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LATTICE MODELS WITH A SOLVABLE SYMMETRY GROUP

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A class of lattice models with a global symmetry characterized by a solvable group are shown to be equivalent to ones having an Abelian symmetry, to which the Kramers-Wannier dual transformation can be applied. Examples are afforded by the permutation groups S_3 and S_4.

The Kramers-Wannier duality [1] for the two-dimensional Ising model has recently been generalized to various statistical and lattice gauge systems, giving a surprising wealth of information [2–7]. Attempts have been made to extend this method in the non-Abelian case [8]. It is soon realized that a non-trivial amount of group theory and topology is required, except in a few examples involving groups with a simple structure such as the permutation group on three objects S_3. The latter belongs to a class of solvable groups for which a general property holds as we shall demonstrate below.

Consider a "classical spin" model on a lattice in arbitrary dimension d. The spin variables g_i take their values in a non-Abelian group G and the latin index $(i, j, ...)$ labels the lattice points. For the applications we have in mind G is likely to be a discrete (and even finite) group. The Hamiltonian reads

$$\mathcal{H} = \sum J_{ij} f(g_i g_j^{-1}) \,. \tag{1}$$

The discussion applies to any type of interaction J_{ij} between a pair of spins although the duality transformation will later require J_{ij} to couple only nearest neighbors. We assume that $\mathcal{H}$ is G-invariant, i.e., that its value is unchanged under a global transformation

$$g_i \to g g_i \,, \qquad \text{all } i \,. \tag{2}$$

Note that $\mathcal{H}$ is also invariant under right transformations $g_i \to g_i g$, by construction. Therefore f is a class-function (or character)

$$f(g(g_i g_j^{-1}) g^{-1}) = f(g_i g_j^{-1}) \,. \tag{3}$$

We will show that, under certain circumstances, the partition function (β: inverse temperature)

$$Z = \sum_{\{g_i\} \in G} \exp(-\beta \mathcal{H}) \tag{4}$$

may be re-expressed as the one pertaining to a different model on the same lattice, with spin variables taking their values in a different group.

Assume that G may be regarded as a semi-direct product $G = A \otimes H$ of an invariant subgroup H and its Abelian factor group A. This means that there exists an homomorphism from A into the group of automorphisms of H which, for any $a \in A$, we denote as

$$h \in H \xrightarrow{a} {}^a h \in H . \tag{5}$$

Any element of G may be written as a pair $g = (a, h)$, with the multiplication law

$$gg' = (a, h)(a', h') = (aa', h\, {}^a h') . \tag{6}$$

Since $(a, e)^{-1} = (a^{-1}, e)$, we have

$$(a, e)(a', h')(a, e)^{-1} = (a', {}^a h) . \tag{7}$$

It is natural to associate with any element $g = (a, h)$ of G an element γ of the direct product $\Gamma = A \times H$ through a bijection α, not a group homomorphism, with β its inverse

$$G \underset{\beta}{\overset{\alpha}{\rightleftharpoons}} \Gamma . \tag{8}$$

This mapping will satisfy a constraint designed to transform the model with symmetry G into a model with symmetry Γ. Let ϕ be a function on Γ defined through

$$\phi(\gamma) = f(\beta(\gamma)) . \tag{9}$$

In order that the interaction (1) be expressible in terms of the new "spins" $\gamma_i = \alpha(g_i)$, $\gamma_j = \alpha(g_j)$ and of the function ϕ we must have

$$f(g_i g_j^{-1}) = \phi(\alpha(g_i)\, \alpha(g_j)^{-1}) = f(\beta[\alpha(g_i)\, \alpha(g_j)^{-1}]) . \tag{10}$$

The minus one superscript on the r.h.s. refers to the inverse in Γ.

If we deal with the most general G-invariant function f, the previous requirement means that the two arguments belong to the same class in G. We denote this equivalence by the symbol $\sim$. The mapping α should therefore fulfill

$$g_i g_j^{-1} \sim \beta[\alpha(g_i)\, \alpha(g_j)^{-1}] . \tag{11}$$

A solution to this condition is, with $g = (a, h)$

$$\alpha(g) = (a, {}^{a^{-1}}h) , \tag{12}$$

where the element on the right is considered in Γ. Indeed α is a bijection and

$$\beta[\alpha(g)\, \alpha(g')^{-1}] = (aa'^{-1}, {}^{a'^{-1}}(h\, {}^{aa'^{-1}}h'^{-1}))$$

$$= (a'^{-1}, e)(aa'^{-1}, h\, {}^{aa'^{-1}}h'^{-1})(a', e)$$

$$\sim (aa'^{-1}, h\, {}^{aa'^{-1}}h'^{-1}) \,. \tag{13}$$

The last element is nothing but gg'^{-1} so that the desired property (11) holds. We conclude that the initial model with symmetry G has been re-expressed as a similar model with symmetry Γ.

If the process can be iterated, i.e., if $G = A_1 \otimes H_1$ and the invariant group H_1 can in turn be written as a semi-direct product $H_1 = A_2 \otimes H_2$, A_2 Abelian, and so on, the initial group G is solvable.

Proceeding as before we can progressively eliminate the non-Abelian features and finally obtain a theory with a purely Abelian symmetry $A_1 \times A_2 \times \ldots$. The class of solvable groups which can be written in cascade as semi-direct products is the one mentioned earlier.

As a consequence the Kramers-Wannier duality may be carried out easily for any spin model based on a solvable group since the transformation is known in the general Abelian case [9].

As already mentioned an example of a solvable group is S_3 which has a structure of semi-direct product $Z_2 \otimes Z_3$. The previous construction maps such models on equivalent ones with $Z_2 \times Z_3$ as a symmetry, among which a self-dual case corresponds to the Potts model [10]. The permutation group of four objects S_4 has a similar structure. We have $S_4 = Z_2 \otimes \Sigma_4$ with Σ_4 the group of even perturbations being itself the semi-direct product $Z_3 \otimes V_4$. The Abelian (dihedral) group V_4 is the set of products of disjoint transpositions.

As is known since the work of Galois, permutation groups of higher order are no longer solvable. It is also clear that the above construction does not extend to lattice models with a local gauge invariance where spins defined on links interact by four.

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