MEAN FIELD WITH CORRECTIONS IN LATTICE GAUGE THEORY

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A systematic expansion of the path integral for lattice gauge theory is performed around the mean field solution. In this letter we present the results for the pure gauge groups Z(2), SU(2) and SO(3). The agreement with Monte Carlo calculations is excellent. For the discrete group the calculation is performed with and without gauge fixing, whereas for the continuous groups gauge fixing is mandatory. In the case of SU(2) the absence of a phase transition is correctly signalled by mean field theory.

1. Introduction. The naive mean field approximation to lattice gauge theory is known to predict a first-order phase transition [1,2]. The reason is that the interaction between the link variables, in the Wilson action, is of fourth order. In spin theories where the action is quadratic in the site variables, a second-order phase transition is predicted.

The mean field method seems at the first glance to contradict Elitzur's theorem [3]. Drouffe [4], however, has argued that the mean field equations [5,6] are equivalent to a saddle-point approximation [7]. This permits a systematic calculation of higher-order corrections. Due to the gauge degeneracy of the saddle points, gauge non-invariant quantities vanish when they are integrated over the degeneracy, thereby satisfying Elitzur's theorem.

In this letter we investigate the lowest-order solutions and their higher-order corrections. We exemplify the method in the case of Z(2), SU(2), and SO(3) pure lattice gauge theories with Wilson action in space–time dimension $D = 4$, with attention paid to the differences between the discrete and continuous cases. The discussion of the more general situation, including other groups, dimensions and variant forms of the action, will be presented in a forthcoming paper.

2. Z(2) without gauge fixing. The link variables take in this case the values $U_\ell = \pm 1$ and the partition function is

$$Z(\beta) = \left(\prod_{\ell} \frac{1}{2} \sum_{U_\ell = \pm 1} \right) \times \exp\left(\beta \sum_{(\ell_1 \ell_2 \ell_3 \ell_4)} U_{\ell_1} U_{\ell_2} U_{\ell_3} U_{\ell_4} \right), \tag{1}$$

where the sum in the exponent is over all distinct plaquettes. The basic tool [8] in obtaining the saddle-point approximation is the conversion of the integral over the compact invariant group measure (here $\frac{1}{2}\Sigma_{U_\ell = \pm 1}$) of the gauge group, into an integral over unconstrained variables by means of Fourier transformation. In the present case

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$$\frac{1}{2}\sum_{U=\pm 1} f(U) = \int_{-\infty}^{\infty} \mathrm{d}U \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{d}\alpha}{2\pi\mathrm{i}} \exp[\omega(\alpha) - \alpha U] f(U), \tag{2}$$

where $\omega(\alpha) = \log \frac{1}{2}\Sigma_{U=\pm 1}\, \mathrm{e}^{\alpha U} = \log\cosh\alpha$. At the expense of introducing one extra variable (α) for each link we may rewrite (1) in the form

$$Z(\beta) = \left(\prod_{\ell} \int_{-\infty}^{\infty} \mathrm{d}U_{\ell} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{d}\alpha_{\ell}}{2\pi\mathrm{i}}\right)$$

$$\times \exp\left(\beta \sum_{(\ell_1\ell_2\ell_3\ell_4)} U_{\ell_1}U_{\ell_2}U_{\ell_3}U_{\ell_4} + \sum_{\ell} [\omega(\alpha_{\ell}) - \alpha_{\ell}U_{\ell}]\right). \tag{3}$$

This integral can now be evaluated by standard saddle-point techniques to be justified later.

In the lowest-order approximation we obtain the saddle-point equations

$$\beta \sum_{(\ell_1\ell_2\ell_3)} U_{\ell_1}U_{\ell_2}U_{\ell_3} = \alpha_{\ell},$$

$$\omega'(\alpha_{\ell})\ (= \tanh\alpha_{\ell}) = U_{\ell}, \tag{4}$$

where the sum is over all sets of three links ($\ell_1\ell_2\ell_3$) that form a plaquette with ℓ. These equations are invariant under (1) translations, (2) rotations by 90°, and (3) gauge transformations $U_{\ell} = U(x,y) \to \sigma(x) \times U(x,y)\sigma(y)$ with $\sigma(x) = \pm 1$. Consequently any solution is highly degenerate if it is not invariant w.r.t. transformations (1)–(3) above. Let us start by looking for a solution which is translationally and rotationally invariant, i.e. $U_{\ell} = U_0$, $\alpha_{\ell} = \alpha_0$. Then we get (eliminating α_0)

$$\omega'(2(D-1)\beta U_0^3) = U_0. \tag{5}$$

For $\beta < \beta_1 = 0.336$ ($D = 4$) this equation has only the trivial, non-degenerate, gauge-invariant solution $U_0 = 0$ for which the free energy vanishes ($F = 0$). For $\beta > \beta_1$ this equation has, besides the trivial solution, also non-trivial solutions with $U_0 > 0$. The free energy per lattice site is in this approximation (fig. 1a)

$$F = \log 2 - \tfrac{3}{2}D(D-1)\beta U_0^4 - \tfrac{1}{2}D\log(1 - U_0^2), \tag{6}$$

where the first term is due to the gauge degeneracy of the solution $U_0 \neq 0$. This degeneracy gives to the partition function a factor 2^N where N is the number of sites in the lattice. (It is easy to see that for $\beta \to \infty$ eq. (6) agrees with the ordinary weak-coupling expansion

$$F = \tfrac{1}{2}D(D-1)\beta - (D-1)\log 2 + D\exp[-4(D-1)\beta] + \mathrm{O}(\exp[-8(D-1)\beta]). \tag{7}$$

Note that due to the discreteness of the groups the next terms are exponentially small).

Zero-order mean field theory does not furnish a good description of the strong-coupling phase. It leads to $F = 0$ which is a poor approximation to the actual behaviour of the free energy in this phase as reflected in the internal energy (see fig. 2a). It is consequently necessary to take corrections into account. When organized according to increasing powers of β the loop expansion around the saddle point $(U_0, \alpha_0) = (0, 0)$ reproduces the strong-coupling expansion [9]. The first non-trivial contribution $F = \frac{1}{4}D(D-1)\beta^2$ arises at the three-loop level. Comparing this with the free energy of the non-trivial solution shows that the transition takes place at $\beta_c = 0.440$ in excellent agreement with the exact result $\beta_c = \frac{1}{2}\log(1+\sqrt{2}) = 0.4407$. Including more terms in the strong-coupling expansion does not change this appreciably (~1%).

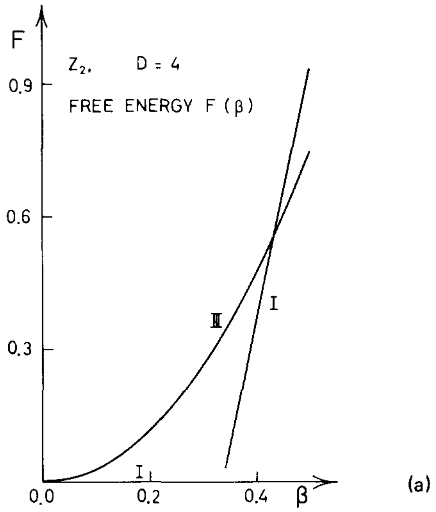
In the weak-coupling phase the corrections already arise at the one-loop level where the change in the free energy is

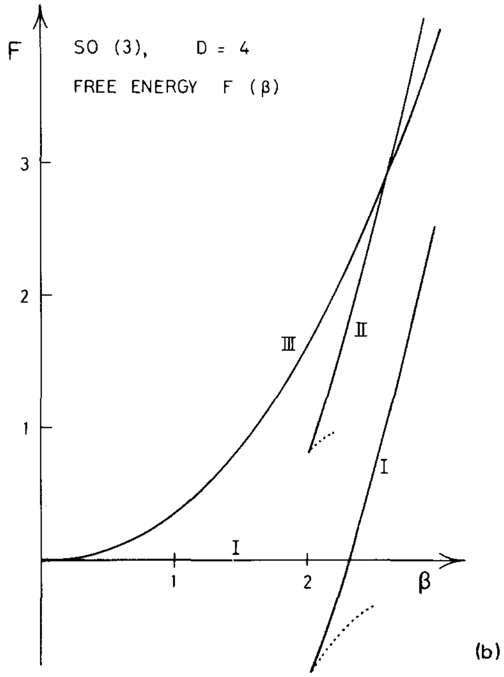
$$\Delta F = -(1/2N)\,\mathrm{tr}\log(1 - \beta W''\Box). \tag{8}$$

Here $\Box_{\ell_1\ell_2} = U_0^2$ if (ℓ_1, ℓ_2) belong to the same plaquette and zero otherwise. The double derivative $W'' = 1 - U_0^2 \simeq \exp[-4\beta(D-1)]$ is already very small (~$\frac{1}{2}$%) at the transition point. Thus $\Delta F = \mathrm{O}(\exp[-8\beta \times (D-1)])$. It may be shown that higher-order diagrams are not further suppressed. We need indeed an infinite number of terms $\beta^n\exp[-8(D-1)\beta]$ to reconstruct the next term $\exp[-8(D-1)\beta](\mathrm{e}^{4\beta} - 1)$ in eq. (7).

The experience with Z(2) shows that the mean field expansion describes both phases of the model with a high degree of accuracy. In the strong-coupling phase it is necessary to go at least to the three-loop level whereas in the weak-coupling phase it is sufficient to stay at tree-level approximation.

3. Z(2) with gauge fixing. The mean field expansion may also be applied to the partition function in





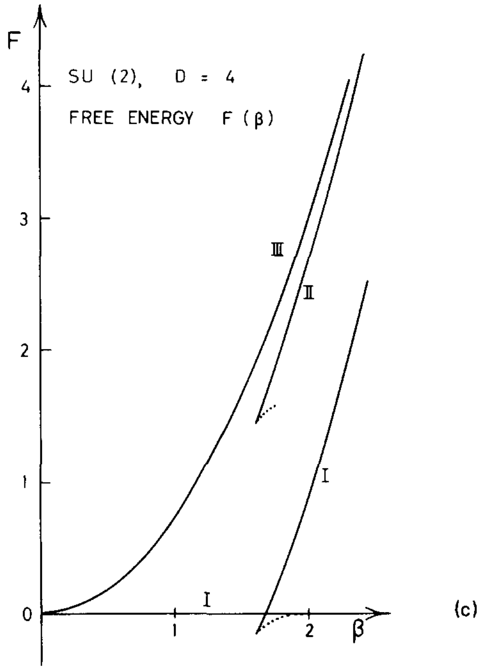


Fig. 1. Free energy for Z_2 (a), SO(3) (b), and SU(2) (c). Graphs labelled I are the zeroth-order mean field results for the strong- and weak-coupling phases. Graphs labelled II are results of the mean field approximation plus one-loop corrections from the isospin one-channel for the weak-coupling phase. Graphs labelled III are results of corrections to the mean field approximation in the strong-coupling phase, for Z_2 the first term in the β-expansion, for SO(3) and SU(2) the first three terms in the character expansion.

a fixed gauge. We no longer have degenerate saddle points and consequently do not obtain a log 2 term due to their entropy. It is most convenient to choose the axial gauge where all links in the D-direction have been fixed to unity. The saddle-point equations (5) now become

$$\omega'(\beta[2(D-2)U_0^3 + 2U_0]) = U_0 \qquad (9)$$

and the corresponding free energy is

$$F_{\text{axial}} = -\beta(D-1)[\tfrac{3}{2}(D-2)U_0^4 + U_0^2] - \tfrac{1}{2}(D-1)\log(1-U_0^2). \qquad (10)$$

Although the free energy is formally independent of the gauge there is a priori no guarantee that the results of the previous and the present section should agree in the lowest order. It is easy to see that $F - F_{\text{axial gauge}} = \exp[-4(D-1)\beta] + \ldots$. This discrep-

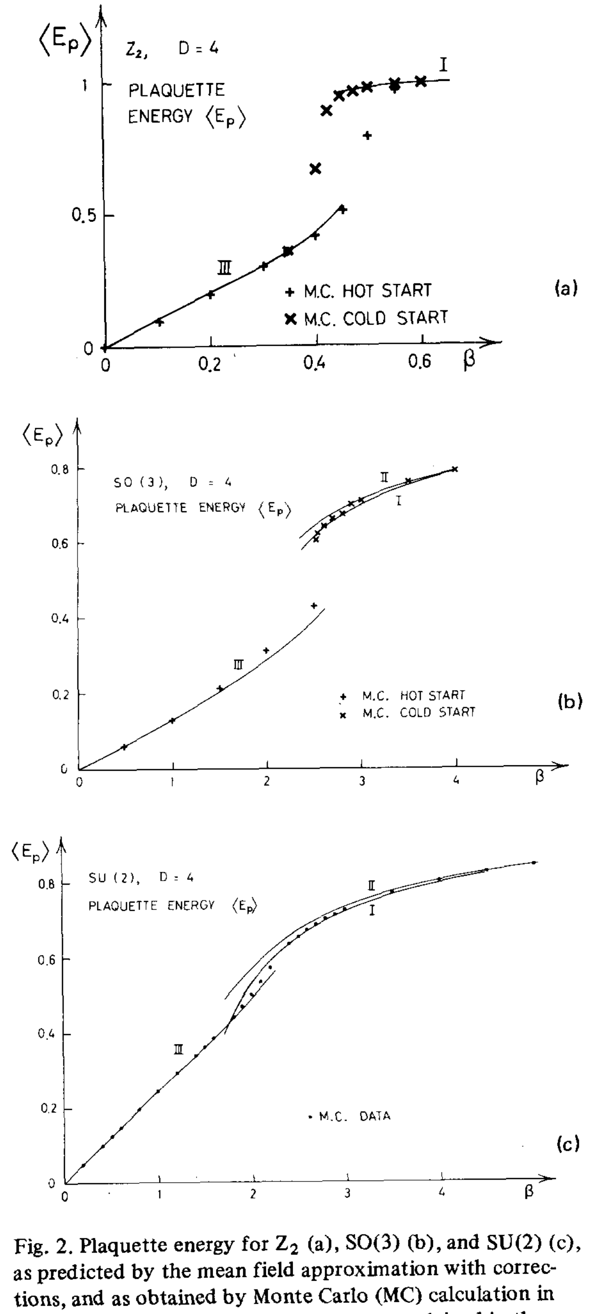


Fig. 2. Plaquette energy for Z_2 (a), SO(3) (b), and SU(2) (c), as predicted by the mean field approximation with corrections, and as obtained by Monte Carlo (MC) calculation in refs. [11,12,14]. Labels I, II, and III are explained in the caption of fig. 1.

ancy is removed by taking into account new configurations with lower action. Such configurations are exponentially suppressed (see fig. 3). We believe that taking these corrections into account will lead to

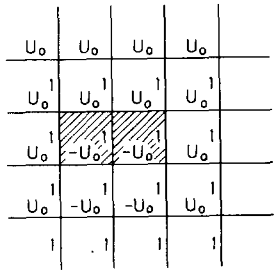


Fig. 3. Configuration contributing to the free energy of Z_2 in axial gauge. Only $2(d-1)$ plaquettes (shaded) are frustrated. A dilute gas of such defects gives a contribution $\Delta F = \exp[-4(d-1)\beta U_0^2]$.

agreement of the two methods to any order and expect the mean field formalism to be gauge invariant ‡1.

4. Continuous groups. We only emphasize the main differences between the treatment of discrete and continuous groups:

(i) Gauge fixing is now mandatory. Otherwise the zero modes corresponding to gauge transformations of a non-invariant solution to the saddle-point equations will give rise to infinite fluctuations. Again we use the axial gauge.

(ii) The previous formalism is extended such that U_ℓ, α_ℓ are matrices in the matrix algebra in which the group is embedded [2×2 complex matrices for SU(2) or 3×3 real matrices for SO(3)]. We look for mean field solutions of the form $U_\ell = U_0 \cdot 1, \alpha_\ell = \alpha_0 \cdot 1$ where U_0 and α_0 are real numbers. Then the mean field equations take exactly the same form as eq. (9) with

$$\omega(\alpha) = \log[I_0(\alpha) - I_2(\alpha)]\,, \qquad \text{for SU(2)}\,,$$
$$= \log[e^{\alpha/3}(I_0 - I_1)(\tfrac{2}{3}\alpha)]\,, \quad \text{for SO(3)}\,. \qquad (11)$$

(iii) The free energy is in the tree approximation

$$F = \beta[\tfrac{1}{2}(D-1)(D-2)U_0^4 + (D-1)U_0^2] + (D-1)[\omega(\alpha_0) - \alpha_0 U_0]\,. \qquad (12)$$

It (and its derivative, the internal energy E) agrees for $\beta \to \infty$ with the first two terms of the weak-coupling

‡1 The problem of gauge invariance of mean field theory has also been discussed by Brout et al. [10].

expansion ($\frac{1}{6}E = 1 - 3/4\beta$) for SU(2). Note that if we had not fixed the gauge we would have obtained the incorrect result $\frac{1}{6}E = 1 - 1/\beta$ for $\beta \to \infty$.

(iv) In calculating the corrections it is convenient to project the matrices $\Delta\alpha = \alpha - \alpha_0$ and $\Delta U = U - U_0$ on irreducible representations [$j = 0, 1$ for SU(2) and $j = 0, 1, 2$ for SO(3)]. The double derivative W'' in (8) then depends on the channel. In the adjoint channel $j = 1$, $W'' \sim 1/\beta$ whereas in other channels $W'' \sim 1/\beta^2$. As a result the contribution to (8) from the latter is very small ($\Delta F \simeq 10^{-2}$, $\Delta E \simeq 10^{-3}$ at β_c). The $j = 1$ channel gives an important contribution ΔF = const + $O(1/\beta)$. Simple power counting shows that in higher orders only a finite number of diagrams contribute to a definite order in $1/\beta$.

5. Results. Let us first present the results of zero-order mean field theory for SO(3) and SU(2). Here the non-trivial free energy passes through zero at $\beta_c = 2.48$ for SO(3) and at $\beta_c = 1.68$ for SU(2). We recall that SO(3) is known [11] ‡2 to have a first-order transition at $\beta = 2.49$ while SU(2) has no phase transition [12] but a sharp peak in the specific heat at $\beta \sim 2.2$. The mean field internal energy reproduces very well the Monte Carlo data in the ordered phase (figs. 2b,c).

The main effect of the one-loop corrections is to shift F by a constant in the ordered phase (~1.8 for both groups at $\beta \approx \beta_c$). For SO(3) we compare this corrected free energy with the first three terms of the character expansion and find that β_c is shifted to $\beta_c = 2.62 \pm 0.10$ where the quoted error bars are estimated from neglected higher-order terms and in the description of the weak-coupling phase. Notice that it makes sense to use a truncated strong-coupling series for $\beta \simeq \beta_c$, since this series is expected to diverge only for larger values of β ($\beta \simeq 2.9$ to 3) [13]. The excellent agreement between the internal energy (fig. 2b) and Monte Carlo data is seen to deteriorate a bit close to the critical point. Presumably two-loop contributions become sizable here.

The case of SU(2) is different. The corrected mean field free energy in the ordered phase seems to be almost degenerate, with or slightly below the strong-coupling estimate (see fig. 1c) in a large region ($2.0 < \beta < 2.4$). Since the strong-coupling series has as radius of convergence $|\beta| \sim 2.2$, its use becomes questionable. Also since the two configurations $U_0 = 0$ and $U_0 \neq 0$ are almost degenerate in free energy in a rather large region we may expect complicated phenomena like tunneling between the two states to take place. This is not taken into account in our approach, but is being investigated now. At any rate we find it gratifying that our simple mean field approach refuses to yield a first-order phase transition for SU(2).

‡2 Halliday and Schwimmer [11] give the value $\beta_c \approx 2.6$, but their figures show $\beta_c = 2.5$.

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