text{div}}$ of Γ_O , which is a local functional, verifies (3.2a), (3.2c), and the following identity:

$$\partial_S \Gamma_{O,\text{div}} = 0. \quad (3.3)$$

The identity (3.3) follows from the absence of divergences of the functional $\Gamma(X=0)$ and from its value in the tree approximation:

$$\hat{\Gamma}_{\text{tree}}(X=0) = \hat{S}.$$

Let us remark that the operator O obviously obeys identity (3.3), since O is gauge invariant:

$$\begin{aligned} \int d^4x \left(\frac{\partial \hat{S}}{\partial J_\mu^i} \frac{\partial}{\partial Q_\mu^i} - L_\mu^i \frac{\partial}{\partial A_\mu^i} \right) O(\tilde{A}_\mu + \tilde{Q}_\mu) \\ = \int d^4x D_\mu^{ij}(A + Q) C^j \frac{\partial O}{\partial Q_\mu^i}(\tilde{A}_\mu + \tilde{Q}_\mu) = 0. \end{aligned}$$

The set of counterterms XO' necessary to renormalize in the one-loop approximation the graphs with one insertion of O contains all polynomials in $\tilde{A}_\mu, \tilde{Q}_\mu, C_i, \bar{C}_i, \tilde{J}_\mu, \bar{K}, \tilde{L}_\mu$ which have the same symmetry properties as O (Lorentz covariance, parity, . . .), a dimension less than or equal to that of the operator, and which verify the identities

$$(a) \quad \mathfrak{D}_x O' = 0, \quad (3.4a)$$

$$(b) \quad \partial_S O' = 0, \quad (3.4b)$$

$$(c) \quad \left(D_\mu^{ij}(A) \frac{\partial}{\partial J_\mu^j(x)} - \frac{\partial}{\partial \bar{C}^i(x)} \right) O' = 0. \quad (3.4c)$$

Equations (3.4a) and (3.4c) are easy to solve. According to Eq. (3.4a) O' is a type-1 invariant function which depends on \tilde{J}_μ and \bar{C}_i only through the combination $[J_\mu^i - D_\mu^{ij}(A)C_j]$ [Eq. (3.4c)]. Concerning identity (3.4b), the linear differential operators ∂_S and ∂_r verify the identities $\partial_S^2 = 0$, $\partial_r^2 = 0$. This follows, after some algebra, from the anti-commuting character of the variables $C_i, \bar{C}_i, \tilde{J}_\mu, \bar{K}, \tilde{L}_\mu$ and from identity (2.9b) for $\hat{\Gamma}$ and \hat{S} . Thus we have found, apart from our classical gauge-invariant operators of class I, other solutions of Eq. (3.4b):

$$O' = \partial_S F \quad \text{class-II operators.}$$

Do there exist any solutions other than the previous ones to Eq. (3.4b) which are type-1 invariant, depend only on $J_\mu^i - D_\mu^{ij}(A)\bar{C}^j$, and have a total ghost γ number equal to zero:

$$\gamma(J_\mu) = \gamma(\bar{C}) = -1, \quad \gamma(K) = -2, \quad \text{and} \quad \gamma(C) = \gamma(L_\mu) = +1.$$

This question is discussed in the Appendix where the identity (3.4b) is simplified with the help of identities (3.4a) and (3.4c) and solved completely

for the case of low dimensions (4, 5, 6) and low twist (2, 3) in a theory including fermions. In general, for arbitrary dimension and *nonzero value of ghost number* for the operator O' , Eq. (3.4b) possesses other solutions than class-II operators. For the operators under consideration ($\gamma = 0$) we will assume in the following that all counterterms of a class-I operator belong either to class I or to class II.

As an example, let us return to the operator $-\frac{1}{4} \int d^4x F_{\mu\nu}^2(x) = O_1$ and perform the same algebraic manipulations as in Ref. 4 to find the counterterms of this operator in the background gauge. They take the following form:

$$\begin{aligned} O_1 &= \left[\left(\alpha \frac{\partial}{\partial \alpha} - \frac{1}{2} g \frac{\partial}{\partial g} \right) \right. \\ &\quad \left. + \int d^4x \left(A_\mu^i \frac{\partial}{\partial A_\mu^i} + Q_\mu^i \frac{\partial}{\partial Q_\mu^i} + K^i \frac{\partial}{\partial K^i} \right) \right] S, \\ O_2' &= \left[g \frac{\partial}{\partial g} - \int d^4x \left(A_\mu^i \frac{\partial}{\partial A_\mu^i} + L_\mu^i \frac{\partial}{\partial L_\mu^i} \right) \right] S, \\ O_3' &= \left[\int d^4x \left(C^i \frac{\partial}{\partial C^i} - K^i \frac{\partial}{\partial K^i} \right) \right] S. \end{aligned} \quad (3.5)$$

($O_1 + \frac{1}{2} O_2'$) and O_3' , as one immediately verifies, are class-II operators:

$$Z(\vec{A}_\mu, \vec{\eta}_\mu, \xi_i, \bar{\xi}_i, \vec{J}_\mu, \vec{K}, \vec{L}_\mu) = \int dQ dC d\bar{C} \exp \left[i \left(\mathcal{S} + \int (\vec{\eta}_\mu \cdot \vec{Q}_\mu + \bar{\xi}_i C_i + \bar{C}_i \xi_i) \right) \right] \quad (3.8)$$

obeys the following identity which expresses the invariance of S under type-2 transformations of Eq. (2.7):

$$\begin{aligned} -\frac{1}{i} \int d^4x L_\mu^i(x) \frac{\partial Z}{\partial A_\mu^i(x)} &= \int dQ dC d\bar{C} \int d^4x \left[\left(\eta_\mu^i(x) + \int d^4z Y(z) \frac{\partial F(z)}{\partial Q_\mu^i(x)} \right) \frac{\partial S}{\partial Q_\mu^i(x)} + \left(\bar{\xi}^i(x) + \int d^4z Y(z) \frac{\partial F(z)}{\partial C^i(x)} \right) \frac{\partial S}{\partial K^i(x)} \right. \\ &\quad \left. + \frac{1}{\alpha} D_\mu^{ij}(A) Q_\mu^j(x) \left(\xi^i(x) - \int d^4z Y(z) \frac{\partial F(z)}{\partial \bar{C}^i(x)} \right) + \int d^4z Y(z) L_\mu^i(x) \frac{\partial F(z)}{\partial A_\mu^i(x)} \right] \\ &\quad \times \exp \left[i \left(\mathcal{S} + \int (\vec{\eta}_\mu \cdot \vec{Q}_\mu + \bar{\xi}_i C_i + \bar{C}_i \xi_i) \right) \right]. \end{aligned} \quad (3.9)$$

Clearly, to complete the right-hand side of Eq. (3.9) to the expression $\partial_S F$, we need terms of the form $[\partial F(z)/\partial J_\mu^i(x)] [\partial S/\partial Q_\mu^i(x)]$ and $[\partial F(z)/\partial K^i(x)] [\partial S/\partial C^i(x)]$ for which information is obtained for the equations of motion:

$$\int dQ dC d\bar{C} \left(\eta_\mu^i(x) + \frac{\partial S}{\partial Q_\mu^i(x)} + \int d^4z Y(z) \frac{\partial F(z)}{\partial Q_\mu^i(x)} \right) \exp \left[i \left(\mathcal{S} + \int (\vec{\eta}_\mu \cdot \vec{Q}_\mu + \bar{\xi}_i C_i + \bar{C}_i \xi_i) \right) \right] = 0, \quad (3.10a)$$

$$\int dQ dC d\bar{C} \left(-\bar{\xi}^i(x) + \frac{\partial S}{\partial C^i(x)} - \int d^4z Y(z) \frac{\partial F(z)}{\partial C^i(x)} \right) \exp \left[i \left(\mathcal{S} + \int (\vec{\eta}_\mu \cdot \vec{Q}_\mu + \bar{\xi}_i C_i + \bar{C}_i \xi_i) \right) \right] = 0. \quad (3.10b)$$

Application of the operators $\int d^4z Y(z) [\partial F(z)/\partial J_\mu^i(x)]$ and $\int d^4z Y(z) [\partial F(z)/\partial K^i(x)]$ on Eqs. (3.10a) and (3.10b), respectively, yields the identities

$$\int dQ dC d\bar{C} \int d^4z Y(z) \left(\eta_\mu^i(x) \frac{\partial F(z)}{\partial J_\mu^i(x)} + \frac{\partial S}{\partial Q_\mu^i(x)} \frac{\partial F}{\partial Q_\mu^i(x)} \right) \exp \left[i \left(\mathcal{S} + \int (\vec{\eta}_\mu \cdot \vec{Q}_\mu + \bar{\xi}_i C_i + \bar{C}_i \xi_i) \right) \right] = 0, \quad (3.11a)$$

$$\begin{aligned} O_1 + \frac{1}{2} O_2' &= \frac{1}{2} \partial_S \int d^4x \{ [J_\mu^i - D_\mu^{ij}(A) \bar{C}^j] Q_\mu^i + K^i C^i \}, \\ O_3' &= \partial_S \int d^4x (K^i C^i). \end{aligned} \quad (3.6)$$

B. Ward identity in the tree approximation for a class-II operator

Instead of a class-I operator, we add to the action S of Eq. (2.5) a source term for an operator F which depends on \bar{C}_i and J_μ^i only through the combination $[J_\mu^i - D_\mu^{ij}(A) \bar{C}_j]$ and which has ghost number equal to -1 :

$$\mathcal{S} = S + \int d^4x Y(x) F(x). \quad (3.7)$$

The associated source $Y(x)$ is of anticommuting character. F is supposed to be invariant under type-1 transformations [Eq. (2.6)]. In the following we derive the WI relating the generating functional Γ_F and $\Gamma_{\partial_S F}$ for 1PI Green's functions with one insertion of the operator F and $\partial_S F$, respectively, in the tree approximation

$$\Gamma_{\partial_S F} = \partial_\Gamma \Gamma_F.$$

The generating functional Z associated with the action \mathcal{S} of Eq. (3.7)

$$\int dQ dC d\bar{C} \int d^4z Y(z) \left(-\bar{\xi}^i(x) \frac{\partial F(z)}{\partial K^i(x)} + \frac{\partial S}{\partial C^i(x)} \frac{\partial F}{\partial K^i(x)} \right) \exp \left[i \left(\int (\vec{\eta}_\mu \cdot \vec{Q}_\mu + \bar{\xi}_i C_i + \bar{C}_i \xi_i) \right) \right] = 0. \quad (3.11b)$$

Note that we have taken advantage of the locality of the operator F and of the anticommuting character of the source Y

$$Y^2(x) = 0 \quad (3.12)$$

to discard the terms quadratic in Y in identity (3.11b) and that we have omitted one-loop contributions of the form $\partial^2 F(z)/\partial Q_\mu^i(x) \partial J_\mu^i(x)$ which will be considered in Sec. IIIC. Combining Eq. (3.9) and (3.11) we derive identity for Z :

$$\begin{aligned} \frac{1}{i} \int d^4x \left(L_\mu^i(x) \frac{\partial}{\partial A_\mu^i(x)} + \eta_\mu^i(x) \frac{\partial}{\partial J_\mu^i(x)} + \bar{\xi}^i(x) \frac{\partial}{\partial K^i(x)} + \frac{1}{\alpha} \xi^i(x) D_\mu^{ij}(A) \frac{\partial}{\partial \eta_\mu^j(x)} \right) Z \\ = \int dQ dC d\bar{C} \int d^4x Y(x) \partial_S F(x) \exp \left[i \left(\int (\vec{\eta}_\mu \cdot \vec{Q}_\mu + \bar{\xi}_i C_i + \bar{C}_i \xi_i) \right) \right]. \end{aligned} \quad (3.13)$$

This identity yields immediately the announced identity between the 1PI generating functionals after a Legendre transformation:

$$\Gamma_{\partial_S F} = \partial_\Gamma \Gamma_F. \quad (3.14)$$

In the tree approximation this result is trivial; however, as we shall see in the following, the same algebra as the one used in the tree approximation works to all orders. This relation implies the absence of coupling of class-II operators to class-I operators.

C. Extension of the Ward identities to all orders for the bare regularized functionals

In the derivation of Eq. (3.13) we have omitted contributions like $\partial^2 F(z)/\partial J_\mu^i(x) \partial Q_\mu^i(x)$ since these terms arising from self-contractions of the operator $F(z)$ are one-loop contributions. These terms are dangerous when we wish to extend the derivation of (3.13) to higher order, since they involve $\delta^4(0)$ in view of the locality of the operator F . In this section we will show that we can subtract the contributions of these self-contractions in a consistent manner and without spoiling the equations of motion and WI; this is achieved by a prescription which we introduce in the following.

Of course, this method is not entirely convincing, since we implicitly assume that the dimensional regularization applied to the class-I operators O_i and to the operators F_i , which sets to zero all contributions from self-contractions, does not spoil the geometric properties which yield the WI. The case of class-II operators $\partial_S F_i$ is somewhat special, however, since in our derivation of equations of motion it is obtained explicitly by calculating the product of two operators. Thus, we start by multiplying operators at different points x and y , subtract the dangerous contributions, and then take the limit $x \rightarrow y$. Our prescriptions are defined by

$$\begin{aligned} \left\langle \frac{\partial \hat{S}}{\partial Q_\mu^i(x)} \frac{\partial F}{\partial J_\mu^i(x)} \right\rangle &= \lim_{x \rightarrow y} \left[\frac{\partial \hat{S}}{\partial Q_\mu^i(x)} \frac{\partial F}{\partial J_\mu^i(y)} - i\hbar \left(g_{\mu\nu} - \frac{\partial_\mu^\nu \partial_\nu^\mu}{\square} \right) \frac{\partial^2 F}{\partial Q_\nu^i(x) \partial J_\mu^i(y)} \right], \\ \left\langle \frac{\partial \hat{S}}{\partial J_\mu^i(x)} \frac{\partial F}{\partial Q_\mu^i(x)} \right\rangle &= \lim_{x \rightarrow y} \left(\frac{\partial \hat{S}}{\partial J_\mu^i(y)} \frac{\partial F}{\partial Q_\mu^i(x)} - i\hbar \frac{\partial_\mu^\nu \partial_\nu^\mu}{\square} \frac{\partial^2 F}{\partial Q_\mu^i(x) \partial J_\nu^i(y)} \right), \\ \left\langle \frac{\partial \hat{S}}{\partial C^i(x)} \frac{\partial F}{\partial K^i(x)} \right\rangle &= \lim_{x \rightarrow y} \left(\frac{\partial \hat{S}}{\partial C^i(x)} \frac{\partial F}{\partial K^i(y)} - i\hbar \frac{\partial^2 F}{\partial C^i(x) \partial K^i(y)} \right), \\ \left\langle \frac{\partial S}{\partial Q_\mu^i(x)} \frac{\partial F}{\partial J_\mu^i(x)} \right\rangle &= \lim_{x \rightarrow y} \left(\frac{\partial S}{\partial Q_\mu^i(x)} \frac{\partial F}{\partial J_\mu^i(y)} - i\hbar \frac{\partial^2 F}{\partial Q_\mu^i(x) \partial J_\mu^i(y)} \right), \\ \left\langle \frac{1}{\alpha} D_\mu^{ij}(A) Q_\mu^j(x) \frac{\partial F}{\partial \bar{C}_i(x)} \right\rangle &= \lim_{x \rightarrow y} \left(\frac{1}{\alpha} D_\mu^{ij}(A) Q_\mu^j(x) \frac{\partial F}{\partial \bar{C}_i(y)} + i\hbar \frac{\partial_\mu^\nu \partial_\nu^\mu}{\square} \frac{\partial^2 F}{\partial Q_\mu^i(x) \partial J_\nu^i(y)} \right), \\ \langle \partial_S F \rangle &= \int d^4x \left(\left\langle \frac{\partial \hat{S}}{\partial Q_\mu^i(x)} \frac{\partial F}{\partial J_\mu^i(x)} \right\rangle + \left\langle \frac{\partial \hat{S}}{\partial J_\mu^i(x)} \frac{\partial F}{\partial Q_\mu^i(x)} \right\rangle - \left\langle \frac{\partial \hat{S}}{\partial C^i(x)} \frac{\partial F}{\partial K^i(x)} \right\rangle - \left\langle \frac{\partial \hat{S}}{\partial K^i(x)} \frac{\partial F}{\partial C^i(x)} \right\rangle - L_\mu^i(x) \frac{\partial F}{\partial A_\mu^i(x)} \right). \end{aligned} \quad (3.15)$$

$$(g_{\mu\nu} \square^x - d_\mu^x d_\nu^x) Q_\nu^i(x) \frac{dF}{dJ_\mu^i(y)} \sim i\hbar \left(g_{\mu\nu} - \frac{d_\mu^x d_\nu^x}{\square} \right) \frac{d^2F}{dQ_\nu^i(x) dJ_\mu^i(y)}$$

FIG. 1. Contribution from $[\partial\hat{S}/\partial Q_\mu^i(x)][\partial F/\partial J_\mu^i(y)]$ eliminated by the prescription.

The graphs subtracted by these prescriptions are given in Figs. 1–5. These prescriptions subtract only the most singular contributions for which the degree of divergence of the closed loop is of degree four at four dimensions of space-time. Other contributions, as that of the graph of Fig. 6, are supposed to be regularized as usual and vanish in the dimensional regularization. Finally, the prescription of Eq. (3.15) makes use of the locality of the polynomial F and of its dependence on \bar{C}_i through $[J_\mu^i - D_\mu^{ij}(A)\bar{C}_j]$.

With these prescriptions, the bare regularized functional Z obeys identity (3.9) where one replaces the quantities

$$\frac{\partial F(z)}{\partial Q_\mu^i(x)} \frac{\partial S}{\partial J_\mu^i(x)} \text{ and } \frac{1}{\alpha} D_\mu^{ij}(A) Q_\mu^j(x) \frac{\partial F}{\partial \bar{C}^i(x)}$$

by the corresponding subtracted operators defined in Eqs. (3.15). Application of the operators $\int d^4z Y(z) [\partial F(z)/\partial J_\mu^i(y)]$ and $\int d^4z Y(z) [\partial F(z)/\partial K^i(y)]$ at point $y \neq x$ on Eqs. (3.10a) and (3.10b) yields precisely in the limit $x \rightarrow y$ the identities (3.11a) and (3.11b) for the subtracted operators

$$\left\langle \frac{\partial S}{\partial Q_\mu^i(x)} \frac{\partial F(z)}{\partial J_\mu^i(x)} \right\rangle \text{ and } \left\langle \frac{\partial S}{\partial \bar{C}^i(x)} \frac{\partial F(z)}{\partial K^i(x)} \right\rangle$$

instead of the unsubtracted operators which appear in these equations. Notice that all this is consistent, since these subtractions arise only in the one-loop approximation and identities (3.11) were derived originally in the tree approximation; factors of \hbar have been made explicit in our prescription to clarify this point. Finally, identity (3.13) is obtained for the operator $\langle \partial_S F \rangle$ instead of $\partial_S F$,

$$d_\mu^\nu \bar{C}_i(y) \frac{dF}{dQ_\mu^i(x)} \sim i\hbar \frac{d_\mu^\nu d_\nu^\nu}{\square} \frac{d^2F}{dJ_\nu^i(y) dQ_\mu^i(x)} + \dots$$

FIG. 2. Contribution from $[\partial S/\partial J_\mu^i(y)][\partial F/\partial Q_\mu^i(x)]$ eliminated by the prescription; the three dots symbolize other terms which are less singular, such as the contribution of the graph of Fig. 6.

by combining similarly the identities (3.9) and (3.11) for subtracted quantities and by taking into account the following identity between subtracted operators:

$$\begin{aligned} \left\langle \frac{\partial S}{\partial Q_\mu^i(x)} \frac{\partial F(z)}{\partial J_\mu^i(x)} \right\rangle &= \left\langle \frac{1}{\alpha} D_\mu^{ij}(A) D_\mu^{jk}(A) Q_\mu^k(x) \frac{\partial F(z)}{\partial J_\nu^i(x)} \right\rangle \\ &= \left\langle \frac{\partial S}{\partial Q_\mu^i(x)} \frac{\partial F(z)}{\partial J_\mu^i(x)} \right\rangle - \left\langle \frac{\partial(S - \hat{S})}{\partial Q_\nu^i(x)} \frac{\partial F(z)}{\partial J_\nu^i(x)} \right\rangle \\ &= \left\langle \frac{\partial \hat{S}}{\partial Q_\mu^i(x)} \frac{\partial F}{\partial J_\mu^i(x)} \right\rangle. \end{aligned}$$

The latter identity holds for the corresponding quantities of the right-hand side of Eq. (3.15) before we take the limit $x \rightarrow y$. With these precautions, we now assume that a gauge-invariant regularization is applied which preserves identities (3.2) and (3.13) for the 1PI bare generating functionals Γ_0 and $\Gamma_{\langle \partial_S F \rangle}$ for single insertions of our subtracted operators, to all orders of perturbation theory, and we are about to exploit these identities for the renormalization procedure.

D. Perturbative proof of Ward identities for renormalized functionals and consequences

We start with the renormalization of class-II operators for which we have derived the identity for the bare functional

$$\Gamma_{\langle \partial_S F_i \rangle} = \partial_\Gamma \Gamma_{F_i}. \quad (3.16)$$

This relation implies that to all orders in perturbation theory we need only counterterms of the form $\langle \partial_S F_i \rangle$ to renormalize the graphs with one insertion of $\langle \partial_S F_i \rangle$. To show this, we proceed by recursion and we suppose that to order n the re-

$$-\alpha^x \bar{C}_i(x) \frac{dF}{dK^i(y)} \sim i\hbar \frac{d^2F}{dC_i(x) dK_i(y)}$$

FIG. 3. Contribution from $[\partial\hat{S}/\partial C^i(x)][\partial F/\partial K^i(y)]$ eliminated by the prescription.

FIG. 4. Contribution from $[\partial S/\partial Q_\mu^i(x)][\partial F/\partial J_\mu^i(y)]$ eliminated by the prescription.

normalized action $S^{R[n]}$ contains only counterterms of the form

$$\sum_{j \leq n} Z_{ij}^{(p)} \langle \partial_S F_j \rangle.$$

The 1PI functional $\Gamma_{\langle \partial_S F_i \rangle}^{R[n]}$ satisfies identity (3.16) and its divergent part at order $(n+1)$, $\Gamma_{\langle \partial_S F_i \rangle_{\text{div}}}^{R[n+1]}$, obeys the identity

$$\Gamma_{\langle \partial_S F_i \rangle}^{R[n+1]} = \partial_S \Gamma_{F_i, \text{div}}^{R[n+1]}. \quad (3.17)$$

At order $(n+1)$ the new counterterm has the expression

$$\sum_j Z_{ij}^{(n+1)} \partial_S F_j.$$

Without altering the counterterm at order $(n+1)$, we introduce in the action instead of the previous expression the subtracted counterterm¹¹

$$\sum_j Z_{ij}^{(n+1)} \langle \partial_S F_j \rangle.$$

The renormalization action $S^{R[n+1]}$ with these counterterms gives rise to a generating functional $\Gamma_{\langle \partial_S F_i \rangle}^{R[n+1]}$ which obeys again identity (3.16). Thus, we have achieved our proof that the operators of class II are coupled only to themselves and not to class-I operators.

Now we are ready to show by recursion that class-I operators generate to all orders of perturbation theory counterterms which are either class-I or class-II operators. Suppose again that this is true

to order n . Since the regularized functionals with one insertion of any class-I operator O_i , $1 \leq i \leq N$ or any class-II operator $\langle \partial_S F_i \rangle$, $N+1 \leq i \leq p$ both satisfy identity (3.2b),

$$\partial_\Gamma \Gamma_{\langle \partial_S F_i \rangle} = (\partial_\Gamma)^2 \Gamma_{F_i} = 0, \quad (3.18)$$

then the 1PI functional for one insertion of a given O_i , say O_1 , with all the counterterms to order n also obeys Eq. (3.2b). Its singular piece at order $(n+1)$ verifies $\partial_S \Gamma_{O_1, \text{div}}^{R[n+1]} = 0$ and, according to our conjecture, the counterterm is of the form

$$\sum_{i=1}^N Z_{1i}^{(n+1)} O_i + \sum_{i=N+1}^p Z_{1i}^{(n+1)} \partial_S F_i.$$

Again, without altering the finiteness at order $(n+1)$ we change the counterterm $\sum Z_{1i} \partial_S F_i$ to $\sum Z_{1i} \langle \partial_S F_i \rangle$.¹¹ This ends the recursion proof.

Returning to our example $\bar{F}_{\mu\nu}^2$, the renormalization matrix for O_1 , $O_1 + \frac{1}{2} O_2'$, and O_3' is identical to the one which we computed in a Fermi gauge in Ref. 4. In this basis the renormalization matrix was indeed triangular.

E. Comments on symmetry after renormalization

Type-1 symmetry is exactly preserved by the renormalization procedure used here: All counterterms are invariant under the initial transformations of Eqs. (2.6), as well as the measure of integration. Type-2 symmetry is, of course, renormalized; however, the algebraic structure of the supergauge transformation is preserved by renormalization. We remark that the generating functional $Z_{\langle \partial_S F \rangle}$ for a single insertion of the operator $\langle \partial_S F \rangle$ obeys the same type-2 identities as the class-I gauge-invariant operators: This is, in fact, an equivalent version of Eq. (3.18). Let us simplify Eq. (3.9) by use of the ghost equation of motion which is unaltered by the presence of the source Y for an operator F depending only on $[J_\mu^i - D_\mu^{ij}(\bar{A}_\mu) \bar{C}_j]$; this yields the following identity:

FIG. 5. Contribution from $(1/\alpha) D_\mu^{ij}(A) Q_\mu^j(x) [\partial F / \partial \bar{C}_i(y)]$ eliminated by the prescription: the three dots denote less singular operators.

FIG. 6. The graph of Fig. 6 is the contribution omitted in the right-hand side of Fig. 2; it diverges less rapidly than the term $i\hbar (\partial_\mu^\alpha \partial_\nu^\alpha / \alpha) [\partial^2 F / \partial J_\nu^i(y) \partial Q_\mu^i(x)]$ in the limit $x \rightarrow y$.

$$\frac{1}{i} \int d^4x \left\{ L_\mu^i(x) \frac{\partial}{\partial A_\mu^i(x)} + \eta_\mu^i(x) \frac{\partial}{\partial J_\mu^i(x)} + \bar{\xi}^i(x) \frac{\partial}{\partial K^i(x)} \right. \\ \left. + \frac{1}{\alpha} D_\mu^{ij}(A) \left[D_\nu^{jk}(A) \frac{1}{i} \frac{\partial}{\partial J_\nu^k(x)} - g \left(\tilde{L}_\nu(x) \times \frac{\partial}{i \partial \eta_\nu^i(x)} \right)_j \right] \frac{\partial}{\partial \eta_\mu^i(x)} \right\} Z = \delta Z = Y Z_{\langle \partial_S F \rangle}, \quad (3.19)$$

where the differential operator δ verifies $\delta^2 = 0$. Thus, the generating functional $Z_{\langle \partial_S F \rangle}$ for one insertion of the operator $\langle \partial_S F \rangle$ satisfies the WI

$$\delta Z_{\langle \partial_S F \rangle} = 0, \quad (3.20)$$

which is equivalent to the following formal property of $\langle \partial_S \rangle$:

$$\langle \partial_S \rangle^2 = 0. \quad (3.21)$$

By use of the equation of motion for the ghost, the generating functional $Z_{O_1}^R$ for a single insertion of O_1 and of its counterterms obeys the following identity:

$$\frac{1}{i} \int d^4x \left(L_\mu^i(x) \frac{\partial}{\partial A_\mu^i(x)} + \eta_\mu^i(x) \frac{\partial}{\partial J_\mu^i(x)} + \bar{\xi}^i(x) \frac{\partial}{\partial K^i(x)} + \frac{1}{\alpha} \xi^i(x) D_\mu^{ij}(A) \frac{\partial}{\partial \eta_\mu^j(x)} \right) Z_{O_1}^R = 0, \quad (3.22)$$

where the expression of the renormalized operator O_1 reads

$$O_1^R = \sum_{i=1}^N Z_{1i} O_i + \sum_{i=N+1}^p Z_{1i} \langle \partial_S F_i \rangle. \quad (3.23)$$

Identity (3.22) expresses the invariance of the generating functional $Z_{O_1}^R$ under the following type-2 transformations, up to source terms:

$$\begin{aligned} \delta Q_\mu^i(x) &= \frac{\partial S^R}{\partial J_\mu^i(x)} \delta \lambda, \\ \delta A_\mu^i(x) &= -L_\mu^i(x) \delta \lambda, \\ \delta C^i(x) &= + \frac{\partial S^R}{\partial K^i(x)} \delta \lambda, \\ \delta \bar{C}^i(x) &= -\frac{1}{\alpha} D_\mu^{ij}(A) Q_\mu^j(x) \delta \lambda, \end{aligned} \quad (3.24)$$

where S^R denotes the action renormalized to first order in the source X ,

$$S^R = S + \int d^4x X(x) O_1^R(x).$$

This geometrical interpretation of Eq. (3.22) is, however, formal, because the Jacobian of the transformation (3.24) is highly singular:

$$\frac{\delta J}{\delta \lambda} = \frac{\partial^2 S^R}{\partial Q_\mu^i(x) \partial J_\mu^i(x)} - \frac{\partial^2 S^R}{\partial C^i(x) \partial K^i(x)}.$$

Finally, let us note that identity (3.22) is not valid for the generating functional $Z^R(A_\mu, \eta_\mu, \xi, \bar{\xi}, J_\mu, K, L_\mu, X)$, where X is the source of the operator O_1^R :

$$\frac{1}{i} \int d^4x \left(L_\mu^i(x) \frac{\partial}{\partial A_\mu^i(x)} + \eta_\mu^i(x) \frac{\partial}{\partial J_\mu^i(x)} + \bar{\xi}^i(x) \frac{\partial}{\partial K^i(x)} + \frac{1}{\alpha} \xi^i(x) D_\mu^{ij}(A) \frac{\partial}{\partial \eta_\mu^j(x)} \right) Z^R(\dots X) = O(X^2).$$

This means that in order to preserve WI for $Z^R(\dots X)$ through renormalization it is necessary to introduce other terms in the action proportional to X^2 or, equivalently, to consider generalized type-2 transformation, analogous to those of Eq. (3.24), which contain X^2 contributions.

A comment concerning the operator ∂_S might be useful. The operator $\langle \partial_S F \rangle$ coincides with the variation of the operator F under a supergauge transformation up to terms which vanish in the absence of sources because of the equations of motion. To see this we introduce again the explicit dependence of F on the ghost field \bar{C}_i and we express ∂_S in terms of the full action S instead of \hat{S} by use of the ghost equation of

motion. This yields

$$\langle \partial_S F \rangle = \left\langle \frac{\delta F}{\delta \lambda} \right\rangle + \int d^4x \left(\left\langle \frac{\partial S}{\partial Q_\mu^i(x)} \frac{\partial F}{\partial J_\mu^i(x)} \right\rangle - \left\langle \frac{\partial S}{\partial C^i(x)} \frac{\partial F}{\partial K^i(x)} \right\rangle \right).$$

For operators of class I, both $\langle \partial_S O \rangle$ and $\langle \delta O / \delta \lambda \rangle$ vanish exactly. The class-II operators are supergauge invariant up to terms which vanish because of the equations of motion in the absence of the non-gauge-invariant sources for the fields Q_μ^i , C^i , and \bar{C}^i .

Finally, let us notice that class II contains operators which are gauge invariant under the classical gauge transformations: $\delta Q_\mu^i(x) = D_\mu^{ij}(A+Q)\delta\omega^j(x)$. These operators vanish in the classical limit if one considers the classical equation of motion of a Yang-Mills field in the absence of the gauge term $\{-(1/2\alpha)[D_\mu^{ij}(A)Q_\mu^j]^2\}$ of the corresponding Faddeev-Popov term and of the source terms; these operators (class-IIa) have the generic expression

$$O' = \left\langle \frac{\partial \mathcal{L}}{\partial Q_\mu^i(x)} (A+Q) G_\mu^i(\bar{A}_\mu + \bar{Q}_\mu)(x) \right\rangle, \quad (3.25)$$

where $G_\mu^i(\bar{A}_\mu + \bar{Q}_\mu)$ transforms according to the adjoint representation of the group. To illustrate this point, it appears more convenient to use a different version for our cohomology operation ∂_S , namely the operation $\hat{\partial}$ which yields an equivalent set of class-II operators (see Appendix):

$$\hat{\partial} = \int d^4x \left(\frac{\partial \mathcal{L}(\bar{B}_\mu)}{\partial B_\mu^i(x)} \frac{\partial}{\partial J_\mu^i(x)} + \frac{g}{2} (\bar{C} \times \bar{C})^i(x) \frac{\partial}{\partial C^i(x)} + J_\mu^i(x) D_\mu^{ij}(\bar{B}_\mu) \frac{\partial}{\partial K^j(x)} - L_\mu^i(x) \frac{\partial}{\partial A_\mu^i(x)} \right), \quad (3.26)$$

$$O' = \langle \hat{\partial} F \rangle \Big|_{\substack{J_\mu^i = J_\mu^i - D_\mu^{ij}(\bar{A}_\mu) \bar{C}_j \\ L_\mu^i = L_\mu^i + D_\mu^{ij}(\bar{A}_\mu) C_j \\ B_\mu^i = A_\mu^i + Q_\mu^i}}$$

for any local type-1-invariant polynomial F in the variables \bar{A}_μ , \bar{B}_μ , C_i , J_μ^i , \bar{K} , \bar{L}_μ^i . With this expression of class-II operators it is clear that the latter corresponds to a polynomial F linear in \bar{J}_μ :

$$O' = \langle \hat{\partial} [J_\mu^i(x) G_\mu^i(B)] \rangle \Big|_{\bar{B}_\mu = \bar{A}_\mu + \bar{Q}_\mu}. \quad (3.27)$$

The other class-II operators, denoted by class IIb, are associated with a polynomial F such as

$$\left. \frac{\partial F}{\partial J_\mu^i(x)} \right|_{\substack{\bar{B}_\mu = \bar{A}_\mu \\ C_i = J_\mu^i = K_i = L_\mu^i = 0}} = 0 \quad (3.28)$$

and vanish for all fields and sources, except \bar{A}_μ , set equal to zero. The presence of class-IIa operators causes several problems which will be discussed further.

F. α dependence of the renormalization matrix

As in Ref. 5, the α dependence of various Green's functions is obtained by examination of Ward identities in the presence of a source term L for the operator $[C^i D_\mu^{ij}(A) Q_\mu^j + \bar{J}_\mu \cdot \bar{Q}^\mu + a \bar{K} \cdot \bar{C}]$. The action \mathcal{S} : $\mathcal{S} = \mathcal{S} + \int d^4x X(x) O_1(x)$ is invariant under the following transformations:

$$\begin{aligned} \delta \alpha &= 2\alpha L \delta \lambda, \\ \delta A_\mu^i(x) &= -L_\mu^i(x) \delta \lambda, \\ \delta Q_\mu^i(x) &= [D_\mu^{ij}(A+Q) C^j(x) + L_\mu^i(x)] \delta \lambda = \frac{\delta \mathcal{S}}{\delta J_\mu^i(x)} \delta \lambda, \\ \delta C^i(x) &= \frac{g}{2} (\bar{C} \times \bar{C})^i(x) \delta \lambda = \frac{\delta \mathcal{S}}{\delta K^i(x)} \delta \lambda, \\ \delta \bar{C}^i(x) &= \left(-\frac{1}{\alpha} D_\mu^{ij}(A) Q_\mu^j(x) - \bar{C}^i(x) L \right) \delta \lambda, \\ \delta J_\mu^i(x) &= -J_\mu^i(x) L \delta \lambda, \\ \delta K^i(x) &= -a K^i(x) L \delta \lambda, \\ \delta L_\mu^i(x) &= \delta L = 0. \end{aligned} \quad (3.29)$$

This invariance property is expressed by the following identity for the regularized 1PI generating functional Γ_{O_1} for graphs with one insertion of the operator O_1 :

$$\delta' \Gamma_{O_1} \equiv L \left\{ 2\alpha \frac{\partial}{\partial \alpha} + \int d^4x \left[Q_\mu^i(x) \frac{\partial}{\partial Q_\mu^i(x)} - J_\mu^i(x) \frac{\partial}{\partial J_\mu^i(x)} - \bar{C}^i(x) \frac{\partial}{\partial \bar{C}^i(x)} + a \left(C^i(x) \frac{\partial}{\partial C^i(x)} - K^i(x) \frac{\partial}{\partial K^i(x)} \right) \right] \right\} \Gamma_{O_1} + \partial_\Gamma \Gamma_{O_1} = 0, \quad (3.30)$$

where δ' verifies

$$\delta'^2 = 0. \quad (3.31)$$

Identity (3.30) for the generating functional $\Gamma_{O_1}(A)$ for Green's functions with only external \tilde{A}_μ legs reads

$$2\alpha \frac{\partial}{\partial \alpha} \Gamma_{O_1}(\tilde{A}_\mu) + \frac{\partial \Gamma}{\partial Q_\mu^i(x)}(\tilde{A}_\mu) \frac{\partial \Gamma_{O_1}}{\partial L \partial J_\mu^i(x)}(\tilde{A}_\mu) = 0. \quad (3.32)$$

This derivation of (3.32) is valid in the one-loop approximation. The counterterms for $\Gamma_{O_1}(\tilde{A}_\mu)$ belong either to class I or to class IIa. Identity (3.32) implies that the α -dependent counterterms are the class-IIa counterterms; thus class-I counterterms are α independent.

To extend the proof to higher orders, we proceed according to the methods of Secs. III A–III D, except that we now take into account the source L . Operators of class II

$$\left\langle \left(\partial_S + 2L\alpha \frac{\partial}{\partial \alpha} \right) ZF \right\rangle$$

obey the following identity in the tree approximation:

$$\Gamma_{[\partial_S + 2L\alpha(\partial/\partial\alpha)]ZF} = \left\{ \partial_{\Gamma_L} + L \left[2\alpha \frac{\partial}{\partial \alpha} + \int d^4x \left(Q_\mu^i(x) \frac{\partial}{\partial Q_\mu^i(x)} - J_\mu^i(x) \frac{\partial}{\partial J_\mu^i(x)} - \bar{C}^i(x) \frac{\partial}{\partial \bar{C}^i(x)} + aC^i(x) \frac{\partial}{\partial C^i(x)} - aK^i(x) \frac{\partial}{\partial K^i(x)} \right) \right] \right\} \Gamma_{ZF}, \quad (3.33)$$

where Γ_L denotes the 1PI generating functional in the absence of the operator F but in the presence of the operator L . The main simplification is the reduction of the bracket of the right-hand side of Eq. (3.33) to $[\partial_S + 2L\alpha(\partial/\partial\alpha)]$ in the tree approximation, where S is the action for $L=0$. To higher orders one first writes identity (3.33) for the bare functional $\Gamma \dots$ before intermediate renormalization; this identity involves the bare variables of Eq. (2.11); then we derive identity (3.33) for $\Gamma \dots$ after our intermediate renormalization, which involves the renormalized variables α, \dots , using the following results of Ref. 4:

$$L^0 = Z_3^{-1/2} \tilde{Z}_3^{-1/2} \left(1 + \alpha \frac{\partial}{\partial \alpha} \ln Z_3 \right) L, \quad (3.34)$$

$$a(g, \alpha) - 1 = [a(g_0, \alpha_0) - 1] [1 + \alpha(\partial/\partial\alpha) \ln Z_3] - 2\alpha(\partial/\partial\alpha) \ln \tilde{Z}_3.$$

All the considerations in this section rest on our choice of a “minimal renormalization” where g_0 is α independent: α dependence and α_0 dependences are thus equivalent. The recursion proof on the number of loops then goes through for class-II operators as in Sec. IIID.

Similarly, identity (3.30) can be derived for the regularized functional Γ_{O_1} after an intermediate renormalization and the singular part of Γ_{O_1} verifies

$$\left(\partial_S + 2L\alpha \frac{\partial}{\partial \alpha} \right) \Gamma_{O_1, \text{div}} = 0. \quad (3.35)$$

With the help of our conjecture for the solution of $\partial_S T = 0$ one shows immediately that the solution of

$$\left(\partial_S + 2L\alpha \frac{\partial}{\partial \alpha} \right) T = 0$$

is of the form

$$T = \sum_{i=1}^N Z_i(g) O_i + \sum_{j=N+1}^p \left\langle \left(\partial_S + 2L\alpha \frac{\partial}{\partial \alpha} \right) Z_j(g, \alpha) F_j \right\rangle, \quad (3.36)$$

where Z_i for $1 \leq i \leq N$ is independent of α . With these ingredients, a recursion proof establishes the validity of Eq. (3.30) for the renormalized functional Γ_{01}^R and the independence of Z_i , $1 \leq i \leq N$, with respect to α to all orders of perturbation theory.

G. Introduction of matter fields

For the sake of definiteness, we consider spinor fields, which transform like some irreducible representation of the gauge group; we denote by $(T^i)_{ab}$ the Hermitian matrices which are the generators of this representation. The classical Lagrangian (2.1) now reads

$$\mathcal{L}(\vec{Q}_\mu, \psi_a, \bar{\psi}_b) = -\frac{1}{4}(\partial_\mu \vec{Q}_\nu - \partial_\nu \vec{Q}_\mu + g \vec{Q}_\mu \times \vec{Q}_\nu)^2 + \bar{\psi}_a (i \not{D}^{ab} - m \delta_{ab}) \psi_b, \quad (3.37)$$

where $D_\mu^{ab} \psi_b$ is the covariant derivative of ψ_a :

$$D_\mu^{ab} \psi_b(x) \equiv \partial_\mu \psi_a(x) - ig Q_\mu^i(x) T_{ab}^i \psi_b(x).$$

This Lagrangian is invariant under the local transformations of both \vec{Q}_μ and $\psi_a, \bar{\psi}_b$ given by (2.2) and

$$\begin{aligned} \delta \psi_a(x) &= ig T_{ab}^i \delta \omega^i(x) \psi_b(x), \\ \delta \bar{\psi}_a(x) &= -ig \bar{\psi}_b(x) T_{ba}^i \delta \omega^i(x). \end{aligned} \quad (3.38)$$

After introduction of the gauge-fixing term, Faddeev-Popov ghosts, and auxiliary sources, the total action takes the form [cf.(2.5)]

$$\begin{aligned} S = \int d^4x \Big(& \mathcal{L}(\vec{A}_\mu + \vec{Q}_\mu, \psi_a, \bar{\psi}_b) - \frac{1}{2\alpha} [D_\mu^{ij}(A) Q_\mu^j]^2 + [J_\mu^i + \bar{C}^j D_\mu^{ji}(A)] [D_\mu^{ik}(A+Q) C^k + L_\mu^k] \\ & + g C_i (\vec{Q}_\mu \times \vec{L}^\mu)_i + \frac{g}{2} K_i (\vec{C} \times \vec{C})_i - ig \bar{\psi}_a T_{ab}^i C^i M_b + ig \bar{M}_a T_{ab}^i C^i \psi_b \Big), \end{aligned} \quad (3.39)$$

where the new sources M_a and \bar{M}_a have been introduced for convenience (M_a and \bar{M}_a will be considered as commuting sources). The results of Sec. IIIB are then readily extended: The action S is invariant under both "type-1" transformations ($\psi_a, M_a, \bar{\psi}_a, \bar{M}_a$ transform according to the same representation T^{ab}) and "type-2" (supergauge) transformations given by (2.7) and

$$\begin{aligned} \delta \psi_a(x) &= ig T_{ab}^i C^i(x) \psi_b(x) \delta \lambda = \frac{\partial S}{\partial \bar{M}_a(x)} \delta \lambda, \\ \delta \bar{\psi}_a(x) &= -ig \bar{\psi}_b(x) T_{ba}^i C^i(x) \delta \lambda = \frac{\partial S}{\partial M_a(x)} \delta \lambda. \end{aligned} \quad (3.40)$$

As a consequence, the 1PI functional Γ satisfies the following identities:

$$\int d^4x \left(\frac{\partial \hat{\Gamma}}{\partial Q_\mu^i(x)} \frac{\partial \Gamma}{\partial J_\mu^i(x)} - \frac{\partial \Gamma}{\partial C^i(x)} \frac{\partial \Gamma}{\partial K^i(x)} - L_\mu^i(x) \frac{\partial \hat{\Gamma}}{\partial A_\mu^i(x)} + \frac{\partial \Gamma}{\partial \psi^a(x)} \frac{\partial \Gamma}{\partial \bar{M}_a(x)} + \frac{\partial \Gamma}{\partial \bar{\psi}_a(x)} \frac{\partial \Gamma}{\partial M_a(x)} \right) = 0, \quad (3.41a)$$

$$\begin{aligned} \mathfrak{D}_x^i \Gamma \equiv \Big[& D_\mu^{ij}(A) \frac{\partial}{\partial A_\mu^j(x)} + g f_{ijk} \left(Q_\mu^j(x) \frac{\partial}{\partial Q_\mu^k(x)} + C^j(x) \frac{\partial}{\partial C^k(x)} + \bar{C}^j(x) \frac{\partial}{\partial \bar{C}^k(x)} \right. \\ & \left. + K^j(x) \frac{\partial}{\partial K^k(x)} + J_\mu^j(x) \frac{\partial}{\partial J_\mu^k(x)} + L_\mu^j(x) \frac{\partial}{\partial L_\mu^k(x)} \right) \\ & + ig T_{ab}^i \left(\bar{\psi}_b(x) \frac{\partial}{\partial \bar{\psi}_a(x)} + \bar{M}_b(x) \frac{\partial}{\partial \bar{M}_a(x)} \right) - ig T_{ab}^i \left(\psi_b(x) \frac{\partial}{\partial \psi_a(x)} + M_b(x) \frac{\partial}{\partial M_a(x)} \right) \Big] \Gamma = 0, \end{aligned} \quad (3.41b)$$

whereas the ghost equation of motion (2.9c) remains unchanged. It is then easy to see that the functional Γ_0 with one insertion of a gauge-invariant operator $O(\vec{A}_\mu + \vec{Q}_\mu, \psi_a, \bar{\psi}_b)$ satisfies the equations

$$\begin{aligned} \mathfrak{D}_x \Gamma_0 &= 0, \\ \partial_r \Gamma_0 &= 0, \\ \left(\mathfrak{D}_\mu^{ij}(A) \frac{\partial}{\partial J_\mu^j(x)} - \frac{\partial}{\partial \bar{C}^i(x)} \right) \Gamma_0 &= 0, \end{aligned} \quad (3.42)$$

where now \mathfrak{D}_x is the differential operator defined in (3.41b) and ∂_r denotes

$$\partial_\Gamma \equiv \int dx \left(\frac{\partial \hat{\Gamma}}{\partial Q_\mu^i(x)} \frac{\partial}{\partial J_\mu^i(x)} + \frac{\partial \Gamma}{\partial J_\mu^i(x)} \frac{\partial}{\partial Q_\mu^i(x)} - \frac{\partial \Gamma}{\partial C^i(x)} \frac{\partial}{\partial K^i(x)} - \frac{\partial \Gamma}{\partial K^i(x)} \frac{\partial}{\partial C^i(x)} \right. \\ \left. - L_\mu^i \frac{\partial}{\partial A_\mu^i(x)} + \frac{\partial \Gamma}{\partial \psi_a(x)} \frac{\partial}{\partial \bar{M}_a(x)} + \frac{\partial \Gamma}{\partial \bar{M}_a(x)} \frac{\partial}{\partial \psi_a(x)} + \frac{\partial \Gamma}{\partial \bar{\psi}_a(x)} \frac{\partial}{\partial M_a(x)} + \frac{\partial \Gamma}{\partial M_a(x)} \frac{\partial}{\partial \bar{\psi}_a(x)} \right). \quad (3.43)$$

The identity $\partial_\Gamma^2 = 0$ remains valid. We are now faced with the problem of solving the system of equations

$$\begin{aligned} (a) \quad \mathfrak{D}_x^i O' &= 0, \\ (b) \quad \partial_S O' &= 0, \\ (c) \quad \left(D_\mu^{ij}(A) \frac{\partial}{\partial J_\mu^j(x)} - \frac{\partial}{\partial \bar{C}_i(x)} \right) O' &= 0 \end{aligned} \quad (3.44)$$

for local polynomial functionals O' of the variables \bar{Q}_μ , \bar{A}_μ , C_i , \bar{C}_i , \bar{J}_μ , \bar{K} , \bar{L}_μ , ψ_a , $\bar{\psi}_a$, M_a , and \bar{M}_a which conserve the total number of ghosts. Some clues to this resolution are given in the Appendix where we prove our conjecture in the simplest cases: Any solution of Eq. (3.44) is of the form

$$O_{\text{inv}}(\bar{A}_\mu + \bar{Q}_\mu, \psi_a, \bar{\psi}_a) + \partial_S F.$$

The generalization of Secs. III B–III F in the presence of matter fields is then straightforward. The operators $\langle \partial_S F \rangle$ [where now contractions between $\partial/\partial M$ ($\partial/\partial \bar{M}_a$) and the kinetic part of $\partial S/\partial \bar{\psi}_a$ ($\partial S/\partial \psi_a$) are also forbidden] form a set of operators stable by renormalization, and the addition of counterterms does not modify the Ward identities (3.42) satisfied by Γ_0 . Class-IIa operators have the form

$$\int dx \frac{\partial \mathcal{L}}{\partial Q_\mu^i(x)} (\bar{A}_\mu + \bar{Q}_\mu, \psi_a, \bar{\psi}_a) G_\mu^i(\bar{A}_\mu + \bar{Q}_\mu, \psi_a, \bar{\psi}_a)(x), \\ \int dx \bar{H}_a(\bar{A}_\mu + \bar{Q}_\mu, \psi_a, \bar{\psi}_a) \frac{\partial \mathcal{L}}{\partial \bar{\psi}_a(x)} (\bar{A}_\mu + \bar{Q}_\mu, \psi_a, \bar{\psi}_a), \text{ or } \int dx \frac{\delta \mathcal{L}}{\delta \psi_a(x)} H_a(\bar{A}_\mu + \bar{Q}_\mu, \psi_a, \bar{\psi}_a),$$

where the assignments for G , H , and \bar{H} to given representations and fermion numbers are obvious. Exactly as in Sec. III F, it can be seen that only these operators contribute to the α dependence of the divergent part of Green's functions with one insertion of a gauge-invariant operator and only external legs of \bar{A}_μ , ψ_a , and $\bar{\psi}_a$. Finally, one can show that class-II operators of the form

$$\frac{\delta \mathcal{L}}{\delta \bar{\psi}_a} H_a(\bar{A}_\mu, \bar{Q}_\mu, \psi_b, \bar{\psi}_b, C_i, \bar{C}_i) \text{ or } \bar{H}_a(\bar{A}_\mu, \bar{Q}_\mu, \psi_b, \bar{\psi}_b, C_i, \bar{C}_i) \frac{\delta \mathcal{L}}{\delta \psi_a}$$

are separately stable by renormalization.

IV. RECIPE FOR THE COMPUTATION OF α -INDEPENDENT ANOMALOUS DIMENSIONS

A. Summary of results

Let us collect first the results of the previous section concerning the renormalization matrix relative to a gauge-invariant operator which does not vanish in the classical limit (class-I operator). We conjecture and prove in the Appendix for operators of dimension $d \leq 6$, or of twist $\tau \leq 5$ for totally symmetric operators, or of twist $\tau \leq 3$ for operators antisymmetric in two indices and symmetric in the other indices, the following assertion. The renormalization matrix $\{Z_{ij}\}$ acts in the space of class-I operators and of class-II operators O' which are defined by the following equation:

$$O'(\bar{A}_\mu + \bar{Q}_\mu, \psi_a, \bar{\psi}_a, C_i, \bar{C}_i) = [\langle \delta F(\bar{A}_\mu, \bar{B}_\mu, \psi_a, \bar{\psi}_a, C_i, \bar{J}_\mu', \bar{K}, \bar{L}_\mu', M_a, \bar{M}_a) \rangle] \begin{cases} \bar{B}_\mu = \bar{A}_\mu + \bar{Q}_\mu \\ J_\mu^i = -D_\mu^{ij}(A) \bar{C}_j \\ L_\mu^{ij} = D^{ij}(A) C_j \\ K_i = M_a = \bar{M}_a = 0 \end{cases} \quad (4.1)$$

The operation $\hat{\partial}$ introduced in the Appendix in the presence of fermions denotes, for M and \bar{M} equal to zero, the expression

$$\hat{\partial} = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial B_\mu^i(x)} \frac{\partial}{\partial J_\mu^i(x)} + \frac{\partial \mathcal{L}}{\partial \psi_a(x)} \frac{\partial}{\partial \bar{M}_a(x)} + \frac{\partial \mathcal{L}}{\partial \bar{\psi}_a(x)} \frac{\partial}{\partial M_a(x)} \right. \\ \left. + \frac{g}{2} [\bar{C}(x) \times \bar{C}(x)]^i \frac{\partial}{\partial C^i(x)} - L_\mu^{ij}(x) \frac{\partial}{\partial A_\mu^i(x)} - D_\mu^{ij}(B) J_j'^\mu(x) \frac{\partial}{\partial K^i(x)} \right) \quad (4.2)$$

and the $\langle \rangle$ product [Eq. (3.7)] subtracts dangerous graphs with a closed loop and no propagator (Fig. 1). The sources $J_\mu^i(x)$, $K^i(x)$, $L_\mu^i(x)$ are elements of the adjoint representation of the Lie algebra and have dimension 2 and ghost number γ : $\gamma(J_\mu^i) = -\gamma(L_\mu^i) = \frac{1}{2}\gamma(K^i) = -1$ with the convention $\gamma(c) = -\gamma(\bar{c}) = +1$. The sources M_a and \bar{M}_a belong to the same representation as the fermion fields ψ and have dimension $\frac{3}{2}$, ghost number $\gamma = -1$, and a fermion number F :

$$F(M_a) = F(\psi_a) = -F(\bar{M}_a) = +1.$$

\mathcal{L} is a shorthand notation for the classical action of $\bar{B}_\mu, \psi_a, \bar{\psi}_a$ [Eq. (3.37)].

To find all operators coupled by renormalization to a given class-I operator O_i with definite dimen-

$$\frac{\partial \mathcal{L}}{\partial \bar{Q}_\mu^i(x)} (\bar{A}_\mu + \bar{Q}_\mu, \psi_a, \bar{\psi}_a) G_i^\mu (\bar{A}_\mu + \bar{Q}_\mu, \psi_a, \bar{\psi}_a) \text{ or } \frac{\partial \mathcal{L}}{\partial \psi_a(x)} (\bar{A}_\mu + \bar{Q}_\mu, \psi_a, \bar{\psi}_a) \bar{H}^a (\bar{A}_\mu + \bar{Q}_\mu, \psi_a, \bar{\psi}_a).$$

Therefore, class I is defined as the set of all gauge-invariant operators which do not belong to class IIa. Other elements of class II are denoted by class IIb. Notice that the described procedure yields a redundant set of operators; the reduction of this set to a basis of independent operators can be achieved by a systematic and tedious use of the Jacobi identity, anticommutation relations (and partial integrations for operators at zero momentum).

With this basis, the matrix $\{Z_{ij}\}$ takes the form

$$\{Z_{ij}\} = \begin{matrix} & \begin{matrix} \text{I} & \text{IIa} & \text{IIb} \end{matrix} \\ \begin{matrix} \text{I} \\ \text{IIa} \\ \text{IIb} \end{matrix} & \begin{pmatrix} \alpha \text{ independent} & & \\ 0 & a \text{ priori} & \\ 0 & \alpha \text{ dependent} & \end{pmatrix} \end{matrix}. \quad (4.3a)$$

The counterterms Z_{ij} are obtained directly from the superficial divergences of various 1PI graphs containing an insertion of an operator O_i or O'_i , after extraction of the wave-function renormalization from each external leg of the graph and coupling-constants renormalization, g_0 and α_0 (see Sec. II); in the following, mention of a superficial divergence refers to the superficial divergence after this intermediate renormalization.

The multiplicatively renormalizable eigenvectors Ω_i associated with the α -independent counterterms of class I, take the expression

$$\Omega_i = \sum_{\text{class-I operators}} \psi_{ij}(g) O_j + \sum_{\text{class-II operators}} \chi_{ij}(\alpha, g) O'_j. \quad (4.3b)$$

Finally, graphs with only external legs of the background field A_μ , ψ , and $\bar{\psi}$ do not give rise to

sion d , Lorentz covariance, and other symmetry properties (like parity, for example), construct (i) all gauge-invariant operators with dimension less or equal to d , the same Lorentz covariance and symmetry properties as O_i , and (ii) all type-1 invariant operators F_i with dimension less or equal to $(d-1)$, the same Lorentz covariance and symmetry properties as O_i , a ghost number equal to (-1) , a fermion number equal to zero, and which depend on the variables $\bar{A}_\mu, \bar{B}_\mu, \psi_a, \bar{\psi}_a, C_i, \bar{J}_\mu', \bar{K}, \bar{L}_\mu', M_a$, and \bar{M}_a . Then operate on each polynomial F_i as indicated in Eqs. (4.1) and (4.2). The $\langle \rangle$ prescription is unessential for the value of O'_i in the tree approximation. This procedure yields a redundant set. For example, there exist gauge-invariant operators which have the expression ∂F (class IIa)

class-IIb counterterms, since these class-IIb operators vanish for all fields and sources, but A_μ , ψ , and $\bar{\psi}$, set to zero. This remark leads directly to the recipe for the computation of the α -independent anomalous dimensions relative to class-I operators in the background gauge.

B. Recipe in the background gauge

The superficial divergence of a 1PI graph with an insertion of a class-I operator O_i and external legs of the background field \bar{A}_μ and fermion fields ψ_a and $\bar{\psi}_a$ gives contributions to counterterms both of class I and class IIa and thus the remaining problem is the extraction of the class I counterterms from the superficial divergences of the mentioned graphs. The first task is, of course, to list all class-I and -IIa operators of the same Lorentz covariance and the same dimension as O_i . In the absence of fermions in the theory, the separation of class-IIa operators would be automatic if we could choose the background field to verify the condition

$$\frac{\partial \mathcal{L}(\bar{A}_\mu)}{\partial A_\mu^i(x)} = 0, \quad (4.4)$$

which is indeed preserved by renormalization; however, we do not know how to realize this condition in a consistent fashion. Therefore, an alternative procedure is to compute the superficial divergence of one of the following:

(i) a given 1PI Green's function with definite Lorentz behavior which may contribute to class-I operators only, because the expression of class-

IIa operators in the tree approximation does not display an element with the same tensor behavior (to perform the computation consistently, you need as many independent tensors as class-I operators), or (ii) a 1PI graph with definite external legs which does not appear in the expression of any class-IIa operator in the tree approximation (again, one needs as many independent graphs as class-I operators), or (iii) if the previous procedures are not possible, either a given 1PI graph with various tensor behaviors in the external momenta or various 1PI graphs and eliminate by hand the contribution to class-IIa operators by performing linear combinations between the various computed counterterms.

Let us illustrate these considerations by the example of traceless twist-two operators at zero momentum. For totally symmetric twist-two operators there exist no class-IIa operators. Indeed, the operator F associated with such a class-IIa operator should be constructed from a function $H_{\mu_2 \dots \mu_n}(B)$ of twist zero, totally symmetric in $\mu_2 \dots \mu_n$:

$$F_{\mu_1 \dots \mu_n}(x) = J_{\mu_1}^i(x) H_{\mu_2 \dots \mu_n}^i(x) + (\text{permutations } \mu_1 \rightarrow \mu_2, \dots, \mu_n). \quad (4.5)$$

A type-1 covariant object $H_{\mu_2 \dots \mu_n}^i(B)$ can be built only from the covariant objects $D_{\mu}^{ij}(B)$ and $F_{\mu\nu}^i(B)$ which is eliminated by the simultaneous requirements of total symmetry and of twist zero (this forbids any contraction which raises the twist); finally, no covariant object can be built only with the tensor $D_{\mu}^{ij}(B)$ at zero momentum. In the presence of fermions, this argument can be implemented. Class-IIa operators constructed with fermion fields have at least twist four, unless γ_{μ} matrices lower the twist; however, the product $\gamma_{\mu}\gamma_{\nu}$, which brings the twist to the value of two, does not contribute to symmetric traceless operators. This result confirms the computation of Sarkar and Strubbe³ and Crewther⁶ of the anomalous dimension of traceless symmetric twist-two operators in the one-loop approximation.

Our next example is given by the operators of twist two and dimension four, antisymmetric in

their indices μ and ν , at nonzero momentum:

$$\begin{aligned} O_1 &= i \bar{\psi} (\gamma_{\mu} \vec{D}_{\nu} - \gamma_{\nu} \vec{D}_{\mu}) \psi, \\ O_2 &= i \bar{\psi} (\gamma_{\mu} \vec{D}_{\nu} - \gamma_{\nu} \vec{D}_{\mu}) \psi, \\ O_3 &= i \bar{\psi} \sigma_{\mu\nu} \vec{D} \psi, \\ O_4 &= i \bar{\psi} \vec{D} \sigma_{\mu\nu} \psi. \end{aligned} \quad (4.6)$$

These operators are independent. No class-I or -IIa operator can be constructed from the gauge field $(\vec{A} + \vec{Q})_{\mu}$ only. O_3 and O_4 are class-IIa operators. In order to determine the renormalization factor Z_{i1} relative to O_1 , we only need to compute the divergent part of the two-point function with one insertion of O_i , of tensorial structure $p_{\mu}\gamma_{\nu} - p_{\nu}\gamma_{\mu}$ (see Fig. 7). Similarly, for O_2 , the divergent part of the 2-point function with tensor structure $q_{\mu}\gamma_{\nu} - q_{\nu}\gamma_{\mu}$ yields the counterterm Z_{i2} relative to the operator O_2 .

C. Recipe in the Fermi gauge

In our approach, the Fermi gauge is just a limiting case of the background gauge where one sets the field \vec{A}_{μ} to zero after all algebraic manipulations. This procedure applied to Eq. (4.1) yields the complete list of class-II operators coupled by renormalization to a class-I operator of a given spin and dimension. The graphs with external legs of the fields \vec{Q}_{μ} , ψ_a , and $\bar{\psi}_a$ which seem suited for the computation of class-I counterterms, need, however, also class-IIb counterterms in contradiction to the graphs selected in the background gauge. A complete list of class-I and -II operators of given spin and dimension must be established and the extraction of class-I counterterms proceeds according to the same general rules as in the background gauge: Either one chooses suitable tensor contributions of certain graphs or computes graphs with different number of legs.

Again, in this Fermi gauge the traceless symmetric twist-two operators at zero momentum are simple. Because of the symmetry and twist constraints, there exist no class-II operators constructed with fermions and thus the superficial divergence of graphs with fermion external legs yields automatically class-I counterterms, depend-

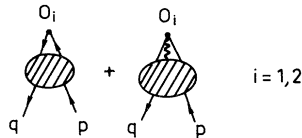


FIG. 7. The contribution of these graphs determines the renormalization factors Z_{i1} and Z_{i2} relative to the operators O_i of Eqs. (4.6).

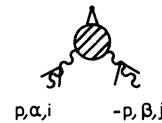


FIG. 8. The contribution of this graph determines the anomalous dimension of symmetric traceless twist-two operators [Eq. (4.7)].

ing on fermion fields. For graphs with external legs of the field \tilde{Q}_μ , the two-point \tilde{Q}_μ function with the following tensor structure (see Fig. 8)

$$\delta_{ij} g_{\alpha\beta} p_{\mu_1} \cdots p_{\mu_n} \quad (n \text{ even}) \quad (4.7)$$

$$F_{\mu_1 \cdots \mu_n} = \int d^4x J_{\mu_1}^i(x) H_{\mu_2 \cdots \mu_n}^i(A, B) + (\text{permutations } \mu_1 \rightarrow \mu_2, \dots, \mu_n). \quad (4.8)$$

The polynomial H can be constructed from the type-1 covariants $D_\mu^{ij}(B)$, $(B-A)_\mu^i$, and $F_{\mu\nu}^i(B)$, which is, however, eliminated by the simultaneous requirements of twist zero and symmetry in all indices. The generic form of class-II operators is therefore given by

$$O' = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \tilde{Q}_\mu^i(x)} H_{\mu_2 \cdots \mu_n}^i(0, \tilde{Q}_\mu) - \int d^4y \left(\partial_{\mu_1} \bar{C}^i(x) \frac{\partial H_{\mu_2 \cdots \mu_n}^i(\tilde{A}_\mu, \tilde{Q}_\mu)}{\partial A_\nu^j(y)} \right) \Big|_{A_\mu=0} \partial_\nu C^j(y) \right] + (\text{permutations } \mu_1 \rightarrow \mu_2, \dots, \mu_n). \quad (4.9)$$

The contribution of these operators to the two-point Q_μ function in the tree approximation

$$\delta_{ij} [p^2 p_{\mu_2} \cdots p_{\mu_{n-1}} (g_{\mu_1\alpha} g_{\mu_n\beta} + g_{\mu_n\alpha} g_{\mu_1\beta}) - p_{\mu_1} \cdots p_{\mu_{n-1}} (p_\alpha g_{\mu_n\beta} + p_\beta g_{\mu_n\alpha}) + (\text{permutations } \mu_1 \rightarrow \mu_2, \dots, \mu_n)] \quad (4.10)$$

vanishes for the peculiar tensor selected in Eq. (4.7). The authors of Ref. 1 seem to have computed the counterterm associated with the particular function of Eq. (4.7), which gives indeed the correct α -independent anomalous dimension.

The operator $\tilde{F}_{\mu\nu}^2$ at zero momentum is an example of a case where there exist no class-IIa operators and thus the prescription is trivial in the background gauge and not *a priori* in the Fermi gauge.

D. Computation of anomalous dimensions at order n

Suppose we are interested only by the value of the anomalous dimensions at some order n of perturbation theory, and not by the complete expression of the corresponding multiplicatively renormalizable operators, and we have picked according to the recipes of the previous sections a convenient graph with definite tensor structure. Then the extraction of the superficial divergence from this graph at order n requires preliminary renormalization of all subgraphs. This implies the computation at order $(n-1)$ of all counterterms Z_{ij} for $i \leq N$ if N denotes the number of independent class-I operators. This includes the class-II counterterms relative to $j > N$. However, the rest of the matrix $\{Z_{ij}\}$ ($i > N$) is irrelevant for the computation of the N α -independent anomalous dimensions. The computation of the counterterms Z_{ij} for $i \leq N$ yields, of course, also the components of the multiplicatively renormalizable operators, associated

needs only class-I counterterms, depending on the field \tilde{Q}_μ . To see this, we notice that the function F [Eq. (4.1)] must be linear in \tilde{J}_μ' and independent of \tilde{K} and \tilde{L}_μ' because of the simultaneous requirement of twist (+1) and ghost number (-1):

with the N α -independent anomalous dimensions, on the space of class-I operators, namely the functions $\psi_{ij}(g)$ of Eq. (4.3b). These simplifications are a direct consequence of the block-triangular structure of the renormalization matrix.

E. Final comments

Let us end with some trivial comments. The presence of class-IIa operators constitutes a restriction of our method since it does not apply to them. These operators might, however, be unphysical, but we dispose of no convincing criteria of "physicality." A necessary, but not sufficient, criteria would be the presence of other α -independent dimensions than the N α -independent ones relative to class-I operators. No general proof could be given concerning this α dependence. Even a counterexample showing explicit α dependence cannot settle the question, since we know of an explicit example of a class-IIb operator with α -independent anomalous dimension: It is the conserved operator $\bar{C}MC$ studied in Ref. 4. This example shows that accidents may occur.

The next point we would like to emphasize is the fact that the variation of all class-II operators under a supergauge transformation (Ref. 8) vanishes in the absence of sources of the quantized fields because of the equations of motion (see Sec. III D4). Therefore, we have at the moment no plausible argument to exclude a contribution to the Wilson expansion of a product of gauge-invariant currents

arising from a class-II operator associated with an α -independent anomalous dimension.

The study of the Wilson expansion involves a preliminary work on the renormalization of a product of two local class-I operators which can presumably be carried out along similar lines to those followed here, namely the exploitation of WI derived from the supergauge transformations. In fact, if the renormalization of a single insertion of a class-I operator can be performed in a way which preserves the form of the bare WI for the regularized functional (as noted in Sec. IIID), then the counterterms for a bilocal product of two operators are again class-II operators, at least at the one-loop level. Thus, the class-II operators seem to play an important role in the renormalization of gauge-invariant operators.

V. CONCLUSIONS

We have achieved by means of Ward-Slavnov identities a systematic study of the renormalization of single insertions of classical gauge-invariant operators of arbitrary momentum which applies also to the compensation of non-“logarithmic” divergences. The classical gauge-invariant operators, which do not vanish in the classical limit in the absence of sources and of the gauge-fixing term and correlated Faddeev-Popov ghosts, are denoted by class-I operators. The latter couple by renormalization to gauge-invariant operators, which vanish in the above mentioned classical limit (they are referred to as class-IIa operators), and to non-gauge-invariant operators which involve gauge and matter fields as well as Faddeev-Popov fields (the latter are referred to as class-IIb operators). Class-II operators which mix with a given class-I operator O of dimension d are generated by the action of a linear differential operator on polynomials which have ghost number (-1) , dimension less than or equal to $(d-1)$, and the same Lorentz covariance as O . The class-II operators obey peculiar Ward identities, which imply for class-II operators the same WI as those for single insertions of class-I operators and which yield a block-triangular form for the renormalization matrix: The renormalization of class-II operators involves only class-II operators. The block-triangular structure of the renormalization matrix simplifies greatly the computation of the α -independent elements of the class-I counterterms. Explicit prescriptions for this computation were given and illustrated by some examples, both in the Fermi gauge and in the background gauge. In this respect the computation in the latter gauge of the superficial divergence of 1PI graphs with external legs of matter fields and of the background field only se-

lects automatically class-I and -IIa operators, eliminating thereby class-IIb operators; the number of vertices in this gauge is, however, larger than in the Fermi gauge.

The above mentioned prescription rests on a conjecture which was proved only in a few cases: Only class-I and -II operators mix with class-I operators under renormalization. This conjecture seems quite reasonable in view of the properties of the class-II operators: same WI as for class-I operators and block-triangular structure for the renormalization matrix. The complete proof of this conjecture was achieved for the following operators, which are involved in the Wilson expansion of currents for a gauge theory including fermions: (i) operators of dimension $d \leq 6$; (ii) totally symmetric traceless operators $O_{\mu_1 \dots \mu_n}$ of twist $\tau \leq 5$; (iii) operators $O_{\mu\nu; \mu_1 \dots \mu_n}$ totally symmetric and traceless in n indices and antisymmetric in the other two indices, of twist $\tau \leq 3$. Finally, the insertion of the class-II counterterms, which is intimately related to the equations of motion, leaves invariant the renormalized action in the presence of a class-I operator under a generalized supergauge invariance which depends on the source of the class-I operator: The situation is similar to the familiar one in the absence of the operator where the renormalization of the coupling constant appears as a renormalization of the supergauge transformation.

The applications of this work to practical computations are numerous. Aside from the original calculations of Ref. 1, calculations of anomalous dimensions are involved in various physical problems of non-Abelian gauge theories; as an example, we may quote the dynamical realization of octet enhancement which was pointed out by Gailard and Lee, and by Altarelli and Maiani (Ref. 12). We hope that our prescription will avoid unnecessary computations of the full renormalization matrix.

There remain several open problems. The technical ones are the general solution of the cohomology problem, which was only conjectured, and the derivation of a basis of operators of given twist which exhausts relations arising from the Jacobi identity, anticommutation relations, and eventually partial space integrations for operators at zero momentum. Next, the existence of class-IIa operators raises the questions of the existence of other α -independent anomalous dimensions than those relative to class-I operators, and of their contribution to Wilson expansions of physical operators. Finally, this work opens a tractable path for the study of Wilson expansion, for which the proper treatment of the renormalization of gauge-invariant operators appeared as a preliminary.^{13,14}

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APPENDIX: SOLUTIONS OF EQS. (3.44)

In this appendix, we want to study the solutions of Eqs. (3.44) satisfied by the counterterms of a gauge-invariant operator. We consider the general case, with possible presence of matter (spinor) fields:

$$(a) \quad \mathfrak{D}_x^i O' = 0,$$

$$(b) \quad \partial_S O' = 0,$$

$$(c) \quad \left(D_\mu^{ij}(A) \frac{\partial}{\partial J_\mu^j(x)} - \frac{\partial}{\partial \bar{C}^i(x)} \right) O' = 0. \quad (3.44)$$

1. Change of variables

We first perform some changes of variables and combinations of these equations. By Eq. (3.44c), O' is a function of \bar{C}^i and J_μ^i only through the combination $J_\mu'^i \equiv J_\mu^i - D_\mu^{ij}(A) \bar{C}^j$. In terms of the variables \bar{A}_μ , $\bar{B}_\mu \equiv \bar{A}_\mu + \bar{Q}_\mu$, C_i , \bar{J}_μ^i , $L_\mu^i \equiv L_\mu^i + D_\mu^{ij}(A) C^j$, ψ_a , $\bar{\psi}_a$, M_a , and \bar{M}_a , it is straightforward to see that the differential operator $\hat{\partial}$

$$\hat{\partial} \equiv \partial_S + \int dx C^i(x) \mathfrak{D}_x^i \quad (A1)$$

takes the form

$$\begin{aligned} \hat{\partial} = \int dx \left[\frac{\partial \mathfrak{L}}{\partial \bar{B}_\mu(x)} \cdot \frac{\partial}{\partial \bar{J}_\mu^i(x)} + \frac{\partial \mathfrak{L}}{\partial \psi_a(x)} \frac{\partial}{\partial \bar{M}_a(x)} + \frac{\partial \mathfrak{L}}{\partial \bar{\psi}_a(x)} \frac{\partial}{\partial M_a(x)} + \frac{g}{2} [\bar{C}(x) \times \bar{C}(x)] \cdot \frac{\partial}{\partial \bar{C}(x)} \right. \\ \left. - \bar{L}_\mu^i(x) \cdot \frac{\partial}{\partial \bar{A}_\mu(x)} - \left(D_\mu^{ij}(B) J_\mu'^i(x) + ig \bar{M}_a(x) T_{ab}^i \psi_b(x) - ig \bar{\psi}_a(x) T_{ab}^i M_b(x) \right) \frac{\partial}{\partial K^i(x)} \right], \end{aligned} \quad (A2)$$

where, here and in the following, \mathfrak{L} is a shorthand notation for $\int dx \mathfrak{L}(B, \psi, \bar{\psi})(z)$, \mathfrak{L} defined in (3.37). In those variables, \mathfrak{D}_x^i reads

$$\begin{aligned} \mathfrak{D}_x^i = D_\mu^{ij}(\bar{A}_\mu) \frac{\partial}{\partial A_\mu^j(x)} + D_\mu^{ij}(\bar{B}_\mu) \frac{\partial}{\partial B_\mu^j(x)} \\ + g \left(\bar{C}(x) \times \frac{\partial}{\partial \bar{C}(x)} + \bar{\psi}_\mu^i(x) \times \frac{\partial}{\partial \bar{\psi}_\mu^i(x)} + \bar{K}(x) \times \frac{\partial}{\partial \bar{K}(x)} + \bar{L}_\mu^i(x) \times \frac{\partial}{\partial \bar{L}_\mu^i(x)} \right)_i \\ + ig \left(\bar{\psi}_a(x) T_{ab}^i \frac{\partial}{\partial \bar{\psi}_b(x)} + \bar{M}_a(x) T_{ab}^i \frac{\partial}{\partial \bar{M}_b(x)} \right) - ig \left(\psi_a(x) T_{ba}^i \frac{\partial}{\partial \psi_b(x)} + M_a(x) T_{ba}^i \frac{\partial}{\partial M_b(x)} \right). \end{aligned} \quad (A3)$$

It is clear that this operator $\hat{\partial}$ still satisfies $\hat{\partial}^2 = 0$

$$\hat{\partial}^2 = \int d^4x \left[\left(-D_\mu^{ij}(B) \frac{\partial}{\partial B_\mu^j(x)} + ig T_{ab}^i \bar{\psi}_a(x) \frac{\partial}{\partial \bar{\psi}_b(x)} - ig T_{ab}^i \psi_b(x) \frac{\partial}{\partial \psi_a(x)} \right) \mathfrak{L} \right] \frac{\partial}{\partial K^i(x)} \quad (A4)$$

and the right-hand side vanishes, because of the local gauge invariance of \mathfrak{L} . If we are now able to prove that some solution of

$$\begin{aligned} (a) \quad \hat{\partial} O' = 0, \\ (b) \quad \mathfrak{D}_x O' = 0 \end{aligned} \quad (A5)$$

is of the form

$$O' = \hat{\partial} F \text{ with } \mathfrak{D}_x F = 0,$$

it follows from the definition (A1) that an equivalent expression for O' is given by $O' = \partial_S F'$. The advantage of this new operator $\hat{\partial}$ is to provide a better separation of variables. Notice that the *local* equation (3.44a), which is valid because we are working in the background gauge, has played an important role in this simplification.

2. Elimination of the variables \bar{L}_μ^i and \bar{A}_μ

The integration of $\hat{\partial} O' = 0$ with respect to variables \bar{A}_μ and \bar{L}_μ^i is straightforward, since the operator $\bar{L}_\mu^i \cdot \partial / \partial \bar{A}_\mu$ anticommutes with the operator $\hat{\partial}$, and is linear in \bar{L}_μ^i . Indeed, the operator F_1 defined by

$$F_1(\vec{A}_\mu, \vec{B}_\mu, \dots, \vec{M}_a) = - \int_0^1 \frac{dt}{t} (\vec{A} - \vec{B})_\mu \cdot \frac{\partial}{\partial \vec{L}'_\mu} [O'(\vec{B}_\mu + t(\vec{A}_\mu - \vec{B}_\mu), \vec{B}_\mu, \vec{C}, \vec{J}'_\mu, \vec{K}, t\vec{L}'_\mu, \psi_a, \bar{\psi}_a, M_a, \bar{M}_a)] \quad (A6)$$

satisfies

$$\begin{aligned} \hat{\partial} F_1 &= \int_0^1 dt \frac{d}{dt} [O'(\vec{B}_\mu + t(\vec{A}_\mu - \vec{B}_\mu), \dots, t\vec{L}'_\mu, \dots)] \\ &= O'(\vec{A}_\mu, \vec{B}_\mu, \vec{C}, \vec{J}'_\mu, \vec{K}, \vec{L}'_\mu, \psi_a, \bar{\psi}_a, M_a, \bar{M}_a) - O'(\vec{B}_\mu, \vec{B}_\mu, \vec{C}, \vec{J}'_\mu, \vec{K}, 0; \psi_a, \bar{\psi}_a, M_a, \bar{M}_a). \end{aligned} \quad (A7)$$

By construction, F_1 is a type-1 invariant: $\mathfrak{D}_x F_1 = 0$, so $\hat{\partial} F_1$ is also a type-1 invariant, and the problem now reduces to the solution of Eq. (A5) for operators O' independent of \vec{A}_μ and \vec{L}'_μ . Unfortunately, the remaining cohomology equation is nontrivial and we have thus checked the validity of our conjecture only in the simplest cases.

3. Low-dimension or low-twist operators

In the case of low-dimension d , or low-twist τ operators, the possible ghost-number conserving solutions O' are polynomials of small degree in C_i ($d=\tau=1$), \vec{J}'_μ ($d=2, \tau=1$), \vec{K} ($d=\tau=2$), M_a , or \bar{M}_a ($d=\tau=\frac{3}{2}$). However, in the presence of *spinor* fields, one can use γ matrices to lower the twist of the operator and the possible degree in C_i depends on the Lorentz structure.

It is easy to see that operators of dimension $d=3$ (4, 5, 6) or twist $\tau=2, 3$ (4, 5), completely symmetric and traceless in their Lorentz indices, are, at most, linear (quadratic) in the C 's. On the other hand, operators of twist $\tau=2$ or 3 antisymmetric in two indices $\mu\nu$ and completely symmetric and traceless in the other indices μ_1, \dots, μ_n may be quadratic in C , for example, the operator

$$\bar{M}_a T_{ab_1}^i \sigma_{\mu\nu} (\gamma_{\mu_1} D_{\mu_2}^{b_1 b_2} \dots D_{\mu_n}^{b_{n-1} b_n} + \text{permutations}) M_{b_n} (\vec{C} \times \vec{C})^i.$$

(a) Operators at most linear in \vec{C}

After elimination of the variables \vec{L}'_μ and \vec{A}_μ , the remaining solutions O' , independent of \vec{L}'_μ and \vec{A}_μ , satisfy the following equation:

$$\begin{aligned} \int d^4x \frac{\partial \mathcal{L}}{\partial \vec{B}_\mu(x)} \cdot \frac{\partial O'}{\partial \vec{J}'_\mu(x)} + \frac{\partial \mathcal{L}}{\partial \psi_a(x)} \frac{\partial O'}{\partial \bar{M}_a(x)} + \frac{\partial \mathcal{L}}{\partial \bar{\psi}_a(x)} \frac{\partial O'}{\partial M_a(x)} + \frac{g}{2} [\vec{C}(x) \times \vec{C}(x)] \cdot \frac{\partial O'}{\partial \vec{C}(x)} \\ - [D_\mu^{ij}(B) J_j'^\mu(x) + ig \bar{M}_a(x) T_{ab}^i \psi_b(x) - ig \bar{\psi}_a(x) T_{ab}^i M_b(x)] \frac{\partial O'}{\partial K^i(x)} = 0, \end{aligned} \quad (A8)$$

which, by application of two derivatives $[\partial/\partial C_i(y)] \times [\partial/\partial C_j(z)]$ implies, for operators at most linear in C_i , the independence of O' with respect to the variable C_i . By ghost-number conservation, O' can therefore depend only on \vec{B}_μ , ψ_a , and $\bar{\psi}_a$ and satisfies automatically Eq. (A8). Equation (A5b) for $O'(\vec{B}_\mu, \psi_a, \bar{\psi}_a)$ tells us that O' is a type-1 invariant, and thus a gauge-invariant polynomial of $\vec{B}_\mu = \vec{A}_\mu + \vec{Q}_\mu$, ψ_a , and $\bar{\psi}_a$.

(b) Operators at most quadratic in \vec{C}

The piece of O' quadratic in the variable C , denoted by O'^2 , satisfies the identity

$$\hat{\partial}_C O'^2 = \int d^4x (\vec{C} \times \vec{C}) \cdot \frac{\partial}{\partial \vec{C}} O'^2 = 0, \quad (A9)$$

where the operator $\hat{\partial}_C$ is again a cohomology operator

$$(\hat{\partial}_C)^2 = 0. \quad (A10)$$

The following proof shows that for operators quadratic in C , $\hat{\partial}_C$ is a trivial cohomology operation; this means that all solutions of equation (A9) take the form

$$O'^2 = \frac{g}{2} \hat{\partial}_C G, \quad (A11)$$

where G denotes a type-1 invariant (Eq. (A5b) local polynomial in the variables \vec{A}_μ , \vec{B}_μ , \vec{J}'_μ , \vec{C} , and \vec{K} . This identity (A11) reduces the problem to the case of operators linear in \vec{C} , which was solved in the previous paragraph. Indeed, the polynomial O''

$$O'' = O' - \hat{\partial}_C G$$

satisfies Eq. (A8), is type-1 invariant, and is, at most, linear in \vec{C} . The main problem is to prove that the polynomial $\partial^2 O'(u)/\partial C_i(x) \partial C_j(y)$ is antisymmetric in the group indices i and j and thus symmetric in space-time variables (x, y) . This property is trivial for operators of dimension $d \leq 6$,

since one may proceed by inspection of the independent polynomials. For operators of given twist, this property is not obvious, and we indicate only the successive steps of the rather intricate derivation. The application of three derivatives on Eq. (A8) yields the following equation:

$$f_{ija}\delta(x-y)\frac{\partial^2 O'}{\partial C_k(z)\partial C_a(x)} + \left[2 \text{ permutations } \begin{pmatrix} x \\ i \end{pmatrix} \rightarrow \begin{pmatrix} y \\ j \end{pmatrix} \rightarrow \begin{pmatrix} z \\ k \end{pmatrix} \right] = 0, \quad (\text{A12})$$

which after some algebraic manipulations says that the ij -antisymmetric part of $\partial^2 O'/\partial C_i(x)\partial C_j(y)$ can be cast in the form

$$\frac{1}{2}C_2 \left(\frac{\partial^2 O'}{\partial C_i(x)\partial C_j(y)} + \frac{\partial^2 O'}{\partial C_i(y)\partial C_j(x)} \right) = f_{ijk}f_{klm}\delta(x-y) \int dz \frac{\partial^2 O'}{\partial C_i(z)\partial C_m(y)}. \quad (\text{A13})$$

C_2 denotes the eigenvalue of the Casimir operator in the adjoint representation:

$$f_{ijk}f_{ijl} = C_2\delta_{kl}. \quad (\text{A14})$$

Equation (A13) yields the announced result (A11) for the antisymmetric part $O_a'^{(2)}$,

$$O_a'^{(2)} = \frac{1}{2} \int dx dy C_j(y)C_i(x) \frac{\partial^2 O_a'^{(2)}}{\partial C_i(x)\partial C_j(y)} = \hat{\delta}_C G, \quad G = \frac{1}{C_2} f_{klm} \int dy dz C_k(y) \frac{\partial^2 O'}{\partial C_m(y)\partial C_l(z)}. \quad (\text{A15})$$

$O_a'^{(2)}$ satisfies all requirements and we are now able to show that the symmetric part $O_s'^{(2)}$ must vanish in view of Eq. (A13) and Eq. (A5b). The

application of two derivatives on the latter equation implies type-1 invariance for $\partial^2 O_s'^{(2)}/\partial C_j(y)\partial C_i(x)$,

$$\mathfrak{D}_z^k \frac{\partial^2 O_s'^{(2)}}{\partial C_j(y)\partial C_i(x)} = 0, \quad (\text{A16})$$

which must thus be proportional to δ_{ij} :

$$\frac{\partial^2 O_s'^{(2)}(u)}{\partial C_j(y)\partial C_i(x)} = \delta_{ij} \Phi(u; x, y). \quad (\text{A17})$$

By insertion of the expression (A17) into identity (A12), the latter reduces to

$$\delta(x-y)\Phi(u; y, z) + (x-y \rightarrow z) = 0$$

and one can convince oneself that this implies the vanishing of the distribution Φ which is of finite order. Thus we have achieved the proof.

(c) Operators of higher degree in C

This method fails for operators of degree three in C . Indeed, there exist solutions of Eqs. (A5) which are not of the form $\hat{\delta}F$; for example, the polynomial $f_{ijk}C_i(x)C_j(x)C_k(y)$ satisfies (A5) because of the Jacobi identity: $\vec{C} \times (\vec{C} \times \vec{C}) = 0$ but cannot be cast in the form $\hat{\delta}F$. However, we have found no simple counterexample of a ghost-number conserving operator which would be a solution of Eq. (A5) and which is not of the form $\hat{\delta}F$; for example, consider O' :

$$O' = f_{ijk}C_i(x)C_j(x)C_k(y)K_l(x)D_\mu^{lm}(B)J_m^\mu(x) = \hat{\delta}F,$$

where F denotes $F = \frac{1}{2}(\vec{C} \times \vec{C}) \cdot \vec{C}(\vec{K} \cdot \vec{K})$. The ghost-number conservation plays a crucial role for the validity of our conjecture, for which a general proof is probably not trivial. The first step, to our view, would be to set up a basis of polynomials which are independent by Jacobi identity and which separate the \vec{C} variables and $(\vec{C} \times \vec{C})$ variables.

*Chargée de Recherches au CNRS.

†Attaché de Recherches au CNRS.

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¹¹In fact, when inserting the counterterm at order $(n+1)$ relative to the insertion of the operators $\langle \partial_s F_i \rangle$ and O_1 , we immediately perform an intermediate renormalization by replacing in their expressions all the variables $\vec{A}_\mu, \vec{Q}_\mu, \dots, g, \alpha$ by the bare variables $\vec{A}_\mu^0, \vec{Q}_\mu^0, \dots, g^0, \alpha^0$ defined in Eq. (2.11). This procedure does not alter the counterterms at order $(n+1)$ and yields the desired Ward identities for the quantities renormalized to order $(n+1)$.

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symmetric twist-two operators at zero momentum, and in this respect, our first example, $\bar{F}_{\mu\nu}^2$, appears as a general case, in our view, rather than as an exception.

¹⁴*Note added in proof.* The insertion of a class-II operator can be absorbed by a change of fields and a change of gauge function which are believed to leave physical quantities invariant. However, the new gauge function is *a priori* of dimension > 2 and involves ghost fields. The renormalization program does not seem immediate in such a gauge. This remark was pointed out to us by J. Zinn-Justin.