# TWO-DIMENSIONAL CONFORMAL INVARIANT THEORIES ON A TORUS 

C. ITZYKSON and J.-B. ZUBER<br>Service de Physique Théorique, CEN-Saclay, 91191 Gif-sur-Yvette Cedex, France

Received 3 March 1986
(Revised 21 April 1986)


#### Abstract

The study of unitary conformal invariant theories on a torus reveals two important properties: the partition function and correlation functions may be expressed in terms of free (gaussian) field modes, and the modular invariance dictates the operator content of the theory: for a generic value of the central charge $c=1-6 / m(m+1)$, there exist at least two distinct models depending whether $m=0,3 \bmod 4$ or $m=1,2 \bmod 4$. The case of non-unitary $c<1$ theories is also briefly discussed.


## 1. Introduction

A major breakthrough has occurred in the study of critical 2-dimensional field theories and their conformal invariance, as a result of the work of Belavin, Polyakov and Zamolodchikov (BPZ) [1]. Relations with string theories and infinite Lie algebras has provided a very rich material. Friedan, Qiu and Shenker [2] have classified unitary representations of the conformal and superconformal Virasoro algebra. Dotsenko, Fateev and Zamolodchikov [3] have computed correlation functions using several techniques including a Coulomb gas representation due to Feigin and Fuks [4]. Cardy, Blöte and Nightingale, and Affleck [5,6] have related finite size effects with the representation theory developed by Kac, Feigin and Fuks [7] and Rocha-Caridi [8] for the conformal algebra, part of which was common lore among string theorists.

It is perhaps not too surprising that the construction of critical 2-dimensional theories can be based on free field theory as advocated by Kadanoff [9]. An interesting aspect is the role of the energy momentum tensor, with its associated Virasoro algebra, a tool not frequently used in the context of statistical mechanics. This introduces the so-called central charge $c$ which characterizes to a large extent the nature of the model. As pointed out in refs. [5,6] the physical significance of $c$ appears in restricted geometries as a kind of Casimir effect, i.e. a finite displacement of the free energy. In complex coordinates $z, \bar{z}$ the energy-momentum tensor is split accordingly in $T \equiv T_{z z}$ and $\bar{T} \equiv T_{\bar{z} \bar{z}}$. A conformal change of coordinates $z \leftrightarrow z^{\prime}$
induces a transformation $T \leftrightarrow T^{\prime}$ such that [1]

$$
\begin{equation*}
T(z)=T^{\prime}\left(z^{\prime}\right)\left(\frac{\mathrm{d} z^{\prime}}{\mathrm{d} z}\right)^{2}+\frac{1}{12} c\left\{z^{\prime}, z\right\} \tag{1.1}
\end{equation*}
$$

where

$$
\begin{equation*}
\left\{z^{\prime}, z\right\}=\frac{\mathrm{d}^{3} z^{\prime} / \mathrm{d} z^{3}}{\mathrm{~d} z^{\prime} / \mathrm{d} z}-\frac{3}{2}\left(\frac{\mathrm{~d}^{2} z^{\prime} / \mathrm{d} z^{2}}{\mathrm{~d} z^{\prime} / \mathrm{d} z}\right)^{2} \tag{1.2}
\end{equation*}
$$

is the schwarzian derivative of the transformation. A similar formula holds for $\bar{T}$. As an example if $z$ varies in a (periodic) strip $|\operatorname{Re} z| \leq \frac{1}{2}$ the transformation $z^{\prime}=$ $\exp (2 \pi i z / L)$ maps the vertical strip in a plane where we may assume $\left\langle T^{\prime}\right\rangle=0$. One should beware of the fact that rotating the strip by $90^{\circ}$ changes the sign of $\langle T\rangle$. As a consequence

$$
\begin{equation*}
\langle T\rangle_{\text {strip }}=\left(\frac{2 \pi}{L}\right)^{2} \frac{c}{24} \tag{1.3}
\end{equation*}
$$

An infinitesimal deformation (which need not be conformal) of a domain $\mathscr{D}$ induces a corresponding change of the free energy according to

$$
\begin{equation*}
\delta \ln Z+\int_{\mathscr{Q}} \frac{\mathrm{d}^{2} z}{2 \pi}\left\langle T_{\mu \nu}(z, \bar{z})\right\rangle \partial^{\mu} \delta r^{\nu}=0 \tag{1.4}
\end{equation*}
$$

We may apply this to a transverse dilatation of a strip $\delta \operatorname{Re} z=\varepsilon \operatorname{Re} z, \delta \operatorname{Im} z=0$. By a procedure familiar to physicists but not mathematically very rigorous, we may extract the free energy per unit longitudinal length (denoted by $T$ ) $[5,6]$

$$
\begin{equation*}
\frac{1}{T} \ln Z=f_{0} L+\frac{1}{6} \pi c \frac{1}{L} \tag{1.5}
\end{equation*}
$$

with $f_{0}$ an unknown constant. Not only does this vindicate the above statement but it also provides a direct access to a numerical determination of $c$ using finite size scaling methods [5,6]. A generalization of (1.4) holds even when the domain $\mathscr{D}$ cannot be obtained by a straightforward one to one map on the plane. Indeed as suggested in ref. [5] a thorough investigation of critical models on tori completely unravels the structure of a given model including its operator content. This will be shown in detail in the sequel together with the relation with characters of the conformal algebra. On the other hand the derivation of (1.5) assumes that the state in which $\left\langle T^{\prime}\right\rangle_{\text {plane }}=0$ corresponds in the strip to the lowest energy level of the corresponding 1 -dimensional quantum hamiltonian and hence that all operators have non-negative dimensions. This may fail for non-unitary models where negative dimensions occur in the spectrum as discussed in sect. 6.

The point of this paper is to study in some detail the connection of free field theory with critical models starting from the gaussian model in a lagrangian formalism. The line of argument is due to Cardy [5]. We will show that sums of (renormalized) partition functions on tori, for free fields subject to specific boundary conditions yield the conformal dimensions of operators, the characters of the associated Virasoro algebra, together with the central charge. The fact that partition functions of conformal theories are given by free field determinants with appropriate boundary conditions was implicit in the work of ref. [6]. What remains an art is to obtain the connection with the statistical models as initially formulated on a lattice away from criticality with their non-universal features. What underlies the above construction is some connection with a Coulomb representation of the observables in a gaussian model [9]. It would be very instructive to obtain some kind of direct formulation which would naturally include such aspects like the role of discrete symmetries and "parafermionic" operators [10].

Interestingly the crucial computations needed to obtain the partition functions date back to Kronecker according to Weil [11], who calls them Kronecker's limit formulas.

These formulas must have been reobtained a great many times, in particular in recent investigations of the string model $[12,13]$. As discussed in the next section the canonical example is provided by the free Bose field on a torus $\mathbb{T}$ characterized by two periods $\omega_{1}, \omega_{2}$ generating a lattice L with $\bar{J}=\mathbb{C} / \mathrm{L}$. With $\omega_{1}$ and $\omega_{2}$ given as complex numbers such that $\tau=\omega_{2} / \omega_{1}$ has a positive imaginary part, the dual lattice $\tilde{\mathrm{L}}$ is generated by $k^{1}, k^{2}$ such that $\operatorname{Re}\left(k^{i} \bar{\omega}_{j}\right)=\delta_{j}^{i}$. The fundamental cell of L , i.e. $\mathbb{T}$, has area $A=\operatorname{Im} \omega_{2} \bar{\omega}_{1}$. The eigenvalues of minus the laplacian in $\mathbb{I}$ are of the form $(2 \pi)^{2}\left|n_{1} k^{1}+n_{2} k^{2}\right|^{2}=(2 \pi / A)^{2}\left|n_{2} \omega_{1}-n_{1} \omega_{2}\right|^{2}$ and the omission of the zero mode is indicated by a prime on sums or products. The meaningless expression $Z_{1}=$ $A^{1 / 2} \Pi^{\prime}(1 / 2 \pi)\left|n_{1} k^{1}+n_{2} k^{2}\right|$ is defined through a procedure of analytic continuation involving the meromorphic function

$$
\begin{equation*}
G(s)=\left(\frac{A}{2 \pi}\right)^{2 s} \sum^{\prime} \frac{1}{\left|n_{2} \omega_{1}-n_{1} \omega_{2}\right|^{2 s}} \tag{1.6}
\end{equation*}
$$

leading to

$$
\begin{equation*}
Z_{1}=A^{1 / 2} \exp \left(\frac{1}{2} G^{\prime}(0)\right)=\frac{\left|\omega_{1}\right|}{A^{1 / 2} \eta[\tau] \overline{\eta[\tau]}} \tag{1.7}
\end{equation*}
$$

where $\eta[\tau]$ is Dedekind's function

$$
\begin{equation*}
\eta[\tau]=\exp (2 i \pi \tau / 24) \prod_{n=1}^{\infty}\left(1-\mathrm{e}^{2 i \pi n \tau}\right) \tag{1.8}
\end{equation*}
$$

When $\omega_{1}=L, \omega_{2}=i T, \tau=i T / L, T \rightarrow \infty$, the torus degenerates into a periodic
strip, $(1 / T) \ln Z_{1} \mapsto \frac{1}{6} \pi(1 / L)$ in agreement with (1.5) for a central charge $c=1$. This discussion will be pursued in sect. 2.

In sect. 3 we reobtain the expression given by Fisher and Ferdinand for a critical Ising model on a torus corresponding to a free (Majorana) Fermi field with $c=\frac{1}{2}$. We exhibit the relation with the conformal characters and discuss correlation functions.

In sect. 4 we study generalizations of the above formulas for the models with central charge in the unitary series of Friedan, Qiu and Shenker. Using modular invariance of the partition functions on a torus, we observe the existence of two sequences of critical models with central charge $c=1-6 / m(m+1), m$ an integer $\geq 3$. In the main sequence all scalar primary conformal operators occur (eqs. (4)-(12)) whereas in the complementary series there appear chiral (i.e. angular momentum carrying) operators, and some typical subset of conformal dimensions (eqs. (4)-(17)).

In sect. 5, generalizing results of Rocha-Caridi [8], the same partition functions are expressed in terms of determinants of Bose and Fermi fields coupled through boundary conditions.

The final section presents a summary, discusses several open problems and contains a short discussion of non-unitary representations exemplified by the case of the Lee-Yang singularity.

## 2. Gaussian model

A free field $\varphi(z, \bar{z})$, assumed real, has a lagrangian given by $\mathscr{L}=\frac{1}{2}\left(\partial_{\mu} \varphi\right)^{2}$ and the action is diagonalized by expanding $\varphi$ in proper modes of (minus) the laplacian with eigenvalues $\lambda \geq 0$. Omitting the zero mode the partition function is the ill-defined expression (because of an ultraviolet divergence) $Z_{1}=A^{1 / 2} \Pi^{\prime} \lambda^{-1 / 2}$. We intend to study this expression for the torus described in the introduction $\mathbb{T}=\mathbb{C} / \mathrm{L}$, where the lattice L is generated by $\omega_{1}, \omega_{2} ; \tau$ is the modular ratio $\tau=\omega_{2} / \omega_{1}$ defined to lie in the upper-half complex plane. To be specific $\omega_{1}$ and $\omega_{2}$ generate the system of closed geodesics on $\mathbb{T}$; any change of basis $\omega_{i}^{\prime}=n_{i j} \omega_{j}$ with the matrix $\left\{n_{i j}\right\}$ in $\mathrm{SL}(2, \mathbb{Z})$ generates a modular transformation on $\tau \mapsto \tau^{\prime}=\left(n_{21}+n_{22} \tau\right) /\left(n_{11}+n_{12} \tau\right)$. In such a transformation the area $A=\operatorname{Im} \omega_{2} \bar{\omega}_{1}$, remains invariant, disclosing the symplectic invariant form. The dual lattice $\tilde{\mathrm{L}}$ is generated by

$$
\begin{gather*}
k^{1}=-i \omega_{2} / A, \quad k^{2}=i \omega_{1} / A, \\
\operatorname{Re} k^{i} \bar{\omega}_{j}=\delta_{j}^{i}, \tag{2.1}
\end{gather*}
$$

is therefore such that $\tau_{k}=k^{2} / k^{1}=-\tau^{-1}$, implying conformal equivalence between L and $\tilde{\mathrm{L}}$. With periodic boundary conditions for the field $\varphi$ the eigenvalues $\lambda$ of minus the laplacian are of the form $(2 \pi)^{2}\left|n_{1} k^{1}+n_{2} k^{2}\right|^{2}=(2 \pi / A)^{2}\left|n_{1} \omega_{2}-n_{2} \omega_{1}\right|^{2}$.

We therefore define $Z_{1}$ using the analytic continuation procedure alluded to in the introduction, as

$$
\begin{equation*}
Z_{1}=A^{1 / 2} \exp \left(\frac{1}{2} G^{\prime}(0)\right) \tag{2.2}
\end{equation*}
$$

with

$$
\begin{equation*}
\left|\frac{2 \pi \omega_{1}}{A}\right|^{2 s} G(s)=\sum_{m, n}^{\prime} \frac{1}{|m+n \tau|^{2 s}} \tag{2.3}
\end{equation*}
$$

The prefactor $A^{1 / 2}$ insures that the renormalized determinant is scale invariant. Eq. (2.3) defines an analytic function for $\operatorname{Re} s>1$ which can be continued using a Mellin transformation following [11]. Summing first over $m$ then over $n$, with the $\zeta$ function given by $\zeta(s)=\sum_{m=1}^{\infty}\left(1 / m^{s}\right), 2 \zeta(0)=-1,2 \zeta^{\prime}(0)=-\ln 2 \pi$, one finds

$$
\left|\frac{2 \pi \omega_{1}}{A}\right|^{2 s} G(s)=2 \zeta(2 s)+\sum_{n}^{\prime} \sum_{m} \frac{1}{|m+n \tau|^{2 s}}
$$

Since the last sum over $m$ is a periodic function in $n \tau$ of unit period it reads

$$
\begin{aligned}
\sum_{m} \frac{1}{|m+n \tau|^{2 s}} & =\sum_{l} \exp (2 i \pi \ln \operatorname{Re} \tau) \int_{0}^{1} \mathrm{~d} y \mathrm{e}^{-2 i \pi l y} \sum_{m} \frac{1}{|m+y+i n \operatorname{Im} \tau|^{2 s}} \\
& =\sum_{l} \int_{-\infty}^{+\infty} \mathrm{d} y \exp (2 i \pi l(n \operatorname{Re} \tau-y)) \frac{1}{|y+i n \operatorname{Im} \tau|^{2 s}} \\
= & \frac{1}{\Gamma(s)} \sum_{l} \int_{-\infty}^{+\infty} \mathrm{d} y \int_{0}^{\infty} \mathrm{d} t t^{s-1} \exp \{2 i \pi l(n \operatorname{Re} \tau-y) \\
& \left.-t\left(y^{2}+n^{2}(\operatorname{Im} \tau)^{2}\right)\right\} \\
= & \frac{\pi^{1 / 2}}{\Gamma(s)} \sum_{l} \int_{0}^{\infty} \mathrm{d} t t^{s-3 / 2} \exp \left(-\left\{t^{2}(\operatorname{Im} \tau)^{2}+\frac{\pi^{2} l^{2}}{t}-2 i \pi \ln \operatorname{Re} \tau\right\}\right)
\end{aligned}
$$

where the last expression is a modified Bessel function. For $l=0$ the integral is $|n \operatorname{Im} \tau|^{1-2 s} \Gamma\left(s-\frac{1}{2}\right)$. For the other terms change $t$ into $|\pi l / n \operatorname{Im} \tau| t$. Using the functional equation for the $\zeta$ function [11] one finds

$$
\begin{align*}
\left|\frac{2 \pi \omega_{1}}{A}\right|^{2 s}\left(\frac{\operatorname{Im} \tau}{\pi}\right)^{s-1 / 2} G(s)= & 2\left(\frac{\operatorname{Im} \tau}{\pi}\right)^{s-1 / 2} \Gamma(s) \zeta(2 S) \\
& +2\left(\frac{\operatorname{Im} \tau}{\pi}\right)^{1 / 2-s} \Gamma(1-s) \zeta(2-2 s) \\
& +\sqrt{\pi} \sum_{l}^{\prime} \sum_{n}^{\prime} \exp (2 i \pi \ln \operatorname{Re} \tau)\left|\frac{l}{n}\right|^{s-1 / 2} \\
& \times \int_{0}^{\infty} \frac{\mathrm{d} t}{t} t^{s-1 / 2} \exp \left(-\pi|\ln | \operatorname{Im} \tau\left(t+t^{-1}\right)\right) \tag{2.4}
\end{align*}
$$

The last double sum is an even entire function of $s-\frac{1}{2}$, so that (2.4) exhibits both the symmetry $s \mapsto 1-s$ of the combination $\left|2 \pi \omega_{1} / A\right|^{2 s}(\operatorname{Im} \tau / \pi)^{s-1 / 2} G(s)$ and its meromorphic properties providing the required analytic continuation. Since for positive $x$

$$
\int_{0}^{1} \frac{\mathrm{~d} t}{t} t^{ \pm 1 / 2} \exp \left\{-x\left(t+t^{-1}\right)\right\}=\sqrt{\frac{\pi}{x}} \mathrm{e}^{-2 x}
$$

using $\zeta(2)=\frac{1}{6} \pi^{2}$, one finds that in the vicinity of $s=0$,

$$
\begin{aligned}
G(s)= & -1-s \ln \left|\frac{A}{2 \pi \omega_{1}}\right|^{2}-s \ln (2 \pi)^{2}+\frac{1}{3} s \pi \operatorname{Im} \tau \\
& +s \sum_{p}^{\prime} \sum_{n}^{\prime} \frac{1}{|p|} \exp (2 i \pi p n \operatorname{Re} \tau-2 \pi|p n| \operatorname{Im} \tau)+\mathrm{O}\left(s^{2}\right)
\end{aligned}
$$

Set

$$
\begin{equation*}
q=\mathrm{e}^{2 i \pi \tau}, \quad P(q)=\prod_{1}^{\infty}\left(1-q^{n}\right) \tag{2.5}
\end{equation*}
$$

with $P(q)^{-1}=\sum_{0}^{\infty} p(n) q^{n}$ the generating function for partitions; then

$$
\begin{align*}
& G(0)=-1 \\
& G^{\prime}(0)=-2 \ln \left\{\left|\frac{A}{\omega_{1}}\right|(q \bar{q})^{1 / 24} P(q) \overline{P(q)}\right\} \tag{2.6}
\end{align*}
$$

and

$$
\begin{equation*}
Z_{1}=A^{1 / 2} \mathrm{e}^{G^{\prime}(0) / 2}=\frac{\left|\omega_{1}\right|}{A^{1 / 2}} \frac{1}{\eta(q) \overline{\eta(q)}} \tag{2.7}
\end{equation*}
$$

with $\eta(q)$ the Dedekind function

$$
\begin{equation*}
\eta(q)=q^{1 / 24} P(q)=q^{1 / 24} \prod_{1}^{\infty}\left(1-q^{n}\right) \tag{2.8}
\end{equation*}
$$

for which we also use the notation $\eta[\tau], q=\mathrm{e}^{2 i \pi \tau}$, as in (1.8).
The free scalar field has an energy-momentum tensor [1] given by

$$
\begin{equation*}
\frac{T}{2 \pi}=\left(\partial_{z} \varphi\right)^{2}, \quad \frac{\bar{T}}{2 \pi}=\left(\partial_{\bar{z}} \varphi\right)^{2} \tag{2.9}
\end{equation*}
$$

with a Wick ordering prescription implied. In infinite space the field propagator is

$$
\begin{equation*}
\left\langle\varphi\left(z_{1}, \bar{z}_{1}\right) \varphi\left(z_{2}, \bar{z}_{2}\right)\right\rangle=\frac{1}{4 \pi} \ln \frac{R^{2}}{z_{12} \bar{z}_{12}}, \tag{2.10}
\end{equation*}
$$

with $z_{12}=z_{1}-z_{2}$ and $R$ an arbitrary scale. The central charge is such that

$$
\begin{equation*}
\left\langle T\left(z_{1}\right) T\left(z_{2}\right)\right\rangle=\left\langle\bar{T}\left(\bar{z}_{1}\right) \bar{T}\left(\bar{z}_{2}\right)\right\rangle^{*}=\frac{c}{2 z_{12}^{4}} . \tag{2.11}
\end{equation*}
$$

From the above this yields

$$
\begin{equation*}
c=1 \tag{2.12}
\end{equation*}
$$

Therefore the behavior of $Z_{1}$ in the limit of a periodic strip agrees with Cardy's expression (1.5). More generally in the limit $q \mapsto 0$

$$
\begin{equation*}
Z_{\mathrm{c}} \underset{q \rightarrow 0}{\sim}(q \bar{q})^{-c / 24} \tag{2.13}
\end{equation*}
$$

The expression (2.7) is remarkable for its modular properties. Indeed the analytic function $G(s)$, and therefore $Z_{1}$ itself is clearly invariant in a modular transformation, i.e. an $\operatorname{SL}(2, \mathbb{Z})$ change of basis vectors in L

$$
\begin{array}{lc}
\omega_{1}^{\prime}=n_{11} \omega_{1}+n_{12} \omega_{2}, &  \tag{2.14a}\\
\omega_{2}^{\prime}=n_{21} \omega_{1}+n_{22} \omega_{2}, & n_{11} n_{22}-n_{12} n_{21}=1, \\
\tau^{\prime}=\frac{n_{21}+n_{22} \tau}{n_{11}+n_{12} \tau}, & A^{\prime}=A .
\end{array}
$$

It follows therefore that

$$
\begin{equation*}
\left(n_{11}+n_{12} \tau\right)^{-1 / 2} \eta\left(\exp \left(2 i \pi \frac{n_{21}+n_{22} \tau}{n_{11}+n_{12} \tau}\right)\right)=\varepsilon \eta(\exp 2 i \pi \tau), \tag{2.14b}
\end{equation*}
$$

where $\varepsilon$ is a phase (in fact $\varepsilon^{24}=1$, see sect. 4).
This property is of course crucial to express that $Z_{1}$ is attached to the torus $\mathbb{T}$ and not to the manner in which we describe it. More generally it will have to hold for any partition function constructed in the sequel.

The modular group is generated by the two transformations $\tau \mapsto \tau+1$ and $\tau \mapsto-\tau^{-1}$, the latter having square one, so that a closed fundamental domain in the upper-half plane is $|\operatorname{Re} \tau| \leq \frac{1}{2},|\tau| \geq 1$. The factorized form observed in (2.7) is also typical. Up to the prefactor $\left|\omega_{1} / A^{1 / 2}\right|$ special to the $c=1$ case, one finds the modulus square of a function of $q$ alone, which is a projective invariant of $\operatorname{SL}(2, \mathbb{Z})$.

This generalizes to the contribution of a so-called "conformal block" in the terminology of BPZ up to slight complications which will appear below. It is also typical that we find a reciprocity relation between two tori, namely $\mathbb{T}$ associated to the lattice L and $\tilde{\mathbb{T}}$ associated to the reciprocal lattice $\tilde{\mathrm{L}}$. Since from (2.14)

$$
\begin{gathered}
\left|\omega_{2}^{-1 / 2} \eta\left(\exp \left(-2 i \pi \tau^{-1}\right)\right)\right|=\left|\omega_{1}^{-1 / 2} \eta(\exp 2 i \pi \tau)\right|, \\
\tilde{A} A=1, \quad\left|k^{1}\right|=\frac{1}{A} \omega_{2}, \quad \frac{k^{2}}{k^{1}}=-\tau^{-1}
\end{gathered}
$$

it follows that

$$
\begin{equation*}
Z_{1}(\mathbb{T})=Z_{1}(\tilde{\mathbb{T}}) \tag{2.15}
\end{equation*}
$$

expressing the conformal equivalence of the two lattices L and $\tilde{\mathrm{L}}$. With $c=1$ we can rewrite $Z_{1}$ as

$$
\begin{align*}
Z_{1} & =\left|\frac{\omega_{1}}{A^{1 / 2}}\right| \operatorname{Tr} q^{L_{0}-c / 24} \bar{q}^{L_{0}-c / 24} \\
& =\left|\frac{\omega_{1}}{A^{1 / 2}}\right| \operatorname{Tr} \exp \left(2 i \pi \operatorname{Re} \tau\left(L_{0}-\bar{L}_{0}\right)-2 \pi \operatorname{Im} \tau\left(L_{0}+\bar{L}_{0}-\frac{1}{12} c\right)\right) \tag{2.16}
\end{align*}
$$

with $L_{0}\left(\bar{L}_{0}\right)$ related to the expansion of the energy-momentum in the plane through (the bar partners obey similar relations)

$$
\begin{equation*}
T(z)=\sum_{-\infty}^{+\infty} z^{-2-p} L_{p}, \quad\left[L_{n}, L_{p}\right]=(n-p) L_{n+p}+\frac{1}{12} c n\left(n^{2}-1\right) \delta_{n+p, 0} \tag{2.17}
\end{equation*}
$$

which define the generators of the commuting isomorphic Virasoro algebras. In a rectangular situation, $q=\bar{q}$, and $L_{0}+\bar{L}_{0}$ is a hamiltonian in a transfer matrix formalism. In agreement with (2.13) $L_{0}\left(\bar{L}_{0}\right)$ is displaced by an amount $-\frac{1}{24} c$. For both $L_{0}$ and $\bar{L}_{0}$ the spectrum is real, starts at $h=0$, is integer spaced and the eigenvalue $n$ is $p(n)$ times degenerated. In general for an irreducible representation of the Virasoro algebra of central charge $c$ with "highest" weight $h$ (i.e. lowest eigenstate of $L_{0}$ with eigenvalue $h$, annihilated by the $L_{p}$ for positive $p$ ) and such that the eigenvalue $h+n$ of level $n$ is $\operatorname{dim}_{n}$ degenerate, one defines the associated character

$$
\begin{equation*}
\chi_{c, h}(q)=\operatorname{Tr} q^{L_{0}}=q^{h} \sum_{n=0}^{\infty} \operatorname{dim}_{n} q^{n}, \tag{2.18}
\end{equation*}
$$

a convergent series for $|q|<1$. With a small abuse of notation (since the free field $\varphi$ is not a primary field), $Z_{1}$ is expressed in terms of

$$
\begin{equation*}
\chi_{1,0}(q)=P(q)^{-1} \tag{2.19}
\end{equation*}
$$

the easiest of a series of results obtained by Kac, Feigin and Fuks, and Rocha-Caridi [7, 8].

Let us quote the important results for the unitary subseries of the so-called "degenerate" representations analysed by Friedan, Qiu and Shenker [2]. For the values of the central charge

$$
\begin{gather*}
c=1-\frac{6}{m(m+1)}, \quad m \text { integer }=3,4, \ldots, \\
h \equiv h_{r s}=\frac{[r(m+1)-s m]^{2}-1}{4 m(m+1)}, \quad 1 \leq s \leq r \leq m-1, \\
P(q) \chi_{c, h}(q)=\sum_{n=-\infty}^{+\infty}\left\{q^{\left([2 n m(m+1)+r(m+1)-s m]^{2}-1\right) / 4 m(m+1)}-(s \leftrightarrow-s)\right\} . \tag{2.20}
\end{gather*}
$$

These expressions will be useful in the sequel.
It is important to realize that there exists no one to one continuous map (let alone conformal) of a torus onto a (Riemann) sphere, i.e. the complex plane completed by a point at infinity and its system of neighborhoods. Their genus $g$ or Euler characteristic $2-2 g$ are distinct. Conformal invariance alone does not allow to derive directly expressions for the torus from similar results in the plane. But because a torus can be viewed as a factor space $\mathbb{C} / L$ the method of images familiar from electrostatics is a powerful tool. For instance an important quantity is the two-point function generalizing (2.10) which will also be useful later on. Because of the zero mode subtraction this quantity cannot be defined as the elementary solution of (minus) the laplacian. Another way to put it is to note that on a compact space a source of given intensity has to find a sink to absorb an equal amount of flux. Therefore on a torus the relevant equation is

$$
\begin{equation*}
-\Delta\left\langle\varphi\left(z_{1}, \bar{z}_{1}\right) \varphi\left(z_{2}, \bar{z}_{2}\right)\right\rangle=\delta^{2}\left(z_{12}\right)-\frac{1}{A}, \tag{2.21}
\end{equation*}
$$

with the zero mode subtraction on the r.h.s. to insure a vanishing integral. The $\delta$-function is understood as a (doubly) periodic function. The symmetric solution is neatly expressed in terms of Jacobi's $\theta$-functions [11]. Set

$$
\begin{align*}
q & =\mathrm{e}^{2 i \pi \tau} \\
y & =\mathrm{e}^{2 i \pi z / \omega_{1}} \\
F(z) & =P(q)^{-2}\left(1-y^{-1}\right) \prod^{\infty}\left(1-y q^{n}\right)\left(1-y^{-1} q^{n}\right) \tag{2.22}
\end{align*}
$$

Then

$$
\begin{align*}
\Gamma_{12} & =\exp \left\{-4 \pi\left\langle\varphi\left(z_{1}, \bar{z}_{1}\right) \varphi\left(z_{2}, \bar{z}_{2}\right)\right\rangle\right\} \\
& =\left|\frac{\omega_{1}}{2 \pi} F\left(z_{12}\right)\right|^{2} \exp \left(-2 \pi\left\{\frac{\left(\left(\operatorname{Im} z_{12}\right) / \omega_{1}\right)^{2}}{\operatorname{Im} \tau}+\left(\operatorname{Im} z_{12}\right) / \omega_{1}\right\rangle\right) \\
& \approx z_{12} \bar{z}_{12} . \tag{2.23}
\end{align*}
$$

The normalization has been chosen such that (2.23) agrees with (2.10) for $R=1$. Observe that (2.23) cannot be written as the modulus square of an analytic function in contrast to what happens in the plane or in a periodic strip. It is tedious to verify, but of course true, that the normalization guarantees modular invariance.

Consider (2.23) when $z_{12}$ is expressed as

$$
\begin{equation*}
z_{12}=-\frac{k}{N} \omega_{1}-\frac{l}{N} \omega_{2} \tag{2.24}
\end{equation*}
$$

Care must now be taken in expressing modular invariance since of course the real coordinates $k / N, l / N$ will be affected by a change of basis. Then one has $y=$ $\mathrm{e}^{-2 i \pi k / N} q^{-1 / N}$

$$
\begin{gather*}
\exp \left\{-4 \pi\left\langle\varphi(0) \varphi\left(\frac{k \omega_{1}+l \omega_{2}}{N}\right)\right)\right\}=\frac{A}{(2 \pi)^{2}} Z_{1}^{2}\left|D_{k / n, l / N}(q)\right|^{2} \\
D_{k / N, l / N}(q)=q^{-\left[6 l(N-l) / N^{2}-1\right] / 12} \prod_{n=0}^{\infty}\left(1-\mathrm{e}^{2 i \pi k / N} q^{n+l / N}\right) \\
\times\left(1-\mathrm{e}^{-2 i \pi k / N} q^{n+(N-l) / N}\right) \tag{2.25}
\end{gather*}
$$

By construction $\left|D_{0,0}\right|^{2}$ vanishes, $Z_{1}^{-2}$ being a "renormalized" version of this quantity. It would therefore appear legitimate to define $\left|D_{0,0}^{\mathrm{R}}\right|^{-2}=A /(2 \pi)^{2} Z_{1}^{2}$ in which case the above expression would appear as the ratio $\left|D_{k / N, t / N} / D_{0,0}^{\mathrm{R}}\right|^{2}$.

As the quantities $D_{k / N, l / N}$ will appear repeatedly in the following, we mention some of their modular properties.

If $\tilde{q}=\exp (-2 i \pi / \tau)$, one has

$$
\begin{equation*}
\left|D_{k / N, l / N}(q)\right|=\left|D_{(N-k) / N,(N-l) / N}(q)\right|=\left|D_{l / N,(N-k) / N}(\tilde{q})\right| \tag{2.26a}
\end{equation*}
$$

Our notation always implies that $q=\mathrm{e}^{2 i \pi \tau}$, so that in the change $\tau \mapsto \tau+1$

$$
\begin{equation*}
D_{k / N, l / N}\left(\mathrm{e}^{2 i \pi} q\right)=D_{(k+l) / N, l / N}(q) \exp \left(-\frac{2}{12} i \pi\left[\frac{6 l(N-l)}{N^{2}}-1\right]\right) \tag{2.26b}
\end{equation*}
$$

and for integer $p$

$$
\begin{align*}
& D_{k / N+p, l / N}(q)=D_{k / N, l / N}(q) \\
& D_{k / N, l / N+p}(q)=(-1)^{p} \mathrm{e}^{-2 i \pi k p / N} D_{k / N, l / N}(q) \tag{2.26c}
\end{align*}
$$

The notation suggested - but did not require - that $k, l$ and $N$ were integers (and $N$ positive). This is now what we assume. Under those circumstances we will now show that $D_{k / N, l / N}(q)$ - when at least one of $k, l$ is not equal to zero $\bmod N$, otherwise we would get zero - can be interpreted in terms of free field path integrals. Namely let us repeat the same calculation that we performed to obtain $Z_{1}$. But we make two modifications:
(i) We sum over modes which in complex notation read

$$
\begin{equation*}
-\left(n_{2}+\frac{l}{N}\right) k^{1}+\left(n_{1}+\frac{k}{N}\right) k^{2}=\frac{i}{A}\left\{\left(n_{1}+\frac{k}{N}\right) \omega_{1}+\left(n_{2}+\frac{l}{N}\right) \omega_{2}\right\} . \tag{2.27}
\end{equation*}
$$

This means that the fields are no longer periodic but are multiplied by a phase $\mathrm{e}^{-2 i \pi / / N}$ as they wind around the torus along the generator $\omega_{1}$ and $\mathrm{e}^{2 i \pi k / N}$ along $\omega_{2}$. To do so we must express a real field as the real part of a complex field.
(ii) Instead of computing a (renormalized) inverse square root determinant, we simply compute the (renormalized) determinant $\exp \left(-G_{k / N, I / N}^{\prime}(0)\right)$ omitting any $A$-dependent prefactor, with

$$
\begin{equation*}
G_{k / N, l / N}(s)=\left(\frac{A}{2 \pi\left|\omega_{1}\right|}\right)^{2 s} \sum_{m, n} \frac{1}{|m+n \tau+(k+l \tau) / N|^{2 s}} \tag{2.28}
\end{equation*}
$$

When $k$ and $l$ are not both zero $\bmod N$ the sum runs over all integers $m$ and $n$. The calculation proceeds as before with the result that $G_{k / N, I / N}(0)=0$, and

$$
\begin{align*}
\exp \left(-G_{k / N, I / N}^{\prime}(0)\right)= & \exp \left\{-\frac{\operatorname{Im} \tau}{\pi} \sum_{n}^{\prime} \frac{\mathrm{e}^{2 i \pi / n / N}}{n^{2}}\right\} \\
& \times\left\{\prod_{n=0}^{\infty}\left(1-\mathrm{e}^{2 i \pi k / N} q^{n+l / N}\right)\right. \\
& \left.\times\left(1-\mathrm{e}^{-2 i \pi k / N} q^{n+(N-l) / N}\right)\right\} \times\{\text { c.c. }\} \\
= & \left|D_{k / n, l / N}(q)\right|^{2} \tag{2.29}
\end{align*}
$$

One recognizes that

$$
\begin{aligned}
& \frac{1}{\pi} \sum_{n}^{\prime} \frac{\mathrm{e}^{2 i \pi \ln / N}}{n^{2}}=\frac{\pi}{N^{2}}\left(\frac{1}{3}+\sum_{s=1}^{N-1} \frac{\mathrm{e}^{2 i \pi / s / N}}{\sin (2 \pi s / N)}\right) \\
& \frac{1}{3} \pi=\frac{1}{\pi} \sum_{n}^{\prime} \frac{1}{n^{2}}=\frac{\pi}{N^{2}}\left(\frac{1}{3}+\sum_{s=1}^{N-1} \frac{1}{\sin (2 \pi s / N)}\right)
\end{aligned}
$$

Hence

$$
\begin{align*}
\frac{1}{\pi} \sum_{n} \frac{\mathrm{e}^{2 i \pi / n / N}}{n^{2}} & =\frac{1}{3} \pi-\frac{2 \pi}{N^{2}} \sum_{s=1}^{N-1}\left(\frac{\sin (\pi l s / N)}{\sin (\pi s / N)}\right)^{2} \\
& =\frac{1}{3} \pi\left[1-\frac{6}{N^{2}}\left(\sum_{s=0}^{N-1}\left(\sum_{m=-j}^{+j} \mathrm{e}^{2 i \pi s m / N}\right)^{2}-l^{2}\right)\right] \\
& =\frac{1}{3} \pi\left[1-\frac{6}{N^{2}} l(N-l)\right] \tag{2.30}
\end{align*}
$$

so that (2.29) agrees with the previous expression (2.25) of $D$. We could of course also identify $|D|^{2}$ as a grassmannian integral over a complex free fermi field obeying the above mentioned boundary conditions.

Before leaving this section we add a remark concerning the energy momentum tensor. Let us apply formula (1.4) to express the result of a small deformation of the torus. In particular we can choose a quasi-conformal transformation $\delta r^{\mu}=\delta_{\varepsilon}^{\mu \sigma} r_{\sigma}$ with a constant $\delta \varepsilon$ matrix. Writing $\omega_{1}$ and $\omega_{2}$ in vector (instead of complex) form $\delta \omega_{i}=\delta \varepsilon \omega_{i}\left\langle T_{\mu \nu}\right\rangle \partial^{\mu} \delta r^{\nu}=\left\langle T_{\mu \nu}\right\rangle \delta \varepsilon^{\mu \nu}=\langle T\rangle\left[\delta \varepsilon^{11}-\delta \varepsilon^{22}+i\left(\delta \varepsilon^{12}+\delta \varepsilon^{21}\right)\right]+$ c.c. . On a torus $\langle T(z)\rangle$ is a constant. Therefore we have

$$
\begin{equation*}
\delta \ln Z=-\frac{A}{2 \pi}\langle T\rangle\left[\delta \varepsilon^{11}-\delta \varepsilon^{22}+i\left(\delta \varepsilon^{12}+\delta \varepsilon^{21}\right)\right]+\text { c.c. } \tag{2.31}
\end{equation*}
$$

This formula applies in the general case; we restrict it here to $c=1$ with

$$
\begin{aligned}
\delta \ln Z_{1} & =\frac{\delta\left|\omega_{1}\right|}{\left|\omega_{1}\right|}-\frac{1}{2} \frac{\delta A}{A}-\frac{\delta \eta(q)}{\eta(q)}-\frac{\overline{\delta \eta(q)}}{\overline{\eta(q)}} \\
& =\left[\delta \varepsilon^{11}-\delta \varepsilon^{22}+i\left(\delta \varepsilon^{12}+\delta \varepsilon^{21}\right)\right]\left\{\frac{1}{4} \frac{\bar{\omega}_{1}}{\omega_{1}}+i \pi \tau\left(\frac{\bar{\omega}_{1}}{\omega_{1}}-\frac{\bar{\omega}_{2}}{\omega_{2}}\right) q \frac{\eta^{\prime}(q)}{\eta(q)}\right\}+\text { с.c.. }
\end{aligned}
$$

Comparing with (2.31) this yields for the free field Bose case

$$
\begin{equation*}
\langle T\rangle_{\text {torus }}=-\frac{\pi}{2 A} \frac{\bar{\omega}_{1}}{\omega_{1}}+\left(\frac{2 \pi}{\omega_{1}}\right)^{2} q \frac{\eta^{\prime}(q)}{\eta(q)}, \quad c=1 \tag{2.32}
\end{equation*}
$$

as a generalization of (1.3). The first term is a peculiarity of the case $c=1$. The second term reads explicitly

$$
\begin{align*}
24 q \frac{\eta^{\prime}(q)}{\eta(q)}=\left[1-24 \sum_{1}^{\infty} \frac{n q^{n}}{1-q^{n}}\right] & =1-24 \sum_{1}^{\infty} \sigma_{1}(p) q^{p} \\
& =1+6 \sum_{1}^{\infty} \frac{1}{(\sin \pi p \tau)^{2}} \tag{2.33}
\end{align*}
$$

where $\sigma_{1}(p)$ is the sum of the divisors of $p$.
Therefore one can also write

$$
\begin{equation*}
\langle T\rangle_{\mathrm{torus}}=-\frac{\pi}{2 A} \frac{\bar{\omega}_{1}}{\omega_{1}}+\frac{1}{2} \sum_{n, p}^{\prime} \frac{1}{\left(n \omega_{1}+p \omega_{2}\right)^{2}} \tag{2.34}
\end{equation*}
$$

where the last sum is understood as a double limit, summing first symmetrically on $p$ then on $n$.

With the help of the previous formalism we turn to specific cases.

$$
\text { 3. Ising model } m=3, c=\frac{1}{2}
$$

At the critical temperature the Ising model reduces to a free massless Majorana field theory with a lagrangian

$$
\begin{equation*}
\mathscr{L}=\frac{1}{2}(\psi \tilde{\partial} \psi+\tilde{\psi} \partial \tilde{\psi}) \tag{3.1}
\end{equation*}
$$

provided one uses anticommuting variables in a grassmannian path integral. The partition function is formally the product of pfaffians $\operatorname{Pf}(\partial) \operatorname{Pf}(\bar{\partial})$ times a constant, i.e. the positive square root $(\operatorname{det}-\Delta)^{1 / 2}$. On a compact torus care must be taken of boundary conditions [14]. The partition function splits into the sum of four distinct terms, each one corresponding to a choice of periodic or antiperiodic boundary conditions for the field. This looks at first slightly puzzling but has to do with the following circumstance. It must be remembered that it is the original Ising spin system which is periodic. The Jordan-Wigner transformation needed to change spins into fermionic variables singles out the last coupling which closes one of the generators of the torus. This has the result of splitting the transfer matrix into two distinct blocks corresponding to even or odd boundary conditions for the fermion
operators (in hamiltonian language). In a path integral for a torus such a splitting of boundary conditions should be applied to both directions since the same hamiltonian is valid in both directions. The required square roots of determinants have just been computed as $\left|D_{0,0}\right|=0,\left|D_{1 / 2,0}\right|,\left|D_{0,1 / 2}\right|$ and $\left|D_{1 / 2,1 / 2}\right|$ and it is therefore no accident that relevant $D$ 's appeared as squares. We write

$$
\begin{align*}
& D_{1 / 2,0}=2 q^{-1 / 24} d_{1 / 2,0}^{2}, \quad D_{0,1 / 2}=q^{-1 / 24} d_{0,1 / 2}^{2}, \quad D_{1 / 2,1 / 2}=q^{-1 / 24} d_{1 / 2,1 / 2}^{2} \\
& d_{1 / 2,0}=q^{1 / 16} \prod_{n=1}^{\infty}\left(1+q^{n}\right)=q^{1 / 16} \frac{P\left(q^{2}\right)}{P(q)}, \\
& d_{0,1 / 2}=\prod_{n=0}^{\infty}\left(1-q^{n+1 / 2}\right)=\frac{P\left(q^{1 / 2}\right)}{P(q)}, \\
& d_{1 / 2,1 / 2}=\prod_{n=0}^{\infty}\left(1+q^{n+1 / 2}\right)=\frac{P(q)^{2}}{P\left(q^{2}\right) P\left(q^{1 / 2}\right)},  \tag{3.2a}\\
& q^{-1 / 16} d_{1 / 2,1 / 2}(q) d_{0,1 / 2}(q) d_{1 / 2,0}(q)=1, \quad d_{1 / 2,1 / 2}^{8}(q)=16 d_{1 / 2,0}^{8}(q)+d_{0,1 / 2}^{8}(q) \tag{3.2b}
\end{align*}
$$

The critical partition function of the Ising model is therefore equal, up to an overall constant, to the sum

$$
\left|D_{1 / 2,0}\right|+\left|D_{0,1 / 2}\right|+\left|D_{1 / 2,1 / 2}\right|,
$$

i.e.

$$
\begin{equation*}
Z_{1 / 2}=(q \bar{q})^{-1 / 48}\left\{\left|d_{1 / 2,1 / 2}(q)\right|^{2}+\left|d_{0.1 / 2}(q)\right|^{2}+2\left|d_{1 / 2,0}(q)\right|^{2}\right\} \tag{3.3}
\end{equation*}
$$

in agreement with the result of Ferdinand and Fisher [15]. Since as $q \mapsto 0$ the bracket goes to 2, we read from (3.3) that

$$
\begin{equation*}
c=\frac{1}{2}, \tag{3.4}
\end{equation*}
$$

a well known fact. More interestingly as shown by Cardy [5] this should also agree with a sum of characters in modulus square. Before we compare with (2.20) it is good to notice that for the two generators of the modular group we have the table of changes

$$
\begin{array}{llll} 
& \left|D_{1 / 2,0}\right| & \left|D_{0,1 / 2}\right| & \left|D_{1 / 2,1 / 2}\right| \\
\tau \mapsto \tau+1 & \left|D_{1 / 2,0}\right| & \left|D_{1 / 2,1 / 2}\right| & \left|D_{0,1 / 2}\right|  \tag{3.5}\\
\tau \mapsto-\tau^{-1} & \left|D_{0,1 / 2}\right| & \left|D_{1 / 2,0}\right| & \left|D_{1 / 2,1 / 2}\right| .
\end{array}
$$

Since permutations on three objects are generated by two transpositions the only linear modular invariant is the sum of the three $|D|$ 's. This offers another direct justification for (3.3).

Eq. (3.3) can be rewritten in the form

$$
\begin{align*}
Z_{1 / 2}(q, \bar{q}) & =2(q \bar{q})^{-1 / 48}\left\{\sum_{ \pm}\left|\frac{1}{2}\left(d_{1 / 2,1 / 2}(q) \pm d_{0,1 / 2}(q)\right)\right|^{2}+\left|d_{1 / 2,0}(q)\right|^{2}\right\} \\
& =2(q \bar{q})^{-1 / 48}\left\{\left|\chi_{1 / 2,0}(q)\right|^{2}+\left|\chi_{1 / 2,1 / 2}(q)\right|^{2}+\left|\chi_{1 / 2,1 / 16}(q)\right|^{2}\right\} \tag{3.6}
\end{align*}
$$

where we recognize the characters of formula (2.20) for $c=\frac{1}{2}$, namely $m=3$. The factor 2 in front could be scaled away.
$h_{11}=0$, related to the identity operator $I$

$$
\begin{aligned}
\chi_{1 / 2,0}(q) & =\frac{1}{2}\left(d_{1 / 2,1 / 2}(q)+d_{0.1 / 2}(q)\right)=\frac{1}{2}\left\{\prod_{0}^{\infty}\left(1+q^{n+1 / 2}\right)+\prod_{0}^{\infty}\left(1-q^{n+1 / 2}\right)\right\} \\
& =\frac{1}{P(q)} \sum_{k}\left\{q^{\left((24 k+1)^{2}-1\right) / 48}-q^{\left((24 k+7)^{2}-1\right) / 48}\right\}_{q \rightarrow 0}^{\sim} 1
\end{aligned}
$$

$h_{2.1}=\frac{1}{2}$, related to the energy operator $\varepsilon$

$$
\begin{aligned}
\chi_{1 / 2,1 / 2}(q) & =\frac{1}{2}\left(d_{1 / 2,1 / 2}(q)-d_{0,1 / 2}(q)\right)=\frac{1}{2}\left\{\prod_{0}^{\infty}\left(1+q^{n+1 / 2}\right)-\prod_{0}^{\infty}\left(1-q^{n+1 / 2}\right)\right\} \\
& =\frac{1}{P(q)} \sum_{k}\left\{q^{\left((24 k+5)^{2}-1\right) / 48}-q^{\left((24 k+11)^{2}-1\right) / 48}\right\} \underset{q \rightarrow 0}{\sim} q^{1 / 2} ;
\end{aligned}
$$

$h_{22}=\frac{1}{16}$, related to the spin operator $\sigma$

$$
\begin{align*}
\chi_{1 / 2,1 / 16}(q) & =d_{1 / 2,0}(q)=q^{1 / 16} \prod_{0}^{\infty}\left(1+q^{n}\right) \\
& =\frac{1}{P(q)} \sum_{k}\left\{q^{\left((24 k-2)^{2}-1\right) / 48}-q^{\left((24 k+10)^{2}-1\right) / 48}\right\} \underset{q \rightarrow 0}{\sim} q^{1 / 16} \tag{3.7}
\end{align*}
$$

These formulas appear in [8] and follow from well known identities on $\theta$ functions. As claimed by Cardy, the partition function on a torus exhibits not only the central charge (see (1.5)), the dimensions of the (finite) set of conformal primary fields [1], or observables but also the number of their "descendants" at a given level $n$. In short it is built on characters $\chi_{c, h}(q) \chi_{c, h}(\bar{q})$ of the direct product of the two commuting Virasoro algebras operating on analytic and antianalytic fields.

The relation between fermions and bosons is slightly involved. Namely a naive view would be that $Z_{1 / 2} Z_{1}$ is a constant reflecting the usual correspondence used for instance as an alternative to replica methods in applications to disordered or constrained systems. Aware of the existence of anomalies we want to obtain the precise form of this relation in the present case. For this purpose associate to the torus $\mathbb{T}$ with periods $\omega_{1}, \omega_{2}$ the two tori $\mathbb{T}^{\prime}$ with periods $2 \omega_{1}$ and $\omega_{2}$ and $\mathbb{T}^{\prime \prime}$ with periods $\omega_{1}$ and $2 \omega_{2}$, then from (2)

$$
\begin{align*}
Z_{1 / 2}(\mathbb{T}) & =\left\{\left|\frac{\eta\left(q^{1 / 2}\right)}{\eta(q)}\right|^{2}+\left|\frac{\eta\left(q^{2}\right)}{\eta(q)}\right|^{2}+\left|\frac{\eta^{2}(q)}{\eta\left(q^{2}\right) \eta\left(q^{1 / 2}\right)}\right|^{2}\right\} \\
& =\left\{\sqrt{2} \frac{Z_{1}(\mathbb{T})}{Z_{1}\left(\mathbb{T}^{\prime}\right)}+\sqrt{2} \frac{Z_{1}(\mathbb{T})}{Z_{1}\left(\mathbb{T}^{\prime \prime}\right)}+\frac{Z_{1}\left(\mathbb{T}^{\prime}\right) Z_{1}\left(\mathbb{T}^{\prime \prime}\right)}{Z_{1}(\mathbb{T})^{2}}\right\} . \tag{3.8a}
\end{align*}
$$

Hence $Z_{1 / 2}(\mathbb{T}) Z_{1}(\mathbb{T})$ which tends to one in the infinite volume limit, differs from unity on the torus by an amount dictated by (3.8), which exhibits what might be termed an anomaly of naive supersymmetry. Using eq. (2.25) an alternative form of (3.8) is

$$
\begin{equation*}
Z_{1 / 2}(\mathbb{T}) Z_{1}(\mathbb{T})=\frac{2 \pi}{A^{1 / 2}} \sum_{k, l=0,1}^{\prime} \exp \left(-2 \pi\left\langle\varphi(0) \varphi\left(\frac{1}{2}\left(k \omega_{1}+l \omega_{2}\right)\right)\right\rangle\right), \tag{3.8b}
\end{equation*}
$$

where we recall that the term with $k=l=0$ vanishes.
In the plane the correlation function of the Fermi field $\psi$ is the Cauchy kernel

$$
\begin{equation*}
\pi\left\langle\psi_{1} \psi_{2}\right\rangle=\frac{1}{z_{12}^{\prime}} \quad \text { (plane) } \tag{3.9}
\end{equation*}
$$

Under the conformal map on a periodic strip of width $L, z^{\prime}=\mathrm{e}^{2 \pi i z / L}$, with $\psi(z) \mathrm{d} z^{1 / 2}$ invariant, one finds

$$
\begin{equation*}
\pi\left\langle\psi_{1} \psi_{2}\right\rangle_{\mathrm{NS}}=\frac{\pi / L}{\sin \left(\pi z_{12} / L\right)} \quad \text { (strip) } \tag{3.10}
\end{equation*}
$$

which obeys antiperiodic boundary conditions across the strip and corresponds to the so-called Neveu-Schwarz (NS) boundary conditions. It would also appear natural to have another propagator corresponding to periodic (Ramond) boundary conditions (but still odd in the interchange $1 \leftrightarrow 2$ ) namely

$$
\begin{equation*}
\pi\left\langle\psi_{1} \psi_{2}\right\rangle_{\mathrm{R}}=\frac{\pi / L}{\operatorname{tg}\left(\pi z_{12} / L\right)} \quad(\text { strip }) \tag{3.11}
\end{equation*}
$$

On a torus, we recall that, in our previous notations ( $\left.\frac{1}{2} k, \frac{1}{2} l\right)$, the first index refers to (anti) periodicity in the direction 2 while the second refers to (anti) periodicity in the direction 1 . We would still like to solve

$$
\begin{equation*}
\bar{\partial}\left\langle\psi_{1} \psi_{2}\right\rangle=\delta^{2}\left(z_{12}\right) . \tag{3.12}
\end{equation*}
$$

If we cut the torus along the generators $\omega_{1}$ and $\omega_{2}$ and integrate $S(z)=\langle\psi(z) \psi(0)\rangle$ (assumed meromorphic in $z$ ) along the closed contour $\mathscr{C}$ generated by $\omega_{1}$ and $\omega_{2}$, in the positive direction, one finds

$$
\begin{align*}
i & =\frac{1}{2} \int_{\mathscr{C}} S(z) \mathrm{d} z \\
& =\frac{1}{2}\left[1-(-1)^{k}\right] \omega_{1} \int_{0}^{1} S\left(z_{0}+t \omega_{1}\right) \mathrm{d} t-\frac{1}{2}\left[1-(-1)^{t}\right] \omega_{2} \int_{0}^{1} \mathrm{~d} t S\left(z_{0}+t \omega_{2}\right) \tag{3.13a}
\end{align*}
$$

and we recall that $k$ and $l$ are not both zero $(\bmod 2)$. One has therefore three possibilities which we denote $S_{k / 2,1 / 2}(z)$ constructed from ratios of Jacobi $\theta$ functions. We use the notations (2.22),

$$
\begin{align*}
\tilde{S}_{1 / 2,0} & =\frac{F\left(z-\frac{1}{2} \omega_{1}\right)}{F(z)}=\frac{y+1}{y-1} \prod_{1}^{\infty} \frac{\left(1+y q^{n}\right)\left(1+y^{-1} q^{n}\right)}{\left(1-y q^{n}\right)\left(1-y^{-1} q^{n}\right)}, \\
\tilde{S}_{0,1 / 2} & =\mathrm{e}^{-i \pi z / \omega_{1}} \frac{F\left(z-\frac{1}{2} \omega_{2}\right)}{F(z)}=\frac{1}{y^{1 / 2}-y^{-1 / 2}} \prod_{1}^{\infty} \frac{\left(1-y q^{n-1 / 2}\right)\left(1-y^{-1} q^{n-1 / 2}\right)}{\left(1-y q^{n}\right)\left(1-y^{-1} q^{n}\right)}, \\
\tilde{S}_{1 / 2,1 / 2} & =\mathrm{e}^{-i \pi z / \omega_{1}} \frac{F\left(z-\frac{1}{2}\left(\omega_{1}+\omega_{2}\right)\right)}{F(z)} \\
& =\frac{1}{y^{1 / 2}-y^{-1 / 2}} \prod_{1}^{\infty} \frac{\left(1+y q^{n-1 / 2}\right)\left(1+y^{-1} q^{n-1 / 2}\right)}{\left(1-y q^{n}\right)\left(1-y^{-1} q^{n}\right)} . \tag{3.13b}
\end{align*}
$$

For the sake of simplicity we did not factor out the normalization; when $z \rightarrow 0$ $(y \rightarrow 1)$

$$
\begin{gather*}
(y-1) \tilde{S}_{1 / 2,0} \rightarrow \sigma_{1 / 2,0}=2 \prod_{1}^{\infty}\left(\frac{1+q^{n}}{1-q^{n}}\right)^{2}=2 \frac{P\left(q^{2}\right)^{2}}{P(q)^{4}}=2 q^{-1 / 8} \frac{d_{1 / 2,0}^{2}(q)}{P(q)^{2}}, \\
(y-1) \tilde{S}_{0,1 / 2} \rightarrow \sigma_{0,1 / 2}=\prod_{1}^{\infty}\left(\frac{1-q^{n-1 / 2}}{1-q^{n}}\right)^{2}=\frac{P\left(q^{1 / 2}\right)^{2}}{P(q)^{4}}=\frac{d_{0,1 / 2}^{2}(q)}{P(q)^{2}}, \\
(y-1) \tilde{S}_{1 / 2,1 / 2} \rightarrow \sigma_{1 / 2,1 / 2}=\prod_{1}^{\infty}\left(\frac{1+q^{n-1 / 2}}{1-q^{n}}\right)^{2}=\frac{P(q)^{2}}{P\left(q^{2}\right)^{2} P\left(q^{1 / 2}\right)^{2}}=\frac{d_{1 / 2,0}^{2}(q)}{P(q)^{2}}, \tag{3.14}
\end{gather*}
$$

so that

$$
\begin{equation*}
S_{k / 2, l / 2}(z)=\frac{2 i}{\omega_{1}} \tilde{S}_{k / 2, l / 2}(z) / \sigma_{k / 2, l / 2}(q) \sim \frac{1}{\pi z} . \tag{3.15}
\end{equation*}
$$

In the limit of a vertical periodic strip ( $q \rightarrow 0$ ), $S_{1 / 2,0}$ approaches (3.10), while $S_{0,1 / 2}$ and $S_{1 / 2,1 / 2}$ tend to (3.11) as they should. The energy operator $\varepsilon$ is described by $\psi \psi$
up to a normalization factor. One could therefore conjecture that the "connected" correlation function reads up to a normalization factor

$$
\begin{align*}
& \left\langle\varepsilon\left(z_{1}\right) \varepsilon\left(z_{2}\right)\right\rangle \\
& \quad=\pi^{2} Z_{1 / 2}^{-1} \sum^{\prime}\left|D_{k / 2, l / 2}\right|\left|S_{k / 2,1 / 2}\left(z_{12}\right)\right|^{2} \\
& \quad=4 \pi^{2} \frac{|P(q)|^{4}}{\left|\omega_{1}\right|^{2}} \frac{\left|\frac{\tilde{S}_{1 / 2,1 / 2}\left(z_{12}\right)}{d_{1 / 2,1 / 2}}\right|^{2}+\left|\frac{\tilde{S}_{0,1 / 2}\left(z_{12}\right)}{d_{0,1 / 2}}\right|^{2}+\frac{1}{2}(q \bar{q})^{1 / 8}\left|\frac{\tilde{S}_{1 / 2,0}\left(z_{12}\right)}{d_{1 / 2,0}}\right|^{2}}{\left|d_{1 / 2,1 / 2}\right|^{2}+\left|d_{0,1 / 2}\right|^{2}+2\left|d_{1 / 2,0}\right|^{2}} . \tag{3.16}
\end{align*}
$$

Each term is the modulus square of an analytic function in contradistinction with (2.23).

It is interesting to compare this with a bosonic expression. From (2.23)

$$
\begin{align*}
\exp 4 \pi\left\{\left\langle\varphi\left(z_{1}\right) \varphi\left(z_{2}\right)\right\rangle-\left\langle\varphi\left(z_{1}\right) \varphi\left(z_{2}+\frac{1}{2} \omega_{1}\right)\right\rangle\right\} & =\left|\tilde{S}_{1 / 2,0}\left(z_{12}\right)\right|^{2} \\
\exp 4 \pi\left\{\left\langle\varphi\left(z_{1}\right) \varphi\left(z_{2}\right)\right\rangle-\left\langle\varphi\left(z_{1}\right) \varphi\left(z_{2}+\frac{1}{2} \omega_{2}\right)\right\rangle\right\} & =(q \bar{q})^{-1 / 8}\left|\tilde{S}_{0,1 / 2}\left(z_{12}\right)\right|^{2} \\
\exp 4 \pi\left\{\left\langle\varphi\left(z_{1}\right) \varphi\left(z_{2}\right)\right\rangle-\left\langle\varphi\left(z_{1}\right) \varphi\left(z_{2}+\frac{1}{2}\left(\omega_{1}+\omega_{2}\right)\right)\right\rangle\right\} & =(q \bar{q})^{-1 / 8}\left|\tilde{S}_{1 / 2,1 / 2}\left(z_{12}\right)\right|^{2} \tag{3.17}
\end{align*}
$$

Hence the conjecture

$$
\begin{align*}
& \left\langle\varepsilon\left(z_{1}\right) \varepsilon\left(z_{2}\right)\right\rangle=\exp \left(4 \pi\left\langle\varphi\left(z_{1}\right) \varphi\left(z_{2}\right)\right\rangle\right) \\
& \quad \times \frac{\sum_{k l}^{\prime} \exp \left(2 \pi\left\langle\varphi(0) \varphi\left(\frac{1}{2}\left(k \omega_{1}+l \omega_{2}\right)\right)\right\rangle-4 \pi\left\langle\varphi\left(z_{1}\right) \varphi\left(z_{2}+\frac{1}{2}\left(k \omega_{1}+l \omega_{2}\right)\right)\right\rangle\right)}{\sum_{k, l}^{\prime} \exp \left(-2 \pi\left\langle\varphi(0) \varphi\left(\frac{1}{2}\left(k \omega_{1}+l \omega_{2}\right)\right)\right\rangle\right)} . \tag{3.18}
\end{align*}
$$

In the plane or a periodic strip

$$
\begin{align*}
\left\langle\varepsilon\left(z_{1}\right) \varepsilon\left(z_{2}\right)\right\rangle & =\frac{1}{z_{12} \bar{z}_{12}} \quad \text { (plane) }  \tag{3.19a}\\
\left\langle\varepsilon\left(z_{1}\right) \varepsilon\left(z_{2}\right)\right\rangle & =\frac{\pi^{2}}{L^{2}} \frac{1}{\sin \left(\pi z_{12} / L\right) \sin \left(\pi \bar{z}_{12} / L\right)} \tag{3.19b}
\end{align*}
$$

One can check that (3.18) agrees with these limits. At half periods it has a very
symmetric structure

$$
\begin{align*}
\left\langle\varepsilon(0) \varepsilon\left(\frac{1}{2} \omega_{1}\right)\right\rangle & =\frac{v^{3}+w^{3}}{u(u v w)(u+v+w)}, \\
\left\langle\varepsilon(0) \varepsilon\left(\frac{1}{2} \omega_{2}\right)\right\rangle & =\frac{w^{3}+u^{3}}{v(u v w)(u+v+w)}, \\
\left\langle\varepsilon(0) \varepsilon\left(\frac{1}{2}\left(\omega_{1}+\omega_{2}\right)\right)\right\rangle & =\frac{u^{3}+v^{3}}{w(u v w)(u+v+w)}, \tag{3.20}
\end{align*}
$$

with

$$
\begin{gathered}
u=\exp \left(-2 \pi\left\langle\varphi(0) \varphi\left(\frac{1}{2} \omega_{1}\right)\right\rangle\right), \quad v=\exp \left(-2 \pi\left\langle\varphi(0) \varphi\left(\frac{1}{2} \omega_{2}\right)\right\rangle\right) \\
w=\exp \left(-2 \pi\left\langle\varphi(0) \varphi\left(\frac{1}{2}\left(\omega_{1}+\omega_{2}\right)\right)\right\rangle\right)
\end{gathered}
$$

If true (3.18) would express an interesting departure from a naive expectation that the energy correlation is given by $\exp \left(4 \pi\left\langle\varphi\left(z_{1}\right) \varphi\left(z_{2}\right)\right\rangle\right)$ which has the correct limiting behavior but wrong analytic structure. Both functions are also positive within the torus.

Pushing the conjecture further this would suggest for an operator $A_{h, h}(z, \bar{z})$ with vanishing mean value a correlation

$$
\begin{align*}
& \left\langle A_{h, h}\left(z_{1}\right) A_{h, h}\left(z_{2}\right)\right\rangle=\exp \left(8 h \pi\left\langle\varphi\left(z_{1}\right) \varphi\left(z_{2}\right)\right\rangle\right) \\
& \times \frac{\sum_{k, l}^{\prime} \exp \left((8 h-2) \pi\left\langle\varphi(0) \varphi\left(\frac{1}{2}\left(k \omega_{1}+l \omega_{2}\right)\right)\right\rangle-8 h \pi\left\langle\varphi\left(z_{1}\right) \varphi\left(z_{2}+\frac{1}{2}\left(k \omega_{1}+l \omega_{2}\right)\right)\right\rangle\right)}{\sum_{k, l}^{\prime} \exp \left(-2 \pi\left\langle\varphi(0) \varphi\left(\frac{1}{2}\left(k \omega_{1}+l \omega_{2}\right)\right)\right\rangle\right)}, \\
& \left\langle A_{h, h}\left(z_{1}\right) A_{h, h}\left(z_{2}\right)\right\rangle \underset{z_{12} \rightarrow 0}{\sim} \frac{1}{\left|z_{12}\right|^{4 h}}, \\
& \left\langle A_{h, h}\left(z_{1}\right) A_{h, h}\left(z_{2}\right)\right\rangle \underset{\text { strip }}{\sim} \frac{1}{\mid(L / \pi) \sin \left(\pi z_{12} / L\right)^{4 h}} \tag{3.21}
\end{align*}
$$

This would apply to the spin operator ( $h=\frac{1}{16}$ ).
A possible mean to check the validity of (3.21) would be to study the susceptibility as a function of the aspect ratio $(q)$. It would certainly also be interesting to obtain the multipoint correlation functions.

## 4. Modular invariance in the general case $c=1-6 / m(m+1)$

Modular invariance of the partition function on a torus, i.e. its independence with respect to the basic periods $\left(\omega_{1}, \omega_{2}\right)$, is a key property, and turns out to put stringent restrictions on the operator content [5].

We have already dwelt on the modular invariance of $Z_{1}$ (eq. (2.7)). It relies on the following modular transformations of $\eta(q)=q^{1 / 24} P(q)$ :

$$
\begin{align*}
\tau \mapsto \tau+1: & \eta\left(\mathrm{e}^{2 i \pi(\tau+1)}\right)=\mathrm{e}^{2 i \pi / 24} \eta\left(\mathrm{e}^{2 i \pi \tau}\right) \\
\tau \mapsto \tau^{-1}: & \eta\left(\mathrm{e}^{-2 i \pi \tau^{-1}}\right)=\left(\tau \mathrm{e}^{-i \pi / 2}\right)^{1 / 2} \eta\left(\mathrm{e}^{2 i \pi \tau}\right) \tag{4.1}
\end{align*}
$$

Similarly, the modular invariance of the Ising partition function $Z_{1 / 2}$ is easy to check on its expansion (3.8a): the two transformations $\tau \mapsto \tau+1$ and $\tau \mapsto \tau^{-1}$ permute the three ierms. In the general case, we proceed to study $\chi_{c, h_{s s}}$ which in the following we also label $\chi_{[r, s]}$. According to (2.20) we have

$$
\begin{align*}
P(q) \chi_{[r, s]}(q)=q^{-1 / 4 m(m+1)} \sum_{n} & \left(q^{[2 n m(m+1)+r(m+1)-s m]^{2} / 4 m(m+1)}\right. \\
& \left.-q^{[2 n m(m+1)+r(m+1)+s m]^{2} / 4 m(m+1)}\right) . \tag{4.2}
\end{align*}
$$

Since

$$
\begin{align*}
h_{r, s} & =\frac{[r(m+1)-s m]^{2}-1}{4 m(m+1)},  \tag{4.3a}\\
h_{r,-s}-h_{r, s} & =h_{-r, s}-h_{r, s}=r s, \tag{4.3b}
\end{align*}
$$

it follows at once that

$$
\begin{equation*}
x_{[r, s]}\left[\mathrm{e}^{2 i \pi(\tau+1)}\right]=\mathrm{e}^{2 i \pi h_{r, s},} x_{[r, s]}\left(\mathrm{e}^{2 i \pi \tau}\right) \tag{4.4}
\end{equation*}
$$

Eq. (4.3b) also explains why $\chi_{[r, s]}$ has non-negative integers in its expansion in powers of $q$. To obtain the transformation law under the second generator, $\tau \rightarrow \tau^{-1}$ one uses Poisson's formula in the form

$$
\begin{aligned}
& \sum_{n} \exp \left\{2 i \pi \tau \frac{[2 n m(m+1)+r(m+1)-s m]^{2}}{4 m(m+1)}\right\} \\
& \quad=\left(2 m(m+1) \tau \mathrm{e}^{-i \pi / 2}\right)^{-1 / 2} \sum_{p} \exp \left\{\frac{-2 i \pi}{\tau} \frac{p^{2}}{4 m(m+1)}+i \pi p\left(\frac{r}{m}-\frac{s}{m+1}\right)\right\} .
\end{aligned}
$$

Combined with (4.1) this gives

$$
\begin{align*}
& \mathrm{e}^{-2 i \pi \tau c / 24} x_{[r, s]}\left(\mathrm{e}^{2 i \pi \tau}\right) \\
& =\frac{\mathrm{e}^{2 i \pi \tau^{-1} c / 24}}{[2 m(m+1)]^{1 / 2} P\left(\mathrm{e}^{-2 i \pi \tau^{-1}}\right)} \sum_{p} \exp \left\{\frac{-2 i \pi}{\tau} \frac{p^{2}-1}{4 m(m+1)}\right\} \\
& \quad \times\left(\exp \left\{i \pi p\left(\frac{r}{m}-\frac{s}{m+1}\right)\right\}-\exp \left\{i \pi p\left(\frac{r}{m}+\frac{s}{m+1}\right)\right\}\right) . \tag{4.5}
\end{align*}
$$

Now $\chi_{[r, s]}(q)$ defined for $r$ and $s$ integers is invariant under translations $(r, s) \rightarrow$ $\left(r+2 k m, s+2 k^{\prime}(m+1)\right)$

$$
\begin{align*}
\chi_{\left[r+2 m k, s+2(m+1) k^{\prime}\right]} & =\chi_{[r, s]}, \quad k, k^{\prime} \text { integers, },  \tag{4.6a}\\
\chi_{[r,-s]} & =\chi_{[-r, s]}=-\chi_{[r, s]},  \tag{4.6b}\\
\chi_{[m-r, m+1-s]} & =\chi_{[r, s]} . \tag{4.6c}
\end{align*}
$$

This implies that $\chi_{[r, s]}$ vanishes for $r=0(\bmod m)$ or $s=0(\bmod (m+1))$ and that it is entirely determined by its $\frac{1}{2} m(m-1)$ values corresponding to $1 \leq s \leq r \leq m-1$. We split the r.h.s. according to residue classes of $p \bmod 2 m(m+1)$ with the understanding that the last two exponentials in (4.5) can be replaced by sines.

Only $p$ 's which are not multiples of $m$ or ( $m+1$ ) need be considered, otherwise the r.h.s. of (4.5) vanishes. Under those circumstances one can write [5] $p^{2}=$ $(2 n m(m+1)+\rho(m+1)-\sigma m)^{2}$ in a unique way, with $\leq|\sigma| \leq \rho \leq m-1$. This means

$$
\begin{align*}
\mathrm{e}^{-2 i \pi \tau c / 24} \chi_{[r, s]}\left(\mathrm{e}^{2 i \pi \tau}\right)= & {\left[\frac{2}{m(m+1)}\right]^{1 / 2} 2 \sum_{1 \leq \sigma \leq \rho \leq m-1}(-1)^{(r+s)(\rho+\sigma)} } \\
& \times \sin \left(\pi \frac{r \rho}{m}\right) \sin \left(\pi \frac{s \sigma}{m+1}\right) \mathrm{e}^{+2 i \pi \tau^{-1} c / 24} \chi_{[\rho, \sigma]}\left(\mathrm{e}^{-2 i \pi \tau^{-1}}\right) \tag{4.7}
\end{align*}
$$

Eq. (4.4) and (4.7) summarize the behavior of $\chi_{[r, s\}}$ under the modular group.
The real symmetric $\frac{1}{2} m(m-1) \times \frac{1}{2} m(m-1)$ matrix

$$
\begin{equation*}
A_{r, s] ;[\rho, \sigma]}=2\left[\frac{2}{m(m+1)}\right]^{1 / 2}(-1)^{(r+s)(\rho+\sigma)} \sin \left(\pi \frac{r \rho}{m}\right) \sin \left(\rho \frac{s \sigma}{m+1}\right) \tag{4.8}
\end{equation*}
$$

is such that in the range $1 \leq s \leq r \leq m-1,1 \leq \sigma \leq \rho \leq m-1$

$$
\begin{equation*}
\sum_{1 \leq \sigma \leq \rho \leq m-1} A_{[r, s],[\rho, \sigma]} A_{\left[r^{\prime}, s^{\prime}\right],[\rho, \sigma]}=\delta_{[r, s] ;\left[r^{\prime}, s^{\prime}\right]} \tag{4.9}
\end{equation*}
$$

i.e. it is orthogonal, with square equal to one as it should, given the meaning of (4.7).

In general

$$
\operatorname{tr} A=\frac{1}{2}\left(1-\cos \frac{1}{2} \pi m-\sin \frac{1}{2} \pi m\right)=\left\{\begin{array}{ll}
0 & \text { if } m \equiv 0,1 \bmod 4  \tag{4.10}\\
1 & \text { if } m=2,3 \bmod 4
\end{array},\right.
$$

and has a set of $\lambda_{+}\left(\lambda_{-}\right)$eigenvalues $+1(-1)$ with

$$
\begin{align*}
& m=4 p, \quad 4 p+1, \quad 4 p+2, \quad 4 p+3, \\
& \lambda_{+}=p(4 p-1), p(4 p+1), p(4 p+3)+1,(p+1)(4 p+1)+1, \\
& \lambda_{-}=p(4 p-1), p(4 p+1), p(4 p+3), \quad(p+1)(4 p+1) . \tag{4.11}
\end{align*}
$$

Eqs. (4.4) and (4.7) imply that the combination

$$
\begin{equation*}
Z_{\mathrm{c}}(q, \bar{q})=(q \bar{q})^{-c / 24} \sum_{1 \leqslant s \leqslant r \leqslant m-1}\left|\chi_{[r, s]}(q)\right|^{2} \tag{4.12}
\end{equation*}
$$

is always a modular invariant. The question is to find out which other similar combinations if any are also invariant. Of course they should qualify as real characters of the direct product of Virasoro algebras, i.e. have non-negative integral coefficients in terms of $\chi_{[r, s]}(q) \chi_{\left[r^{\prime}, s^{\prime}\right]}(\bar{q})$, and be such that if $[r, s] \neq\left[r^{\prime}, s^{\prime}\right]$ the corresponding dimensions differ by an integer, so that the associated operators have integer spin. In fact this is equivalent to the invariance under the transformation (4.4). In summary, one looks for combinations $\sum Q_{\left.[r, s], \mid r^{\prime}, s^{\prime}\right]} \chi_{[r, s]}(q) \overline{\chi_{\left[r^{\prime}, s^{\prime}\right]}(q)}$, where $Q$ is symmetric, integer-valued (up to an overall positive scale) invariant under (4.4) and (4.7) and the summation implies that the pair $[r, s]$ is in the range $1 \leqslant s \leqslant r \leqslant m-1$. By integral $Q$ 's we mean non-negative integers. Non-vanishing off-diagonal elements imply that $h_{[r, s]}=h_{\left[r^{\prime}, s^{\prime}\right]} \bmod 1$. This does not occur for $m=3,4$ but does occur for $m=5$ (see below).

In general it is possible to enumerate all possible circumstances when $h_{[r, s]}$ $h_{\left[r^{\prime}, s^{\prime}\right]}$ is an integer. We assume $1 \leqslant s \leqslant r \leqslant m-1,1 \leqslant s^{\prime} \leqslant r^{\prime} \leqslant m-1$ and look for $h_{[r, s]}-h_{\left[r^{\prime}, s^{\prime}\right]}=k$ an integer, i.e.

$$
4 k m(m+1)=\left\{\left(r+r^{\prime}\right)(m+1)-\left(s+s^{\prime}\right) m\right\}\left\{\left(r-r^{\prime}\right)(m+1)-\left(s-s^{\prime}\right) m\right\}
$$

which implies

$$
\begin{aligned}
& r^{2} \equiv r^{\prime 2} \bmod m \\
& s^{2}=s^{\prime 2} \bmod (m+1)
\end{aligned}
$$

Among the possible solutions, we have either (i) $r^{\prime}=m-r$ and $s^{\prime}=s$, or (ii) $r^{\prime}=r$ and $s^{\prime}=m+1-s$. The combination $r^{\prime}=m-r, s^{\prime}=m+1-s$ leads to identical dimensions. In both cases we have

$$
\begin{equation*}
h_{[r, s]}-h_{[m-r, s]}=h_{[r, s]}-h_{[r, m+1-s]}=\frac{1}{4}(m+1-2 s)(2 r-m) . \tag{4.13}
\end{equation*}
$$

Since both factors cannot be even, one of them is a multiple of 4 . Hence the various possibilities for the first case:

$$
\begin{array}{llll}
m \equiv 0(4), & 1 \leqslant s \leqslant r_{\mathrm{even}}<\frac{1}{2} m, & r^{\prime}=m-r, & s^{\prime}=s, \\
m \equiv 1(4), & 1 \leqslant s_{\mathrm{odd}} \leqslant r<\frac{1}{2} m, & r^{\prime}=m-r, & s^{\prime}=s, \\
m \equiv 2(4), & 1 \leqslant s \leqslant r_{\mathrm{odd}}<\frac{1}{2} m, & r^{\prime}=m-r, & s^{\prime}=s, \\
m \equiv 3(4), & 1 \leqslant s_{\mathrm{even}} \leqslant r<\frac{1}{2} m, & r^{\prime}=m-r, & s^{\prime}=s, \tag{4.14}
\end{array}
$$

and the second case is obtained by changing $\left(r^{\prime}, s^{\prime}\right)$ into $\left(m-r^{\prime}, m+1-s^{\prime}\right)$. This
has a typical periodicity 4 , with the first possibility of integral difference of dimensions occurring for $m=5$ as noticed already.

Instead of working with $r$ and $s$ restricted by $1 \leqslant s \leqslant r \leqslant m-1$ it will be more convenient to consider the full range $1 \leqslant r \leqslant m-1,1 \leqslant s \leqslant m$ with the points $[r, s]$ and $[m-r, m+1-s$ ] identified. With this convention we need not distinguish the two transformations $[r, s] \mapsto[m-r, s]$ and $[r, s] \mapsto[r, m+1-s]$. We shall therefore use the symbol

$$
[r, s] \mapsto[\widetilde{r, s}]
$$

for any one of those. Inspired by (4.14) it is suggested to study the action of the matrix $A$ on the combinations

$$
\begin{equation*}
\mathrm{e}^{-2 i \pi \tau c / 24}\left\{\chi_{[r, s]}\left(\mathrm{e}^{2 i \pi \tau}\right) \pm \chi_{[\widetilde{r, s},}\left(\mathrm{e}^{2 i \pi \tau}\right)\right\}=\psi_{[r, s]}^{( \pm)}(\tau) \tag{4.15}
\end{equation*}
$$

A case-by-case study shows that according to the residue of $m$ modulo 4 one finds two complementary orthogonal subspaces $\mathrm{V}_{1}$ and $\mathrm{V}_{2}$ invariant up to a phase under modular transformations, i.e. under both (4.4) and (4.7). They are generated by

$$
m \equiv O(4)
$$

$$
V_{1}:\left\{\psi_{\left[r_{\mathrm{reve}}, s\right]}^{(+)}, \psi_{\left[r_{\text {rad }}, s\right]}^{(+)}, \psi_{\left[r_{\text {cdd }}, s\right]}^{(-)}\right\}, \quad V_{2}:\left\{\psi_{\left[r_{\mathrm{cen}}, s\right]}^{(-)}\right\},
$$

$$
m \equiv 1(4)
$$

$$
V_{1}:\left\{\psi_{\left[r, s_{\text {edd }}\right]}^{(-)}, \phi_{\left[r, s_{\text {ceen }}^{(+)}\right]}^{\left(\phi_{\left[r, s_{\text {even }}\right]}^{(-)}\right], \quad V_{2}:\left\{\psi_{\left[r, s_{\text {odd }}\right]}^{(+)}\right\}, ~}\right.
$$

$m \equiv 2(4)$

$$
V_{1}:\left\{\psi_{\left[r_{\text {cosd }}, s\right]}^{(-)}, \psi_{\left[r_{\text {even }}, s\right]}^{(+)}, \psi_{\left[r_{\text {even }}, s\right]}^{(-)}\right\}, \quad V_{2}:\left\{\psi_{\left[r_{\text {codd }}, s\right]}^{(+)}\right\}
$$

$m=3(4)$

$$
\begin{equation*}
V_{1}:\left\{\psi_{\left[r, s_{\text {cecn }}\right]}^{(+)}, \psi_{\left[r, s_{\text {codd }}\right]}^{(+)}, \psi_{\left[r, s_{\text {odd }}\right]}^{(-)}\right\}, \quad V_{2}:\left\{\psi_{\left[r, s_{\text {ceve }}\right]}^{(-)}\right] . \tag{4.16}
\end{equation*}
$$

The splittings $r$ or $s$ even or odd are compatible with the identification $[r, s] \equiv$ [ $m-r, m+1-s$ ], according to whether $m$ is even (split the $r$ 's) or $m$ is odd (split the $s$ 's). The combinations occurring in $\psi^{ \pm}$are such that the differences of dimensions $h_{[r, s]}-h_{[r, s]}$ is an integer according to (4.14). Moreover the $\psi^{(-)}$ combinations involve minus signs which must be compensated for a physical partition function. This means that besides $Z_{\mathrm{c}}$ given by (4.12) valid for any $m \geqslant 2$, we have for $m>4, m \equiv 1,2 \bmod 4$, the following new invariant qualifying as a partition function, and generalizing Cardy's result for $m=5$, i.e. essentially the
hermitian form restricted to $\mathrm{V}_{2}$. Specifically

$$
\begin{align*}
& m=4 p+1 \\
& Z_{\mathrm{c}}^{(+)}(q, \bar{q})=(q \bar{q})^{-c / 24}\left\{\sum_{\substack{1 \leqslant r \leqslant 2 p \\
1 \leqslant s_{\text {odd }} \leqslant 2 p-1}}\left|\chi_{[r, s]}(q)+\chi_{[4 p+1-r, s]}(q)\right|^{2}\right. \\
& \left.+2 \sum_{2 p+1 \leqslant r \leqslant 4 p}\left|\chi_{[r, 2 p+1]}(q)\right|^{2}\right\}, \\
& m=4 p+2 \\
& Z_{\mathrm{c}}^{(+)}(q, \bar{q})=(q \bar{q})^{-c / 24}\left\{\sum_{\substack{1 \leqslant r_{\text {odd }} \leqslant 2 p-1 \\
1 \leqslant s \leqslant 2 p+1}}\left|\chi_{[r, s]}(q)+\chi_{[4 p+2-r, s]}(q)\right|^{2}\right. \\
& \left.+2 \sum_{1 \leqslant s \leqslant 2 p+1}\left|\chi_{[2 p+1, s]}(q)\right|^{2}\right\}, \tag{4.17}
\end{align*}
$$

For $m=5, c=\frac{4}{5}$ (3-state Potts model), and for $m=6, c=\frac{6}{7}$ (tricritical 3-state Potts model), one finds

$$
\begin{align*}
(q \bar{q})^{1 / 30} Z_{4 / 5}^{(+)}(q, \bar{q})= & \left|\chi_{[1,1]}(q)+\chi_{[4,1]}(q)\right|^{2}+\left|\chi_{[2,1]}(q)+\chi_{[3,1]}(q)\right|^{2} \\
& +2\left|\chi_{[3,3]}(q)\right|^{2}+2\left|\chi_{[4,3]}(q)\right|^{2} \\
= & \left|\chi_{4 / 5,0}(q)+\chi_{4 / 5,3}(q)\right|^{2}+\left|\chi_{4 / 5,2 / 5}(q)+\chi_{4 / 5,7 / 5}(q)\right|^{2} \\
& +2\left|\chi_{4 / 5,1 / 15}(q)\right|^{2}+2\left|\chi_{4 / 5,2 / 3}(q)\right|^{2} \tag{4.18}
\end{align*}
$$

and

$$
\begin{align*}
(q \bar{q})^{1 / 28} & Z_{6 / 7}^{(+)}(q, \bar{q}) \\
= & \left|\chi_{[1,1]}(q)+\chi_{[5,1]}(q)\right|^{2}+\left|\chi_{[5,5]}(q)+\chi_{[5,2]}(q)\right|^{2} \\
& +\left|\chi_{[5,4]}(q)+\chi_{[5,3]}(q)\right|^{2}+2\left|\chi_{[3,1]}(q)\right|^{2}+2\left|\chi_{[3,2]}(q)\right|^{2}+2\left|\chi_{[3,3]}(q)\right|^{2} \\
= & \left|\chi_{6 / 7,0}(q)+\chi_{6 / 7,5}(q)\right|^{2}+\left|\chi_{6 / 7,1 / 7}+\chi_{6 / 7,22 / 7}(q)\right|^{2} \\
& +\left|\chi_{6 / 7,5 / 7}(q)+\chi_{6 / 7,12 / 7}(q)\right|^{2}+2\left|\chi_{6 / 7,4 / 3}(q)\right|^{2} \\
& +2\left|\chi_{6 / 7,10 / 21}(q)\right|^{2}+2\left|\chi_{6 / 7,1 / 21}(q)\right|^{2} \tag{4.19}
\end{align*}
$$

where use has been made of the two alternative notations $\chi_{[r, s]}$ and $\chi_{c, h}$. These expressions involve primary operators $A_{h, \bar{h}^{\prime}}$ which are either scalar (real) (e.g. $A_{0,0}, A_{2 / 5,2 / 5}, \ldots$ for $c=4 / 5$ ), or "chiral" (e.g. $A_{0,3}, A_{3,0}, A_{2 / 5,7 / 5}, \ldots$ ). The "spins" of the latter are $\pm 3, \pm 1$ for $c=\frac{4}{5}, \pm 5, \pm 3, \pm 1$ for $c=\frac{6}{7}$. For the general case, we see from eq. (4.13) that the spin $\sigma=h-h$ may take the following (odd or even) values:

$$
\begin{array}{llll}
\text { if } m=4 p+1, & \sigma= \pm(p+1-t)(m-2 r), & 1 \leqslant t \leqslant p, & 1 \leqslant r \leqslant 2 p \\
\text { if } m=4 p+2, & \sigma= \pm(p+1)(m+1-2 s), & 1 \leqslant t \leqslant p, & 1 \leqslant s \leqslant 2 p+1
\end{array}
$$

It would seem that for $m \equiv 0,3 \bmod 4, m>4$, the traces restricted to $V_{1}$ would also fulfill the requirements of invariance and non-negative integer coefficients, but in contradistinction to the previous solutions (4.13) and (4.20) they involve a coefficient 2 in front of $\left|\chi_{[1,1]}(q)\right|^{2}$, and seem to imply a degenerate ground state. For this reason they seem questionable. These combinations read

$$
\begin{align*}
& m=4 p \\
& Z_{\mathrm{c}}^{(+)}(q, \bar{q})=(q \bar{q})^{-c / 24}\left\{2 \sum_{1 \leqslant s \leqslant r_{\text {odd }} \leqslant 4 p-1}\left|\chi_{[r, s \mid}(q)\right|^{2}\right. \\
& +\sum_{\substack{2 \leqslant r_{\text {cenen }} \leqslant 2 p-1 \\
1 \leqslant s \leqslant 2 p}}\left|\chi_{[r, s]}(q)+\chi_{[4 p-r, s]}(q)\right|^{2} \\
& \left.+2 \sum_{1 \leqslant s \leqslant 2 p}\left|\chi_{\{2 p, s \mid}(q)\right|^{2}\right\}, \\
& m=4 p+3 \\
& Z_{\mathrm{c}}^{(+)}(q, \bar{q})=(q \bar{q})^{-c / 24}\left\{2 \sum_{1 \leqslant s_{\text {odd }} \leqslant r \leqslant 4 p+2}\left|\chi_{[r, s]}(q)\right|^{2}\right. \\
& +\sum_{\substack{1 \leqslant r \leqslant 2 p+1 \\
2 \leqslant s_{\text {even }} \leqslant 2 p}}\left|\chi_{[r, s]}(q)+\chi_{[4 p+3-r, s]}(q)\right|^{2} \\
& \left.+2 \sum_{1 \leqslant r \leqslant 2 p+1}\left|\chi_{[r, 2 p+2]}(q)\right|^{2}\right\} . \tag{4.20}
\end{align*}
$$

While our analysis cannot claim to be exhaustive and does not bar a possible further splitting of the solutions, it shows at least the existence of two sequences of candidates to physical unitary statistical models. One which we might call the main sequence is defined for all $m \geqslant 3$ and corresponds to the partition function $Z_{c}$ given by eq. (4.12) with only real (non-chiral) primary operators. The complementary sequence is defined for $m \geqslant 5, m=1,2$ modulo 4 , with a partition function $Z_{c}^{(+)}$
given by (4.17) and involve pairs of chiral operators. The cases $m=0,3$ modulo 4 of the complementary sequence look doubtful ${ }^{\star}$. The main sequence starts with the Ising model ( $m=3$ ), the complementary one with the three-state Potts model ( $m=5$ ). While no chiral operators appear as primary in the main sequence it is nevertheless true that they underlie the construction as in the case of free fermions for the Ising model. Similarly, the fact that some operators do not appear in the partition function of the models in the complementary sequence does not mean that they do not play any role in the physics of those models. For example, Nienhuis and Knops [16] have constructed in the Potts models some of the operators that do not contribute to (4.18) or (4.19).

## 5. Free field realization

From the preceding sections it is suggested that the combinations $\psi_{[r, s]}^{( \pm)}(q)$ of characters defined in (4.15) are natural candidates for an expression in terms of free field determinants $D_{k / N, I / N}(q)$.

We recall from (2.29) that $\left|D_{k / N, 1 / N}(q)\right|^{2}$ stands for the functional determinant of $-\Delta$ with corresponding boundary conditions

$$
\begin{equation*}
D_{k / N, l / N}(q)=q^{-\lambda_{1} / N} \prod_{n=0}^{\infty}\left(1-\mathrm{e}^{2 i \pi k / N} q^{n+l / N}\right)\left(1-\mathrm{e}^{-2 i \pi k / N} q^{n+(N-l) / N}\right) \tag{5.1}
\end{equation*}
$$

where we have introduced the notation (for fixed $N$ )

$$
\begin{equation*}
\lambda_{l}=\frac{1}{12}\left(\frac{6 l(N-l)}{N^{2}}-1\right) . \tag{5.2}
\end{equation*}
$$

We set

$$
\begin{equation*}
D_{\cdot, /}(q ; N)=\prod_{0 \leqslant k \leqslant N-1} D_{k / N, I / N}(q)=q^{-\lambda_{l}} \prod_{n=0}^{\infty}\left(1-q^{N n+l}\right)\left(1-q^{N n+N-l}\right) \tag{5.3a}
\end{equation*}
$$

and for $N$ even

$$
\begin{align*}
D_{+, l}(q ; N) & =\prod_{0 \leqslant k_{\text {even }} \leqslant N-2} D_{k / N, l / N}(q) \\
& =q^{-\lambda_{l} / 2} \prod_{n=0}^{\infty}\left(1-q^{N n / 2+l / 2}\right)\left(1-q^{N n / 2+(N-l) / 2}\right),  \tag{5.3b}\\
D_{-, l}(q ; N) & =\prod_{0 \leqslant k_{\text {odd }} \leqslant N-1} D_{k / N, l / N}(q) \\
& =q^{-\lambda_{l} / 2} \prod_{n=0}^{\infty}\left(1+q^{N n / 2+l / 2}\right)\left(1+q^{N n / 2+(N-l) / 2}\right) . \tag{5.3c}
\end{align*}
$$

[^0]For a given $m$ and $c=1-6 / m(m+1)$, we have the following factorized forms for $\psi^{( \pm)}$:

$$
\begin{align*}
\psi_{[r, s]}^{( \pm)}(q) \equiv & q^{-c / 24}\left\{\chi_{[r, s]}(q) \pm \chi_{[m-r, s]}(q)\right\} \\
= & q^{h_{r, s}-c / 24} \frac{P\left(q^{m(m+1) / 2}\right)}{P(q)}\left\{\prod_{n=0}^{\infty}\left(1 \mp q^{(n+1 / 2) m(m+1) / 2-r(m+1) / 2+s m / 2}\right)\right. \\
& \left.\times\left(1 \mp q^{(n+1 / 2) m(m+1) / 2+r(m+1) / 2-s m / 2}\right)-q^{r s}(s \leftrightarrow-s)\right\} . \tag{5.4}
\end{align*}
$$

Using various arithmetic identities (see appendix A) it is readily checked that the following results hold. First concentrate on the complementary series. Assume according to (5.17) that if

$$
\begin{array}{ll}
m \equiv 0(4), & r \text { is even } \\
m \equiv 1(4), & s \text { is odd } \\
m \equiv 2(4), & r \text { is odd } \\
m \equiv 3(4), & s \text { is even } \tag{5.5}
\end{array}
$$

Then the combinations

$$
\begin{align*}
k & =\frac{1}{2} m(m+1)-r(m+1)+s m \\
k^{\prime} & =\frac{1}{2} m(m+1)-r(m+1)-s m \tag{5.6}
\end{align*}
$$

are both even, and

$$
\begin{equation*}
\psi_{[r, s]}^{(+)}(q)=\frac{D_{\cdot, k / 2}\left(q, \frac{1}{2} m(m+1)\right)+D_{\cdot,-k^{\prime} / 2}\left(q, \frac{1}{2} m(m+1)\right)}{\left\{\prod_{l=1}^{m(m+1) / 2-1} D_{\cdot, l}\left(q, \frac{1}{2} m(m+1)\right)\right\}^{1 / 2}} \tag{5.7}
\end{equation*}
$$

where $D_{., I}$ is defined in (5.3a). Inserting this in formulas (4.20) one obtains a free field realization of the partition function in the complementary sequence. Notice also that in this complementary sequence, the integer $N=\frac{1}{2} m(m+1)$ takes only odd values.

In the case of the main sequence we note that $N=m(m+1)$ is even, hence (5.3b) and (5.3c) make sense. Then with $k, k^{\prime}$ still given by (5.6), we find

$$
\begin{align*}
\psi_{[r, s]}^{( \pm)}(q)= & \left\{D_{ \pm, k}(q ; m(m+1))-D_{ \pm, k^{\prime}}(q ; m(m+1))\right\} \sqrt{\frac{2}{m(m+1)}} \\
& \times \prod_{2 \leqslant p_{\mathrm{even}} \leqslant m(m+1)-2} D_{p / m(m+1), 0}^{1 / 2}(q) \tag{5.8}
\end{align*}
$$

The infinite product representations of characters provided by eqs. (5.4), (5.7), (5.8) generalize expressions obtained by Rocha-Caridi [8] for $m=3,4$. For $m=3$, eq. (5.8) seems to imply boundary conditions modulo $N=m(m+1)=12$. Using arithmetic identities, it is easy to recast it in the forms displayed in (3.7). More generally, we have found a host of such alternative expressions of the $\chi$ 's as sums of infinite products. In the absence of a good interpretation of these formulae, we refrain from presenting them.

It seems worth writing explicitly these expressions in the cases $m=4,5,6$.
Various authors have suggested that the tricritical Ising model should be described by a value $c=\frac{7}{10}$ of the central charge, i.e. $m=4$ in the parametrization (2.20). In this case, the formulae (5.5) above seem to imply determinants of free fields with boundary conditions twisted by multiples of $2 \pi / m(m+1)=\frac{2}{20} \pi$. It is possible to rewrite them only in terms of twists of $\frac{2}{10} \pi$. Writing for compactness $D_{ \pm, l}$ for $D_{ \pm, l}(q, 10)$ we have

$$
\begin{align*}
q^{-7 / 240}\left(\chi_{7 / 10,0}+\chi_{7 / 10,3 / 2}\right) & =\frac{D_{1 / 2,1 / 2}^{1 / 2}}{D_{-.1} D_{+, 4}}, \\
q^{-7 / 240}\left(\chi_{7 / 10,0}-\chi_{7 / 10,3 / 2}\right) & =\frac{D_{0,1 / 2}^{1 / 2}}{D_{+.1} D_{+, 4}}, \\
q^{-7 / 240}\left(\chi_{7 / 10,1 / 10}+\chi_{7 / 10,3 / 5}\right) & =\frac{D_{1 / 2,1 / 2}^{1 / 2}}{D_{+.2} D_{-, 3}}, \\
q^{-7 / 240}\left(\chi_{7 / 10.1 / 10}-\chi_{7 / 10,3 / 5}\right) & =\frac{D_{0,1 / 2}^{1 / 2}}{D_{+, 2} D_{+, 3}}, \\
q^{-7 / 240} \chi_{7 / 10,7 / 16} & =\frac{D_{1 / 2,0}^{1 / 2}}{\sqrt{2} D_{+, 4} D_{-, 4}}, \\
q^{-7 / 240} \chi_{7 / 10,3 / 80} & =\frac{D_{1 / 2,0}^{1 / 2}}{\sqrt{2} D_{+, 2} D_{-, 2}} \tag{5.9}
\end{align*}
$$

The partition function of the tricritical Ising model is therefore, up to an overall normalization

$$
\begin{align*}
Z_{7 / 10}= & 2(q \bar{q})^{-7 / 240}\left\{\left|\chi_{7 / 10,0}\right|^{2}+\left|\chi_{7 / 10,3 / 2}\right|^{2}+\left|\chi_{7 / 10,1 / 10}\right|^{2}\right. \\
& \left.+\left|\chi_{7 / 10,3 / 5}\right|^{2}\left|\chi_{7 / 10,7 / 16}\right|^{2}+\left|\chi_{7 / 10,3 / 80}\right|^{2}\right\} \\
= & \frac{\left|D_{1 / 2,1 / 2}\right|}{\left|D_{-, 1} D_{+, 4}\right|^{2}}+\frac{\left|D_{0,1 / 2}\right|}{\left|D_{+, 1} D_{+, 4}\right|^{2}}+\frac{\left|D_{1 / 2,1 / 2}\right|}{\left|D_{+, 2} D_{-, 3}\right|^{2}} \\
& +\frac{\left|D_{0,1 / 2}\right|}{\left|D_{+, 2} D_{+, 3}\right|^{2}}+\frac{\left|D_{1 / 2,0}\right|}{\left|D_{+, 4} D_{-, 4}\right|^{2}}+\frac{\left|D_{1 / 2,0}\right|}{\left|D_{+, 2} D_{-, 2}\right|^{2}} . \tag{5.10}
\end{align*}
$$

The free field realization of this model is quite puzzling. Massless Bose and Fermi fields, which contribute respectively to the denominators and numerators in (5.10), are "coupled" by boundary conditions modulo 10 , a fact which would require some physical explanation, and which might throw some light on the hidden supersymmetry of the model [17]. One also notices some relationship with the case $c=\frac{1}{2}$, (3.3) where each term has been split into two contributions.

Such a relationship does not seem to appear between the expressions relative to the critical ( $m=5, c=\frac{4}{5}$ ) and tricritical ( $m=6, c=\frac{6}{7}$ ) Potts model. Their respective partition functions read:

$$
\begin{align*}
Z_{4 / 5}= & \left\{\left|D_{\cdot, 7}(q, 15)-D_{\cdot, 2}(q, 15)\right|^{2}+\left|D_{\cdot, 4}(q, 15)+D_{\cdot, 1}(q, 15)\right|^{2}\right. \\
& \left.+2\left|D_{\cdot, 3}(q, 15)\right|^{2}+2\left|D_{\cdot, 6}(q, 15)\right|^{2}\right\} \\
& \times \prod_{l=1}^{14}\left|D_{\cdot, l}^{-1}(q, 15)\right|,  \tag{5.11}\\
Z_{6 / 7}= & \left\{\left|D_{\cdot, 10}(q, 21)-D_{\cdot, 4}(q, 21)\right|^{2}+\left|D_{\cdot, 8}(q, 21)-D_{\cdot, 1}(q, 21)\right|^{2}\right. \\
& +\left|D_{\cdot, 5}(q, 21)+D_{\cdot, 2}(q, 21)\right|^{2}+2\left|D_{\cdot, 3}(q, 21)\right|^{2}+2\left|D_{\cdot, 6}(q, 21)\right|^{2} \\
& \left.+2\left|D_{\cdot, 9}(q, 21)\right|^{2}\right\} \prod_{l=1}^{20}\left|D_{\cdot, l}^{-1}(q, 21)\right| . \tag{5.12}
\end{align*}
$$

Here also, a physical interpretation of these expressions and a clear understanding of their connection with the symmetries of the model would be most desirable. One notices that the contributions of chiral operators in (5.11), (5.12) involves linear combinations of $D_{, l}$ where $l$ takes all the integral values between 1 and $\frac{1}{2}\left[\frac{1}{2} m(m+\right.$ 1 ) -1 ], that are multiples neither of $m$ nor of $\frac{1}{2}(m+1)$ if $m=4 p+1$ (resp. of $\frac{1}{2} m$ and $m+1$ if $m=4 p+2$ ). Scalar operators (that come with a factor 2) contribute $D_{,, l}$ with $l$ multiples of $\frac{1}{2}(m+1)$ (resp. of $\left.\frac{1}{2} m\right)$.

Besides their series expansion (2.2a), and product expressions (5.4), (5.7), (5.8), the characters $\chi_{c, h}$ still have another representation, in terms of appropriate correlation functions of the gaussian model. This has already been exemplified on the Ising model in eq. (3.8b).

## 6. Summary and conjectures - remarks on the non-unitarity case

In this paper, we have seen that the study of a conformal theory in a finite geometry, in particular on a torus, may be extremely fruitful in elucidating the content of that theory.

On the one hand, we have shown that all partition functions of 2-dimensional unitary conformal theories with $c<$ may be expressed in terms of free field modes with appropriate boundary conditions. For correlation functions, the construction is quite involved and requires a more detailed analysis.

On the other hand, modular invariance of the partition function has proved to be a very stringent constraint [5]. This has led us to two series of unitary models with $c<1$ : the principal one, involving all the scalar primary operators, starts with the Ising model: the complementary series involves some chiral primary operators, and starts with the 3 -state Potts model. One may wonder about the nature of these two series of conformal models. There exist speculations that some "continuations" of $\mathrm{O}(n)$ invariant models $(1 \leqslant n<2$ ) or of $q$-state critical (or multicritical) Potts models $(2 \leqslant q \leqslant 4)$ such that $n=2 \cos (\pi / m)$ or $q=4 \cos ^{2}(\pi /(m+1))$ (resp. $4 \cos ^{2}(\pi / m)$ ) have some relation with conformal theories at $c=1-6 / m(m+1)$. These models have already been discussed by a number of authors [3,18]: in particular, Dotsenko and Fateev have proposed to identify the thermal and magnetic operators of the Potts model ( $m$ odd) with $A_{[n+1,1]}, n=1,2, \ldots$, and $A_{[(m-1) / 2-n,(m+1) / 2]}, n=0,1,2, \ldots$.

Similarly, Gehlen, Rittenberg and Ruegg suggest that the energy and order operators of the tricritical Potts model ( $m$ even) are $A_{1,2}$, and $A_{m / 2, m / 2}$ respectively. As for the principal series we also recall that Huse [19] has proposed to interpret the multicritical point of the RSOS model of Andrews, Baxter and Forrester [20] as a realization of the generic conformal model. Their magnetic exponents are $h_{[1,1]}, h_{[2,2]}, \ldots, h_{[m-1, m-1]}$, with even and odd labels and therefore cannot originate from the complementary series.

The expressions (5.1)-(5.5) of the unitary characters of the Virasoro algebra entail the following formula for the conformal weight $h_{[r, s]}$ :

$$
\begin{equation*}
2 h_{[r, s]}=-\frac{1}{2} \gamma_{k}+\frac{N^{2}-4}{8 N} \tag{6.1}
\end{equation*}
$$

where $N=m(m+1), k=\frac{1}{2} N-r(m+1)+s m$ and

$$
\begin{equation*}
\gamma_{k}=\frac{k(N-k)}{N} \tag{6.2}
\end{equation*}
$$

This expression is readily checked against the Kac formula (2.20). We find it intriguing and suggestive in view of the interpretation of $\gamma_{k}$ as the square length of the $k$ th fundamental weight of $\mathrm{SU}(N)$. This might suggest that the corresponding representation can be constructed from exponentials of scalar products of free fields with weights of $\mathrm{SU}(N)$. Such realizations are known to exist for some representations of the Virasoro algebra [21], but not in the unitary case for $m \geqslant 5$, to the best of our knowledge.

One crucial point missing in all this discussion is the relation between the discrete group of symmetries broken at the phase transition and the structure of the critical models.

A further direction of study, not unrelated to string theory, is the study of free fields on compact riemannian 2-dimensional manifolds of higher genus.

Finally, it would be interesting and physically important to extend these results to other cases: unitary representations with $c>1$, superconformal models...

We want to finish this paper with a short discussion of the $c<1$ non-unitary representations, where new difficulties arise. Belavin, Polyakov and Zamolodchikov [1] have shown that there exist minimal degenerate representations of the Virasoro algebra admitting a finite number of primary conformal operators $A_{[r, s]}$ (see appendix B for a brief summary).

An important case discussed by Cardy [22] pertains to the Lee-Yang edge singularity or scalar field theory with an (imaginary) cubic coupling. This is the only case with a unique scalar operator $\varphi$ beside the identity $\mathbf{1}$. The dimensions are $h_{\varphi}=-\frac{1}{5}, h_{1}=0$ and the central charge is $c=-\frac{22}{5}$. Following the lines of sect. 4, one may write expressions for the corresponding characters $\chi$ and study their modular transformations. We relegate all general formulae to appendix B and only quote the result for the Lee-Yang case. The only modular invariant partition function involving $\chi_{-22 / 5,0}$ and $\chi_{-22 / 5,-1 / 5}$ is

$$
\begin{equation*}
Z_{-22 / 5}=(q \bar{q})^{11 / 60}\left\{\left|\chi_{-22 / 5,0}\right|^{2}+\left|\chi_{-22 / 5,-1 / 5}\right|^{2}\right\} \tag{6.3}
\end{equation*}
$$

Since $c$ is negative, the exponent of the prefactor is positive. On the other hand, due to the negative dimensions $h=\bar{h}=-1 / 5$ of the field $\varphi$, the leading term in the bracket for small $q$ is $(q \bar{q})^{-1 / 5}$. Therefore $Z_{-22 / 5}(q, \bar{q}) \sim_{q \rightarrow 0}(q \bar{q})^{-1 / 60}$ and this is different from $(q \bar{q})^{-c / 24}$ as claimed in the introduction. This is clearly due to fields with negative dimensions (hence growing correlations at large distances), a fact which explains why the derivation of the "Casimir effect" on strips is not valid, since one should apply the same reasoning to the state with the lowest negative dimension $h_{0}$. This has the effect of shifting the "effective" central charge to $c^{\prime}=c-24 h_{0}$ [24].

More generally, even with $0<c<1$, a non-unitary degenerate model of the type discussed in appendix $B$ always has fields with negative dimensions, and the
partition function no longer behaves as $(q \bar{q})^{-c / 24}$. This looks rather puzzling in view of the numerical experiments which tested this leading behavior [6,23]. We can see two possible ways out of this puzzle, which might be realized in different situations:
either negative dimension fields effectively contribute to the partition function and the small $q$ behavior of $Z$ is no longer given by $(q \bar{q})^{-c / 24}$, but by $(q q)^{-c^{\prime} / 24}$,
or a modular invariant partition function may be constructed in which only fields of non-negative dimension contribute; this is analogous to the situation discussed above in sect. 4. In the case $c=-\frac{22}{5}$, this cannot occur if only the minimal set of representations $h=0,-\frac{1}{5}$ is used. It seems plausible that the introduction of higher representations will force one to have an infinite number of fields. This looks much more difficult than the situation analyzed in this paper, and we hope to return to this question [24].

## Appendix $A$

We gather here a few useful formulae and arithmetic identities. Here $q$ denotes a complex number satisfying $|q|<1$ :

Euler identity

$$
\begin{equation*}
\prod_{n=1}^{\infty}\left(1-q^{2 n-1}\right)\left(1+q^{n}\right)=1 \tag{A.1}
\end{equation*}
$$

Jacobi triple product identity

$$
\begin{equation*}
\Theta(y, q) \equiv \sum_{k=-\infty}^{\infty} y^{k} q^{k^{2}}=\prod_{n=1}^{\infty}\left(1-q^{2 n}\right)\left(1+y q^{2 n-1}\right)\left(1+y^{-1} q^{2 n-1}\right) \tag{A.2}
\end{equation*}
$$

where $y$ is an arbitrary complex number.
Euler's identity (A.1) is easily proved; for a physicist's proof of (A.2), see [25]. A famous particular case of (A.2) is Euler pentagonal identity

$$
\begin{equation*}
\sum_{k=-\infty}^{\infty}(-1)^{k} q^{(3 k+1) k / 2}=\prod_{n=1}^{\infty}\left(1-q^{n}\right)=P(q) \tag{A.3}
\end{equation*}
$$

In this article, use is made of (A.2) in the form

$$
\begin{equation*}
\sum_{k=-\infty}^{\infty}( \pm 1)^{k} q^{\left(N k^{2}+r k\right) / 2}=\prod_{n=0}^{\infty}\left(1-q^{(n+1) N}\right)\left(1 \pm q^{n N+(N-r) / 2}\right)\left(1 \pm q^{n N+(N+r) / 2}\right) \tag{A.4}
\end{equation*}
$$

Finally, if $\omega$ denotes the $N$ th root of $1, \omega=\exp (2 \pi i / N)$, Gauss' sum is

$$
\begin{equation*}
\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \omega^{n^{2}}=\frac{1}{2}(1+i)\left(1+(-1)^{N}\right) \tag{A.5}
\end{equation*}
$$

## Appendix B

According to BPZ [1], whenever $p$ and $p^{\prime}$ are coprimes, there exist minimal degenerate representations of the Virasoro algebra, with a central charge

$$
\begin{equation*}
c=1-\frac{6\left(p-p^{\prime}\right)^{2}}{p p^{\prime}} \tag{B.1}
\end{equation*}
$$

and the dimensions of primary operators are, assuming $p>p^{\prime}$

$$
\begin{align*}
h_{r s} & =\frac{1}{4 p p^{\prime}}\left\{\left(r p-s p^{\prime}\right)^{2}-\left(p-p^{\prime}\right)^{2}\right\} \\
& =h_{p^{\prime}-r, p-s}, \quad r, s \text { positive integers } . \tag{B.2}
\end{align*}
$$

BPZ observe that among this infinite set of primary operators, it is consistent to restrict one self to the finite set labelled by

$$
\begin{equation*}
1 \leqslant r \leqslant p^{\prime}-1, \quad 1 \leqslant s \leqslant p-1 \tag{B.3}
\end{equation*}
$$

(whence the denomination "minimal"). The unitary $c<1$ case corresponds to $p^{\prime}=m, p=m+1$. From the analysis of the embeddings of Verma modules $[7,8]$, the corresponding character may be derived and reads, for $(r, s)$ in the rectangle (B.3)

$$
\begin{equation*}
\chi_{[r, s]}(q)=\frac{1}{P(q)} \sum_{n=-\infty}^{\infty}\left\{q^{\left(\left(2 n p p^{\prime}+r p-s p^{\prime}\right)^{2}-\left(p-p^{\prime}\right)^{2}\right) / 4 p p^{\prime}}-(s \mapsto-s)\right\} \tag{B.4}
\end{equation*}
$$

This satisfies

$$
\begin{align*}
\chi_{[r, s]}(q) & =\chi_{\left[p^{\prime}-r, p-s\right]}(q),  \tag{B.5a}\\
\chi_{[r,-s]}(q) & =\chi_{[-r, s]}(q)=-\chi_{[r, s]}(q),  \tag{B.5b}\\
\chi_{\left[r+2 k^{\prime} p^{\prime}, s+2 k p\right]}(q) & =\chi_{[r, s]}(q) . \tag{B.5c}
\end{align*}
$$

Caution! As the reflections or translations of $(r, s)$ implied by (B.5b, c) take them out of the domain (B.3), these relations must be understood as satisfied by the continuation of $\chi_{[r, s]}$ as a function of $r$ and $s$. Expressions for the characters $\chi$ beyond the domain (B.3) will be given below, see (B.16).

Modular transformations of the $\chi$ 's are derived as in sect. 4. Since $h_{r,-s}-h_{r, s}=r s$ is an integer,

$$
\begin{equation*}
\chi_{[r, s]}\left(\mathrm{e}^{2 i \pi} q\right)=\mathrm{e}^{2 i \pi h_{r s}} \chi_{[r, s]}(q) \tag{B.6}
\end{equation*}
$$

The transformation under $q=\mathrm{e}^{2 i \pi \tau} \rightarrow \tilde{q}=\mathrm{e}^{-2 i \pi \tau^{-1}}$ is

$$
\begin{align*}
& \sum_{n} \exp \left(2 i \pi \tau \frac{\left(2 n p p^{\prime}+r p-s p^{\prime}\right)^{2}}{4 p p^{\prime}}\right) \\
& \quad=\left(2 p p^{\prime} \tau \mathrm{e}^{-i \pi / 2}\right)^{-1 / 2} \sum_{n=-\infty}^{+\infty} \exp \left(-\frac{2 i \pi}{\tau} \frac{n^{2}}{4 p p^{\prime}}+i \pi n\left(\frac{r}{p^{\prime}}-\frac{s}{p}\right)\right) \tag{B.7}
\end{align*}
$$

Combined with (4.1), with $c$ given by (1), $q=\mathrm{e}^{2 i \pi \tau}, \tilde{q}=\mathrm{e}^{-2 i \pi \tau^{-1}}$

$$
q^{-c / 24} \chi_{[r, s]}(q)=2 \frac{\tilde{q}^{-c / 24}}{\sqrt{2 p p^{\prime}} P(\tilde{q})} \sum_{n=-\infty}^{+\infty} \tilde{q}^{\left(n^{2}-\left(p-p^{\prime}\right)^{2}\right) / 4 p p^{\prime}} \sin \left(\pi \frac{n r}{p^{\prime}}\right) \sin \left(\pi \frac{n s}{p}\right)
$$

Since ( $p, p^{\prime}$ ) are coprimes, integers $a$ and $b$ exist such that

$$
\begin{equation*}
a p-b p^{\prime}=1 \tag{B.8}
\end{equation*}
$$

and the pair $(a, b)$ can be replaced by ( $a+k p^{\prime}, b+k p$ ) and chosen so as to satisfy $1 \leqslant a \leqslant p^{\prime}-1,1 \leqslant b \leqslant p-1$. In the r.h.s. one can sum over integers $n$ which are neither multiples of $p$ nor of $p^{\prime}$. This means that

$$
\begin{align*}
n & =\rho+\alpha p^{\prime}, & & 1 \leqslant \rho \leqslant p^{\prime}-1 \\
& =\sigma+\beta p, & & 1 \leqslant \sigma \leqslant p-1 \tag{B.9}
\end{align*}
$$

hence $n$ can be uniquely written for a given choice $(a, b)$ as

$$
\begin{array}{ll}
n=a \rho p-b \sigma p^{\prime}+k p p^{\prime}, & 1 \leqslant \rho \leqslant p^{\prime}-1 \\
& 1 \leqslant \sigma \leqslant p-1 . \tag{B.10}
\end{array}
$$

Inserting this in eq. (B.8), distinguishing even and odd $k$, and noticing that not both
$a$ and $b$ may be even yields

$$
\begin{equation*}
q^{-c / 24} \chi_{[r, s]}(q)=\sum_{\substack{1 \leqslant \tilde{r} \leqslant p^{\prime}-1 \\ 1 \leqslant \bar{s} \leqslant p-1 \\ \tilde{s} p^{\prime}<\tilde{r} p}} A_{[r, s],[\tilde{r}, \tilde{s} \mid} \tilde{q}^{-c / 24} \chi_{[\tilde{r}, \tilde{s}]}(\tilde{q}) \tag{B.11}
\end{equation*}
$$

where

$$
\begin{equation*}
A_{[r, s],[\tilde{r}, \tilde{s}]}=\sqrt{\frac{8}{p p^{\prime}}}(-1)^{(r+s)(\tilde{r}+\tilde{s})} \sin \left(\frac{\pi r \tilde{r}}{p^{\prime}}\right)\left(p-p^{\prime}\right) \sin \left(\frac{\pi s \tilde{s}}{p}\right)\left(p-p^{\prime}\right) \tag{B.12}
\end{equation*}
$$

which agrees with (4.8) in the case $\left(p, p^{\prime}\right)=(m+1, m)$.
The $\frac{1}{2}(p-1)\left(p^{\prime}-1\right) \times \frac{1}{2}(p-1)\left(p^{\prime}-1\right)$ real symmetric matrix $A$ is orthogonal. Hence, the expression

$$
\begin{equation*}
Z=(q \bar{q})^{-c / 24} \sum_{\substack{1 \leqslant r \leqslant p^{\prime}-1 \\ 1 \leqslant s \leqslant \leq r-1 \\ s p^{\prime} \leqslant p}} \mid \chi_{\{r, s]^{2}} \tag{B.13}
\end{equation*}
$$

where all the scalar primary operators of the minimal set contribute, is always a modular invariant, therefore a candidate for a partition function on a torus. In the Lee-Yang case, $p=5, p^{\prime}=2, c=-\frac{22}{5}$, eqs. (B.11), (B.12) boil down to

$$
\begin{align*}
q^{-c / 24}\binom{\chi_{[1,1]}(q)}{\chi_{[1,2]}(q)} & =\tilde{q}^{-c / 24}\left(\begin{array}{rr}
-s_{1} & s_{2} \\
s_{2} & s_{1}
\end{array}\right)\binom{\chi_{[1,1]}(\tilde{q})}{\chi_{[1,2]}(\tilde{q})}, \\
s_{k} & =\sin \frac{2}{5} \pi k, \quad s_{1_{2}}^{2}=\frac{1}{8}(5 \pm \sqrt{5}), \tag{B.14}
\end{align*}
$$

and it is easy to see that

$$
\begin{equation*}
Z_{-22 / 5}(q, \bar{q})=(q \bar{q})^{11 / 60}\left(\left|\chi_{[1,1]}(q)\right|^{2}+\left|\chi_{[1,2]}(q)\right|^{2}\right) \tag{B.15}
\end{equation*}
$$

is the only invariant.
In general, among the dimensions given by (B.2), (B.3), some are negative (for example, take $r=a, s=b$ from eq. (B.8)) and their contribution dominates the small $q$ behavior of (B.13). In a way similar to the method of sect. 4 , one may try to construct other modular invariants than (B.13). This is possible in particular whenever $p$ and $p^{\prime}$ are of opposite parities (and larger than 2), but it seems impossible in this way to get rid of all negative dimension operators. As discussed
at the end of sect. 6, this suggests that operators outside the minimal set may be coupled. We end up with a list of the corresponding characters.

$$
\begin{align*}
& \chi_{\left[\rho+2 n p^{\prime}, \sigma\right]}(q)=\frac{1}{P(q)} \sum_{k \notin[-n, n-1]}\left\{q^{\left(\left(2 k p p^{\prime}+\rho p-\sigma p^{\prime}\right)^{2}-\left(p-p^{\prime}\right)^{2}\right) / 4 p p^{\prime}}-(\sigma \mapsto-\sigma)\right\}, \\
& \chi_{\left[\rho+(2 n+1) p^{\prime}, \sigma\right]}(q) \\
& \quad=\frac{1}{P(q)} \sum_{k \notin[-n-1, n-1]}\left\{q^{\left.\left(\left((2 k+1) p p^{\prime}+\rho p-\sigma p^{\prime}\right)^{2}-\left(p-p^{\prime}\right)^{2}\right) / 4 p p^{\prime}-(\sigma \mapsto-\sigma)\right\},}\right. \\
& \chi_{\left[(2 n+1) p^{\prime}-\rho, \sigma\right]}(q) \\
& \quad=\frac{1}{P(q)} \sum_{k \notin[-n, n-1]}\left\{q^{\left.\left(\left((2 k+1) p p^{\prime}+\rho p+\sigma p^{\prime}\right)^{2}-\left(p-p^{\prime}\right)^{2}\right) / 4 p p^{\prime}-(\sigma \mapsto-\sigma)\right\},}\right. \\
& \chi_{\left[2(n+1) p^{\prime}-\rho, \sigma\right]}(q) \\
& \quad=\frac{1}{P(q)} \sum_{k \notin[-n-1, n-1]}\left\{q^{\left.\left(\left(2(k+1) p p^{\prime}+\rho p+\sigma p^{\prime}\right)^{2}-\left(p-p^{\prime}\right)^{2}\right) / 4 p p^{\prime}-(\sigma \mapsto-\sigma)\right\} .}\right. \tag{B.16}
\end{align*}
$$

In all these expressions, $1 \leqslant \rho \leqslant p^{\prime}-1,1 \leqslant \sigma \leqslant p-1$. The excluded interval in the summation over the integer $n$ makes the study of the modular transformations of these quantities more complicated.

In the course of this work we have benefited from interesting discussions with J.M. Luck, Th. Nieuwenhuizen, and H. Saleur.

## References

[1] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Nucl. Phys. B241 (1984) 333
[2] D. Friedan, Z. Qiu and S. Shenker, Phys. Rev. Lett. 52 (1984) 1575; in Vertex operators in mathematics and physics, ed. J. Lepowsky, S. Mandelstam and I. Singer (Springer, NY, 1985)
[3] V.S. Dotsenko, Nucl. Phys. B235 [FS11] (1984) 54;
V.S. Dotsenko and V.A. Fateev, Nucl. Phys. B240 [FS12] (1984) 312, B251 [FS13] (1985) 691:
V.A. Fateev, A.B. Zamolodchikov, Landau Institute preprint (1985)
[4] B.L. Feigin and D.B. Fuks, Moscow preprint;
J.L. Gervais and A. Neveu, Nucl. Phys. B257 [FS14] (1985) 59
[5] J.L. Cardy, Nucl. Phys. B270 [FS16] (1986) 186
[6] H.W. Blöte, J.L. Cardy and M.P. Nightingale, Phys. Rev. Lett. 56 (1986) 742; I. Affleck, Phys. Rev. Lett. 56 (1986) 746
[7] V.G. Kac, Lecture Notes in Phys. 94 (1979) 441 ;
B.L. Feigin and D.B. Fuks, Functs. Anal. Pril. 16 (1982) 47 (Funct. Anal. Pril. 17 (1983) 91; Funct. Anal. Appl. 16 (1982) 114, 17 (1983) 241)
[8] A. Rocha-Caridi, in Vertex operators in mathematics and physics (Springer, NY 1985)
[9] L.P. Kadanoff, J. Phys. A11 (1978) 1399; Ann. of Phys. 120 (1979) 39;
L.P. Kadanoff and A. Brown, Ann. of Phys. 121 (1979) 318
[10] E. Fradkin and L.P. Kadanoff, Nucl. Phys. B170 [FS1] (1980) 1
[11] A. Weil, Elliptic functions according to Einstein and Kronecker (Springer, 1976) ch. 7, 8
[12] C.B. Thorn, Phys. Reports 67 (1980) 487 and references therein.
[13] J. Polchinski, Austin preprint UTTG-13 (1985)
[14] B. Kaufman, Phys. Rev. 76 (1949) 1232;
T.D. Schultz, D.G. Mattis and E.H. Lieb, Rev. of Mod. Phys. 36 (1964) 856;
B. McCoy and T.T. Wu, The 2-dimensional Ising model (Harvard Univ. Press, Cambridge, Mass, 1973) and references therein
[15] A.E. Ferdinand and M.E. Fisher, Phys. Rev. 185 (1967) 832
[16] B. Nienhuis and H.J.F. Knops, Phys. Rev. B32 (1985) 1872
[17] D. Friedan, Z. Qiu and S. Shenker, Phys. Lett. 151B (1985) 37
[18] G.V. Gehlen, V. Rittenberg and H. Ruegg, J. Phys. A19 (1986) 107
[19] D.A. Huse, Phys. Rev. B30 (1984) 3908
[20] G.E. Andrews, R.J. Baxter and P.J. Forrester, J. Stat. Phys. 35 (1984) 193
[21] P. Goddard and D. Olive, Nucl. Phys. B257 (1985) 226;
T. Eguchi and K. Higashijima, in Recent developments in quantum field theory, Niels Bohr Centennial Conference 1985, ed. J. Ambjørn, B. Durhuus and J.L. Petersen (North-Holland, 1985)
[22] J.L. Cardy, Phys. Rev. Lett. 54 (1985) 1354
[23] H. Saleur, J. Phys. A., to appear
[24] C. Itzykson, H. Saleur and J.-B. Zuber, Europhysics Lett. 2 (1986) 91
[25] C. Itzykson and J.M. Luck, J. Phys. A19 (1986) 211
[26] A. Cappelli, C. Itzykson and J.-B. Zuber, to be published


[^0]:    *Note added in proof: For $m=0,3 \bmod 4$, the trace restricted to $V_{2}$ may be subtracted from (4.12), yielding a physically sensible modular invariant. This and still other invariants will be discussed elsewhere [26].

