# ROUGHENING TRANSITION IN LATTICE GAUGE THEORIES IN ARBITRARY DIMENSION (II). The groups $\mathbf{Z}_{3}, \mathbf{U}(\mathbf{1}), \mathbf{S U}(\mathbf{2 )}, \mathbf{S U}(\mathbf{3})$ 

J.M. DROUFFE and J.B. ZUBER<br>D.Ph.T, CEA-Saclay, 91191 Gif-sur-Yvette Cedex, France

Received 15 October 1980


#### Abstract

Strong coupling expansions for the string tension and other quantities in lattice gauge theories are computed in arbitrary dimension for the groups $Z_{3}, \mathrm{U}(1), \mathrm{SU}(2), \mathrm{SU}(3)$. This enables us to determine the location of the roughening transition, which seems to be group independent when measured in an appropriate variable. In four dimensions, the strong coupling expansion of string tension calculated up to fourteenth order for $\operatorname{SU}(2)$, and twelfth order for $\operatorname{SU}(3)$ agrees nicely with Monte Carlo data up to the roughening point.


## 1. Introduction

In the preceding paper [1], the roughening transition for the $Z_{2}$ lattice gauge theories has been analyzed using strong coupling expansions up to fourteenth order in $t=\tanh \beta$. We now present the result of the same analysis for other groups. The expansions have been pushed to fourteenth order for the groups $U(1)$ and $S U(2)$, and to twelfth order in the more intricate cases of $Z_{3}$ and $\operatorname{SU}(3)$. The group $Z_{3}$ has been considered because it is the center of $\mathrm{SU}(3)$ and because the corresponding model is self-dual in four dimensions. In sect. 2, we introduce our notations, and present the series for the string tension and two possible indicators of roughening, which are mere generalizations of the observable $\mathscr{G}$ introduced for $Z_{2}[1,2]$. The nearest singularity of these indicators signals the roughening transition. Our results for the location of this singularity in various groups and dimensions are discussed in sect. 3.

In four dimensions, it has been shown by Monte Carlo calculations [3] and strong coupling expansions $[4,5]$ that there is a break in the behavior of the tension in a narrow region. We show that roughening takes place in this region and that strong coupling series reproduce very nicely the Monte Carlo data, up to the roughening point. This had already been noticed for $\operatorname{SU}(2)[2,6]$ and is now established for $\operatorname{SU}(3)$. The roughening singularity is therefore confirmed to limit the domain of validity of the strong coupling expansion for observables attached to the surface.

The appendix gathers some technical details and displays the string tension series up to twelfth order for an arbitrary group.

## 2. Notations and expansions

For a general group, we choose the Wilson action to be given in terms of the character of the fundamental representation:

$$
\begin{equation*}
S=\sum_{\text {plaq }} s\left(U_{\mathrm{P}}\right)=\beta \sum_{\text {plaq }} \nu_{\mathrm{f}}\left(\chi_{\mathrm{f}}\left(U_{\mathrm{P}}\right)+\chi_{\overline{\mathrm{i}}}\left(U_{\mathrm{P}}\right)\right) \tag{2.1}
\end{equation*}
$$

Here and in the following, $\nu_{r}$ denotes the dimension of the representation $r$. The second term on the r.h.s. of (2.1) will be omitted whenever the fundamental representation is equivalent to its conjugate (e.g., in $\operatorname{SU}(2)$ ). The exponential of the action for each plaquette is then expanded on irreducible characters [7]

$$
\begin{equation*}
\mathrm{e}^{s\left(U_{\mathbf{P}}\right)}=f(\beta)\left[1+\sum_{r \neq 0} \nu_{r} \beta_{r} \chi_{r}\left(U_{\mathbf{P}}\right)\right] \tag{2.2}
\end{equation*}
$$

where the $\beta_{r}$ are functions of $\beta$. We will actually expand them in powers of

$$
\begin{equation*}
t=\beta_{\mathrm{f}} \tag{2.3}
\end{equation*}
$$

which is the natural expansion parameter. Indeed with the characters normalized according to

$$
\begin{equation*}
\int \mathrm{D} U \chi_{r}(U) \chi_{s}\left(U^{-1} V\right)=\frac{\delta_{r s}}{\nu_{r}} \chi_{s}(V), \tag{2.4}
\end{equation*}
$$

integration over non-singular links (i.e., links shared by two plaquettes) are easily performed and do not introduce new group theoretic factors. Those appear only when singular lines are present (see the appendix). It is just a matter of patience to expand the desired quantities to a given order.

What we want to compute is the expectation value of the Wilson loop:

$$
\begin{gather*}
\langle W\rangle=Z^{-1} \sum_{\left\{U_{\ell}\right\}} \mathrm{e}^{s} W, \\
W=\chi_{\mathrm{f}}\left(\prod U_{\ell}\right), \tag{2.5}
\end{gather*}
$$

or averages of various observables in the presence of $W$. We have computed here the expansion of the string tension

$$
\begin{equation*}
k=\lim _{A \rightarrow \infty}\left\{\frac{-\ln \langle W\rangle}{A}\right\} \tag{2.6}
\end{equation*}
$$

and of the "pinch operators" introduced in [2, 1]:

$$
\begin{align*}
p_{\mathrm{W}}(c) & =\frac{\left\langle W \mathrm{e}^{s\left(c U_{\mathrm{P}}\right)-s\left(U_{\mathrm{P}}\right)}\right\rangle}{\langle W\rangle}  \tag{2.7}\\
p_{0}(c) & =\left\langle\mathrm{e}^{s\left(c U_{\mathrm{P}}\right)-s\left(U_{\mathrm{P}}\right)}\right\rangle \tag{2.8}
\end{align*}
$$

Here, $c$ denotes an element of the center of the gauge group. We recall [1, 2] that $p_{\mathrm{w}}$ and $p_{0}$ measure the effect of "frustrating" a test plaquette P of the minimal plane in the presence or in the absence of the Wilson loop. The indicator of roughening is

$$
\begin{equation*}
\mathscr{g}_{c}=\frac{2}{p_{0}(c)-p_{\mathrm{W}}(c)} \tag{2.9}
\end{equation*}
$$

However, it is possible to introduce a slight modification in the definition of the pinch operator. Instead of frustrating the plaquette $P$, we may just delete it, i.e., subtract its contribution to the action. The corresponding observables read

$$
\begin{align*}
q_{\mathrm{w}} & =\frac{\left\langle W \mathrm{e}^{-s\left(U_{P}\right)}\right\rangle}{\langle W\rangle},  \tag{2.10}\\
q_{0} & =\left\langle\mathrm{e}^{-s\left(U_{P}\right)}\right\rangle \tag{2.11}
\end{align*}
$$

The new indicator,

$$
\begin{equation*}
g_{0}=\frac{1}{q_{0}-q_{w}} \tag{2.12}
\end{equation*}
$$

is as good a candidate as $\mathscr{G}_{c}$ to signal roughening, since both are expected to diverge at $t_{\mathrm{R}}$. Notice that $p_{\mathrm{W}}(c)$ receives non-trivial contributions from all diagrams where the test plaquette P bears an "even" representation: $\chi_{r}(c)=\chi_{r}(1)=\nu_{r}$, whereas only diagrams which do not contain $P$ contribute to $q_{w}$. It is easy to show that, for $Z_{2}, \mathscr{I}_{-1}=\mathscr{g}_{0}$. We will see that for higher groups, $\mathscr{g}_{0}$ seems to provide more reliable results than $\mathscr{g}_{\mathrm{c}}$, as the coefficients of its series have smaller oscillations. Moreover, for $S U(3)$ or $Z_{3}, \mathscr{G}_{0}$ is real while $\mathscr{G}_{e} \pm 2 \pi / 3$ is not.

We now turn to the results. More detailed expressions are given in the appendix, enabling the reader to write the expansion of $k$ up to twelfth order for an arbitrary group. We content ourselves here with the actual series for the groups $Z_{3}, U(1)$, $S U(2)$ and $S U(3)$. They are displayed in tables 1 and 2 . For the sake of brevity we do not present the expansion for $\mathscr{g}_{c}$. The $\mathrm{U}(1)$ and $\mathrm{SU}(2)$ series are even in $t$, while the $\operatorname{SU}(3)$ and $Z_{3}$ series are not. The former have been worked out up to fourteenth order, whereas the latter have been truncated to twelfth order, because of the host

Table 1
Series coefficients for the string tension in $d$ dimensions

| $\text { group }^{n}$ |  |  | 6 |  |  | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathrm{Z}_{3}$ | 1 |  | 0 | 0 |  | - 16 | $19 d-\frac{121}{2}$ | $18 d-\frac{79}{2}$ | $2 d-3$ | $\begin{aligned} & \frac{364}{3} d^{2}-\frac{1270}{3} d \\ & +\frac{748}{3} \end{aligned}$ |  |  |
| U(1) | 1 | 0 | $\frac{1}{2}$ | 0 |  | $d-\frac{131}{6}$ | 0 | $18 d-\frac{4787}{6}$ | 0 | $\begin{aligned} & \frac{364}{3} d^{2}-\frac{7115}{12} d \\ & +\frac{2101999}{2250} \end{aligned}$ | 0 | $\begin{aligned} & 448 d^{2}-\frac{120491}{49} d \\ & +\frac{30824011}{8601} \end{aligned}$ |
| SU(2) | 1 | 0 | 0 | 0 |  | $d-\frac{64}{3}$ | 0 | $8 d-\frac{1026}{405}$ | 0 | $\begin{aligned} & \frac{364}{3} d^{2}-\frac{1754}{3} d \\ & +\frac{80571}{1215} \end{aligned}$ | 0 | $\begin{aligned} & \frac{712}{3} d^{2}-\frac{400081}{201} d \\ & +\frac{116050511}{15030} \end{aligned}$ |
| SU(3) | 1 | 3 |  | - |  | $+\frac{103}{8}$ | $57 d-\frac{7989}{40}$ | $48 d-\frac{1381323}{20480}$ | $\begin{aligned} & -\frac{681}{2} d \\ & +\frac{993217}{8122} \end{aligned}$ | $\begin{aligned} & \frac{364}{3} d^{2}-\frac{676}{3} d \\ & +\frac{2989179}{2255760} \end{aligned}$ |  |  |

Writing $(-k-\ln t) / 2(d-2)=\Sigma_{n} k_{n} t^{n}$, we display the $k_{n}$ up to $n=12$ for $\mathrm{Z}_{3}$ and $\mathrm{SU}(3)$ and $n=14$ for $U(1)$ and $S U(2)$.
of new diagrams contributing to the order $t^{13}$ and $t^{14}$ for $\mathrm{SU}(3)$ and $Z_{3}$. By contrast, the contribution of diagrams with fourteen plaquettes to the $U(1)$ and $\mathrm{SU}(2)$ theories are easily obtained from the $\mathrm{Z}_{2}$ case (see the appendix). The reader will notice in tables 1 and 2 that the coefficients of highest degree in $d$ in the terms of the form $t^{4 n}$ is group independent. This is no surprise; ref. [8] shows that the large- $d$ limit does not depend on the group. Expansions to fourteenth order have been previously obtained for the $\mathrm{SU}(2)$ and $\mathrm{U}(1)$ string tensions in $d=3$ by Duncan and Vaidya [9], and to twelfth order in $d=3,4$ for $Z_{3}, \mathrm{SU}(3)$ and $\mathrm{U}(1)$ by Münster and Weisz [10]. We have found small discrepancies between the results of [9] and ours [to order $t^{14}$ for 3D $\mathrm{U}(1)$ and $\mathrm{SU}(2)$ ].

Table 2
Series coefficients for the indicator $\mathscr{g}_{0}=1 /\left(q_{0}-q_{W}\right)$

| $n$ |  |  |  |  |  |  |  |  |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| group | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |

We write $\mathscr{G}_{0}=1+2(d-2) \Sigma_{n} \mathscr{G}_{0, n} t^{n}$ and list the coefficients $\mathscr{G}_{0, n}$.

## 3. Roughening transition for $\boldsymbol{d}=3,4,5$

Using the strong coupling expansions for the various indicators, we may now try to locate their nearest singularity, identified with the roughening point $t_{\mathrm{R}}$. For $\mathrm{U}(1)$ and $\operatorname{SU}(2)$, the coefficients of the expansions of $\mathscr{g}_{0}$ and $\mathscr{G}_{-1}$ are real and positive; computing the ratio of successive terms leads to consistent determinations of the nearest (real positive) singularity. We find that $\mathscr{I}_{0}$ yields more stable results than $\mathscr{G}_{-1}$, and, as in the $Z_{2}$ case, that the ratios $\left(G_{n} / \mathscr{G}_{n+4}\right)^{1 / 4}$ oscillate less than $\left(\mathscr{G}_{n} / \mathscr{G}_{n+2}\right)^{1 / 2}$ : this has been shown [1] to be a reflection of the large-d behavior of the coefficients of the series. We illustrate this on the 4D SU(2) indicators:

$$
\begin{align*}
& \mathscr{g}_{0}=1+4 t^{4}+32 t^{6}+116 t^{8}+\frac{2776}{3} t^{10}+\frac{341404}{81} t^{12}+\frac{281522}{9} t^{14}  \tag{3.1}\\
& \mathcal{G}_{-1}=1+4 t^{4}+40 t^{6}+\frac{356}{3} t^{8}+\frac{47584}{45} t^{10}+\frac{610556}{135} t^{12}+\frac{292002826}{8505} t^{14} \tag{3.2}
\end{align*}
$$

The ratios are

$$
\begin{align*}
\left(\mathscr{I}_{0, n} / G_{0, n+2}\right)^{1 / 2} & =\{0.353,0.525,0.354,0.468,0.367\}, \\
\left(\mathscr{G}_{0, n} / G_{0, n+4}\right)^{1 / 4} & =\{0.431,0.431,0.407,0.415\} ;  \tag{3.3}\\
\left(\mathcal{G}_{-1, n} / G_{-1, n+2}\right)^{1 / 2} & =\{0.316,0.581,0.335,0.483,0.363\}, \\
\left(\mathscr{G}_{-1, n} / G_{-1, n+4}\right)^{1 / 4} & =\{0.428,0.441,0.402,0.419\} ; \tag{3.4}
\end{align*}
$$

pointing to a singularity at

$$
\begin{equation*}
t_{\mathrm{R}} \approx 0.41 \pm 0.01 \tag{3.5}
\end{equation*}
$$

The cases of $\mathrm{SU}(3)$ and $\mathrm{Z}_{3}$ are more difficult to analyze, because terms of odd degree in $t$ show up to order $t^{9}$ and cause wild oscillations of the ratios. On the other hand, the quantity $\mathscr{G}_{\mathrm{e} \pm 2 i \pi / 3}$ is complex and the corresponding singularities lie in the complex plane, rather close to the real axis and to the singularity of $\mathscr{G}_{0}$.

The results of this analysis for $d=3,4,5$ are gathered in table 3 , where we have also reproduced our estimates in the case of $\mathbf{Z}_{2}$, for an easier comparison. The errors are estimated from the oscillations of the last ratios and are not to be taken too seriously.

The most striking feature is the group independence of the location of the roughening point, when measured in the parameter $t$. This had already been noticed by other authors [11, 10] on a different sample of groups and dimensions. It was expected to some extent, as roughening comes from large scale fluctuations of

Table 3
Location of the roughening point measured in the variable $t$ for various groups and dimensions; we also give the corresponding value of $\beta$ for four-dimensional non-abelian theories

|  | $d=3$ | $d=4$ | $d=5$ |
| :---: | :---: | :---: | :---: |
| $\mathrm{Z}_{3}$ | $0.46 \pm 0.01$ | $0.40 \pm 0.01$ | $0.37 \pm 0.01$ |
| $\mathrm{U}(1)$ | $0.465 \pm 0.01$ | $0.41 \pm 0.01$ | $0.37 \pm 0.01$ |
| $\mathrm{SU}(2)$ | $0.465 \pm 0.01$ | $0.41 \pm 0.01$ | $0.375 \pm 0.010$ |
|  |  | $\beta \simeq 0.47 \pm 0.01$ | $\beta \approx 0.415$ |
| $\mathrm{SU}(3)$ | $0.47 \pm 0.04$ | $0.41 \pm 0.02$ | $0.37 \pm 0.03$ |
|  |  | $\beta \simeq 0.33 \pm 0.02$ | $0.364 \pm 0.015$ |
| $\mathrm{Z}_{2}$ | $0.46 \pm 0.01$ | $0.40 \pm 0.01$ | 0.3 |

the surface which are insensitive to the symmetry group. However, small groupdependent corrections might have shifted the roughening point. This is seemingly not the case, and in view of table 3, we can talk of universality of the roughening point. It may be worth recalling that all observables attached to the surface, not only our indicators, must possess this roughening singularity. We find it gratifying that our determination of $t_{\mathrm{R}}$ for $\mathrm{Z}_{3}, \mathrm{U}(1)$ or $\mathrm{SU}(2)$ agrees with other methods [6, $10,11]$. We have also checked that the string tension, extrapolated from its series, has a singularity in the roughening region.

Next, it is interesting to compare the location of this roughening point with the usual transition point when it occurs at a non-trivial coupling. Estimates of the latter come either from self-duality ( $Z_{3}$ in four dimensions), or from Monte Carlo experiments, for $\mathrm{U}(1)$ in $4 \mathrm{D}[12,13]$ or $\mathrm{SU}(2)$ in 5 D [12]. We recall that in the 4D $Z_{2}$ model, the roughening point was found $[2,6,1]$ hardly distinguishable from $t_{\mathrm{c}}=\sqrt{2}-1$. This is clearly not a general feature, as all possible cases seem to occur:

$$
\begin{aligned}
& d=4, \mathrm{Z}_{3}: t_{\mathrm{c}}=\frac{1}{2}(\sqrt{3}-1)=0.366<t_{\mathrm{R}}, \\
& \mathrm{U}(1): \beta_{\mathrm{c}} \simeq 0.5[12,13] \text { or } t_{\mathrm{c}} \simeq 0.45>t_{\mathrm{R}} ; \\
& d=5, \mathrm{SU}(2): \beta_{\mathrm{c}} \simeq 0.41[12] \text { or } t_{\mathrm{c}} \simeq 0.37 \approx t_{\mathrm{R}} .
\end{aligned}
$$

We finally turn to the most interesting cases of $\mathrm{SU}(2)$ and $\mathrm{SU}(3)$ in four dimensions, and compare our strong coupling expansions for the string tension to Monte Carlo data [3] (see fig. 1). In the case of $\operatorname{SU}(2)$, the series or its extrapolation agree very well [5] with the data, up to $\beta=1 / g_{0}^{2} \approx 0.5$. (Here $g_{0}^{2}$ is the conventional coupling constant of continuum gauge field theories.) Beyond this point, the strong coupling expansion seems to indicate a zero of $k$ and becomes unreliable. This fits nicely with the observation that the roughening singularity at $\beta_{\mathrm{R}} \approx 0.47$ limits the convergence domain of the series. For $\operatorname{SU}(3)$, the strong coupling series departs

Fig. 1. The string tension from the truncated series or its Pade extrapolation compared with Monte Carlo data [3].
from its asymptotic behavior $-\ln t \approx-\ln \beta=\ln 3 g_{0}^{2}$ much faster than for $\mathrm{SU}(2)$. However, the series truncated to order 12 or its Padé approximants reproduce the existing data very well up to a value $\beta=1 / 3 g_{0}^{2} \approx 0.35$, to be compared with our estimate of the roughening point $\beta_{\mathrm{R}} \approx 0.33$.

## 4. Conclusion

In this paper we have presented new strong coupling expansions for the string tension and indicators of roughening for various groups and in any dimensions. We have found good evidence of roughening at a universal value of the parameter $t=\int \mathrm{D} U \exp \{s(U)\} \chi_{\overline{\mathrm{f}}}(U)$, a property which remains to be fully elucidated.

For four-dimensional $\operatorname{SU}(2)$ and $\mathrm{SU}(3)$ theories, the string tension agrees very well with Monte Carlo data, up to the roughening point. We think that this is a further evidence of the roughening singularity which limits the radius of convergence of the strong coupling series. As already stressed, we are left with the important task of finding an effective theory beyond the roughening point [14].

We acknowledge fruitful discussions with C. Itzykson, N. Sourlas and K. Wilson.

## Appendix

The contribution of any diagram to the strong coupling expansion of a quantity such as $k$ or $p_{\mathrm{W}}, p_{0}$, etc., is the product of two factors, a geometrical configuration number independent of the underlying symmetry group and a group theoretic factor. We will here give some indications on these group theoretic factors, and give the expression of the string tension to twelfth order for an arbitrary group.

We use the notations and normalizations of eqs. (2.2)-(2.4). Let $N_{r s t}$... denote the number of times the trivial representation appears in the decomposition of the product $r \otimes s \otimes t \ldots$.

$$
\begin{equation*}
N_{r s t \ldots}=\int \mathrm{D} U \chi_{r}(U) \chi_{s}(U) \chi_{t}(U) \ldots \tag{A.1}
\end{equation*}
$$

We introduce the following expressions:

$$
\begin{align*}
A_{i} & =\sum_{r \neq 0} \nu_{r}^{2} \beta_{r}^{i}  \tag{A.2}\\
A_{i j k} & =\sum_{r, s, t \neq 0} N_{r s t} \nu_{r} \nu_{s} \nu_{t} \beta_{r}^{i} \beta_{s}^{j} \beta_{t}^{k},  \tag{A.3}\\
B_{i j} & =\sum_{r, s \neq 0} N_{f r s} \frac{\nu_{r} \nu_{s}}{\nu_{\mathrm{f}}} \beta_{r}^{i} \beta_{s}^{j}, \tag{A.4}
\end{align*}
$$

$$
\begin{align*}
B_{i j k}= & \sum_{r, s, t \neq 0} N_{\mathrm{f} r s t} \frac{\nu_{r} \nu_{s} \nu_{t}}{\nu_{\mathrm{f}}} \beta_{r}^{i} \beta_{s}^{j} \beta_{t}^{k},  \tag{A.5}\\
C_{i j k l}= & \sum_{r, s, t, u \neq 0} N_{\mathrm{f} r s} N_{s t u} \frac{\nu_{r} \nu_{t} \nu_{u}}{\nu_{\mathrm{f}}} \beta_{r}^{i} \beta_{s}^{j} \beta_{t}^{k} \beta_{u}^{l}  \tag{A.6}\\
D_{k}^{i} m_{l}^{j}= & \sum_{r, s, t, u, v \neq 0} \frac{\nu_{r} \nu_{s} \nu_{t} \nu_{u} \nu_{v}}{\nu_{\mathrm{f}}} \beta_{r}^{i} \beta_{s}^{j} \beta_{t}^{k} \beta_{u}^{l} \beta_{v}^{m} \\
& \times \int \mathrm{D} R \mathrm{D} S \mathrm{D} T \mathrm{D} U_{\chi_{r}}\left(R T^{-1}\right) \chi_{s}\left(T S^{-1}\right) \chi_{\mathrm{f}}\left(S R^{-1}\right) \\
& \times \chi_{t}\left(R U^{-1}\right) \chi_{u}\left(S U^{-1}\right) \chi_{v}\left(T U^{-1}\right) \tag{A.7}
\end{align*}
$$

These coefficients may be pictorially represented as shown in fig. 2. The computation of our observables $p_{0}$ and $p_{W}$ also involves slight modifications of these expressions, where the value $c_{r}$ of the selected element of the center in some representation may appear under the summation sign in the r.h.s. of eqs. (A.2)-


Fig. 2. Pictorial representation of group theoretic factors.
(A.7). We are now in a position to give the expression of the string tension up to twelfth order for an arbitrary group in any dimension. We write

$$
\begin{equation*}
\frac{k+\ln t}{2(d-2)}=-\sum_{n=4}^{\infty} \hat{K}_{n} \tag{A.8}
\end{equation*}
$$

where $\hat{K}_{n}$ is the contribution from diagrams with $n$ plaquettes. We find

$$
\begin{align*}
\hat{K}_{4}= & t^{4} \\
\hat{K}_{5}= & B_{51} t^{-1} \\
\hat{K}_{6}= & 2 t^{6}-A_{6} \\
\hat{K}_{7}= & 0 \\
\hat{K}_{8}= & (9 d-22) t^{8}+2 B_{82} t^{-2}+4 t^{2} B_{42}, \\
\hat{K}_{9}= & (10 d-29) B_{91} t^{-1}+\left(d-\frac{5}{2}\right) B_{55} t^{-1}+2 D_{1}^{4} 1_{1}^{4} t^{-2}+(8 d-29) t^{3} B_{51}, \\
\hat{K}_{10}= & (48 d-132) t^{10}-\left(11 d-\frac{59}{2}\right) A_{10}+\left(d-\frac{5}{2}\right) B_{551} t^{-1} \\
& +(10 d-29)\left(C_{1451} t^{-1}-t^{4} A_{6}\right)-d B_{51}^{2} t^{-2}+12 t^{4} B_{42}  \tag{A.9}\\
\hat{K}_{11}= & (24 d-88) t^{5} B_{51}-\left(11 d-\frac{59}{2}\right) A_{551}-(10 d-29) B_{51} A_{6} t^{-1}+20(d-3) B_{11,1^{t}} t^{-1} \\
& +12 t B_{73}+6 B_{11,3} t^{-3}+10 t^{5} B_{33}, \\
\hat{K}_{12}= & \left(121 \frac{1}{3} d^{2}-555 \frac{1}{3} d+610 \frac{1}{3}\right)+t^{5} A_{6}(-24 d+72)+(11 d-29) A_{6}^{2} \\
& -20(d-3) A_{12}+12 D_{2}^{7} 1_{1}^{4} t^{-3}+12 t D_{1}^{4} 1_{2}^{3}+6 B_{42}^{2}+B_{12,4} t^{-4}+(16 d-40) t^{2} B_{64} \\
& +t^{2} B_{82}(48 d-160)+t^{6} B_{42}(60 d-168)+8(d-3) B_{84} \\
& +4(2 d-5) B_{86} t^{-2}+4(9 d-26) B_{12,2} t^{-2}
\end{align*}
$$

Of course, these expressions have then to be expanded to the required order in $t$, using series expansions of the higher $\beta_{r}$ in $t=\beta_{\mathrm{f}}$. Needless to say, use of a computer is almost compulsory. We have throughout this work used the Algebraic Manipulation Program (AMP) written by one of us [J.M.D.].

To push these expansions to order fourteen in $t$, in the case of $\mathrm{SU}(2)$ or $\mathrm{U}(1)$, we just have to add the contribution of diagrams with thirteen and fourteen plaquettes.

Diagrams with 13 plaquettes have one and only one plaquette bearing the representation of spin 1 for $S U(2)$ or $\mathrm{e}^{ \pm 2 i \theta}$ for $\mathrm{U}(1)$. They contribute

$$
\begin{align*}
\hat{K}_{13}= & \left(144 d^{2}-864 d+1225\right) t^{7} B_{51}+\left(80 d^{2}-538 d+893\right) t^{3} B_{91} \\
& +\left(140 d^{2}-822 d+1211\right) B_{13,1} t^{-1} \\
& +(8 d-20) C_{8151} t^{-2}+(12 d-32) t^{2} C_{4151}+\mathrm{O}\left(t^{16}\right) \tag{A.10}
\end{align*}
$$

Finally diagrams with fourteen plaquettes are those already encountered in $\mathbf{Z}_{2}$, affected by group theoretical factors:

$$
\begin{align*}
\hat{K}_{14}= & {\left[\left(16570-11934 d+2160 d^{2}\right)+2 \tau(d-3)+\left(566-\frac{1090}{3} d+\frac{172}{3} d^{2}\right) \alpha\right.} \\
& \left.+\left(-7136+5062 d-900 d^{2}\right) \gamma\right] t^{14}+\mathrm{O}\left(t^{16}\right) \tag{A.11}
\end{align*}
$$

where the four terms come respectively from diagrams topologically equivalent to a plane, to a plane with a handle, from diagrams with one closed singular line along which four plaquettes meet, and from disconnected parts. The weights are $\tau=\alpha=$ $\gamma=1$ for $Z_{2}, \tau=\frac{1}{4}, \alpha=8, \gamma=4$ for $\mathrm{SU}(2), \tau=1, \alpha=3, \gamma=2$ for $\mathrm{U}(1)$.

Putting everything together, one readily derives the expression listed in table 1.

## References

[1] J.M. Drouffe and J.B. Zuber, Nucl Phys. B180[FS2] (1981) 253; and references therein
[2] C. Itzykson, M. Peskin and J.B. Zuber, Phys. Lett. 95B (1980) 259
[3] M. Creutz, Phys. Rev. D21 (1980) 2308; Phys. Rev. Lett. 45 (1980) 313
[4] J. Kogut, R.P. Pearson and J. Shigemitsu, Phys. Rev. Lett. 43 (1979) 484;
J. Kogut and J. Shigemitsu, Phys. Rev. Lett. 45 (1980) 410;
J. Kogut, R.P. Pearson and J. Shigemitsu, Illinois preprint TH 80.41
[5] G. Münster, Phys. Lett. 95B (1980) 59; Nucl. Phys. B180[FS2] (1981) 23
[6] M. Lüscher, G. Münster and P. Weisz, Nucl. Phys. B180[FS2] (1981) 1
[7] J.M. Drouffe, Nucl. Phys. B170 [FSI] (1980) 91
[8] J.M. Drouffe, G. Parisi and N. Sourlas, Nucl. Phys. B161 (1979) 397
[9] A. Duncan and H. Vaidya, Phys. Rev. D20 (1979) 903
[10] G. Münster and P. Weisz, Desy preprints 80/57; Nucl. Phys. B180[FS2] (1981) 13
[11] A. Hasenfratz, E. Hasenfratz and P. Hasenfratz, Nucl. Phys. B180[FS2] (1981) 253
[12] M. Creutz, Phys. Rev. Lett. 43 (1979) 553
[13] M. Creutz, L. Jacobs and C. Rebbi, Phys. Rev. D20 (1979) 1915;
B. Lautrup and M. Nauenberg, Phys. Lett. 95B (1980) 63
[14] M. Lüscher, Nucl. Phys. B180[FS2] (1981) 317

