LARGE q EXPANSIONS FOR q-STATE GAUGE-MATTER POTTS MODELS IN LAGRANGIAN FORM

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We consider the lagrangian form of a q-state generalization of Ising gauge theories with matter fields in d = 3 and 4 dimensions. The theory is exactly soluble in the limit $q \rightarrow \infty$ and corrections are easily calculable in power series in $1/q^{1/d}$. Extrapolating the series for the free energies and latent heats by the method of Padé approximants, we have constructed the phase diagrams for all values of q. Our results agree well with known results for pure spin systems and, for the case q = 2, with Ising Monte Carlo data.

1. Introduction

Recent interest in the phase diagram of Ising lattice gauge theory with matter fields has led Kogut [1] to investigate a q-state Potts generalization. The model, despite its relative simplicity, possesses some important features in common with lattice quantum chromodynamics. The Wilson correlation function, for example, exhibits a perimeter law behavior in both theories due to matter screening and no longer serves as a signature distinguishing confinement and free charge phases. In general, we are led to try to understand as much as possible about the phase diagram of combined gauge-matter systems.

The interior of phase diagrams is generally inaccessible to conventional expansion methods, typically valid only near the extremes of parameter domains. Monte Carlo simulations, however, not suffering from this limitation, have recently been used to map out the entire phase diagram for Ising gauge theories [2, 3]. We are thus provided with a basis with which to compare other, hopefully more intuitive or less ponderous, methods of calculation. One such method was given by Kogut [1], who found that the $q \rightarrow \infty$ limit of the hamiltonian version of the q-state model is exactly soluble, with corrections to the free energy for finite q occurring as power series in 1/q. Since the coefficients of each power of 1/q can be computed exactly, the large q limit is the only approximation which need be made and the results, to any given order in 1/q, are valid throughout the phase diagram. Kogut found in addition that his results even made qualitative sense for values of q as low as q = 2.

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P. Ginsparg et al. / Gauge-matter Potts models

In this paper we consider the class of models suggested by Kogut but in their lagrangian form. We are thus able to make a detailed comparison between our calculations for the coupled gauge-matter systems and Monte Carlo data, a comparison not possible for calculations performed in the hamiltonian version of the theory. We are also able to compare our results with series expansions of conventional *q*-state spin and gauge systems. Moreover, we have found it extremely straightforward in this formulation of the theory to calculate higher-order corrections, and we thus display results to a comparatively high order in the expansion parameter, here $1/q^{1/d}$ (d = number of space-time dimensions). In particular, no infinite summation of graphs is required as in Kogut's calculation; each order in $1/q^{1/d}$ receives contributions from only a finite and easily enumerated set of graphs. That we find an expansion parameter $1/q^{1/d}$ whereas Kogut finds 1/q, independent of dimension, need not be surprising given that the hamiltonian form of the theory is derived from a strongly anisotropic limit of the lagrangian form and hence, at least for large q where the transitions are first order, corresponds to a different theory, with qualitatively different properties. We have calculated the free energies and latent heats for these models in d = 3 and 4 dimensions and used them to construct the phase diagrams for all q. Our results are for the most part in good quantitative agreement with known results in those parts of the phase diagrams and for those values of q where comparison is possible, thereby providing confidence in the validity of the entire phase diagrams for all q.

Before proceeding to the calculation, let us describe briefly some properties of the phase diagrams of the class of models [defined in eqs. (2.3), (2.4)] under consideration. We refer the reader to Kogut's article [1] for a more complete presentation. There are two temperature-like variables, κ and β , controlling respectively the fluctuations of the gauge and matter fields. The phase diagram in the (β, κ) plane (represented schematically in fig. 1) is known to have the following limiting behaviors:

(i) on the $\beta = 0$ boundary, the matter fields decouple and the theory reduces to a pure gauge theory which undergoes a confining-non-confining transition at some κ_c ;

(ii) on the $\kappa = \infty$ boundary, the gauge degrees of freedom freeze out and the theory reduces to a pure spin system which undergoes an order-disorder transition at some β_c .

(iii) along the $\kappa = 0$ and $\beta = \infty$ boundaries, the theory is trivial.

The transitions at the $\beta = 0$ and $\kappa = \infty$ boundaries are known to persist at least a small distance into the phase diagram [4, 5] and the lines of transitions thus implied are determined to curve, very near the boundary, as exaggerated in fig. 1. Finally, along the $\kappa = 0$ and $\beta = \infty$ boundary, there is known to exist, for q finite, a strip where the free energy is analytic [5] so there is a path in the phase diagram between the confinement and Higgs regions, denoted I and II, respectively, along which no phase transition occurs. Fig. 1 is drawn consistent with the above considerations and also incorporates the assumption that the free charge region (no confinement along the



Fig. 1. Schematic phase diagram for q-state coupled gauge-matter systems in the (β, κ) plane. The dotted lines represent phases transitions of unspecified order.

 $\beta = 0$ axis for $\kappa > \kappa_c$), denoted by III, is isolated by a phase boundary. The order of the phase transition along the dotted lines depends in general, as will be discussed, on q and the dimensionality of the system. The line extending out towards the lower right-hand corner is suggested by Ising Monte Carlo results and was first suggested by the mean field approximation [8] (see also the recent results of [20]).

The aim of this paper is to learn as much as possible about these phase diagrams from the standpoint of a large q expansion. In sect. 2, we define the models of interest and review the relevant formalism. In sect. 3, we assess the validity of our expansion method by applying it to the two-dimensional Potts spin system for which there exist exact results with which to compare. In sect. 4, we treat the coupled gauge-matter in three dimensions and in sect. 5, the analogous system in four dimensions. Sect. 6 summarizes our results and the appendix deals with the identification of the leading contributions to the free energy in the lower right corner of the phase diagram.

2. Basic formalism

The conventional q-state Potts [6] model is defined as follows: to each site i of the lattice is attached a spin z_i taking its value in Z_q , the group of the qth roots of unity. The partition function

$$Z = \prod_{\text{sites}} \frac{1}{q} \sum_{z_i \in \mathbb{Z}_q} \exp\left(-\beta H_{\text{m}}\right)$$
(2.1)

defines the spin, or pure matter, model where

$$-\beta H_{\rm m} = \beta \sum_{\langle ij \rangle} \delta_{z_i, z_j}, \qquad (2.2)$$

and the sum is over nearest neighbor sites on the lattice.

With the addition of link variables $U_{ij} \in \mathbb{Z}_q$, we may define, in the usual way [7, 8], a model of gauge fields coupled to matter fields with a local \mathbb{Z}_q gauge invariance:

$$Z = \prod_{\substack{\text{sites}\\\text{and links}}} \frac{1}{q} \sum_{U_{i_i}} \frac{1}{q} \sum_{z_i} \exp\left(-\beta H_{\text{mg}} - \kappa H_{g}\right), \qquad (2.3)$$

where

$$-\beta H_{mg} = \beta \sum_{\langle ij \rangle} \delta_{z_i U_{il} z_j^{-1}, 1},$$

$$-\kappa H_g = \kappa \sum \delta_{U_p, 1}, \qquad U_p = U_{ij} U_{ik} U_{kl} U_{li}. \qquad (2.4)$$

The models defined by (2.1), (2.2) and (2.3), (2.4), are generalizations of the Ising model and are known to have analogous self-duality properties [9–11]. In two dimensions, the model (2.1) is self-dual with the duality relation for the free energy per site $F = -(1/S) \ln Z$ given by

$$F(\beta) = F(\tilde{\beta}) + 2 \ln \left[(e^{\beta} - 1) / \sqrt{q} \right], \qquad (2.5a)$$

where the dual coupling $\vec{\beta}$ satisfies

$$(e^{\beta} - 1)(e^{\tilde{\beta}} - 1) = q$$
. (2.5b)

In three dimensions, the model (2.3) is self-dual with

$$F(\beta, \kappa) = F(\tilde{\beta}, \tilde{\kappa}) + 3 \ln \left[(e^{\tilde{\beta}} - 1)(e^{\tilde{\kappa}} - 1)/q \right],$$
$$(e^{\beta} - 1)(e^{\tilde{\kappa}} - 1) = (e^{\tilde{\beta}} - 1)(e^{\kappa} - 1) = q.$$
(2.6)

Finally, in four dimensions, the pure gauge model ($\beta \equiv 0$) is self-dual with

$$F(\kappa) = F(\tilde{\kappa}) + 6 \ln \left[(e^{\kappa} - 1)/\sqrt{q} \right],$$

(e^{\kappa} - 1)(e^{\kappa} - 1) = q. (2.7)

In order to define a sensible large q limit, it is convenient to parametrize the theory in terms of new couplings v and w defined by

$$e^{\beta} = 1 + q^{a}v, \qquad e^{\kappa} = 1 + q^{b}w.$$
 (2.8)

We shall fix a and b by requiring that both the high- and low-temperature expansions have smooth limits as $q \to \infty$ for v and w fixed. Considering first the high-temperature (region I: small β , κ) limit, we recall briefly the method of expansion, following the notation of [10, 12]. The Boltzmann weight in (2.3) is expanded in terms of the (q-1) non-trivial characters $\chi_r(z) = z'(1 \le r \le q-1)$ of the group Z_q :

$$q\delta_{z,1} = 1 + \sum_{r=1}^{q-1} \chi_r(z) , \qquad (2.9)$$

P. Ginsparg et al. / Gauge-matter Potts models

$$\exp\left(\beta\delta_{z_{i}U_{ij}z_{j}^{-1},1}\right) = (e^{\beta} - 1)\delta_{z_{i}U_{ij}z_{j}^{-1},1} + 1$$

$$= \frac{e^{\beta} - 1}{q} \left[1 + \sum_{r=1}^{q-1} \chi_{r}(z_{i}U_{ij}z_{j}^{-1})\right] + 1 \qquad (2.10)$$

$$= \left(1 + \frac{e^{\beta} - 1}{q}\right) \left[1 + A\sum_{r} \chi_{r}(z_{i})\chi_{r}(U_{ij})\chi_{r}^{*}(z_{j})\right],$$

where

$$A = \frac{e^{\beta} - 1}{e^{\beta} - 1 + q} = \frac{q^{a}v}{q + q^{a}v} = \frac{vq^{a-1}}{1 + vq^{a-1}}.$$
 (2.11)

Similarly, for each plaquette

$$\exp\left(\kappa\delta_{U_{p},1}\right) = \left(1 + \frac{e^{\kappa} - 1}{q}\right) \left[1 + B\sum_{r}\chi_{r}(U_{ij})\chi_{r}(U_{jk})\chi_{r}(U_{kl})\chi_{r}(U_{li})\right], \quad (2.12)$$

with

$$B = \frac{wq^{b-1}}{1 + wq^{b-1}}.$$
 (2.13)

We then use the orthogonality of characters,

$$\frac{1}{q}\sum_{z\in \mathbb{Z}_q}\chi_r(z)\chi_s(z)\cdots = \frac{1}{q}\sum_{z}\chi_{r+s+\cdots}(z) = \delta_{r+s+\cdots,0}$$
(2.14)

(the sum $r+s+\cdots$ is understood modulo q), to perform the summation over the matter and gauge variables $\{z_i\}$ and $\{U_{ij}\}$ yielding an expansion of the partition function in powers of v and w. The diagrams corresponding to the terms in the series are composed of matter bonds and gauge plaquettes. To each oriented matter bond and plaquette, respectively, are attached integers r_{ij} and r_p $(1 \le r \le q-1)$ satisfying the conservation laws:

for each site *i*:
$$\sum_{i} \varepsilon_{ij}^{(i)} r_{ij} = 0$$
, (mod *q*), (2.15a)

for each link:
$$r_{ij} + \sum_{\mathbf{p}:(ij)} \varepsilon_{\mathbf{p}}^{(ij)} r_{\mathbf{p}} = 0$$
, $(\text{mod } q)$, (2.15b)

where the last sum is over plaquettes incident upon the link (ij) and $\varepsilon = \pm 1$ according to relative orientations. To a given diagram there is associated a combinatorial weight counting the number of ways of satisfying (2.15), and also a factor of A for each matter bond and a factor of B for each gauge plaquette. Including the contribution from the prefactors in (2.10) and (2.12), we find, for the partition function on a d-dimensional lattice with S sites:

$$Z = (1 + q^{a-1}v)^{dS} (1 + q^{b-1}w)^{d(d-1)S/2} \left[1 + \sum_{\text{diagrams}} (\text{weight}) A^{L_m} B^{P_g} \right]. \quad (2.16)$$

Here and in the following, $S_g(S_m)$, $L_g(L_m)$, and P_g stand for the number of sites or vertices, of links, and of plaquettes in the gauge (matter) part of the diagram. We are now in a position to identify the leading q dependence of a given diagram. The weight in (2.16) behaves for large q as q' with

$$r = \sum_{\substack{\text{connected} \\ \text{parts}}} (L_{\rm m} - S_{\rm m} + 1) + (P_{\rm g} - R_{\rm g}) .$$
(2.17)

The first bracket, the matter contribution, counts the number of independent choices of integers satisfying (2.15a), i.e., the number of independent circuits in the matter part of the diagram. In the second bracket, R_g is the number of independent constraints coming from eq. $(2.15b)^*$. Including the large q behavior of A and B we find the leading q dependence of a diagram to be q^a with

$$s = r + (a - 1)L_{\rm m} + (b - 1)P_{\rm g}.$$
(2.18)

For a and b too large, we would find the power of q to increase arbitrarily with the volume of the diagram, preventing the definition of a sensible large q limit. Considering the pure matter $(w = \infty)$ and pure gauge (v = 0) models separately, and taking in each case the "worst" configuration (i.e., a diagram of given size with the highest possible power of q) of a large connected diagram containing all the matter links or all the gauge plaquettes in a given region, we find

$$L_{\rm m} \sim dS_{\rm m} ,$$

$$L_{\rm g} \sim dS_{\rm g} , \qquad P_{\rm g} \sim \frac{1}{2} d(d-1)S_{\rm g} , \qquad R_{\rm g} \sim L_{\rm g} - S_{\rm g} ,$$

and therefore

$$s \sim (ad-1)S_{\rm m}$$
, (matter)

or

$$s \sim (d-1)(\frac{1}{2}bd-1)S_g$$
, (gauge). (2.19)

To prevent the volume-dependent parts from blowing up, we thus need to impose

$$a \leq \frac{1}{d}, \qquad b \leq \frac{2}{d}. \tag{2.20}$$

We now turn to the low temperature (region II: large β , κ) expansion of the model. In this limit, we expand from a ground state with all link variables U_{ij} and all site variables z_i equal to unity up to a local gauge rotation. Corrections to the partition

^{*} As pointed out to us by Jean-Michel Drouffe, this counting is only valid for configurations of plaquettes which form an orientable surface. Non-orientable surface may contribute, but only when q is even, since the plaquettes must carry the representation $\chi_{q/2}$ in order to satisfy (2.15b). This implies that we should actually consider two distinct large q limits, according to the parity of q. However, the first non-orientable diagrams correspond to a power in our expansion parameter much higher than will be considered in the following and so this effect will not concern us.

function come from perturbations of this state with successively more flipped spins and/or links. These corrections may be represented diagrammatically, the contribution of a diagram being proportional to $\exp(-\beta L_m - \kappa P_g)$, where P_g and L_m count the number of plaquettes and bonds whose associated U_p and $z_i U_{ij} z_j^{-1}$ are different from unity, times a weight factor counting the number of possible orientations of the selected flipped spins and bonds.

The partition function can therefore be written

$$Z = \frac{q^S}{q^{S(1+d)}} \exp\left(d\beta S + \frac{1}{2}d(d-1)\kappa S\right) \left[1 + \sum_{\text{diagrams}} (\text{weight}) e^{-\kappa P_g - \beta L_m}\right]. \quad (2.21)$$

Again with the intent of requiring that only a finite number of diagrams contribute to a given order in q, we examine for the pure spin (pure gauge) model the "worst case" behavior of a diagram containing all the spins (links) flipped in a given volume. The $O(q^{S_m})$ ($O(q^{L_g-S_g})$ non gauge-equivalent) configurations contribute a factor

$$q^{s_{m}}(1+q^{a}v)^{-L_{m}} \sim q^{s_{m}-aL_{m}},$$
 (matter), (2.22)

$$q^{L_{g}-S_{g}}(1+q^{b}w)^{-P_{g}} \sim q^{-bP_{g}+L_{g}-S_{g}},$$
 (gauge). (2.23)

Suppression of large volume effects now leads to the requirements

$$a \ge \frac{1}{d}, \qquad b \ge \frac{2}{d},$$
 (2.24)

so comparison with (2.20) fixes

$$a = \frac{1}{d}, \qquad b = \frac{2}{d},$$
$$e^{\beta} = 1 + q^{1/d}v, \qquad e^{\kappa} = 1 + q^{2/d}w. \qquad (2.25)$$

With the volume-dependent behavior cancelled by the prescription (2.25), it is easy to see that the surface-dependent effects will always act to suppress the contributions of large diagrams by factors of $1/q^{1/d}$. Although the gauge-matter systems have been treated independently in the above, it is easy to convince oneself that (2.25) ensures that only a finite number of diagrams will contribute to any given order $q^{-1/d}$ also in the interacting gauge-matter system. Moreover, a third possible expansion, at low temperature in the gauge sector and high temperature in the matter sector (region III: large κ , small β) also has a smooth $q \rightarrow \infty$ limit. We will make extensive use of these three expansions in the following sections.

In summary, the rescaling (2.25) allows a reordering of the usual high and low temperature expansions into expansions in the parameter

$$z = q^{-1/d} \,. \tag{2.26}$$

To any given order in z, the expansions are exact to all orders in v and w and so can be reliably used to probe interior regions of the v, w phase diagram. These expansions

also respect the duality properties (2.5)-(2.7) of the model order by order in z (with the duality relations for the couplings particularly naturally expressed in terms of v and w) and moreover lead to critical couplings v_c , w_c of order unity.

3. The two-dimensional pure spin model

The two-dimensional pure spin model, for which there exists a variety of exact results [9, 13], serves as a good laboratory to assess the validity of extrapolating a large q expansion to low values of q. We recall that this model is self-dual, and hence that its transition, if unique, is located at v = 1 [see eq. (2.25)]. Moreover, Baxter [13] has given an exact expression for the latent heat at this point. Defining the latent heat by

$$L = \frac{1}{2} \left(\frac{\mathrm{d}F}{\mathrm{d}v} \Big|_{v=1} - \frac{\mathrm{d}F}{\mathrm{d}v} \Big|_{v=1} \right), \tag{3.1}$$

then for $q \leq 4$, L vanishes and the transition is second order, while for q > 4

$$L = \tanh \frac{1}{2}\theta \prod_{m=1}^{\infty} \tanh^2 m\theta, \qquad (3.2)$$

where θ is related to q by

$$2\cosh\theta = \sqrt{q} \,. \tag{3.3}$$

From eq. (3.2) it may be shown that as q (now considered as a continuous parameter) approaches 4 from above, L vanishes with an essential singularity:

$$L \sim \sqrt{q-4} \exp -\frac{c}{\sqrt{q-4}}.$$
(3.4)

How are these features reproduced by a large q expansion? The expression (3.2) for the latent heat is easily expanded to any desired order in the parameter $z = q^{-1/2}$. For example, to order 11, we find

$$L = 1 - 2z - 2z^{2} + 4z^{3} - 6z^{4} + 12z^{5} - 16z^{6} + 32z^{7} - 38z^{8} + 76z^{9} - 76z^{10} + 152z^{11}$$

= (1 - 2z)(1 - 2z^{2} - 6z^{4} - 16z^{6} - 38z^{8} - 76z^{10}). (3.5)

This corresponds exactly to what is obtained using (3.1) and the high temperature expansion of the free energy, supplemented by the duality relation between F(v > 1) and F(v < 1). The diagrams contributing up to fourth order in z are depicted in fig. 2.

Let us try now to reproduce the critical value $q_c = 4$ and the numerical value of L for low q, say q = 5 to 10, using only the truncated series (3.5). We can either use this series as it stands, or extrapolate it in some way, by Padé approximants for example. It turns out that in this particular case, both approaches give comparable results, due to certain properties of the series (3.5). Truncated to any *odd* order, the series has an



Fig. 2. Diagrams contributing to the high temperature expansion of the free energy of the twodimensional pure spin system up to fourth order in $z = q^{-1/2}$; diagrams (a), (b), (c, d) give respective contributions of order z^2 , z^3 , z^4 .

overall factor of (1-2z) and therefore yields the correct value $q_c = 1/z_c^2 = 4$. Secondly, the radius of convergence of the series (3.5) is $|z| = \frac{1}{2}$, and hence it is safe to compute its value for q as low as 5. We shall see that these properties do not persist in higher dimensions and we shall therefore rely more generally on Padé extrapolations. In the present case, we have also computed the sequence of diagonal and upper-diagonal Padé approximants, looked for the first zero either on or slightly off the positive real axis, (expected to approach $z_c = \frac{1}{2}$), and computed their values for low q. The results are displayed in table 1. We see that the Padé extrapolations approximate the strong zero of (3.4) at $z = \frac{1}{2}$ by one (or more) earlier zero(s) at $z < \frac{1}{2}$. On the other hand, their values at $q = 5, 6, \ldots$ converge slowly, but nicely, to the exact value.

We conclude that the $1/\sqrt{q}$ expansion has indeed been able to reproduce qualitatively and even quantitatively all the important low q features of the twodimensional model. We thus feel justified in extrapolating a large q expansion to small q in 3D and 4D cases where there exist no analogous exact results with which to compare.

	Zc	L(q=5)	L(q=6)	L(q=7)	L(q=10)
Exact	0.5	0.01828	0.07153	0.12819	0.26441
Truncation order 7	0.5	0.02449	0.07816	0.13306	0.26610
Truncation order 9	0.5	0.01807	0.07278	0.12920	0.26471
Truncation order 11	0.5	0.01551	0.07098	0.12810	0.26443
[4, 3]	$0.4407 \pm 0.0436i$	0.105	0.0917	0.135	0.2654
[4, 4]	$0.4541 \pm 0.0366i$	0.046	0.0783	0.131	0.2647
[5, 4]	0.46534 and 0.6	0.0151	0.0706	0.12785	0.26437
[5, 5]	$0.46984 \pm 0.02221i$	0.0215	0.0721	0.12837	0.26442
[6, 5]	$0.47348 \pm 0.01716i$	0.0193	0.07161	0.12820	0.26441

TABLE 1

Zero nearest the origin, z_c , in the $q^{-1/2}$ plane and value for low q of the latent heat of the two-dimensional pure spin model, either truncated, or extrapolated by Padé approximants

4. The three-dimensional phase diagram

In three dimensions our expansion parameter is $z \equiv 1/q^{1/3}$. In this case we define v and w by

$$v = (e^{\beta} - 1)z, \qquad (4.1)$$

$$w = (\mathbf{e}^{\kappa} - 1)z^2 \,. \tag{4.2}$$

It will be necessary to carry out the calculation in two regimes only, since the high temperature (small v and w) regime is dual to the low temperature regime. Using the variables v and w, the duality relation (2.6) takes the form

$$F(v, w) = F\left(\frac{1}{w}, \frac{1}{v}\right) - 3 \ln vw, \qquad (4.3)$$

and the self-dual line is given by

$$vw = 1. (4.4)$$

It is convenient to define additional variables x and y, which take their values between 0 and 1, related to v and w by

$$x \equiv \frac{v}{1+v}, \tag{4.5}$$

$$y \equiv \frac{w}{1+w} \,. \tag{4.6}$$

In terms of x and y the duality relation (4.4) takes the form

$$x + y = 1$$
. (4.7)

The diagrams contributing up to order z^8 to the partition function in the small v, w region denoted by I, are depicted in fig. 3. The free energy is given by

$$-\frac{1}{3}F_{I} = \ln (1 + vz^{2}) + \ln (1 + wz) + \frac{1}{3}z^{3}(1 - z^{3})w^{6}(1 + wz)^{-6} + z^{5}(1 - z^{3})(1 - 2z^{3})w^{11}(1 + wz)^{-11} + z^{6}(1 - z^{3})v^{4}w(1 + vz^{2})^{-4}(1 + wz)^{-1} - \frac{7}{6}z^{6}(1 - z^{3})^{2}w^{12}(1 + wz)^{-12} + z^{7}(1 - z^{3})w^{10}(1 + wz)^{-10} + 5z^{7}(1 - z^{3})(1 - 2z^{3})^{2}w^{16}(1 + wz)^{-16} - 4z^{8}(1 - z^{3})^{2}(1 - 2z^{3})w^{17}(1 + wz)^{-17} + 2z^{8}(1 - z^{3})(1 - 2z^{3})w^{6}(1 + wz)^{-6}v^{4}(1 + vz^{2})^{-4} + z^{8}(1 - z^{3})(1 - 2z^{3})(1 - 3z^{3})^{2}w^{20}(1 + wz)^{-20} + O(z^{9}).$$
(4.8)

Region II, defined by large v and w, is the region dual to I, and the free energy in this region is obtained from (4.8) by using eq. (4.3).

P. Ginsparg et al. / Gauge-matter Potts models



Fig. 3. Diagrams contributing up to order 8 in $z = q^{-1/3}$ to the region I expansion of the free energy of the 3D gauge-matter model. Solid lines represent edges of gauge plaquettes, dotted lines matter links, and in diagrams (b), (f), (g), (i) the inner plaquettes have been selected, while not in (e).

Finally, we calculated the free energy in region III where v < 1, w > 1, i.e., high matter temperature, low gauge temperature. The lowest order diagrams contributing to the partition function are depicted in fig. 4. The free energy is given by

$$-\frac{1}{3}F_{\rm III} = \ln w + \ln (1 + w^{-1}z^2) + \ln (1 + vz^2) + z^5(1 - z^3)v^4(1 + vz^2)^{-4} + z^5(1 - z^3)w^{-4}(1 + w^{-1}z^2)^{-4} + O(z^7).$$
(4.9)

Our next task is to find the phase boundaries. Since the ground state of the system is the state of minimal free energy, the phase boundaries are the lines along which the free energies calculated in neighbouring regions are equal.



Fig. 4. Contributions to the high-w, small-v expansion of the 3D and 4D models. Diagram (a) is a box of high temperature matter bonds, diagram (b) represents a flipped gauge link.

Padé	4 2	3	4	5	6	10	20	
[3, 2]	1.84	1.73	1.66	1.61	1.58	1.49	1.38	
[4, 2]	2.12	1.90	1.78	1.70	1.65	1.52	1.39	
[3, 3]	2.22	1.94	1.80	1.71	1.65	1.52	1.30	
High temp. expansions	2.26 [14]	2.03 [15]	1.75 [16]					

TABLE 2

Value of w_c for v = 0 in three dimensions for various values of q and various Padé approximants: for comparison, results obtained from high temperature expansion of spin systems are also displayed

The line separating regions I and III is readily found to be

$$w = 1 + z + \frac{1}{3}z^{3} + \frac{2}{3}z^{4} - (1 + v^{4})z^{5} + (\frac{20}{9} - v^{4})z^{6} + O(z^{7}).$$
(4.10)

The line separating regions II and III is given by applying to (4.10) the duality transformation

$$(v, w) \rightarrow (1/w, 1/v)$$
. (4.11)

Before considering the full line (4.10), let us first concentrate on its end point at v = 0. This point is the transition point of the pure gauge system, dual to the critical point of the pure matter system. We have constructed a set of Padé approximants to locate the position of this point. The results of these approximants evaluated for various values of $q (\equiv 1/z^3)$, are given in table 2. For q = 2, 3, 4 we also list for comparison results obtained in refs. [14–16] using high and low temperature series for the dual pure spin system. The agreement is impressive considering the low order of our calculation.

Returning now to the full line (4.10) and using again Padé approximants to evaluate its position, we find that the line is nearly straight and the value of w at the point at which it intersects the self-dual line (4.4) deviates from the value at the boundary only by about 1%. The shape of the phase boundaries for q = 2 and q = 10 are given in fig. 5. In these figures we display also the end point of the self-dual line, a subject to be discussed shortly.

We proceed to evaluate the discontinuity of the derivative of the free energy across the phase boundaries, a quantity which will be referred to as the "latent heat". Consider first the latent heat across the I to III phase boundary. This is given by

$$\frac{1}{3}L = \frac{1}{3} \frac{d}{dw} (F_1 - F_{III}) = w^{-1} - z + z^2(w - w^{-2}) - z^3(w^2 + 2w^5) + z^4(w^3 + w^{-3} + 14w^6) + z^5(-4w^{-5} - w^4 - 56w^7 - 11w^{10}) + z^6(-w^{-4} + 3w^5 + 168w^8 + 146w^{11} - v^4) + \cdots$$
(4.12)



Fig. 5. Phase diagrams of the 3D model for q = 2 and 10. The vertical and horizontal transition lines result from the [4, 2] or [3, 3] Padé approximants, while the endpoints along the diagonal line comes from the zero of the [4, 2] (dotted line) or [4, 4] approximants. Monte Carlo experiments [3] suggest that the endpoint occurs within the shaded region.

Substituting for w the expression (4.10) and working out the Padé approximants for the resulting expression, we find for the end point v = 0 that L vanishes at $q \approx 4.5$ for the [3, 3] Padé. Considering instead the quantity wL, the same Padé approximant gives $q \approx 4.1$. Although these values seem a little high for 3-dimensional Potts spin systems where the critical value of q is below four and is believed even to be below three, this is presumably due to the slow convergence of the Padé extrapolation in this case. For similar reasons, we were unable to determine whether the phase transition, which for large q is first order along the entire length of the I-III phase boundary, has for low q a change in its order somewhere along this boundary.

Let us turn now to the phase boundary lying along the self-dual line (4.4). At one end, this line terminates at the triple point at which the three phases meet. At the other end, it terminates, for q finite, due to the vanishing of the latent heat across it. For $q = \infty$ the line does not terminate and separates the confined phase I from the Higgs phase II.

Before calculating the latent heat let us introduce the coordinates ξ and η defined by

$$\xi = \frac{1}{2}(x - y), \qquad (4.13a)$$

$$\eta = \frac{1}{2}(x + y - 1) . \tag{4.13b}$$

Using these variables the duality relation (4.3) takes the form

$$F(\xi,\eta) = F(\xi,-\eta) - 3 \ln \frac{(\frac{1}{2}+\eta)^2 - \xi^2}{(\frac{1}{2}-\eta)^2 - \xi^2}.$$
 (4.14)

Defining also

$$\gamma = \frac{\frac{1}{2} + \xi}{\frac{1}{2} - \xi},$$
 (4.15)

the latent heat is given by

$$\frac{1}{6}L_{\eta} = \frac{1}{\frac{1}{4} - \xi^2} \left[1 - \frac{1}{3}\gamma \frac{\partial F_{\mathrm{I}}}{\partial v} - \frac{1}{3}\gamma^{-1} \frac{\partial F_{\mathrm{I}}}{\partial w} \right] \Big|_{v = w^{-1} = \gamma}.$$
(4.16)

Using the expression (4.8) for F_1 we find

$$\frac{1}{6}L_{\eta} = \frac{1}{\frac{1}{4} - \xi^{2}} \left[1 - \gamma^{-1}z - (\gamma - \gamma^{-2})z^{2} - (\gamma^{-3} + 2\gamma^{-6})z^{3} + (\gamma^{2} + \gamma^{-4} + 14\gamma^{-7})z^{4} - (\gamma^{-5} + 56\gamma^{-8} + 11\gamma^{-11})z^{5} - (6\gamma^{3} - 3\gamma^{-6} - 168\gamma^{-9} - 146\gamma^{-12})z^{6} + (6\gamma^{2} - 15\gamma^{-7} - 430\gamma^{-10} - 1040\gamma^{-13} - 80\gamma^{-16})z^{7} + (25\gamma^{4} - 7\gamma - 20\gamma^{-2} + 57\gamma^{-8} + 1067\gamma^{-11} + 5278\gamma^{-14} + 1428\gamma^{-17} - 20\gamma^{-20})z^{8} + O(z^{9}) \right]. \quad (4.17)$$

Before evaluating this expression let us note that its leading terms for large γ and small z are

$$\frac{1}{6}(\frac{1}{4} - \xi^2)L_{\eta} = 1 - \gamma z^2 + (\gamma z^2)^2 - 6(\gamma z^2)^3 + 25(\gamma z^2)^4 + \text{higher order terms}.$$
(4.18)

These are the only terms which survive when $\gamma \to \infty$ and $z \to 0$ such that γz^2 is finite.

The diagrams appearing in the free energy which survive this limit are just those diagrams for which each gauge plaquette has a matter plaquette on top of it: see fig. 6. (The corresponding diagrams in the dual low temperature expansion are just those with only link variables flipped). This is elaborated in the appendix. The contribution of these diagrams to the free energy is

$$-F_1 \sim 3 \ln (1+z^2 v) + \sum_{\text{diagrams}} a_{LP} z^{2(L-P)} \left(\frac{v}{1+v z^2}\right)^L w^P.$$
(4.19)



Fig. 6. Examples of diagrams of the high temperature expansion contributing in the limit $q \rightarrow \infty$, $vq^{-(d-1)/d}$ fixed, $wq^{2/d}$ fixed. Here solid lines represent matter bonds, and selected plaquettes have been shaded.

where L is the number of links, P is the number of plaquettes and a_{LP} are numerical coefficients associated with each diagram. Using eq. (4.16), one finds

$$\frac{1}{6}(\frac{1}{4}-\xi^2)L_{\eta} = \frac{1}{1+\gamma z^2} - \frac{1}{3}\sum_{\text{diagrams}} a_{LP} \frac{(\gamma z^2)^{L-P}}{(1+\gamma z^2)^L} \left(\frac{L}{1+\gamma z^2}+P\right).$$
(4.20)

From (4.20) it is evident that for large q (small z) the end point of the line separating phases I and II will have

$$\gamma_{\rm c} \sim {\rm const} \cdot q^{2/3} \,, \tag{4.21}$$

implying

$$v_{\rm c} \sim {\rm const} \cdot q^{2/3} \,, \tag{4.22a}$$

$$w_c \sim \operatorname{const} \cdot q^{-2/3} \,. \tag{4.22b}$$

Note that (4.22b) implies that κ_c is finite as q becomes large. One of us [18] has shown that even for finite q a suitable low fugacity-high temperature expansion for an equivalent lattice gas theory can be constructed in this corner of the phase diagram (β large, κ small). A van-der-Waals type approximation yields a result consistent with (4.22), namely that κ_c remains finite as q becomes large, together with an evaluation of the constant in (4.21) or (4.22).

We have calculated various Padé approximants for the expression inside the square brackets of eq. (4.17). We show in fig. 5 the location of the end point of the self-dual line. As q becomes large, (4.21) is found to be fulfilled with the constant ≈ 1.16 using a [4, 2] and ≈ 0.8 using a [4, 4] Padé approximant.

Finally we compare the shape of the phase boundaries obtained above and displayed in fig. 5 with the results of Jongeward, Stack and Jayaprakash [3] obtained by Monte Carlo simulations of the Z_2 gauge-matter system. The shape of the free charge region coincides exactly with our results. The phase boundary along the self-dual line terminates in the interior of the phase diagram at a point whose exact location is hard to evaluate exactly from the Monte Carlo results but which corresponds roughly to $-0.11 \le \xi \le -0.05$. Our results for the endpoint are $\xi \sim 0.084$ using a [4, 2] Padé approximant and $\xi \sim 0.016$ using a [4, 4] Padé, so we see at least a tendency to converge towards the Monte Carlo result.

5. The four-dimensional phase diagram

In four dimensions the expansion parameter is $z \equiv 1/q^{1/4}$. The couplings v and w are still defined by

$$v = z(e^{\beta} - 1),$$
 (5.1)

$$w = z^{2}(e^{\kappa} - 1) .$$
 (5.2)

In this case we must calculate the expansion for the free energy in all three regions since regions I and III are no longer dual as in the three dimensional model.



Fig. 7. Contributions up to order $z^{12} = q^{-3}$ to the high temperature expansion of the 4D model. Same conventions as in fig. 3.

The diagrams contributing to the partition function in region I(v, w small) to $O(z^{12})$ are depicted in fig. 7. The free energy to this order is given by

$$-F_{I} = 6 \ln (1 + wz^{2}) + 4 \ln (1 + vz^{3}) + 4z^{8}(1 - z^{4})w^{6}(1 + wz^{2})^{-6} + 6z^{10}(1 - z^{4})v^{4}(1 + vz^{3})^{-4}w(1 + wz^{2})^{-1} + O(z^{13}).$$
(5.3)

The diagrams contributing to the partition function in the low temperature region - region II (v, w large) - are depicted in fig. 8. The expression for the free energy in this region is given by

$$-F_{\rm II} = 4 \ln v + 6 \ln w + 4 \ln (1 + v^{-1}z) + 6 \ln (1 + w^{-1}z^{2}) + z^{4}(1 - z^{4})v^{-8}(1 + v^{-1}z)^{-8} + 4z^{7}(1 - z^{4})(1 - 2z^{4})v^{-15}(1 + v^{-1}z)^{-15} - \frac{9}{2}z^{8}(1 - z^{4})^{2}v^{-16}(1 + v^{-1}z)^{-16} + 4z^{9}(1 - z^{4})v^{-1}(1 + v^{-1}z)^{-1}w^{-6}(1 + w^{-1}z^{2})^{-6} + 4z^{10}(1 - z^{4})v^{-14}(1 + v^{-1}z)^{-14} + 28z^{10}(1 - z^{4})(1 - 2z^{4})^{2}v^{-22}(1 + v^{-1}z)^{-22} - 64z^{11}(1 - z^{4})^{2}(1 - 2z^{4})v^{-23}(1 + v^{-1}z)^{-23} + 6z^{12}(1 - z^{4})(1 - 2z^{4})(1 - 3z^{4})^{2}v^{-28}(1 + v^{-1}z)^{-28} + 8z^{12}(1 - z^{4})(1 - 2z^{4})v^{-8}(1 + v^{-1}z)^{-8}w^{-6}(1 + w^{-1}z^{2})^{-6} + \frac{109}{3}z^{12}(1 - z^{4})^{3}v^{-24}(1 + v^{-1}z)^{-24} + O(z^{13}).$$
(5.4)

Finally, in region III – high temperature matter and low temperature gauge (v small, w large) – the relevant diagrams up to $O(z^{12})$ are the same as those depicted in fig. 4. The free energy in this region is given by

$$-F_{\rm III} = 6 \ln w + 6 \ln (1 + w^{-1}z^2) + 4 \ln (1 + vz^3) + 6z^8 (1 - z^4) v^4 (1 + vz^3)^{-4} + 4z^8 (1 - z^4) w^{-6} (1 + w^{-1}z^2)^{-6} + O(z^{-13}).$$
(5.5)

We now proceed to locate the phase boundaries by equating the expressions for the free energy in the various regions. The equation for the boundary between phases I and III is

$$w = 1 - v^{4}z^{8} - v^{4}z^{10} + 4v^{5}z^{11} + O(z^{13}).$$
(5.6)

This line is almost straight but does exhibit the proper curvature indicated in sect. 1.

P. Ginsparg et al. / Gauge-matter Potts models



Fig. 8. Diagrammatic expansion up to order $z^{12} = q^{-4}$ of the low temperature 4D free energy. Small open dots represent flipped spins. The bonds incident upon these spins are usually different from unity (broken line) except when the neighboring spin is also flipped into the same state [e.g., diagram (e)]. Solid lines represent flipped gauge links.

Substituting v = 0, we obtain w = 1 which is the self-dual point of the pure gauge model.

The equation for the boundary between phases II and III is

$$v^{-1} = 1 + z + z^{2} + \frac{1}{4}z^{4} + \frac{1}{2}z^{5} + \frac{3}{4}z^{6} + (-w^{-6} - \frac{43}{32})z^{8} + (-w^{-6} + \frac{27}{8})z^{9} + (-w^{-6} + 6w^{-7} - \frac{59}{32})z^{10} + (-w^{-6} + 6w^{-7} - 9)z^{11} + (-\frac{1}{4}w^{-6} + 6w^{-7} - 21w^{-8} - \frac{5281}{128})z^{12} + O(z^{13}).$$
 (5.7)

For $w = \infty(y = 1)$, we recover the phase transition of the pure spin model. The value of v_c^{-1} for various values of q and for a number of Padé approximants is given in table

	7					
	2	3	4	5	6	10
[5, 5]	3.266	2.723	2.464	2.305	2.195	1.955
[6, 5]	3.164	2.689	2.477	2.295	2.189	1.953
[6, 6]	3.277	2.727	2.466	2.306	2.196	1.955
High temperature	3.402 [14]		2.460 [16]			

Value of v_c^{-1} for the four-dimensional pure spin model (w = 0), for various values of q and various Padé approximants, to be compared with results obtained from high temperature expansions

3. We also compare with the results of Fisher and Gaunt [14] for q = 2 and of Kadanoff and Ditzian [16] for q = 4 obtained from high temperature series.

The third line, separating regions I and II, was calculated to $O(z^{12})$. Since the expression is very long, we give here only the first few terms:

$$w = v^{-2/3} \alpha , \qquad (5.8)$$

where

$$\alpha = 1 - \frac{2}{3}v^{-1}z + (v^{-2/3} + \frac{5}{9}v^{-2} - v^{2/3})z^{2} + (-\frac{4}{3}v^{-5/3} - \frac{4}{81}v^{-3} + \frac{2}{3}v)z^{3} + (-\frac{13}{9} + v^{-4/3} + \frac{14}{9}v^{-8/3} + \frac{110}{243}v^{-4} - \frac{1}{6}v^{-8})z^{4} + O(z^{5}).$$
(5.9)

The position of this line was also evaluated using Padé approximants up to the [6, 6] Padé. The phase diagrams for q = 2 and q = 10 are displayed in fig. 9. The remarkable feature is that the three phase boundaries meet at a single point. (This is evident for q = 10. For q = 2, 'the results of the Padé extrapolation for the I-II phase boundary cannot be trusted very close to the triple point but this line certainly appears to lead to the crossing point of the two other phase boundaries.) The phase diagram for q = 2 compares very well with the Monte Carlo results [2]: the three phase boundaries coincide exactly; in particular the endpoint of the I-II boundary found by Creutz lies in the shaded region of fig. 9a.

We now discuss the latent heats in this model, defined as before as the discontinuities in the derivatives of the free energy across the phase boundaries. Consider first the far end of the boundary line between regions I and II. The expected endpoint should be detected by the "latent heats" $L_v = v (d/dv)(F_I - F_{II})$ and $L_w = w(d/dw) (F_I - F_{II})$ becoming negative. Notice that in contrast with the threedimensional case of sect. 4, there is no natural choice of latent heat "across' the phase boundary. As already discussed in sect. 4, as q grows this endpoint must approach the right lower corner of the phase diagram. But again according to the appendix for large q, the only diagrams which need be considered there are easily characterized in both the high and low temperature expansions. Repeating the steps of sect. 4, we find



Fig. 9. Phase diagram of the 4D model for q = 2 and q = 10, obtained using [6, 6] Padé. For q = 2, along the boundary between regions I and II, we also give vertical bars indicating estimates of the error. Since this error becomes very large near the triple point, this segment of curve has not been drawn. For q = 10, this estimated error, which is about ten times smaller has not been plotted. Monte Carlo experiments [2] suggest that the endpoint occurs within the shaded region.

that the two Padé extrapolations of the latent heats L_v and L_w change sign at values of v and w obeying asymptotically

$$v_{\rm c} \sim q^{3/4}$$
, $w_{\rm c} \sim q^{-1/2}$. (5.10)

Eq. (5.10) implies again, as for d = 3, that κ_c is independent of q, as q becomes large. It follows that the endpoint here is further away from the triple point than in the three-dimensional case, and enters the region where the errors in our extrapolations are large. Moreover, to the order we have calculated the free energy, only one diagram survives in the regime (5.10) in both the high and low temperature expansions: clearly retaining only this term is a very crude approximation. Finally, we notice that the subdominant terms in the regime (5.10) are only down by inverse powers of $q^{1/4}$: as a consequence the value of the (extrapolated) latent heats L_v and L_w for low values of q, q = 2 - 10, is very different from the dominant term in regime (5.10). All these reasons explain why we could not find any clear signal of an endpoint for these low values of q. We see rather our extrapolated latent heat becoming small, with large error estimates.

We now return to the phase transition of the pure spin model and discuss its nature as we did in the three-dimensional case. We thus compute the latent heat

$$L_{\rm II-III} = \frac{1}{4} \frac{\rm d}{{\rm d}v^{-1}} \left(F_{\rm II} - F_{\rm III}\right) = v - z + \cdots, \qquad (5.11)$$

up to order x^{12} . As discussed in sect. 4, we can substitute for v^{-1} the expression (5.7) either in L_{II-III} or in $v^{-1}L_{II-III}$, and look for the first zero of the Padé approximants as

428

 $q^{-1/4}$ grows from 0 to 1. The results are again rather sensitive to the method, showing that the convergence is slow. The critical value q_c where the transition goes from first to second order ranges between 4.2 and 4.9 depending on the approximant, when considering L_{II-III} and between 3.5 and 3.6 when we take $v^{-1}L_{II-III}$. Notice that relative errors in z_c are multiplied by four when we calculate q_c . We conclude that our large q expansion is not a very efficient method for determining q_c but is still in rough agreement with the consensus that $q_c \leq 3$.

The four-dimensional pure gauge model (v = 0) is of interest in and of itself. Self-duality implies a transition, if unique, at $w_c = 1$, but does not specify the nature of this transition. However, Monte Carlo experiments [19] as well as the analysis of the high temperature expansion [12] indicate that it is first order, even for the Z_2 model, and yield estimates of the latent heat. How does the $1/\sqrt{q}$ expansion reproduce these features? The expansion (5.3) restricted to v = 0 but pushed up to order 8 in the parameter $\zeta = z^2 = q^{-1/2}$ gives (see fig. 10)

$$-\frac{1}{6}F_{1} = w\zeta - \frac{1}{2}w^{2}\zeta^{2} + \frac{1}{3}w^{2}\zeta^{3} + \zeta^{4}(\frac{2}{3}w^{6} - \frac{1}{4}w^{4}) + \zeta^{5}(\frac{1}{5}w^{5} - 4w^{7}) + \zeta^{6}(14w^{8} - \frac{5}{6}w^{6}) + \zeta^{7}(\frac{29}{7}w^{7} - \frac{112}{3}w^{9} + 6w^{11}) + \zeta^{8}(-\frac{113}{8}w^{8} + 90w^{10} - \frac{217}{3}w^{12}),$$
(5.12)

and therefore for the latent heat at the critical temperature we have

$$L = \frac{1}{6} \frac{d}{dw} (F_1 - F_{II})|_{w=1} = 1 + 2 \frac{d}{dw} ({}_6^1 F_1)$$

= 1 - 2\zeta + 2\zeta^2 - 2\zeta^3 - 6\zeta^4 + 54\zeta^5 - 214\zeta^6 + 482\zeta^7 + 162\zeta^8. (5.13)

The analysis of the Padé extrapolations of these series reveals that L differs substantially from its two-dimensional analog (see table 4).

(i) The zero nearest the origin along (or close to) the positive real axis is located in the range $\zeta \sim 0.83$, corresponding to $q \approx 1.4-1.5$. In view of our past experience in lower dimensions (sects. 3, 4), this may mean that the critical value of q is as low as $q_c = 1$.



Fig. 10. Diagrams contributing to the free energy of the four-dimensional pure gauge model, up to order q^{-4} .

TABLE 4	
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Zero nearest the origin, z_c , along the positive real axis in the $q^{-1/2}$ plane, and the value for low q of the latent heat for the four-dimensional pure gauge model extrapolated by the [3, 3] to [4, 4] Padé approximants

	z _c	L(q=2)	L(q=3)	L(q=4)	L(q=10)
[3, 3]	0.8226 (q = 1.48)	0.097	0.217	0.298	0.5069
[4, 3]	0.8388 (<i>q</i> = 1.42)	0.104	0.221	0.296	0.5066
[4, 4]	0.8191 (<i>q</i> = 1.49)	0.095	0.216	0.295	0.5063
Monte Carlo or high temperature		0.124 [12, 19]	0.229 [12, 19]		

(ii) The analytic structure of L in the complex $\zeta = 1/\sqrt{q}$ plane seems more complicated. This manifests itself in the appearance of pairs of poles and zeros, located almost on top of each other at $\zeta \approx -0.2 \ (1 \pm i)$, very stable against successive extrapolations. This, in turn, limits the radius of convergence of the series (5.13) to a low value $|\zeta| \sim 0.3$ corresponding to $|q| \sim 12$. In any case, our approximants give unambiguously first order transitions for all the models $q \ge 2$, and yield in the two cases q = 2 and 3 numerical values for L in qualitative agreement with the "data" [12, 19] (see table 4).

6. Summary

To reiterate, we have used a large q expansion to construct the phase diagrams for a class of q-state coupled gauge-matter systems in d=3 and 4 dimensions. Extrapolating our results to q=2, we find the predicted location of the transition lines in good quantitative agreement with Ising Monte Carlo data for both d=3 and 4. On the border of the phase diagram where the theory reduces to a Potts spin system, we have found the predicted critical coupling also to agree very well, quantitatively, with the results of conventional series expansions for q=2, 3, 4 in d=3 dimensions and q=2, 4 in d=4 dimensions. Our results for the latent heat of the pure gauge theory for d=4 and q=2, 3 agree reasonably well with the available Monte Carlo and high temperature series data and the agreement would be expected to improve for larger q. On the other hand, although detailed comparisons are not yet possible, we suspect that we are unable to predict the critical values of q for which transitions change from first to second order at critical points and lines nearly as reliably as their locations.

The d = 3, 4 phase diagrams have a pocket of free charge isolated by a phase boundary which persists for all q. In d = 3 dimensions this phase boundary is expected, on the basis of Ising Monte Carlo and series expansion data, to shift from its first order nature at $q = \infty$ to a line of second order transitions for some q_c between 2 and 4. In d = 4 dimensions, however, the Monte Carlo simulations indicate that the II-III phase boundary becomes second order above q = 2 but the I-III boundary remains first order as low as q = 2. On the basis of our results for the pure gauge theory, we speculate that the I-III boundary for d = 4 remains first order down to a $q_c \approx 1$, but are unable as yet to similarly estimate the critical q of the II-III boundary for d = 4.

At $q = \infty$ there is also a line of first order transitions extending from the boundary of the free charge region to the lower right corner of the phase diagram and separating the Higgs from the confinement phase. The results of ref. [5], applying in general to the case of matter fields in the fundamental representation of the gauge group and applying in this case for q finite, imply a strip of analyticity along the far border of the phase diagram. We have indeed found, for d = 3, that the finite qcorrections act to terminate this line before it reaches the corner and our phase diagram results are thus nicely in accord with the picture of ref. [5]. (It would be interesting to find an order parameter associated with this first order transition line since it should provide, in the $q \rightarrow \infty$ limit, a symmetry criterion distinguishing between the Higgs and confinement phases of the theory.) For d = 4, however, it unfortunately seems that a higher order calculation is necessary to unambiguously locate the low q endpoint of this first order transition line. The specific nature of the endpoint itself is also of a certain interest and can be treated independently of a large q expansion as a liquid-vapor critical point in dilute gas approximation [18].

Finally, having established a certain measure of confidence in the utility of a large q expansion for understanding the properties of coupled gauge-matter systems, we may proceed to investigate other quantities of interest perhaps accessible by the methods of this paper including the detailed behavior of the Wilson loop correlation function and the spectrum of the theory.

We wish to thank M. Peskin for instigating our interest in this problem. We are most grateful to J.M. Drouffe for discussions and for kindly assisting us in use of his Algebraic Manipulations Program, greatly simplifying analysis of our power series expansions. We have also enjoyed conversations with E. Brézin and J. Zinn-Justin.

Appendix

This appendix is devoted to a study of the leading contribution to the free energy in the approach to the right lower corner of the phase diagram.

More precisely, in d dimensions, we assume that v grows and w goes to zero according to

$$v \sim q^{(d-1)/d} = z^{-(d-1)}, \qquad w \sim z^2.$$
 (A.1)

Let us show that in this regime, the high and low temperature expansions have well defined limits. In the high temperature limit, first, a typical contribution is

$$q' \left(\frac{z^{d-1}v}{1+z^{d-1}v}\right)^{L_m} \left(\frac{z^{d-2}w}{1+z^{d-2}w}\right)^{P_a}$$
(A.2)

with the same notations as in sect. 2. Consider the class of diagrams all of whose gauge plaquettes are bordered by matter links; we have

$$r = P_{g}, \qquad (A.3)$$

since the choice of P_g independent integers r_p determines completely the link integers r_{ij} by (2.15b), and since (2.15a) is then automatically satisfied. Now in the regime (A.1)

$$z^{d-1}v \sim O(1), \qquad z^{d-2}w \sim q^{-1},$$
 (A.4)

and hence a diagram of this class contributes a power q^0 . It is then clear that any other diagram is obtained by adding further plaquettes to a diagram of this class, in such a way that (2.15b) remains satisfied. But in this addition of ΔP_g plaquettes, r differs from (A.3) by an amount

$$\Delta r = \Delta P_{g} - \Delta R_{g},$$

where $\Delta R_g > 0$ is the number of constraints along the new links. Therefore, the diagram has a net power $q^{-\Delta R_x}$ and is negligible as $q \to \infty$. To summarize, the surviving diagrams in the regime (A.1) have as many gauge links as matter links. Examples are depicted on fig. 6.

We next turn to the low temperature expansion. A typical diagram will contribute

$$q^{\alpha} e^{-\beta L_{\rm m}-\kappa P_{\rm g}}, \qquad (A.5)$$

where

$$\alpha \leq S_{\rm m} + L_{\rm g}$$

and L_m , P_g , S_m , L_g are defined as in the discussion of the low temperature expansion in Sect. 2. To avoid overcounting gauge-equivalent configurations, we impose some gauge-fixing condition. In the regime (A.1)

$$e^{\beta} = 1 + q^{1/d} v \sim q$$
,
 $e^{\kappa} = 1 + q^{2/d} w = O(1)$, (A.6)

and hence the power of q associated with the diagram reads

$$S_{\rm m}+L_{\rm g}-L_{\rm m}$$
,

which may be seen to be non-positive. The leading contribution comes from diagrams where only gauge links have been flipped, and hence $S_m = 0$, $L_m = L_g$.

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