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ROUGHENING TRANSITION IN LATTICE GAUGE THEORIES IN ARBITRARY DIMENSION (I). The Z_2 case

J.M. DROUFFE and J.B. ZUBER

DPhT, CEA-Saclay, 91191 Gif-sur-Yvette Cedex, France

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We present new evidence for a roughening transition in the Z_2 lattice gauge theory in arbitrary dimension, using strong coupling expansions to fourteenth order and an analysis at large dimension.

1. Introduction

Lattice gauge theories [1] are well suited for the study of the strong coupling behavior of physical quantities such as the Wilson loop $\langle W \rangle$ and of the coefficient of its area law, the string tension $k = -\ln \langle W \rangle / A$ [2, 3]. However, it has recently been pointed out [4, 5]* that strong coupling expansions are affected by a singularity at some roughening temperature. Roughening is a well-known phenomenon in three-dimensional statistical mechanics [7], and describes the delocalization of the interface between two phases which occurs at a high enough temperature. In our context, the Wilson operator $\langle W \rangle$ has a strong coupling behavior given by an average over surfaces bounded by the loop. At infinite coupling, only the minimal surface contributes. As the coupling decreases, the surface starts fluctuating. Ultimately, at the roughening temperature t_R, the system has no remembrance of the minimal surface, even when examined on a large scale [7]. This manifests itself by a non-analytical behavior at this point of quantities related to the surface, but not of bulk quantities such as the free energy.

That this singularity exists and is reached before the usual transition temperature is now clear in the case of the 3D Z_2 gauge theory, owing to the extensive study of the phenomenon in the dual Ising model [7]. In higher dimensions or for other groups, the situation is less clear [4, 5, 8]. In this article, we concentrate on the Z_2 models in arbitrary dimension, leaving the case of other groups to a forthcoming publication. From Monte Carlo calculations [9], and strong coupling expansions

* That roughening may play a role in lattice gauge theories was first suggested by Parisi [6].

[10], it is believed that the Z_2 model undergoes a first-order transition if $d > 3$. The issue is then to determine whether the roughening transition still exists and whether it takes place before ($t_R = \tanh \beta_R < t_c$) the first-order transition or in the metastable region ($t_R > t_c$). Answering these questions requires an accurate determination of t_R. One may look for the nearest singularity of suitable extrapolations of the series for k. However, as k is expected to have an essential singularity on top of a smooth background [4], it may be safer to look at other quantities attached to the surface. In [4], the following observable has been introduced:

$$p_W = \frac{\langle W \exp - 2\beta U_P \rangle}{\langle W \rangle}, \tag{1.1}$$

where P is a plaquette in the minimal surface taken as a reference plane. Here and in the following, expectation values are defined as

$$\langle X \rangle = Z^{-1} \sum_{\{U_l = \pm 1\}} X \exp \beta \sum_{\text{plaquettes}} U_P \tag{1.2}$$

with

$$Z = \sum_{\{U_l = \pm 1\}} \exp \beta \sum_{\text{plaquettes}} U_P .$$

The quantity p_W measures the probability that the surface contains the test plaquette P. At the roughening temperature, p_W must become equal to the same expectation value in the absence of the loop:

$$p_0 = \langle \exp - 2\beta U_P \rangle . \tag{1.3}$$

Thus, t_R is determined as the singularity of the indicator

$$\mathcal{G} = \frac{2}{p_0 - p_W} . \tag{1.4}$$

This is by no means the only operator which may be introduced to determine t_R. In [8], the width σ of the flux tube between two charges has been calculated in units of k^{-1} and shown to diverge at t_R. In four dimensions, all quantities computed using strong coupling expansions up to the twelfth order in $t = \tanh \beta$ point to a roughening temperature $t_R \lesssim t_c$, either just below or on top of t_c. In this paper we first present the calculation of k and $\mathcal{G}$ to fourteenth order for an arbitrary dimension d. The results presented in sect. 2 confirm nicely the 12th-order computation. Roughening does take place for $d \geqslant 3$; at $d = 4$, t_R seems to lie just below t_c, while at $d = 5$, it is beyond t_c, hence in the metastable region. For higher

d, the results are less reliable, due to oscillations of the coefficients of the series. This leads us to investigate the behavior at large d (sect. 3) using an expansion in powers of $d^{-1/2}$ [11]. In this large-d approximation, the indicator $\mathcal{G}$ still exhibits singularities, which seem to connect smoothly with those calculated for low dimension. A discussion of what might happen at intermediate d, and our concluding remarks are contained in sect. 4. The appendix deals with some technicalities of the $d^{-1/2}$ expansion.

2. Fourteenth-order calculation

The calculation of the fourteenth order uses the by now familiar diagrammatic techniques [2, 3]. It is, of course, impossible to display here the 302 diagrams contributing to that order, but their table may be obtained on request. In d dimensions, $-k = \ln\langle W\rangle/A$, p_{W}, p_0 and $\mathcal{G}$ read:

$$\begin{aligned}-k = \ln t + 2(d-2)\Big[&t^4 + t^6 + (9d-22)t^8 \\ &+ (28d-76)t^{10} + \left(\tfrac{364}{3}d^2 - \tfrac{1765}{3}d + \tfrac{2140}{3}\right)t^{12} \\ &+ \left(\tfrac{1976}{3}d^2 - \tfrac{10850}{3}d + 4997\right)t^{14}\Big],\end{aligned} \tag{2.1}$$

$$\begin{aligned}p_{\mathrm{W}} = -1 + 4(d-2)\Big[&t^4 + 4t^6 + (8d-7)t^8 \\ &+ (58d-103)t^{10} + (104d^2 - 262d + 165)t^{12} \\ &+ \left(\tfrac{3152}{3}d^2 - \tfrac{12722}{3}d + 4668\right)t^{14}\Big],\end{aligned} \tag{2.2}$$

$$\begin{aligned}p_0 &= 1 - 2t\frac{\partial \tilde{F}}{\partial t} \\ &= 1 - 4(d-2)\Big[t^6 + (10d-25)t^{10} \\ &\qquad + (8d-31)t^{12} + (140d^2 - 742d + 1001)t^{14}\Big],\end{aligned} \tag{2.3}$$

$$\begin{aligned}\mathcal{G} = \frac{2}{p_0 - p_{\mathrm{W}}} = 1 + 2(d-2)\Big[&t^4 + 5t^6 + (10d-11)t^8 \\ &+ (88d-168)t^{10} + (140d^2 - 312d + 106)t^{12} \\ &+ \left(\tfrac{5048}{3}d^2 - \tfrac{20216}{3}d + 7213\right)t^{14}\Big].\end{aligned} \tag{2.4}$$

In eq. (2.3), $\tilde{F}$ denotes the non-trivial part of the free energy per plaquette

$$\frac{2}{Nd(d-1)}\ln Z = \ln\cosh t + \tilde{F}. \tag{2.5}$$

The expansion of $-k$ at $d=3$ was already known [2, 6]. On the other hand, the terms of highest degree in d may be calculated by an independent method, using the expansion around $d=\infty$ in the strong coupling phase (see ref. [11] and sect. 3). This provides a useful cross-check of our computation.

The series (2.4) for $\mathcal{G}$ may be analyzed by Padé approximants, or by other generalizations of the method of successive ratios. Their nearest positive singularity turns out to be very stable with respect to the various procedures, for $3 \leqslant d \leqslant 5$. This is displayed in table 1. The value at $d=3$ is to be compared with the value $t_R = 0.4593$ obtained by Weeks [7] through a careful analysis of various 18th order series.

At $d=4$, $t_R \simeq 0.40 \pm 0.01$ seems to lie below $t_c = \sqrt{2}-1 = 0.414$. Of course, we cannot bar possible oscillations of the further terms in the series, and it is therefore premature to assert that $t_R \neq t_c$. At $d=5$, in spite of the larger uncertainty on the location of the roughening point, there is some evidence that it lies beyond the critical point $t_R > t_c$. Indeed the first-order transition point may be as estimated as

$$t_c = \tanh\left[\frac{1}{2(d-1)}\left(2.755205 - \frac{0.912561}{d} + \frac{0.601169}{d^2} + \cdots\right)\right]. \tag{2.6}$$

This expression is obtained by a systematic $1/d$ expansion about the mean field approximation [13], not to be confused with the previously mentioned $d^{-1/2}$ expansion. For $d=5$, this yields:

$$t_c \simeq 0.313 \pm 0.01\,, \tag{2.7}$$

to be compared with

$$t_R \simeq 0.364 \pm 0.015 \tag{2.8}$$

TABLE 1

Nearest singularity of the series $\mathcal{G} = \sum \mathcal{G}_k t^{2k}$, as determined by various procedures

	$(\mathcal{G}_8/\mathcal{G}_{10})^{1/2}$	$(\mathcal{G}_{10}/\mathcal{G}_{12})^{1/2}$	$(\mathcal{G}_{12}/\mathcal{G}_{14})^{1/2}$	$(\mathcal{G}_8/\mathcal{G}_{12})^{1/4}$	$(\mathcal{G}_{10}/\mathcal{G}_{14})^{1/4}$	Pole of [5/2] Padé	Pole of [4/3] Padé
$d=3$	0.4449	0.4725	0.4482	0.4585	0.4602	0.4594	0.4522
$d=4$	0.3970	0.4094	0.3910	0.4031	0.4001	0.4019	0.4011
$d=5$	0.3787	0.3646	0.3623	0.3716	0.3635	0.3619	0.3461

As we let the dimension grow, the nearest singularity of (2.4) becomes increasingly unstable with respect to the extrapolation procedure. The ratios of consecutive terms $(\mathcal{G}_n/\mathcal{G}_{n+2})^{1/2}$ oscillate wildly, while the ratios $(\mathcal{G}_n/\mathcal{G}_{n+4})^{1/4}$ are stable. This comes from the structure of the series

$$\mathcal{G}_{4n} \sim \mathcal{G}_{4n+2} \sim d^n \tag{2.9}$$

which will be at the center of the forthcoming discussion of the behaviour at large d.

3. Roughening at large dimension

Let us first review the method for studying the large-d limit [11, 12]. In the strong coupling phase, it is easy to select the leading contributions to F or $\langle W \rangle$ as $d \to \infty$: they are tree-like or "hydra-like" diagrams, where an arbitrary number of three-dimensional cubes are glued together, a new direction being chosen at each step. This implies that $t^4 d$ is the relevant parameter in this limit, and that subdominant terms are suppressed by powers of $t^2 \sim d^{-1/2}$. Introducing the variable u through

$$u(1-u)^4 = 2dt^4, \tag{3.1}$$

the free energy $\tilde{F}$ is found to be:

$$\tilde{F} = \frac{d^{-1/2}}{6\sqrt{2}} u^{3/2}(1-3u) + \mathrm{O}(d^{-1}). \tag{3.2}$$

The resulting swallow-tail shape of $\tilde{F}$ is depicted in fig. 1. The parametrization $u(t)$

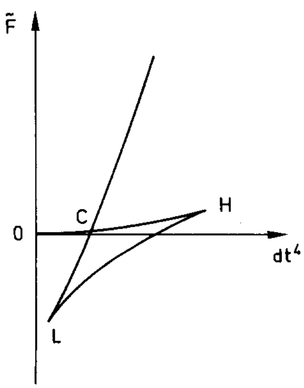

Fig. 1. Qualitative behavior of the free energy $\tilde{F}$ for large d.

is singular at $u_{\mathrm{H}}=\frac{1}{5}$ and $u_{\mathrm{L}}=1$. This introduces singularities in $F(t)$ at

$$t_{\mathrm{H}}=\frac{4}{5}\left(\frac{1}{10d}\right)^{1/4}, \qquad t_{\mathrm{L}}=0. \tag{3.3}$$

However, these singularities are unphysical: they occur on the boundaries of the metastable regions. As t increases from zero or decreases from 1 (low coupling region), the system undergoes a first-order transition at t_c, with $t_{\mathrm{L}}<t_c<t_{\mathrm{H}}$. As mentioned above, t_c may be determined by a different method and turns out to be of order d^{-1}.

The effect of $d^{-1/2}$ corrections in the expansion of $\tilde{F}$ is to shift slightly u_{H} and u_{L} but does not upset the previous picture.

We now turn to the calculation of the surface tension k and of the operator p_{W}, in this $d^{-1/2}$ expansion. The leading contribution comes again from trees of cubes glued to the plaquettes of the minimal surface. The next-to-leading terms are also easily characterized (see the appendix). This leads to

$$k=-\tfrac{1}{4}\ln\frac{u}{2d}+\sqrt{\frac{2u}{d}}\,\frac{1}{1-5u}\left(\tfrac{1}{2}u+\tfrac{3}{2}u^2-\tfrac{1}{6}u^3\right)+\mathrm{O}\left(\frac{1}{d}\right), \tag{3.4}$$

$$p_{\mathrm{W}}=-1+2u+2\sqrt{\frac{2u}{d}}\,\frac{1}{1-5u}\left(2u-\tfrac{15}{2}u^2+\tfrac{23}{6}u^3-\tfrac{7}{3}u^4\right), \tag{3.5}$$

$$p_0=1-\sqrt{\frac{2u}{d}}\,u(1-u)+\mathrm{O}\left(\frac{1}{d}\right). \tag{3.6}$$

When re-expanding these expressions in powers of t at fixed d, we recover the leading terms of the form $t^{4n}d^n$ and $t^{4n+2}d^n$ in eqs. (2.1)–(2.3): this provides the cross-check mentioned above.

It is now possible to compute the expression of the indicator $\mathcal{G}=2(p_0-p_{\mathrm{W}})^{-1}$ to order $d^{-1/2}$:

$$\mathcal{G}=\frac{1}{1-u}+\sqrt{\frac{2u}{d}}\,\frac{u(15-63u+38u^2-14u^3)}{6(1-u)^2(1-5u)}. \tag{3.7}$$

To this low order in $d^{-1/2}$, a proper analysis of the singularities u_{R} of $\mathcal{G}$ requires further assumptions, as we have to disentangle u_{H}, u_{L} from u_{R}. For instance, the simplest hypothesis is that the leading singularity is of the form

$$\mathcal{G}=(\bar{u}_{\mathrm{R}}-u)^{-\alpha}. \tag{3.8}$$

This leads to

$$\alpha = 1\,, \tag{3.9}$$

and

$$\bar{u}_{\mathrm{R}} = 1 - \sqrt{\frac{2}{d}} + \mathrm{O}\left(\frac{1}{d}\right), \tag{3.10}$$

or

$$\bar{t}_{\mathrm{R}} = \frac{2^{1/4}}{d^{3/4}}\,. \tag{3.11}$$

The knowledge of one more term in the expansion (3.7) would allow one to assert the consistency of the assumption (3.8), or to discriminate among other possible schemes.

4. Discussion and conclusion

How are these results at large d to be interpreted and how do they fit with those obtained at low dimension, from the strong coupling expansion of sect. 2? Under the assumption (3.8), we can follow the singularity $\bar{t}_{\mathrm{R}}(d)$ of (3.11) as d decreases. A remarkable feature is that this extrapolation seems to be smoothly connected to the curve $t_{\mathrm{R}}(d)$ for low d. This is clear in fig. 2 which illustrates the following scenario. As d grows from 3, the roughening point $t_{\mathrm{R}}(d)$, defined as the nearest singularity of $\mathcal{G}$, increases from $t_{\mathrm{R}}(d=3) < t_{\mathrm{c}}(d=3)$, crosses the first-order transition and enters the metastable region at $d \gtrsim 4$, and reaches H, the end of the strong coupling metastable phase at some $d_{\mathrm{c}} \approx 6-8$. This particular value d_{c} might have something to do with the upper critical dimension appearing in polymer physics [6, 14]. It then bounces off H and goes on the unphysical branch HL, reaching L at $d=\infty$. It must be clear that what happens above d_{c} is a very academic problem, and that many other schemes may be suggested.

It is physically more interesting to investigate the features of roughening and to wonder whether the same mechanism takes place in higher dimensions. For $d=3$, there is clear evidence [7] that the delocalization of the surface results from long-wavelength fluctuations rather than local (short-wavelength) deformations. On the other hand, we have seen that as $d \to \infty$, local deformations play a dominant role. It is then legitimate to wonder about the behavior at intermediate values of d: $4 \leqslant d \leqslant d_{\mathrm{c}}$.

To summarize, we have re-examined in this paper the roughening transition of Z_2 lattice gauge theory in arbitrary dimension. New information comes from the strong coupling expansion pushed up to order $\tanh^{14}\beta$ and from a study of the

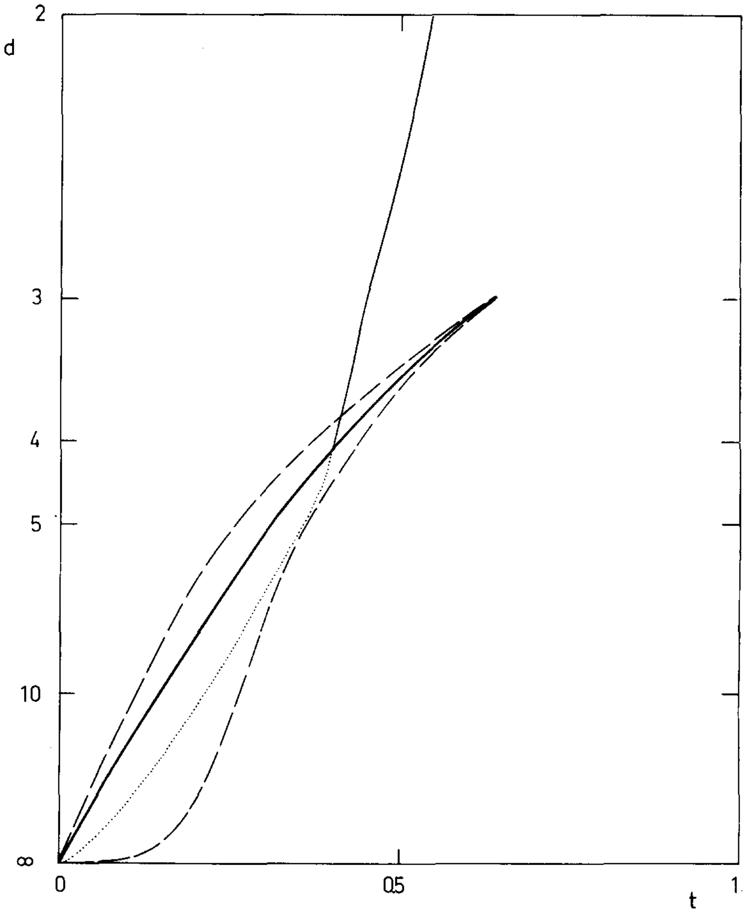

Fig. 2. Plot of $t_{\mathrm{H}}, t_{\mathrm{L}}$ (broken lines), t_{c} (heavy solid line), t_{R} (solid and dotted line) as functions of $1/d$. The roughening line is obtained for $d \lesssim 5$ by the analysis of strong coupling series, and for $d \gtrsim 8$ by the large-d expansion.

large-d behavior. The strong coupling series, through suitable extrapolation, leads to a roughening singularity in good agreement with the results of lower order calculations, for $d \lesssim 6$. This stability makes us rather confident in the determination of the location of this singularity. The large-d expansion provides a cross-check on the strong coupling series. Moreover, the analysis of its singularities, though not unambiguous, seems in reasonable agreement, for intermediate values of d with the previous method. The resulting picture is that the roughening transition lies in the physical region up to $d \approx 4$, and in the metastable region for $4 \lesssim d \lesssim d_{\mathrm{c}} \approx 6-8$. It is now a theoretical challenge to establish firmly the existence of this roughening transition, to find how to compute beyond this point, and to see its effect on numerical simulations.

Stimulating discussions with C. Itzykson are gratefully acknowledged.

Appendix

This appendix gives some indications on the computation of various expectation values in the large-d limit. The appendices of ref. [11] describe the basic techniques needed here, and in particular present the determination and the summation of the dominant contributions and of the first $d^{-1/2}$ corrections to the free energy. Therefore, we restrict ourselves to the specific features of the computation of the string tension k and of the operator p_W.

We recall that the diagrams contributing to k and p_W have the topology of surfaces made of plaquettes, bounded by the Wilson loop. Each contribution is a polynomial in N, the number of sites, and in A, the minimal area. The string tension is obtained by retaining in the summation over all the diagrams (connected or not), the linear term in A and by setting N to zero. Similarly, $\frac{1}{2}(1+p_W)$ is computed by setting to zero both N and A in the sum over all diagrams which do not contain the test plaquette P [cf. eq. (1.1)].

In the large-d limit, the fundamental objects are the boundary surfaces of trees made of adjacent three-dimensional cubes. This kind of diagrams has been shown to give a dominant contribution in the strong coupling region for large dimensions. They can be summed up and they contribute, for example, to the free energy *per site* an amount

$$\frac{d^{3/2}}{12\sqrt{2}}u^{3/2}(1-3u)\,, \tag{A.1}$$

where the parameter u has been defined in (3.1). We may define a dressing operation by replacing each plaquette of a given diagram by a tree of cubes. This yields a multiplicative factor

$$f=\frac{1}{1-u} \tag{A.2}$$

for each dressed plaquette. For example,

$$\langle W\rangle=(tf)^A \tag{A.3}$$

and therefore

$$k=-\ln ft=\tfrac{1}{4}\ln\frac{2d}{u}\,. \tag{A.4}$$

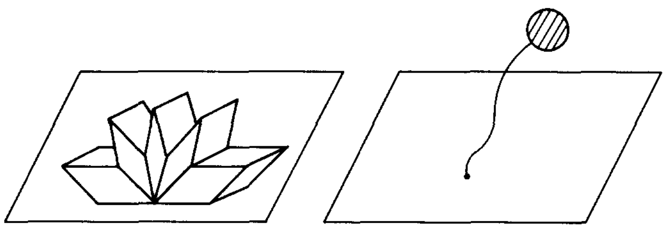

Fig. 3. Skeleton diagrams contributing to $d^{-1/2}$ corrections.

For the computation of p_W, we have to distinguish whether the test plaquette P is not dressed [no contribution to $\frac{1}{2}(p_W+1)$] or dressed with a non-trivial tree, whence the leading contribution $(f-1)/f=u$ to $\frac{1}{2}(p_W+1)$.

We now turn to the $d^{-1/2}$ correction. Let us first consider a diagram made of two connected pieces: the dressed Wilson surface and some disconnected tree. In view of the factor $d^{3/2}$ in (A.1), this seems to give a dominant contribution to $\langle W \rangle$ [cf. (A.3)]. However, such a term contains a factor N coming from the translations of the disconnected tree and hence does not contribute to $\langle W \rangle$ or p_W. We only have to subtract the forbidden configurations where two (or more) plaquettes of the two disconnected pieces coincide, and to count the diagrams when these disconnected pieces intersect along a closed singular line (though allowed, these diagrams must be considered separately to avoid multiple counting). As such coincidences or crossings occur with a probability $1/d^2$, we see that disconnected diagrams and diagrams with singular lines contribute to the $d^{-2}\times d^{3/2}=d^{-1/2}$ order.

Apart from these contributions, the $d^{-1/2}$ correction involves two types of connected diagrams depicted in fig. 3. First, n three-dimensional cubes ($n \geqslant 2$) sharing one link may be put on two plaquettes on the Wilson surface, as illustrated for $n=5$ in fig. 3a. Second, a closed diagram generating the $d^{-1/2}$ correction to the free energy (see [11]), depicted as a shaded blob in fig. 3b, may be connected to the surface by a string of cubes. In either case, the resulting skeleton diagram have to be dressed by "hydra-like" trees.

Using these indications and the techniques of ref. [11], the reader can easily recover our results (3.4)–(3.7).

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