ROUGHENING OF WILSON'S SURFACE

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We suggest that surface roughening, well known in three dimensions, also affects the computation of the Wilson loop in 4-D gauge theories. We propose an observable to locate this singularity. We also discuss the consequences of this singularity for actual calculations.

The lattice gauge theory [1] is now rapidly evolving into a formalism which allows quantitative computations in strong-interaction dynamics. Recent developments of strong-coupling [2–5] and Monte-Carlo methods [6] offer the possibility of actually computing the string tension of QCD, the coefficient of the linearly rising term in the quark-confining potential. However, both methods of calculation make strong use of an assumption that the physics of the lattice gauge theory is smoothly varying from the strong-coupling region, where these methods are most accurate, to the weak-coupling continuum limit. We suggest, in this letter, for the specific case of the string tension, this assumption is not correct. We will argue that the string tension, as a function of the gauge coupling, has an essential singularity at a finite value of g, even if the vacuum state shows no phase transition between strong and weak coupling.

The origin of this singularity is a phenomenon, well known in the statistical mechanics of interfaces in three dimensions, known as surface roughening [7]. It concerns the fluctuations of a two-dimensional interface between ordered media; at a certain roughening temperature, the mean square fluctuation of the position of the interface diverges. This introduces non-analyticity into all quantities which characterize the interface, including the surface tension.

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We will briefly review interface roughening in three dimensions and we will introduce a new observable useful for locating the roughening transition. Then, we will show that this whole analysis generalizes to the case of the surface which spans the Wilson loop in the quark-confining phase of an abelian or non-abelian 4-D gauge theory. We will locate the roughening transition approximately in the Z_2 and SU(2) pure gauge theories by extrapolations of high temperature expansions. Finally, we will offer an argument which determines the form of the singularity in the string tension at the roughening transition.

We begin our study by considering the 3-D Ising model, known to be dual to a Z_2 gauge theory [8]. Let us write the partition function of this model as

$$Z = \sum_{\{\sigma_i = \pm 1\}} \exp\left(\frac{1}{T} \sum_{\substack{\langle i,j\rangle \\ \text{neighbors}}} \sigma_i \sigma_j\right). \tag{1}$$

In the low temperature phase of this model, corresponding to the strong-coupling phase of the gauge theory, it is possible to set up boundary conditions in such a way that an interface between two oppositely magnetized regions forms on a surface perpendicular to, say, the $\hat{3}$-axis. For future use, we call $\mathcal{F}_W = \ln Z$ the free energy with these boundary conditions, $\mathcal{F}$ the free energy with uniform boundary conditions. At $T = 0$, $\sigma_i = +1$ for $x_3 \geqslant 0$, $\sigma_i = -1$ for $x_3 < 0$. At finite temperature, thermal fluctuations move the position of the surface. Eventually, at some temperature, these thermal fluctuations destroy the magnetic ordering: this is the critical point $T = T_c$. However, even if we restrict ourselves to $T < T_c$, there are two distinct regimes characterized by qualitatively different behavior of the fluctuations in the position of the interface. This can be seen as follows.

Consider the fluctuations Δh of the height of the surface at a given point [‡1]. We can imagine computing this quantity in two different approximation schemes, one valid at very low temperature, the other valid near T_c. At low temperature, one may derive an expansion by beginning from the zero-temperature state just described and then, successively, overturning spins. From (1), the expansion parameter for this series is $t = e^{-2/T}$, the relative weight of a mismatched bond. Each excursion of the surface parallel to the 3-direction then costs powers of t for each step of a lattice spacing. For sufficiently small t, such excursions are suppressed and the thermal average $\langle(\Delta h)^2\rangle$ is obviously finite. Near T_c, however, one should calculate from a different picture. For values of T still in the low-temperature phase, but sufficiently close to T_c, the interface should be viewed as a continuum surface. Its fluctuations may be computed from the continuous integral:

$$Z_{\text{surface}} \sim \int \mathcal{D}h(x_1, x_2) \exp(-kA[h]) , \tag{2}$$

where (x_1, x_2) are coordinates of a point on the zero-temperature surface, $h(x_1, x_2)$ is the height of the surface at this point, $A[h]$ is the area of the surface of height h, and k is the surface tension. Eq. (2) implies that $\langle(\Delta h)^2\rangle$ diverges logarithmically due to long wavelength fluctuations:

$$\langle \Delta h^2 \rangle \sim k^{-1} \int \frac{d^2p}{p^2} . \tag{3}$$

Similarly, the correlation function between heights at different points diverges logarithmically with the separation for T close enough to T_c. Crudely speaking the surface becomes delocalized. The boundary between these two regimes occurs at some temperature T_r, the roughening temperature.

Before presenting numerical evidence for the existence of this T_r and for the determination of its location, let us make contact with the 3-D Z_2 gauge theory. The partition function of this theory is:

$$Z = \sum_{\{U_{ij} = \pm 1\}} \exp\left[\beta \sum_{\text{plaquettes}} (U_{ij} U_{jk} U_{kl} U_{li})\right] . \tag{4}$$

We will be interested in the expectation value of the Wilson loop W:

$$\langle W \rangle = Z^{-1} \sum_{\{U\}} \left(\prod U\right) \exp\left[\beta \sum_{\mathrm{P}} (UUUU)\right] , \tag{5}$$

which decreases exponentially with the minimal area enclosed by W.

$$\langle W \rangle \sim \exp(-kA) . \tag{6}$$

Duality [8] relates the models (1) and (4) through

$$t = \tanh \beta = e^{-2/T} , \tag{7}$$

and also enables us to write

$$\langle W \rangle = \exp(\mathcal{F}_W - \mathcal{F}) , \tag{8}$$

using the Ising free energies defined above.

A strong coupling expansion of $-k$ (or a low temperature series in the dual Ising model) yields [9]

$$-k = \ln t + 2t^4 + 2t^6 + 10t^8 + 16t^{10} + 80\tfrac{2}{3}t^{12} + 150t^{14} + 734t^{16} + 1444\tfrac{2}{3}t^{18} + \dots . \tag{9}$$

We expect k to be a decreasing function of t which vanishes only at $t_c = 0.6418$ with a behavior suggested by the renormalization group:

$$k \approx \text{const.}\, a^2/\xi^2 , \tag{10}$$

where ξ is the bulk correlation length,

$$\xi \sim (t_c - t)^{-\nu} , \quad \nu \approx 0.64 . \tag{11}$$

Any extrapolation of the series (9) is inconsistent with this picture (see fig. 1), showing that k must be singular at some t_r less than t_c. However, it is hard to locate t_r from the analysis of the expansion (9). Indeed, we shall see that the singularity in k at t_r is expected to be very weak.

Weeks and collaborators [7,9] have proposed various observables which have a stronger singularity at t_r, thus providing one a better signal of the transition. They consider, for example, the susceptibility χ of the Ising model with respect to a uniform magnetic field, in the presence of the boundary conditions defining $\mathcal{F}_W$.

‡1 This can only be an approximation since the surface can have "overhangs", and disconnected islands of reversed spins can appear at a distance of the minimal surface, thus h should be multi-valued. We shall see how to remedy this later.

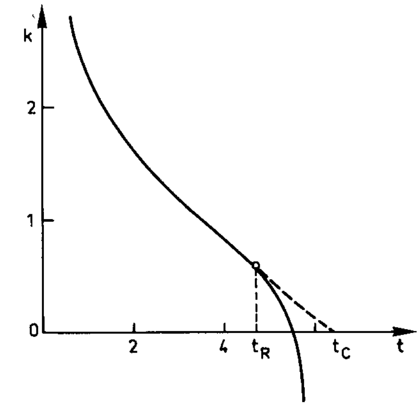

Fig. 1. $k(t)$ in the 3-D Ising model. The solid line is the result of extrapolating the series expansion (9); the dotted line shows the expected behavior near t_c.

They find:

$$\chi = 2t^4 + 14t^6 + 128t^8 + 864t^{10} + 6178t^{12} + 40086t^{14} + 263792t^{16} + 1671444t^{18} + \ldots . \tag{12}$$

Analysis of this and of analogous series reveals a singularity at

$$t_r \approx 0.4593 . \tag{13}$$

We propose now a new observable which generalizes to the case of other groups and higher dimensions, and admits a gauge invariant definition even for a finite loop where the language of phase separation is inadequate. Consider in the gauge version of the model the effect of changing the sign of the coupling of one plaquette belonging to the minimal surface by W. The ratio of the new to the old expectation value of W is

$$p_W = \langle W\, e^{-2\beta(UUUU)} \rangle / \langle W \rangle . \tag{14}$$

Let p_0 be the equivalent quantity in the absence of W:

$$p_0 = \langle e^{-2\beta(UUUU)} \rangle . \tag{15}$$

The "pinch operator" p_W measures the probability that the surface that spans the Wilson loop includes the selected plaquette on the minimal surface. In the dual Ising model, p_W and p_0 read:

$$p_W = \langle \sigma_+ \sigma_- \rangle_W , \quad p_0 = \langle \sigma_+ \sigma_- \rangle , \tag{16}$$

where σ_+ and σ_- are two spins just above and below the minimal surface. At $t = 0$, $p_W = -1$, while above t_r, where the surface is delocalized, p_W should equal p_0. Thus t_r may be identified as the singularity of $(p_0 - p_W)^{-1}$. The series expansions for p_W, p_0 are:

$$p_W = -1 + 4t^4 + 16t^6 + 68t^8 + 284t^{10} + 1260t^{12} + \ldots ,$$

$$p_0 = 1 - 4t^6 - 20t^{10} + 28t^{12} + \ldots , \tag{17}$$

so that

$$2(p_0 - p_W)^{-1} = 1 + 2t^4 + 10t^6 + 38t^8 + 192t^{10} + 860t^{12} + \ldots . \tag{18}$$

The ratios of successive coefficients show a singularity at

$$t_r \approx 0.47 , \tag{19}$$

a value in reasonable agreement with the more precise one quoted above.

It is of utmost importance to understand better the nature of the singularity at t_r. In three dimensions, it is possible to define a simpler model, the so-called solid-on-solid (SOS) model [7], by considering an anisotropic Z_2 gauge theory with $\beta = \beta_{13} = \beta_{23}$ finite, $\beta_{12} \to 0$. For a Wilson loop in the 12-plane, the tension reads

$$k = -\ln \tanh \beta_{12} - \sigma_{SOS}(t) . \tag{20}$$

In this limit, the properties of the bulk system become trivial, and $t_c = 1$. Then $\langle W \rangle$ is given in terms of surfaces bounded by W, without overhangs or disconnected parts, hence described by the height h_i of the actual surface above each plaquette p_i of the minimal surface:

$$\exp[A\sigma_{SOS}(t)] = \sum_{h_i}{}' \exp\Bigl(\ln t \sum_{\substack{\langle i,j \rangle \\ \text{neighbors}}} |h_i - h_j|\Bigr) . \tag{21}$$

Here the summation Σ' runs over all integer heights h_i at all but one plaquettes of the minimal surface. This surface model is close to a dual form of the 2-D $X-Y$ model [10]; this latter would have $|h_i - h_j|$ replaced by $|h_i - h_j|^2$. The critical properties of these two models are expected to be identical; this is shown by numerical studies, renormalization group arguments, by the exact solution, due to van Beijeren [11], of the SOS model on a certain face of a body centered cubic lattice. If t_r denotes the roughening transition point of the SOS model, σ_{SOS} in (20) behaves for $t \approx t_r$ as

$$\sigma_{SOS}(t) \sim \text{(smooth background)} + \text{const.} \exp[-\text{const.}/(t-t_r)^{1/2}] \,. \qquad (22)$$

This form is familiar as the free energy singularity in the 2-D $X-Y$ model. In the van Beijeren model, connected to the six-vertex model, σ is described by two distinct analytic functions for $t < t_r$ and $t > t_r$, which coincide with all their derivatives at $t = t_r$, thus exhibiting a transition of infinite order. Numerically, $\sigma(t)$ and $\sigma_{SOS}(t)$ are rather close, and there is evidence [7] justifying our expectation that they have the same critical behavior (22). One can prove [11] the inequalities

$$t_c^{(2)} < t_r^{SOS} \leqslant t_r \,, \qquad (23)$$

where $t_c^{(2)} = \sqrt{2} - 1$ is the critical point of the 2-D Ising model. Numerically [7,9],

$$t_r^{SOS} \approx 0.4256 \,, \qquad (24)$$

so all three points are rather close. Our indicator $2(1 - p_W^{SOS})^{-1}$ calculated to order 12 gives $t_r^{SOS} \approx 0.45$.

The roughening transition which we have seen in the 3-D Z_2 gauge theory is likely to occur also in other 3-D gauge theories. It is possible to define an SOS approximation to k for any gauge group; the resulting theories are identical, up to a redefinition of t. Similarly, the observable p can be generalized. We postpone this systematic analysis to a future publication.

We now turn to the case of 4-D gauge theories. We notice first that, even though the embedding of 2-D surfaces in 4-D space cannot be interpreted as separating two distinct regions, the qualitative arguments given above about the size of surface fluctuations carry over to this case whenever the bulk transition is a continuous one. When the model has a first order transition, as in the case of Z_2, it is a numerical matter to determine the relative location of t_r and t_c. However, for continuous groups all indications point to the fact that the only transition is a continuous one occurring at vanishing coupling; our earlier argument therefore predicts a roughening transition at a finite value of the coupling.

The simplest observable (of course not the only one) designed to give information about roughening is a simple generalization of the quantity p defined in three dimensions. For the case of a Z_2 gauge theory the definition (14) makes sense in any number of dimensions. In four dimensions, the selected plaquette P of the original lattice has associated with it a plaquette $\tilde{\mathrm{P}}$ of the dual lattice; in this case, our construction corresponds precisely to computing the expectation value of an 't Hooft frustration operator [12] defined on the loop which is the boundary of $\tilde{\mathrm{P}}$ (see fig. 2). We may define p for a general group G as the expectation value of the 't Hooft operator on $\tilde{\mathrm{P}}$; this entails modifying the coupling on P by an element of the center of G. For SU(2), we replace $\beta \to -\beta$ on P, as in the Z_2 case.

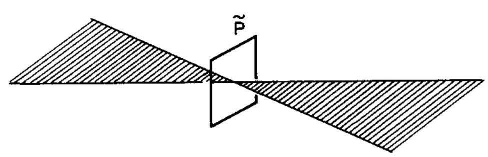

Fig. 2. The plaquette $\tilde{\mathrm{P}}$ dual to a plaquette of the minimal (shaded) surface.

The series expansions for p_W and p_0 are easily obtained to a given order in t by inspection of the diagrams contributing to the string tension and the free energy to that order. To compute the series presented below, we have used, and verified, the tabulations to order t^{12} given by Munster [5].

In the Z_2 case

$$p_W = -1 + 8\,t^4 + 32\,t^6 + 200\,t^8 + 1032\,t^{10} + 6248\,t^{12} + \dots \,, \qquad (25)$$

$$p_0 = 1 - 8\,t^6 - 120\,t^{10} - 8\,t^{12} + \dots \,. \qquad (26)$$

Our indicator

$$2(p_0 - p_W)^{-1} = 1 + 4\,t^4 + 20\,t^6 + 116\,t^8 + 736\,t^{10} + 4392\,t^{12} + \dots \,, \qquad (27)$$

points to a value of t_r which is hardly distinguishable of the bulk critical point $t_c = \sqrt{2} - 1$. More accurate numerical work is required to locate t_r with respect to t_c, that is, to tell whether the roughening transition occurs in the metastable or the physical region of t.

For the group SU(2), we use the Wilson action

$$A = \tfrac{1}{2}\beta\chi_{1/2}(UUUU) \,,$$

where $\chi_{1/2}$ is the spin 1/2 character. It is useful to expand in the parameter

$$u = \frac{1}{2}\,\frac{\int \mathrm{d}U\,\chi_{1/2}(U)\exp[\frac{1}{2}\beta\chi_{1/2}(U)]}{\int \mathrm{d}U \exp[\frac{1}{2}\beta\chi_{1/2}(U)]} \,, \qquad (28)$$

which varies between 0 and 1. We find

$$p_W = -1 + 2(4u^4 + 24u^6 + 102\tfrac{2}{3}u^8 + 497\tfrac{19}{45}u^{10} + 2741\tfrac{41}{135}u^{12} + ...)\,, \qquad (29)$$

$$p_0 = 1 - 2(16u^6 + 240u^{10} - 416u^{12} + ...)\,, \qquad (30)$$

and

$$2(p_0 - p_W)^{-1} = 1 + 4u^4 + 40u^6 + 118\tfrac{2}{3}u^8 + 1057\tfrac{19}{45}u^{10} + 4522\tfrac{86}{135}u^{12} + ... \,. \qquad (31)$$

This series is short and still rather irregular but it seems to indicate a roughening singularity around $u_r \approx 6^{-1/2}$ with a large error. It is amusing to note that this value would correspond to β_r around 2 which coincides with the shoulder in the numerical plot of k versus β found by Creutz [6] where the behaviour of the tension departs from its strong-coupling regime.

We have now motivated the occurrence of a roughening transition in 4-D gauge theories and attempted to find its location in two examples. We turn to the question of the behavior of k at t_r. In 3-D we saw that k has an essential singularity of the type (22). A rough argument can be given in 4-D which yields the same type of singularity.

Assume that the fluctuations of the surface can be characterized by assigning to a point $n \equiv (n_1, n_2)$ of the minimal surface two integers (h_3, h_4) which give the position in the 3 and 4 directions to which this point has moved. We ignore overhangs and vacuum diagrams by assuming that (h_3, h_4) is a single-valued function of (n_1, n_2) and of course we disregard all group theoretic decoration. In 4-D space the values (h_3, h_4) do not uniquely specify the fluctuating surface, but we may sum over all possibilities keeping the sets (h_3, h_4) fixed to obtain an effective partition function

$$Z_{\text{eff}} = \sum_{h_3(n),h_4(n)}{}' \times \exp\left[\sum_{n,n'} f(h_3(n), h_3(n'), h_4(n), h_4(n'))\right]\,, \qquad (32)$$

where f is short range and falls off rapidly when $|h(n) - h(n')|$ becomes large. It is globally invariant separately under $h_i \to -h_i$ and translations of h_i for each of $i = 3, 4$, and under the interchange $h_3 \leftrightarrow h_4$. Under a duality transformation the variables $h_i(n)$ are replaced by angular variables θ_i with an action invariant under the corresponding transformations: translations and reflections of θ_i and the interchange $\theta_3 \leftrightarrow \theta_4$. This is a model for two interacting 2-D $X-Y$ systems with the original strong-coupling region of the gauge theory mapped onto the high temperature regime of these $X-Y$ models. Near the roughening transition the long-range behavior is determined by a universal renormalizable model possessing the above symmetries; the only candidate is described by a lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\partial\theta_3)^2 + \tfrac{1}{2}(\partial\theta_4)^2\,, \qquad (33)$$

a theory of two-decoupled $X-Y$ models. The critical point of the $X-Y$ models is an effect not of the spin waves but rather of the vortices. However, at the critical point vortices of quantum number higher than the minimal value and vortex–vortex interactions are negligible. Hence this decoupling extends to vortices as well. We conclude that the roughening singularity in 4-D is of the same type (22) as in 3-D.

The roughening transition is a natural barrier to the extrapolation of strong-coupling series for the string tension. This we saw explicitly in the Z_2 gauge theory in 3-D. The same might be true for gauge theories in 4-D. This is the most important consequence of our analysis. For Monte-Carlo computations of the string tension, the influence of roughening is more difficult to quantify, but it is clear that such computations are endangered by two phenomena: an increase in the number of configurations building the equilibrium state of the Wilson surface, and an increase of the range of finite size effects beyond t_r. In neither of these cases does the location of t_r make itself felt in the process of computing k, precisely because the singularity is so weak. We suggest that those who attempt computations of k use a measure such as our quantity p_W to locate t_r accurately and exercise great care in interpreting results in the region $t > t_r$.

In this paper, then, we have argued for the existence of a singularity in k at a finite coupling t_r. Our arguments were, in 4-D gauge theories, for the most part, intuitive; clearly more work is required to establish definitively the presence of this transition. But it is important, as well, to already explore the consequences of this idea, and especially to investigate how one can compute numerically in the region between t_r and the continuum limit.

While this work was being completed we heard that A., E., and P. Hasenfratz have developed similar considerations about the possibility of roughening in lattice gauge theories.

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