An IOP and SISSA journal

# Sum rules for the ground states of the $O(1)$ loop model on a cylinder and the XXZ spin chain 

P Di Francesco ${ }^{1}$, $\mathbf{P}$ Zinn-Justin ${ }^{2}$ and J-B Zuber ${ }^{3}$<br>${ }^{1}$ Service de Physique Théorique de Saclay, CEA/DSM/SPhT, URA 2306 du CNRS, CEA-Saclay, F-91191 Gif sur Yvette Cedex, France<br>${ }^{2}$ Laboratoire de Physique Théorique et Modèles Statistiques, UMR 8626 du CNRS, Université Paris-Sud, Bâtiment 100, F-91405 Orsay Cedex, France<br>${ }^{3}$ Laboratoire de Physique Théorique et Hautes Énergies, Tour 24-25 5ème étage, Université Pierre et Marie Curie-Paris6, UMR 7589 du CNRS, Université Denis Diderot-Paris7, Boîte 126, 4 pl Jussieu, 75252 Paris Cedex 05, France E-mail: zuber@lpthe.jussieu.fr

Received 13 April 2006
Accepted 23 June 2006
Published 11 August 2006
Online at stacks.iop.org/JSTAT/2006/P08011
doi:10.1088/1742-5468/2006/08/P08011


#### Abstract

The sums of components of the ground states of the $O(1)$ loop model on a cylinder or of the XXZ quantum spin chain at $\Delta=-1 / 2$ of size $L$ are expressed in terms of combinatorial numbers. The methods include the introduction of spectral parameters and the use of integrability, a mapping from size $L$ to $L+1$, and knot-theoretic skein relations.


Keywords: integrable spin chains (vertex models), loop models and polymers, quantum integrability (Bethe ansatz)

ArXiv ePrint: math-ph/0603009

## Contents

1. Introduction and summary of results ..... 2
2. The inhomogeneous $O(1)$ loop model in odd size ..... 7
2.1. Link patterns, transfer matrix and $R$-matrices ..... 7
2.2. The quantum Knikhnik-Zamolodchikov equation ..... 9
2.3. Projections to link patterns of larger size ..... 10
2.4. Projections and transfer matrices ..... 12
2.5. Ground state sum rule ..... 13
3. The inhomogeneous $O(1)$ loop model in even size ..... 14
4. The XXZ spin chain and six-vertex model at $\Delta=-1 / 2$ ..... 15
4.1. Definition of the XXZ spin chain ..... 15
4.2. Equivalence with the $O(1)$ loop model ..... 15
4.3. Mapping of odd size to even size ..... 16
4.4. Polynomial eigenvector ..... 17
4.5. Sum rule ..... 18
4.6. Bilinear form and sum of squares ..... 18
5. Conclusion ..... 19
Acknowledgments ..... 21
References ..... 21

## 1. Introduction and summary of results

The observation by Razumov and Stroganov [1] that the components of the ground state of integrable quantum spin chains in some adequate basis enjoy integrality conditions and are connected to known combinatorial problems, namely alternating sign matrices (ASMs) and their avatars, has been the source of an amazing burst of new developments. It was soon realized that in many cases these integers in fact count some configurations of other lattice models $[2,3]$. To be more explicit, the components of the ground state of the Temperley-Lieb Hamiltonian describing the $O(1)$ loop model on an even number of sites with periodic boundary conditions are conjectured to count the numbers of configurations of so-called fully packed loop (FPL) models: this is the now celebrated Razumov-Stroganov (RS) conjecture [3]. Other types of boundary conditions have also been considered [4]-[6].

After these original observations and conjectures, a major progress has been the introduction of inhomogeneities in the original problem, in the form of spectral parameters, thus enabling one to use the full machinery of integrable models $[7,8]$. In particular, this has led to recursion formulae between components of the ground state and ultimately to a proof of the ground state sum rule, a weak but non-trivial version of the RS conjecture. These recursion formulae have been shown to follow from an underlying algebraic structure rooted in the affine Hecke algebra (AHA) [9], or alternatively, related to the so-called
quantum Knizhnik-Zamolodchikov equation (see [10] and further references therein). This has then been extended into several distinct directions. Loop models with crossings [11] have been shown to display an amazing relationship to the algebraic geometry of matrix varieties $[12,13]$. Extensions to other types of boundary conditions [14], to higher rank algebras [10], or to both [15], have been considered in turn.

In the present paper, we return to a case first tackled in the original papers in this domain, that of the $O(1)$ loop model on a square lattice wrapped on a semi-infinite cylinder of odd integer perimeter $[2,5]$, or of the associated periodic XXZ spin chain [1]. In the loop model, the sites of the boundary of the cylinder are pairwise connected via nonintersecting links. The oddness of the perimeter implies that one site remains unmatched, hence giving rise to a defect line, connecting it to the point at infinity on the cylinder. In the spin chain on $L=2 n+1$ sites, likewise, the ground state is made of $n+1$ spins pointing upward and $n$ downward (or vice versa). It is also interesting to consider the case of the $O(1)$ loop model on a cylinder of even perimeter, when the loops wrapping around the cylinder are not allowed to contract, or equivalently, when the corresponding link pattern is drawn on a punctured disc. This case was called periodic with 'distinct connectivities' in [6]. It must be distinguished from the more usual loop model with 'identified connectivities' which was the subject of [8], which will be also discussed in what follows since we need it to analyse the other cases. The related XXZ spin chain of even size $L=2 n$ has twisted boundary conditions [2], as we recall below in section 4 . See also [16] for more data on these different boundary conditions.

Let $A_{n}=\prod_{j=1}^{n}(3 j-2)!/(n+j-1)$ ! denote the number of alternating sign matrices of size $n$ (see [17] and further references therein) and $\mathcal{N}_{n}$ stand for

$$
\begin{equation*}
\mathcal{N}_{n}=\frac{3^{n / 2}}{2^{n}} \frac{2 \times 5 \times \cdots \times(3 n-1)}{1 \times 3 \cdots \times(2 n-1)} A_{n}=3^{n / 2} \frac{\prod_{j=1}^{n}(3 j-1)!}{\prod_{j=1}^{n}(n+j)!} . \tag{1.1}
\end{equation*}
$$

Let us also introduce the number $A_{\mathrm{HT}}(L)$ of 'half-turn symmetric' alternating sign matrices [17]-[20]

$$
A_{\mathrm{HT}}(L)= \begin{cases}\prod_{j=0}^{n-1} \frac{3 j+2}{3 j+1}\left(\frac{(3 j+1)!}{(n+j)!}\right)^{2} & \text { if } L=2 n \text { is even }  \tag{1.2}\\ \prod_{j=1}^{n} \frac{4}{3}\left(\frac{(3 j)!j!}{(2 j)!^{2}}\right)^{2} & \text { if } L=2 n+1 \text { is odd. }\end{cases}
$$

Okada [21] showed that this number $A_{\mathrm{HT}}(L)$ may be expressed in terms of the dimension of certain representations of $\mathrm{GL}(L)$

$$
\begin{align*}
& A_{\mathrm{HT}}(2 n+1)=3^{-n^{2}}\left(\operatorname{dim}_{Y_{n}^{\prime}(2 n+1)}^{\mathrm{GL}}\right)^{2}  \tag{1.3}\\
& A_{\mathrm{HT}}(2 n)=3^{-n(n-1)} \operatorname{dim}_{Y_{n}}^{\mathrm{GL}(2 n)} \operatorname{dim}_{Y_{n}^{\prime}}^{\mathrm{GL}(2 n)}
\end{align*}
$$

The Young diagram $Y_{n}^{\prime}$ which defines that representation is made of one row of length $n$, two rows of length $n-1$, two of length $n-2, \ldots$, two of length 1 , while the Young diagram $Y_{n}$ results from the ablation of the first row of $Y_{n}^{\prime}$.

In [2], it was conjectured that for the $O(1)$ loop model on a cylinder of perimeter $L=2 n+1$, all components of the Perron-Frobenius eigenvector $\Psi$ are integers if the
smallest one is normalized to be 1 , that the sum of components of $\Psi$ over all loop configurations $\pi$ is
$\sum_{\pi} \Psi_{\pi}=A_{\mathrm{HT}}(2 n+1)=\mathcal{N}_{n}^{2}=1,3,25,588,39204 \ldots \quad L=2 n+1=1,3, \ldots$
and that the largest component is

$$
\begin{equation*}
\Psi_{\max }=A_{n}^{2}=1,1,4,49,1764, \ldots \quad L=2 n+1=1,3, \ldots \tag{1.5}
\end{equation*}
$$

As for the case of even $L=2 n$ with distinct connectivities, it was conjectured in [6] that the sum of components of the Perron-Frobenius eigenvector $\Phi^{*}$ over all loop configurations $\pi$ is

$$
\begin{equation*}
\sum_{\pi} \Phi_{\pi}^{*}=A_{\mathrm{HT}}(2 n)=2,10,140,5544 \ldots \quad L=2 n=2,4,6,8, \ldots \tag{1.6}
\end{equation*}
$$

The largest component was also conjectured to be

$$
\begin{equation*}
\Phi_{\max }^{*}=A_{\mathrm{HT}}(2 n-1)=1,3,25,588, \ldots \quad L=2 n=2,4,6,8, \ldots \tag{1.7}
\end{equation*}
$$

In both the odd and the even cases, it was further conjectured in [5] and in [6], respectively, that the individual components $\Psi_{\pi}$, resp. $\Phi_{\pi}^{*}$, count the number of HTSFPLs, that is, half-turn symmetric FPL configurations drawn on an $L \times L$ grid, whose connectivity pattern is described by the arch pattern $\pi$. For example, the unique HTSFPL pertaining to $L=3$ and the link pattern $\pi=\{\frown \cdot\}$ is as shown below:

while further examples for $L=4,5$ are depicted in figure 1 .
For completeness and future use in this paper we also recall here the sum rule written in [2] for the loop model of even size $L=2 n$ with 'identified connectivities'. The PerronFrobenius eigenvector denoted $\Phi$ satisfies

$$
\begin{equation*}
\sum \Phi_{\pi}=A_{n} \tag{1.8}
\end{equation*}
$$

together with $\Phi_{\max }=A_{n-1}$. While the latter remains a conjecture, (1.8) has now been established [8], as we recall below.

Turning now to the XXZ spin chain of odd size, the parallel (and prior!) conjectures read [1]

$$
\begin{equation*}
\sum_{\alpha} \tilde{\Psi}_{\alpha}=3^{n / 2} \mathcal{N}_{n}=1,3,15,126,1782 \ldots \quad L=2 n+1=1,3 \ldots \tag{1.9}
\end{equation*}
$$

for the sum over all spin configurations of total spin $1 / 2$, the largest component is

$$
\begin{equation*}
\tilde{\Psi}_{\max }=A_{n}=1,1,2,7,42 \ldots \quad L=2 n+1=1,3, \ldots \tag{1.10}
\end{equation*}
$$



Figure 1. A sample of half-turn symmetric FPL configurations of size 4 and 5. The lower half of these configurations has a link pattern described by, respectively, $\{\frown \frown\},\{\backslash \frown \subset\}$ and $\{\frown \frown \cdot\}$.
and the square norm of $\Psi$ is

$$
\begin{equation*}
\sum_{\alpha} \tilde{\Psi}_{\alpha}^{2}=\mathcal{N}_{n}^{2}=A_{\mathrm{HT}}(2 n+1) \tag{1.11}
\end{equation*}
$$

when the normalization is such that $\tilde{\Psi}_{+\ldots+\ldots \ldots}=1$. For an even size $L=2 n$, (with twisted boundary conditions, see section 4 ), the ground state wavefunction is complex, and the parallel conjectures of [2] and [22] read

$$
\begin{align*}
& \sum_{\alpha} \tilde{\Phi}_{\alpha}=3^{n / 2} A_{n}  \tag{1.12a}\\
& \sum_{\alpha} \tilde{\Phi}_{\alpha}^{2}=A_{n}^{2}  \tag{1.12b}\\
& \sum_{\alpha}^{\alpha}\left|\tilde{\Phi}_{\alpha}\right|^{2}=A_{\mathrm{HT}}(2 n)  \tag{1.12c}\\
& \tilde{\Phi}_{\max }=\tilde{\Phi}_{(+-)^{n}}=\mathcal{N}_{n-1} \mathrm{e}^{\mathrm{i} \pi / 6} \tag{1.12d}
\end{align*}
$$

with the normalization that $\tilde{\Phi}_{\text {min }}=\tilde{\Phi}_{++\ldots+\ldots \ldots-}=\mathrm{e}^{\mathrm{i} \pi n / 6}$. The first of these conjectures is now established, since it is a corollary of the sum rule (1.8), as will be reexplained below.
Examples. $L=5(n=2)$ : $\Psi=(1,1,1,1,1,4,4,4,4,4)$ in the loop basis, $\tilde{\Psi}=$ $(1,1,1,1,1,2,2,2,2,2)$ after change to the spin basis. Note that for $L$ odd, there are always $L$ repeats due to the breaking of rotational symmetry by the defect.

Table 1. Notations for the different models and boundary conditions considered in this paper.

| Size $L$ | Model | Space | Dimension | Perron-Frobenius <br> eigenvector | Transfer <br> matrix |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $2 n+1$ | Loop | $L P_{2 n+1}$ | $\binom{2 n+1}{n}$ | $\Psi_{\pi}$ | $T$ |
| $2 n$ | Loop ('I.C.') | $L P_{2 n}$ | $C_{n}=\frac{(2 n)!}{n!(n+1)!}$ | $\Phi_{\pi}$ | $\Phi_{\pi}^{*}$ |
| $2 n$ | Loop ('D.C.') | $L P_{2 n}^{*}$ | $\binom{2 n}{n}$ | $\tilde{\Psi}_{\alpha}$ | $T$ |
| $2 n+1$ | Spin | $\left\{s_{z}=\frac{1}{2}\right\}$ | $C_{n+1}$ | $\tilde{\Phi}_{\alpha}$ | $T^{*}$ |
| $2 n$ | Spin | $\left\{s_{z}=0\right\}$ | $C_{n}$ | $\tilde{T}$ |  |

Indicating these repeats with a superscript, at $L=7, \Psi=\left(1^{(7)}, 6^{(7)}, 14^{(14)}, 49^{(7)}\right)$ for link patterns, $\tilde{\Psi}=\left(1^{(7)}, 3^{(14)}, 4^{(7)}, 7^{(7)}\right)$ for spins. In even sizes, for $L=4$, $\Phi^{*}=\left(3^{(2)}, 1^{(4)}\right)$ and $\tilde{\Phi}=\left(\sqrt{3} \mathrm{e}^{ \pm i \pi / 6}, \mathrm{e}^{ \pm i \pi / 3}, 1^{(2)}\right)$; for $L=6, \Phi^{*}=\left(25^{(2)}, 9^{(6)}, 5^{(6)}, 1^{(6)}\right)$, $\tilde{\Phi}=\left(5 \mathrm{e}^{ \pm \mathrm{i} \pi / 6},\left(\sqrt{7} \mathrm{e}^{ \pm \mathrm{i} \alpha}\right)^{(3)},\left(\sqrt{7} \mathrm{e}^{ \pm \mathrm{i} \beta}\right)^{(2)}, \sqrt{7} \mathrm{e}^{ \pm \mathrm{i} \gamma},\left(\mathrm{e}^{ \pm \mathrm{i} \pi / 6}\right)^{(2)}, \mathrm{e}^{ \pm \mathrm{i} \pi / 2}\right)$ with $\alpha=\arctan 1 / 3 \sqrt{3}$, $\beta=(\pi / 3)-\alpha, \gamma=(\pi / 3)+\alpha$.

In view of previous experience, it is very natural to extend the discussion to the inhomogeneous version of the loop model (whose precise definition is recalled below in section 2). Indeed the main result of [8] (Theorem 5) is that for the even-size (IC) loop model,

$$
\begin{equation*}
\sum_{\pi \in L P_{2 n}} \Phi_{\pi}\left(z_{1}, \ldots, z_{2 n}\right)=3^{-n(n-1) / 2} s_{Y_{n}}\left(z_{1}, \ldots, z_{2 n}\right) \tag{1.13}
\end{equation*}
$$

so that in the homogeneous limit $z_{i} \rightarrow 1,(1.8)$ follows. The main result of this paper is that

$$
\begin{array}{ll}
\sum_{\pi \in L P_{2 n+1}} \Psi_{\pi}\left(z_{1}, \ldots, z_{2 n+1}\right)=3^{-n^{2}} s_{Y_{n+1}}(z) s_{Y_{n}^{\prime}}(z) & \text { if } L=2 n+1 \\
\sum_{\pi \in L P_{2 n}^{*}} \Phi_{\pi}^{*}\left(z_{1}, \ldots, z_{2 n}\right)=3^{-n(n-1)} s_{Y_{n}^{\prime}}(z) s_{Y_{n}}(z) & \text { if } L=2 n, \tag{1.14}
\end{array}
$$

as we will show in sections 2.5 and 3. In these expressions, $s_{Y}(z)$ stands for the Schur function labelled by a Young diagram $Y$, a symmetric function of $z_{1}, \ldots, z_{L}$. The diagrams $Y_{n}^{\prime}$ and $Y_{n}$ have been defined above, after (1.3), thus $Y_{n+1}$ has two rows of length $n$, two of length $n-1$, etc, two of length 1 . From this follow the proofs of the sum rules (1.4) and (1.6). The sum rule (1.9) for the odd spin chains will be also derived in section 4.5, and (1.11) and (1.12b) in section 4.6. On the conceptual level, it may be interesting to notice that our derivation makes use not only of the previously mentioned techniques, introduction of spectral parameters, recursion equations and $q \mathrm{KZ}$ equation, etc, but also of a new idea borrowed from knot theory, namely the use of skein relations.

The educated reader will recognize in (1.14) formulae equivalent to those written by Razumov and Stroganov [20] for the partition function of the square-ice model with boundary conditions appropriate to the enumeration of half-turn symmetric ASMs. We return to this in our conclusion.

For the sake of the reader, we summarize in tables 1 and 2 some notations and data for the various situations that we consider in this paper.

Sum rules for the ground states of the $O(1)$ loop model on a cylinder and the XXZ spin chain
Table 2. Some other notations with the section or equation number where they first appear.

| $B$ | Equation (4.10) | $B_{0}, B_{\infty}$ | Equation (3.2) |
| :--- | :--- | :--- | :--- |
| $L_{2 n+1}$ | Section 2.1 | $L_{2 n+2}$ | Section 2.3 |
| $P_{0}, P_{\infty}$ | Section 2.3 | $Q, P_{ \pm}$ | Section 4.3 |
| $S$ | Section 4.2 | $S_{Y_{n}}, s_{Y_{n}^{\prime}}, Y_{n}, Y_{n}^{\prime}$ | Equations (1.3), (1.14) |

## 2. The inhomogeneous $O(1)$ loop model in odd size

### 2.1. Link patterns, transfer matrix and $R$-matrices

The model is defined on a semi-infinite cylinder of a square lattice of odd perimeter $2 n+1$ whose faces are covered by either of the two following face configurations:


In a given configuration, labelling cyclically $i=1,2, \ldots, 2 n+1$ the centres of the boundary edges (and with the convention that $i+2 n+1 \equiv i$ ), we note that these points are connected among themselves or to the point at infinity via non-intersecting curves. Actually, drawing on each face the two configurations (2.1) with probabilities $p$ and $1-p, p \in(0,1)$, leads only to situations where one boundary point is connected to infinity, while the remaining $2 n$ are pairwise connected. Forgetting about the underlying lattice, the connection pattern, also called the link pattern, is simply a chord diagram, namely a configuration of $2 n+1$ points on a circle, pairwise connected by non-intersecting arcs within the interior disc, with one unmatched point that we connect to the centre of the disc. The set of such link patterns is denoted by $L P_{2 n+1}$ and has cardinality $d_{n}=\binom{2 n+1}{n}$. Later on, we regard this set as a linear space, spanned by the previous link patterns, and thus of dimension $d_{n}$.

Here we study the inhomogeneous version of this model, in which plaquettes above the boundary point $i$ are picked among the two faces (2.1) with respective probabilities $p_{i}$ and $1-p_{i}, p_{i} \in(0,1)$. We wish to compute the probabilities $\mathrm{P}_{\pi}$ that random configurations connect the boundary point according to a given $\pi \in L P_{2 n+1}$. Clearly, these probabilities are invariant under the addition of a row of $2 n+1$ plaquettes to the original semi-infinite cylinder, which amounts to an equation of the form

$$
\begin{equation*}
\sum_{\pi^{\prime} \in L P_{2 n+1}} T\left(p_{1}, \ldots, p_{2 n+1}\right)_{\pi, \pi^{\prime}} \mathrm{P}_{\pi^{\prime}}=\mathrm{P}_{\pi} \tag{2.2}
\end{equation*}
$$

where the transfer matrix $T$ acts on link patterns in an obvious way, by concatenation. The vector $\mathrm{P}=\left\{\mathrm{P}_{\pi}\right\}_{\pi \in L P_{2 n+1}}$ is determined as the properly normalized Perron-Frobenius eigenvector of $T$, with eigenvalue 1 .

This system is known to be integrable, as $T$ may be constructed by multiplying and then tracing plaquette operators that satisfy the Yang-Baxter equation. Indeed, parameterizing the probabilities as

$$
\begin{equation*}
p_{i}=\frac{q z_{i}-q^{-1} t}{q t-q^{-1} z_{i}} \tag{2.3}
\end{equation*}
$$

where $q$ is a complex cubic root of unity $q=-\mathrm{e}^{\mathrm{i} \pi / 3}$, we may identify the $i$ th plaquette operator (or $R$-matrix) as

$$
\begin{equation*}
R_{i}\left(z_{i}, t\right)=\frac{q z_{i}-q^{-1} t}{q t-q^{-1} z_{i}} \triangle+\frac{z_{i}-t}{q t-q^{-1} z_{i}} \square \tag{2.4}
\end{equation*}
$$

where $z_{i}$ and $t$ are spectral parameters attached respectively to the vertical line above point $i$, and to the horizontal one running around the cylinder. With the parameterization (2.3), the probability vector P is clearly a rational fraction of the $z \mathrm{~s}$. In what follows, we will use a different normalization $\Psi \propto \mathrm{P}$ in which the entries of $\Psi$ are coprime polynomials of the $z \mathrm{~s}$.

In $[8,12]$, it was shown that the transfer matrix relation (2.2) may be equivalently replaced by a system of relations of the form

$$
\begin{equation*}
\check{R}_{i, i+1}\left(z_{i}, z_{i+1}\right) \Psi=\tau_{i} \Psi, \quad i=1,2, \ldots, 2 n+1 \tag{2.5}
\end{equation*}
$$

where $\tau_{i}$ is simply the interchange of spectral parameters $z_{i} \leftrightarrow z_{i+1}$ and $\check{R}$ is the tilted plaquette operator

$$
\begin{equation*}
\check{R}_{i, i+1}(z, w)=\frac{q z-q^{-1} w}{q w-q^{-1} z} I+\frac{z-w}{q w-q^{-1} z} e_{i} \tag{2.6}
\end{equation*}
$$

in terms of the local identity $I$ and Temperley-Lieb operators $e_{i}, i=1,2, \ldots, 2 n+1$ defined pictorially as

$$
I=\widehat{\nabla} \quad \text { and } \quad e_{i}=\widehat{\nabla}
$$

and acting at points $i$ and $i+1$ on link patterns by concatenation as indicated schematically below:


Note that if $k=i$ and $j=i+1, e_{i}$ acts by creating a loop which we allow ourselves to erase, and therefore this action leaves the pattern invariant.

The $e_{i}$ obey the usual conditions $e_{i} e_{i \pm 1} e_{i}=e_{i}$ and $e_{i}^{2}=e_{i}$ that define the (cyclic) Temperley-Lieb algebra $T L(1)$.

The last equation $i=2 n+1$ of (2.5) may be replaced by a cyclic invariance condition

$$
\begin{equation*}
\Psi\left(z_{2}, z_{3}, \ldots, z_{2 n+1}, z_{1}\right)=\Psi\left(z_{1}, z_{2}, \ldots, z_{2 n+1}\right) \tag{2.8}
\end{equation*}
$$

Finally, we will also be using $\check{R}$ matrices with the second spectral parameter sent either to zero or to infinity. Up to a multiplicative redefinition by $-q^{ \pm 3 / 2}$, the two corresponding
plaquette operators ${ }^{4}$

$$
\begin{align*}
& t_{i}^{-1} \equiv-q^{3 / 2} \check{R}_{i, i+1}(z, 0)=q^{-1 / 2} I+q^{1 / 2} e_{i}  \tag{2.9}\\
& t_{i}=-q^{-3 / 2} \check{R}_{i, i+1}(z, \infty)=q^{1 / 2} I+q^{-1 / 2} e_{i}
\end{align*}
$$

may be interpreted respectively as under- and over-crossings of links, with pictorial representation

$$
t_{i}=\left\langle\quad \text { and } \quad t_{i}^{-1}=\langle\right.
$$

and the two pieces of equation (2.9) are nothing else than the celebrated skein relations for knots, which read pictorially

$$
\begin{align*}
& \Delta>=q^{-1 / 2} \Delta q^{1 / 2} \measuredangle  \tag{2.10}\\
& \Delta>=q^{1 / 2} \Delta D+q^{-1 / 2} \square
\end{align*}
$$

### 2.2. The quantum Knikhnik-Zamolodchikov equation

More generally, it was noted in [9] that the condition that $q$ be a cubic root of unity could be relaxed and that equations (2.5) and (2.8) are a particular case of quantum Knizhnik-Zamolodchikov ( $q \mathrm{KZ}$ ) equation $[10,15]$. The latter amounts to the system

$$
\begin{align*}
& \check{R}_{i, i+1}\left(z_{i}, z_{i+1}\right) \Psi=\tau_{i} \Psi \quad i=1,2, \ldots, 2 n  \tag{2.11}\\
& \sigma \Psi\left(z_{2}, \ldots, z_{2 n+1}, s z_{1}\right)=c \Psi\left(z_{1}, \ldots, z_{2 n+1}\right)
\end{align*}
$$

where $s, c$ are scalars and the operator $\sigma$ acts on link patterns via the cyclic rotation of labels $i \rightarrow i+1$. Here $\check{R}$ denotes the same plaquette operator as before, except that its definition involves the generators $e_{i}$ of the Temperley-Lieb algebra $T L(\tau)$, (i.e. satisfying the relation $e_{i}^{2}=\tau e_{i}$ ), with $\tau=-q-q^{-1}$. Note that we may replace the second line of (2.11) by an equation of the form

$$
\begin{equation*}
\check{R}_{2 n+1,1}\left(z_{2 n+1}, s z_{1}\right) \Psi\left(z_{1}, \ldots, z_{2 n+1}\right)=\Psi\left(s^{-1} z_{2 n+1}, \ldots, s z_{1}\right) \tag{2.12}
\end{equation*}
$$

where the omitted $z s$ are left unchanged.
We now look for polynomial solutions $\Psi\left(z_{1}, \ldots, z_{2 n+1}\right)$ of this system, with minimal degree. From the first line of equation (2.11) we learn that whenever points $i$ and $i+1$ are not connected in $\pi, \Psi_{\pi}$ factors out a term $q z_{i}-q^{-1} z_{i+1}$, and more generally if no two points between $i$ and $j>i$ are connected, then $\Psi_{\pi}$ factors out $q z_{i}-q^{-1} z_{j}$. For the link pattern $\pi_{0}$ connecting points $i \leftrightarrow 2 n+2-i, i=1,2, \ldots, n$ while $n+1$ is unmatched, this fixes the base component $\Psi_{\pi_{0}}$ to be

$$
\begin{equation*}
\Psi_{\pi_{0}}=\prod_{1 \leq i<j \leq 2 n+1} \frac{q z_{i}-q^{-1} z_{j}}{q-q^{-1}} \tag{2.13}
\end{equation*}
$$

[^0]up to multiplication by a symmetric polynomial, which we pick to be 1 for the sake of minimality. This in turn determines both scalars in (2.11) to be $s=q^{3}$ and $c=q^{3 n}$. The other entries of $\Psi$ are then determined from equation (2.11), which reads in components
\[

$$
\begin{align*}
& -\left(q^{-1} z_{i+1}-q z_{i}\right) \partial_{i} \Psi_{\pi}=\sum_{\substack{\pi^{\prime} \neq \pi \\
e_{i} \pi^{\prime}=\pi}} \Psi_{\pi^{\prime}},  \tag{2.14}\\
& \Psi_{\pi}\left(z_{2}, \ldots, z_{2 n+1}, s z_{1}\right)=c \Psi_{\sigma^{-1} \pi}\left(z_{1}, \ldots, z_{2 n+1}\right),
\end{align*}
$$
\]

where

$$
\begin{align*}
& \partial_{i} f\left(z_{1}, \ldots, z_{i}, z_{i+1}, \ldots, z_{2 n+1}\right) \\
& \quad:=\frac{f\left(z_{1}, \ldots, z_{i+1}, z_{i}, \ldots, z_{2 n+1}\right)-f\left(z_{1}, \ldots, z_{i}, z_{i+1}, \ldots, z_{2 n+1}\right)}{z_{i+1}-z_{i}} . \tag{2.15}
\end{align*}
$$

Also true for any $q$ is the recursion relation:

$$
\begin{align*}
\Psi_{\phi_{i}(\pi)}\left(z_{1}, \ldots,\right. & \left.z_{L}\right)\left.\right|_{z_{i+1}=q^{2} z_{i}}=\frac{1}{\left(q-q^{-1}\right)^{2 L-3}} z_{i} \prod_{j=1}^{i-1}\left(q z_{j}-q^{-1} z_{i}\right)\left(z_{j}-q z_{i}\right) \\
& \times \prod_{j=i+2}^{L}\left(q z_{i+1}-q^{-1} z_{j}\right)\left(z_{i+1}-q z_{j}\right) \Psi_{\pi}\left(z_{1} \ldots z_{i-1}, z_{i+1} \ldots z_{L}\right) \tag{2.16}
\end{align*}
$$

where $\phi_{i}$ inserts two consecutive points connected via a 'little arch' between points $i-1$ and $i$ in any link pattern $\pi$ of size $2 n-1$; all other components vanish when $z_{i+1}=q^{2} z_{i}$.

### 2.3. Projections to link patterns of larger size

In this section, we introduce two projection operators $P_{0}$ and $P_{\infty}$ from $L P_{2 n+1}$ into the vector space generated by the link patterns with $2 n+2$ points without unmatched points, denoted $L P_{2 n+2}$. These two projections would allow us to relate the solutions of the $q \mathrm{KZ}$ equation for punctured discs with $2 n+1$ points on their perimeter to those for unpunctured discs of perimeter $2 n+2$. We decide from now on to restrict ourselves to the particular value $q=-\mathrm{e}^{\mathrm{i} \pi / 3}$ (RS point), as we will be mostly reasoning on transfer matrices for the loop model on a punctured disc, for which the periodicity is requested.

The projections $P_{0}$ and $P_{\infty}$ are defined as follows. For any link pattern of size $2 n+1$, let us first add an extra point labelled $2 n+2$ on the boundary of the disc, between points 1 and $2 n+1$. Next this point is linked to the centre of the disc, which is itself connected to the (unique) unmatched point of the link pattern. But by doing so, the added link may have to cross existing links. As illustrated in figure 2, we define $P_{0}$ by imposing that all these crossings be undercrossings, and $P_{\infty}$ by imposing that they all be overcrossings. We then simply have to use the skein relations (2.10) at each crossing to obtain a linear combination of non-crossing link patterns of size $2 n+2$.

It is also clear that the mappings $P_{0}$ and $P_{\infty}$ are surjective: every $\pi^{\prime}$ in $L P_{2 n+2}$ in which the point $2 n+2$ is matched with some $i$ is the image by $P_{0}$ or $P_{\infty}$ of a $\pi \in L P_{2 n+1}$ in which $i$ is unmatched and the other points form the same pairs. This justifies calling $P_{0}$ and $P_{\infty}$ 'projections' from $L P_{2 n+1}$ to $L P_{2 n+2}$.


Figure 2. Example of projections $P_{0}$ and $P_{\infty}$ of a link pattern of size 13. We first add a point 14 between 1 and 13 on the boundary of the link pattern, and then connect the latter to the centre, itself linked to the unmatched point (labelled 5 here), via a link passing under ( $P_{0}$ ) or over $\left(P_{\infty}\right)$ those separating the two points. Finally, we must use the skein relations (2.9) to express the latter as linear combinations of non-crossing link patterns of size 14.

For illustration, there are 3 link patterns of size 3 , and their projections in the vector space of non-crossing link patterns of size 4 read:





$P_{\infty}$



Figure 3. The projection $P_{0}$ intertwines the transfer matrices for size $2 n+1$ and $2 n+2$, the latter with $z_{2 n+2}=0$. We have represented the transfer matrices pictorially as adding an extra row to the disc. Each intersection between lines stands for an $\check{R}$ matrix operator. That $z_{2 n+2}=0$ imposes here that the added intersection be an undercrossing, in agreement with the action of $P_{0}$.

### 2.4. Projections and transfer matrices

Let $T\left(t \mid z_{1}, \ldots, z_{2 n+1}\right)$ denote the transfer matrix for the loop model on a punctured disc of perimeter $2 n+1$, obtained from (2.2) via the parameterization (2.3), and likewise $T\left(t \mid z_{1}, \ldots, z_{2 n+2}\right)$ for a non-punctured disc of perimeter $2 n+2$. Then the projection operators $P_{0}$ and $P_{\infty}$ of previous section act as intertwiners between odd and even transfer matrices, where in the latter the additional spectral parameter is taken to zero or infinity respectively, namely:

$$
\begin{align*}
& P_{0} T\left(t \mid z_{1}, \ldots, z_{2 n+1}\right)=T\left(t \mid z_{1}, \ldots, z_{2 n+1}, 0\right) P_{0}  \tag{2.19}\\
& P_{\infty} T\left(t \mid z_{1}, \ldots, z_{2 n+1}\right)=T\left(t \mid z_{1}, \ldots, z_{2 n+1}, \infty\right) P_{\infty}
\end{align*}
$$

This equality is easily proved graphically; see figure 3 .
Now apply equations (2.19) to $\Psi$ : we find that $P_{\bullet} \Psi$ is an eigenvector of $T\left(t \mid z_{1}, \ldots, z_{2 n+1} \bullet \bullet\right)$ with eigenvalue $1, \bullet=0, \infty$. This eigenvalue being generically nondegenerate, we conclude that $P \bullet \Psi$ is proportional to the vector $\Phi\left(z_{1}, \ldots, z_{2 n+1}, \bullet\right)$, defined as the ground state eigenvector of the system on a disc of even perimeter $2 n+2$, as introduced in [8]:

$$
\begin{align*}
& P_{0} \Psi=A_{0}\left(z_{1}, \ldots, z_{2 n+1}\right) \Phi\left(z_{1}, \ldots, z_{2 n+1}, 0\right) \\
& P_{\infty} \Psi=A_{\infty}\left(z_{1}, \ldots, z_{2 n+1}\right) \Phi\left(z_{1}, \ldots, z_{2 n+1}, \infty\right) \tag{2.20}
\end{align*}
$$

Here the notation $\Phi\left(z_{1}, \ldots, z_{2 n+1}, \infty\right)$ is slightly abusive, and stands for the limit $\lim _{z_{2 n+2} \rightarrow \infty} \Phi\left(z_{1}, \ldots, z_{2 n+1}, z_{2 n+2}\right) / z_{2 n+2}^{n}$. The polynomial character of the quantities $A_{0}$ and $A_{\infty}$ may be deduced from the recursion relations (2.16) by sending spectral parameters to 0 or $\infty$; these polynomials will be determined in the next section.

### 2.5. Ground state sum rule

The fundamental remark of Razumov and Stroganov is that at the special point $q=-\mathrm{e}^{\mathrm{i} \pi / 3}$ (RS point) the problem simplifies drastically, and leads to all sorts of combinatorial wonders. This point may be characterized as the unique one (up to conjugation) where there exists a non-trivial common left eigenvector to all operators $e_{i}$, or equivalently to all plaquette operators $\check{R}$. In view of the action (2.7) of $e_{i}$ on link patterns, it is clear that, under $e_{i}$, each link pattern gives rise to exactly one link pattern. However, if $i$ is already connected to $i+1$ in the link pattern, the action of $e_{i}$ creates a loop, which may be removed at the expense of a multiplicative factor $\tau$. The existence of a common left eigenvector to all $e_{i}$ imposes therefore that $\tau=1$, for which we pick the root $q=-\mathrm{e}^{\mathrm{i} \pi / 3}, q^{1 / 2}=\mathrm{e}^{-\mathrm{i} \pi / 3}$. This eigenvector is then simply the sum of components over all link patterns, namely the covector $v$ with all entries equal to 1 in the link pattern basis.

As a consequence of this definition, we have $v \check{R}=v$ for all arguments, including when the second spectral parameter is zero or infinity; hence $v t_{i}=v t_{i}^{-1}=v$ as well (as $q^{3 / 2}=-1$ at the RS point). Moreover, let $v^{\prime}$ be the vector with all entries 1 in the link pattern basis of $L P_{2 n+2}$. This implies that

$$
\begin{equation*}
v^{\prime} P_{0}=v \quad \text { and } \quad v^{\prime} P_{\infty}=v ; \tag{2.21}
\end{equation*}
$$

hence we finally get

$$
\begin{align*}
& v \cdot \Psi=A_{0}\left(z_{1}, \ldots, z_{2 n+1}\right) v^{\prime} \cdot \Phi\left(z_{1}, \ldots, z_{2 n+1}, 0\right) \\
& v \cdot \Psi=A_{\infty}\left(z_{1}, \ldots, z_{2 n+1}\right) v^{\prime} \cdot \Phi\left(z_{1}, \ldots, z_{2 n+1}, \infty\right) \tag{2.22}
\end{align*}
$$

In [8], the quantity $v^{\prime} \cdot \Phi\left(z_{1}, \ldots, z_{2 n+2}\right)$ has been identified with the so-called IzerginKorepin determinant, which, at the RS point, is equal to the GL $(2 n+2)$ Schur function $s_{Y_{n+1}}\left(z_{1}, \ldots, z_{2 n+2}\right)$ for the Young diagram $Y_{n+1}$ with two rows of $n$ boxes, two rows of $n-1$ boxes, etc, two rows of one box, as defined in section 1 . Sending $z_{2 n+2}$ to 0 amounts to restricting the Schur function to $\mathrm{GL}(2 n+1)$, while keeping the same tableau $Y_{n+1}$. Sending $z_{2 n+2}$ to infinity amounts to restricting the Schur function to GL $(2 n+1)$, while truncating the Young tableau into $Y_{n}^{\prime}$ equal to $Y_{n+1}$ with the first row removed. Hence $v \cdot \Psi=A_{0} s_{Y_{n+1}}\left(z_{1}, \ldots, z_{2 n+1}\right)=A_{\infty} s_{Y_{n}^{\prime}}\left(z_{1}, \ldots, z_{2 n+1}\right)$. Finally, we make the hypothesis that the Schur functions $s_{Y_{n+1}}\left(z_{1}, \ldots, z_{2 n+1}\right)$ and $s_{Y_{n}^{\prime}}\left(z_{1}, \ldots, z_{2 n+1}\right)$ have no common factor. From this hypothesis, which is supported by the examination of the first cases but would require a complete proof, we deduce that $v \cdot \Psi$ is a polynomial multiple of $s_{Y_{n+1}} s_{Y_{n}^{\prime}}$. Now the degree of $s_{Y_{n+1}}$ is the total number of boxes in $Y_{n+1}$; hence $s_{Y_{n+1}} s_{Y_{n}^{\prime}}$ has degree $n(n+1)+n^{2}=n(2 n+1)$. This coincides with the degree of $\Psi$ as given by that of $\Psi_{\pi_{0}}$ in equation (2.13); hence we conclude that $v \cdot \Psi$ is proportional to $s_{Y_{n+1}} s_{Y_{n}^{\prime}}$ by a scalar factor, fixed to be $3^{-n^{2}}$ by the recursion relation (2.16).

We therefore identify the polynomials $A_{0} \propto s_{Y_{n}^{\prime}}$ and $A_{\infty} \propto s_{Y_{n+1}}$, and get the multiparameter sum rule
$\sum_{\pi \in L P_{2 n+1}} \Psi_{\pi}\left(z_{1}, \ldots, z_{2 n+1}\right)=3^{-n^{2}} s_{Y_{n+1}}\left(z_{1}, \ldots, z_{2 n+1}\right) s_{Y_{n}^{\prime}}\left(z_{1}, \ldots, z_{2 n+1}\right)$.
In the homogeneous limit where all the $z_{i}$ tend to 1 , the Schur functions $s_{Y_{n+1}}$ and $s_{Y_{n}^{\prime}}$ reduce to the dimensions of the corresponding representations of $\mathrm{GL}(2 n+1)$, and we note that $\operatorname{dim}_{Y_{n+1}}=\operatorname{dim}_{Y_{n}^{\prime}}$ because the two Young diagrams form together a rectangle of size $(2 n+1) \times n$, (and thus $\left.s_{Y_{n}^{\prime}}(1 / z)=\prod_{j} z_{j}^{3 n} s_{Y_{n+1}}(z)\right)$. This leads to (1.4) via (1.3).

## 3. The inhomogeneous $O(1)$ loop model in even size

As discussed in section 1, the inhomogeneous $O(1)$ loop model may also be considered on a punctured disc of even perimeter $2 n$ : link patterns now have an isolated puncture in the centre of the disc, and we denote by $L P_{2 n}^{*}$ their set, of cardinality $\binom{2 n}{n}$. Applying the same line of thought as in the previous section, we now define mappings $P_{0}$ and $P_{\infty}$ from $L P_{2 n}^{*}$ to $L P_{2 n+1}$, by simply creating an extra point between the points 1 and $2 n$, and connecting it to the puncture via a link passing below or above all the crossed ones. Comparing the dimensions it is clear that $P_{0}$ and $P_{\infty}$ cannot be surjective, though they are presumably injective. Using the same type of arguments as in section 2, we find that there is an intertwining relation

$$
\begin{align*}
& P_{0} T^{*}\left(t \mid z_{1}, \ldots, z_{2 n}\right)=T\left(t \mid z_{1}, \ldots, z_{2 n}, 0\right) P_{0} \\
& P_{\infty} T^{*}\left(t \mid z_{1}, \ldots, z_{2 n}\right)=T\left(t \mid z_{1}, \ldots, z_{2 n}, \infty\right) P_{\infty} \tag{3.1}
\end{align*}
$$

where $T^{*}$ is the inhomogeneous transfer matrix acting on the span of $L P_{2 n}^{*}$. This now leads to

$$
\begin{align*}
& B_{0}\left(z_{1}, \ldots, z_{2 n}\right) P_{0} \Phi^{*}\left(z_{1}, \ldots, z_{2 n}\right)=\Psi\left(z_{1}, \ldots, z_{2 n}, 0\right) \\
& B_{\infty}\left(z_{1}, \ldots, z_{2 n}\right) P_{\infty} \Phi^{*}\left(z_{1}, \ldots, z_{2 n}\right)=\Psi\left(z_{1}, \ldots, z_{2 n}, \infty\right) \tag{3.2}
\end{align*}
$$

where once again $\Psi\left(z_{1}, \ldots, z_{2 n}, \infty\right)$ denotes the highest degree $(=2 n)$ terms in $z_{2 n+1}$ of $\Psi$ and $B_{0}$ and $B_{\infty}$ will be determined soon.

As a consequence, we find that the even case sum rule $B_{0} v \cdot \Psi\left(z_{1}, \ldots, z_{2 n}\right)$ must be equal to the quantities (2.23) taken for $z_{2 n+1}=0$, while $B_{\infty} v \cdot \Psi(z)$ equals that for $z_{2 n+1}=\infty$ (with the above-mentioned division by an appropriate power of $z_{2 n+1}$ ). These restrictions on the last spectral parameter have the effect of truncating the Schur functions. More precisely, when $z_{2 n+1} \rightarrow 0$, the factor $s_{Y_{n+1}}\left(z_{1}, \ldots, z_{2 n+1}\right)$ tends to $s_{Y_{n+1}}\left(z_{1}, \ldots, z_{2 n}\right)=\left(z_{1} \ldots z_{2 n}\right) s_{Y_{n}}\left(z_{1}, \ldots, z_{2 n}\right)$, where $Y_{n}$ now has two rows of length $n-1$, two rows of length $n-2$, etc, two rows of length 1 . Meanwhile, the factor $s_{Y_{n}^{\prime}}\left(z_{1}, \ldots, z_{2 n+1}\right)$ tends to $s_{Y_{n}^{\prime}}\left(z_{1}, \ldots, z_{2 n}\right)$. When $z_{2 n+1} \rightarrow \infty$, the leading coefficient in $s_{Y_{n+1}}\left(z_{1}, \ldots, z_{2 n+1}\right)$ truncates to $s_{Y_{n}^{\prime}}\left(z_{1}, \ldots, z_{2 n+1}\right)$ as explained before, while that in $s_{Y_{n}^{\prime}}\left(z_{1}, \ldots, z_{2 n+1}\right)$ truncates to $s_{Y_{n}}\left(z_{1}, \ldots, z_{2 n}\right)$. This is consistent for $B_{0} \propto z_{1} \ldots z_{2 n}$ and $B_{\infty}$ a constant. In both limits, we reach the same sum rule:

$$
\begin{equation*}
\sum_{\pi \in L P_{2 n}^{*}} \Phi_{\pi}^{*}\left(z_{1}, \ldots, z_{2 n}\right)=3^{-n(n-1)} s_{Y_{n}}\left(z_{1}, \ldots, z_{2 n}\right) s_{Y_{n}^{\prime}}\left(z_{1}, \ldots, z_{2 n}\right) . \tag{3.3}
\end{equation*}
$$

The proportionality factor is fixed by the even analogue of the recursion relation (2.16), together with the normalization of the component

$$
\begin{equation*}
\Phi_{\pi_{0}}^{*}\left(z_{1}, \ldots, z_{2 n}\right)=\prod_{1 \leq i<j \leq 2 n} \frac{q z_{i}-q^{-1} z_{j}}{q-q^{-1}} \tag{3.4}
\end{equation*}
$$

for the fully nested link pattern $\pi_{0}$ connecting points $i$ and $2 n+1-i, i=1,2, \ldots, n$, while the puncture sits in the face delimited by the little arch connecting $n$ to $n+1$ and the disc boundary.

## 4. The $X X Z$ spin chain and six-vertex model at $\Delta=-1 / 2$

We present here results concerning another closely related model: the XXZ spin chain at the value $\Delta=\left(q+q^{-1}\right) / 2=-1 / 2$ of the anisotropy. As above we shall need an inhomogeneous version of the model, which is the inhomogeneous six-vertex model. New results are found mostly in the case of an odd-size chain, since the even case is already covered (if a little implicitly) by [8], but we shall need to introduce the model for arbitrary size anyway.

### 4.1. Definition of the $X X Z$ spin chain

The XXZ spin chain is given by the Hamiltonian

$$
\begin{equation*}
\tilde{H}=-\frac{1}{2} \sum_{i=1}^{L}\left(\sigma_{i}^{x} \sigma_{i+1}^{x}+\sigma_{i}^{y} \sigma_{i+1}^{y}+\Delta \sigma_{i}^{z} \sigma_{i+1}^{z}\right) \tag{4.1}
\end{equation*}
$$

acting on $\left(\mathbb{C}^{2}\right)^{\otimes L}$, each $\mathbb{C}^{2}$ being a single spin space: $|+\rangle \equiv\binom{1}{0},|-\rangle \equiv\binom{0}{1}$. We first take $\Delta$ and $q$ generic, $\Delta=\left(q+q^{-1}\right) / 2$. We choose here periodic boundary conditions for $L$ odd: $\sigma_{L+1} \equiv \sigma_{1}$, and twisted periodic boundary conditions for $L$ even: $\sigma_{L+1}^{z} \equiv \sigma_{1}^{z}$, $\sigma_{L+1}^{ \pm} \equiv q^{ \pm 2} \sigma_{1}^{ \pm}$, where $\sigma^{ \pm}=\sigma^{x} \pm \mathrm{i} \sigma^{y}$. We can also write these as $\sigma_{L+1} \equiv \Omega \sigma_{1} \Omega^{-1}$ with $\Omega=\left(\begin{array}{cc}-q & 0 \\ 0 & -q^{-1}\end{array}\right)$ for $L$ even and $\Omega=1$ for $L$ odd.

We also define the transfer matrix of the inhomogeneous six-vertex model (see also appendix B of [8] where the case $L$ even is treated). For spectral parameters $z_{1}, \ldots, z_{L}$, it is given by

$$
\begin{equation*}
\tilde{T} \equiv \tilde{T}\left(t \mid z_{1}, \ldots, z_{L}\right):=\operatorname{tr}_{0}\left(R_{L, 0}\left(z_{L}, t\right) \cdots R_{1,0}\left(z_{1}, t\right) \Omega\right) \tag{4.2}
\end{equation*}
$$

where the $R$ matrices act on the tensor product of the physical space $\left(\mathbb{C}^{2}\right)^{\otimes L}$, each factor being labelled by $i=1, \ldots, L$, and of another $\mathbb{C}^{2}$, labelled by 0 . The expression of $R$ acting in $\mathbb{C}^{2} \otimes \mathbb{C}^{2}$ reads (in the so-called homogeneous gradation)

$$
R(z, t)=\frac{1}{q t-q^{-1} z}\left(\begin{array}{cccc}
q z-q^{-1} t & 0 & 0 & 0  \tag{4.3}\\
0 & z-t & \left(q-q^{-1}\right) t & 0 \\
0 & \left(q-q^{-1}\right) z & z-t & 0 \\
0 & 0 & 0 & q z-q^{-1} t
\end{array}\right)
$$

The twist $\Omega$ acts on the auxiliary space, which is consistent with the chosen boundary conditions for the spin chain since $\Omega^{-1} R_{1,0} \Omega=R\left(\sigma_{1}, \Omega^{-1} \sigma_{0} \Omega\right)=R\left(\Omega \sigma_{1} \Omega^{-1}, \sigma_{0}\right)=R_{L+1,0}$, where the second equality follows from $U(1)$ invariance of the $R$-matrix.

When all $z$ s are equal, $\tilde{T}$ commutes with the Hamiltonian $\tilde{H}$. In particular their ground states are identical.

### 4.2. Equivalence with the $O(1)$ loop model

For any $q$ there is a mapping from the $O(1)$ loop model to the XXZ/six-vertex model. See also [16] for a similar construction. Call $\omega^{1 / 2}$ a square root of $\omega:=-q$. To a link pattern $\pi$ in $L P_{L}$ associate the tensor product over the set of arches of $\pi$, of the vectors $\omega^{1 / 2}|+\rangle_{j} \otimes|-\rangle_{k}+\omega^{-1 / 2}|-\rangle_{j} \otimes|+\rangle_{k}$, where the indices $j, k$ are the endpoints of the arch (and indicate the labels of the two spaces $\mathbb{C}^{2}$ in which these vectors live) ordered in the
following way: $j<k$ for $L$ even, while for $L$ odd, $j<k<\ell$ or $\ell<j<k$ or $k<\ell<j$, $\ell$ being the label of the unmatched point. In the case $L$ odd we also choose the spin of the unmatched point to be $|+\rangle_{\ell}$. The result is a vector in $\left(\mathbb{C}^{2}\right)^{\otimes L}$ which satisfies that its total $z \operatorname{spin} s^{z}=0$ or $s^{z}=1 / 2$ depending on parity. For generic $q$ this mapping, which we call $S$, is injective. In the case $L$ odd its image is in fact the whole $s^{z}=1 / 2$ subspace, of dimension $\binom{L}{(L-1) / 2}$. In the case $L$ even it is a subspace $\left(U_{q}(\mathfrak{s l}(2))\right.$ singlets) of dimension $L!/(L / 2)!(L / 2+1)!$. Furthermore, it is easy to show that the transfer matrices of the $O(1)$ loop model and of the six-vertex model are intertwined by $S$ :

$$
\begin{equation*}
\tilde{T}\left(t \mid z_{1}, \ldots, z_{L}\right) S=S T\left(t \mid z_{1}, \ldots, z_{L}\right) \tag{4.4}
\end{equation*}
$$

This is essentially a consequence of the fact that the $R$-matrix (4.3) involves just another representation of the Temperley-Lieb algebra; explicitly, if $\check{R}=R \mathcal{P}$, where $\mathcal{P}$ switches factors of tensor products, then the formula (2.6) holds with the Temperley-Lieb generator

$$
e=\left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & \omega & 1 & 0 \\
0 & 1 & \omega^{-1} & 0 \\
0 & 0 & 0 & 0
\end{array}\right)
$$

The twist, necessary for $L$ even to take care of the arbitrariness in the ordering $j<k$ of the indices, disappears for $L$ odd thanks to the unmatched point. Henceforth, the loop model is equivalent to a sector of the six-vertex model.

At $q=-\mathrm{e}^{\mathrm{i} \pi / 3}$, however, the mapping $S$ is no longer injective for $L$ odd. In fact, one can show that the kernel of $S$ is exactly the same as the kernel of the projector $P_{0}$ of section 2.3 , so that the dimension of the image is nothing else than $(2 n+2)!/(n+1)!(n+2)!$, the dimension of the space of link patterns of size $2 n+2$. So at this special point, the odd six-vertex model is equivalent to the even loop model of size one more.

### 4.3. Mapping of odd size to even size

In analogy with the loop model, we now define a mapping of the six-vertex model from $L=2 n+1$ to $L=2 n+2$. It is very simple: to a basis element $\alpha$, that is a sequence of $2 n+1$ spins, we associate the new vector $\alpha$ - obtained by concatenating $\alpha$ and an extra minus spin. Call $Q$ this mapping. We also need the projections $P_{ \pm}$within the model of size $2 n+2$ which project onto the subspaces where the last spin is $\pm$, orthogonally to the subspace where it is $\mp$, so that $P_{+}+P_{-}=1, P_{+} P_{-}=P_{-} P_{+}=0$.

We now have the two following properties of the $P_{ \pm}$and $Q$ operators:
(i) The subspace $\operatorname{Im}\left(P_{+}\right)$is stable under $\tilde{T}\left(z_{1}, \ldots, z_{2 n+1}, \infty\right)$ :

$$
\begin{equation*}
P_{+} \tilde{T}\left(t \mid z_{1}, \ldots, z_{2 n+1}, 0\right)=\tilde{T}\left(t \mid z_{1}, \ldots, z_{2 n+1}, 0\right) P_{+} \tag{4.5}
\end{equation*}
$$

Indeed, if the last spectral parameter is zero, the corresponding matrix $R_{2 n+2,0}$ becomes

$$
R(0, t)=-q^{-1}\left(\begin{array}{cccc}
q^{-1} & 0 & 0 & 0  \tag{4.6}\\
0 & 1 & q^{-1}-q & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & q^{-1}
\end{array}\right)
$$

which means that if the spin $2 n+2$ is + it will stay so.
(ii) At $q^{3}=1, Q$ intertwines the odd and even spin-chain transfer matrices, up to the projector $P_{-}$:

$$
\begin{equation*}
Q \tilde{T}\left(t \mid z_{1}, \ldots, z_{2 n+1}\right)=P_{-} \tilde{T}\left(t \mid z_{1}, \ldots, z_{2 n+1}, 0\right) Q \tag{4.7}
\end{equation*}
$$

This time we assume that the last spin is -. According to equation (4.6), it might become + after action of $\tilde{T}$; so we project again with $P_{-}$. Now the action of $R_{2 n+2,0}(0, t)$ becomes simply the twist of the auxiliary space with its lower right submatrix, that is the diagonal matrix $\left(\begin{array}{cc}-q^{-1} & 0 \\ 0 & -q^{-2}\end{array}\right)$ : at $q^{3}=1$ this compensates exactly the twist $\Omega=\left(\begin{array}{cc}-q & 0 \\ 0 & -q^{-1}\end{array}\right)$ and we find that the right-hand side of equation (4.7) is simply the transfer matrix of odd size with periodic boundary conditions and an additional spin at site $2 n+2$ which is - .

### 4.4. Polynomial eigenvector

From now on we always set $q=-\mathrm{e}^{\mathrm{i} \pi / 3}, \omega^{1 / 2}=\mathrm{e}^{\mathrm{i} \pi / 6}$. It is easy to check that $\tilde{T}$, just like $T$, possesses the eigenvalue 1 ; we denote by $\tilde{\Psi}$ (resp. $\tilde{\Phi}$ ) the corresponding polynomial eigenvector for $L$ odd (resp. even), normalized so that its entries are coprime. This leaves an arbitrary constant in the normalization which will be fixed below.

The various maps defined above allow us to write several relations between the various eigenvectors. First and foremost, using properties (i) and (ii) of section 4.3 we find that

$$
\begin{equation*}
Q \tilde{\Psi}\left(z_{1}, \ldots, z_{2 n+1}\right)=C P_{-} \tilde{\Phi}\left(z_{1}, \ldots, z_{2 n+1}, 0\right) \tag{4.8}
\end{equation*}
$$

or in components, $\tilde{\Psi}_{\alpha}\left(z_{1}, \ldots, z_{2 n+1}\right)=C \tilde{\Phi}_{\alpha-}\left(z_{1}, \ldots, z_{2 n+1}, 0\right)$, with $C$ a normalization constant. Let us prove this. In this paragraph parameters are omitted with the assumption that $z_{2 n+2}=0$. On the one hand, one can apply $\tilde{\Psi}$ to equation (4.7). We find $Q \tilde{\Psi}=P_{-} \tilde{T} Q \tilde{\Psi}$. On the other hand, decompose $\tilde{\Phi}=P_{+} \tilde{\Phi}+P_{-} \tilde{\Phi}$ and apply $P_{-} \tilde{T}$ : using equation (4.5) we find $P_{-} \tilde{T} P_{-} \tilde{\Phi}=P_{-} \tilde{\Phi}$. Comparing, we find that $Q \tilde{\Psi}$ and $P_{-} \tilde{\Phi}$ must be proportional. The proportionality factor must be a constant because components on both sides of the equation are coprime polynomials. Its numerical value will be determined below.

We also have the connection with loop models. For $L$ even this was already discussed in appendix B of $[8]$ and we find simply

$$
\begin{equation*}
S \Phi=\tilde{\Phi} \tag{4.9}
\end{equation*}
$$

(up to a constant, which we fix to be 1). However, for $L$ odd, we only have

$$
\begin{equation*}
S \Psi=B\left(z_{1}, \ldots, z_{2 n+1}\right) \tilde{\Psi} \tag{4.10}
\end{equation*}
$$

where $B$ is a polynomial to be determined in next section. $B$ appears because $S$ is not injective (at $q^{3}=1$ ), so that the components of $S \Psi$ can have a non-trivial GCD. (Such a situation does not arise for $L$ even since $S$ is then injective.)

### 4.5. Sum rule

We now wish to compute the sum of entries of $\tilde{\Psi}$. In order to do so we can rely on the mapping to the even-size system:

$$
\begin{equation*}
\sum_{\alpha} \tilde{\Psi}_{\alpha}\left(z_{1}, \ldots, z_{2 n+1}\right)=C \sum_{\alpha} \tilde{\Phi}_{\alpha-}\left(z_{1}, \ldots, z_{2 n+1}, 0\right) \tag{4.11}
\end{equation*}
$$

and then use the mapping $S$ to the loop model. The normalization constant $C$ will be adjusted at the end of the computation. Via $S$, each link pattern of size $2 n+2$ contributes $\omega^{1 / 2}\left(\omega^{1 / 2}+\omega^{-1 / 2}\right)^{n}$ to the right-hand side of equation (4.11), so $\sum_{\alpha} \tilde{\Phi}_{\alpha-}=$ $\omega^{1 / 2}\left(\omega^{1 / 2}+\omega^{-1 / 2}\right)^{n} \sum_{\pi} \Phi_{\pi}=\omega^{1 / 2} 3^{n / 2} 3^{-n(n+1) / 2} s_{Y_{n+1}}\left(z_{1}, \ldots, z_{2 n+2}\right)$ according to (1.13). So we have

$$
\begin{equation*}
\sum_{\alpha} \tilde{\Psi}_{\alpha}\left(z_{1}, \ldots, z_{2 n+1}\right)=3^{-n^{2} / 2} \omega^{1 / 2} C s_{Y_{n+1}}\left(z_{1}, \ldots, z_{2 n+1}\right) \tag{4.12}
\end{equation*}
$$

To adjust the constant $C$ in (4.11), we finally impose that the 'base component' of $\tilde{\Psi}$, i.e. its smallest component in the homogeneous limit, be 1 . By (4.8), it is given by that of $\tilde{\Phi}$, itself proportional to that of $\Phi$,

$$
\begin{equation*}
\tilde{\Psi}_{+\cdots+-\cdots-}=\omega^{(n+1) / 2} C \prod_{\substack{1 \leq i<j \leq n+1 \\ n+2 \leq i<j \leq 2 n+1}} \frac{q z_{i}-q^{-1} z_{j}}{q-q^{-1}} \prod_{i=n+2}^{2 n+1} \frac{q z_{i}}{q-q^{-1}}, \tag{4.13}
\end{equation*}
$$

so at $z_{i}=1, C^{-1}=(-\mathrm{i} \sqrt{3})^{-n} \omega^{(n+1) / 2} q^{n}=3^{-n / 2} \omega^{1 / 2}$ and the homogeneous sum rule is

$$
\begin{equation*}
\sum_{\alpha} \tilde{\Psi}_{\alpha}=3^{-n(n-1) / 2} s_{Y_{n+1}}\left(1^{2 n+1}\right)=3^{n / 2} \mathcal{N}_{n} \tag{4.14}
\end{equation*}
$$

As a side-product, by using $S$ directly on an odd-size system, we can also compute the proportionality polynomial factor $B$ :

$$
\begin{equation*}
B \sum_{\alpha} \tilde{\Psi}_{\alpha}=\left(\omega^{1 / 2}+\omega^{-1 / 2}\right)^{n} \sum_{\pi} \Psi_{\pi}=3^{n / 2} 3^{-n^{2}} s_{Y_{n+1}} s_{Y_{n}^{\prime}} \tag{4.15}
\end{equation*}
$$

so that $B=3^{-n(n-1) / 2} \omega^{-1 / 2} C^{-1} s_{Y_{n}^{\prime}}=3^{-n^{2} / 2} s_{Y_{n}^{\prime}}(z)$.

### 4.6. Bilinear form and sum of squares

These sum rules possess obvious corollaries dealing with the sum of squares of components. Indeed, for generic $q$ there is a bilinear form which to a pair of link patterns $\pi$ and $\pi^{\prime}$ of $L P_{2 n+2}$ associates $\left\langle\pi \mid \pi^{\prime}\right\rangle=(-q-1 / q)^{\#}$ where $\#$ is the number of loops obtained by pasting together the two diagrams along their common boundary. Via $S$, this bilinear form becomes simply diagonal in the spin basis:

$$
\begin{equation*}
\left\langle\pi \mid \pi^{\prime}\right\rangle=\sum_{\alpha}(S \pi)_{\alpha}\left(S \pi^{\prime}\right)_{\alpha} . \tag{4.16}
\end{equation*}
$$

We now specialize to $q=-\mathrm{e}^{\mathrm{i} \pi / 3}$. Since $-q-1 / q=1$, we have the identity $v \cdot \Phi=\left\langle\Phi \mid \pi_{0}\right\rangle$, where $\pi_{0}$ is here any fixed link pattern. More generally, the bilinear
form is degenerate of rank 1 so that we can write (in size $L=2 n+2$ )

$$
\begin{equation*}
\sum_{\alpha} \tilde{\Phi}_{\alpha}^{2}=\langle\Phi \mid \Phi\rangle=\left\langle\Phi \mid \pi_{0}\right\rangle\left\langle\pi_{0} \mid \Phi\right\rangle=(v \cdot \Phi)^{2}=3^{-n(n+1)} s_{Y_{n+1}}^{2} \tag{4.17}
\end{equation*}
$$

and in particular in the homogeneous limit, $\sum_{\alpha} \tilde{\Phi}_{\alpha}^{2}=A_{n+1}^{2}$, which is (1.12b).
As noticed in [22], the even-size twisted XXZ Hamiltonian is invariant under simultaneous complex conjugation and spin reversal, and the ground state $\tilde{\Phi}$ is chosen to be invariant under this operation. Thus $\tilde{\Phi}_{\alpha+}=\bar{\Phi}_{\bar{\alpha}-}$, and (4.17) may be rewritten as

$$
\langle\Phi \mid \Phi\rangle=(v \cdot \Phi)^{2}=3^{-n(n+1)} s_{Y_{n+1}}^{2}=\sum_{\alpha} \tilde{\Phi}_{\alpha}^{2}=\sum_{\alpha^{\prime}}\left(\tilde{\Phi}_{\alpha^{\prime}-}^{2}+\tilde{\Phi}_{\alpha^{\prime}+}^{2}\right)=2 \Re\left(\sum_{\alpha^{\prime}} \tilde{\Phi}_{\alpha^{\prime}-}^{2}\right) .
$$

Using equation (4.8), we can then derive the sum rule for the squares of the $\tilde{\Psi}$, in odd size. By taking the real part of $C^{-2} \sum_{\alpha^{\prime}} \tilde{\Psi}_{\alpha^{\prime}}^{2}(z)=\sum_{\alpha^{\prime}} \tilde{\Phi}_{\alpha^{\prime}-}^{2}(z, 0)$, (the $\tilde{\Psi}$ are real), and using the value of $C$ determined above, we get

$$
\begin{equation*}
\sum_{\alpha^{\prime}} \tilde{\Psi}_{\alpha^{\prime}}^{2}=\frac{\Re \sum_{\alpha^{\prime}} \tilde{\Phi}_{\alpha^{\prime}-}^{2}}{\Re C^{-2}}=3^{-n^{2}} s_{Y_{n+1}}^{2}\left(z_{1}, \ldots, z_{2 n+1}, 0\right) \tag{4.18}
\end{equation*}
$$

In the homogeneous limit, we get

$$
\begin{equation*}
\sum_{\alpha^{\prime}} \tilde{\Psi}_{\alpha^{\prime}}^{2}=\mathcal{N}_{n}^{2}=A_{\mathrm{HT}}(2 n+1) \tag{4.19}
\end{equation*}
$$

as announced in (1.11).
In contrast, the alleged sum rule (1.12c) seems to be of a different nature, since its deformation by spectral parameters does not involve symmetric functions of $z_{1}, \ldots, z_{2 n}$.

## 5. Conclusion

In this paper, we have proved multiparameter sum rules for the components of the ground state vector of the inhomogeneous $O(1)$ loop model on a punctured disc of odd or even perimeter, or of its XXZ chain counterpart. Our strategy has relied on the construction of projection operators onto components of the ground state vector of the same model, but on a non-punctured disc of perimeter one more, in which the extra inhomogeneity (spectral parameter) is taken to either zero or infinity. The new ingredient here is the use of knot-theoretic crossing operators to relate the non-crossing link patterns of both models. Admittedly, our proof in the odd case still relies on a technical assumption that we have been unable to prove in general, namely the fact that the two Schur functions $s_{Y_{n+1}}$ and $s_{Y_{n}^{\prime}}$ of section 2.5 have no common factor. It would be highly desirable to fill this gap.

In the derivations of sections 2 and 3 we made an assumption of minimal degree, namely that the base component of $\Psi$ or $\Phi$ is given by (2.13), (3.4). The justification of this assumption could be done as in [8], through the use of the Bethe Ansatz, or by appealing to the representation theory of the $q \mathrm{KZ}$ equation.

As already mentioned in section 1, our formulae for the sums of components of the $O(1)$ loop model lead to the same expressions as those derived by Razumov and Stroganov for the partition functions of the square-ice model with adequate boundary conditions.

The latter are designed so as to make a bijection between the states of the lattice model and half turn symmetric ASMs of size $L \times L$. In [20], the partition function of the six-vertex model on a $L \times\left\lfloor\frac{1}{2}(L+1)\right\rfloor$ grid with such 'half-turn symmetric' boundary conditions [19] was computed as a function of spectral parameters $x_{i}$ and $y_{j}, i, j=1, \ldots,\lfloor(L+1) / 2\rfloor$. It is a symmetric function of the $x \mathrm{~s}$ and the $y \mathrm{~s}$, thanks to the Yang-Baxter equation, and it was shown to be a product of two factors. For the special value of the crossing parameter corresponding to our $q=-\mathrm{e}^{\mathrm{i} \pi / 3}$ (their $a=-q$ ), and in the odd-size case ( $L=2 n+1$ ), upon specialization of the last spectral parameter $y_{n+1}=x_{n+1}$, it becomes a completely symmetric function of all its spectral parameters and it reads (in our notations)

$$
\begin{align*}
& Z_{\mathrm{HT}}\left(x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{n}\right) \\
& \quad=(-3)^{n} \prod_{i=1}^{n}\left(x_{i} y_{i}\right)^{1-2 n} s_{Y_{n}}\left(x_{1}^{2}, \ldots, y_{n}^{2}\right) s_{Y_{n}^{\prime}}\left(x_{1}^{2}, \ldots, y_{n}^{2}\right) \quad L=2 n \\
& Z_{\mathrm{HT}}\left(x_{1}, \ldots, x_{n+1}, y_{1}, \ldots, y_{n}, y_{n+1}=x_{n+1}\right)  \tag{5.1}\\
& =3^{n} \prod_{i=1}^{n+1} x_{i}^{-2 n} \prod_{i=1}^{n} y_{i}^{-2 n} s_{Y_{n+1}}\left(x_{1}^{2}, \ldots, x_{n+1}^{2}, y_{1}^{2}, \ldots, y_{n}^{2}\right) s_{Y_{n}^{\prime}}\left(x_{1}^{2}, \cdots, y_{n}^{2}\right) \\
& \quad L=2 n+1
\end{align*}
$$

that is, up to a trivial factor $\left((-3)^{n^{2}} \Pi\left(x_{j} y_{j}\right)^{1-2 n}\right.$, resp. $\left.3^{n(n+1)} \Pi x_{j}^{-2 n} \Pi y_{j}^{-2 n}\right)$, the same expression as $\sum_{\pi} \Phi^{*}$, resp. $\sum_{\pi} \Psi$. Recall that the partition function, upon some specialization of its spectral parameters, yields the 'refined $x$-enumerations' of HTSASMs. The agreement between $Z_{\mathrm{HT}}(z)$ and the sum of the components is a further indication of the detailed connection between the two sets of problems: the determination of the Perron-Frobenius eigenvectors of the $O(1)$ loop models and the counting of HTSASMs. More precisely it gives support to the strong RS conjecture that $\Psi_{\pi}$ or $\Phi_{\pi}^{*}$ count the number of HTSFPL configurations of link pattern $\pi$ (see section 1).

At this stage, our results do not, however, provide any direct way to verify the conjectures about the largest components of the Perron-Frobenius eigenvectors (see section 1), nor to even confirm the empirical observation that in all cases (but the twisted $\tilde{\Phi})$ all components are positive integers once the smallest one is normalized to 1 .

Another interesting question is what survives of these constructions for generic values of $q$ (not at the RS point). One expects in particular from previous experience [10,23] that the point $q=-1$ should have some geometrical interpretation, involving (multi-)degrees of some matrix schemes, and we indeed checked for odd sizes that all components of $\Psi$ become (possibly vanishing) non-negative integers in the homogeneous limit $z_{i} \rightarrow 1$. It is not too difficult to convince oneself that the correct definition of the projection operators $P_{0}$ and $P_{\infty}$ of section 2 requires switching to different conventions for the crossing operators $T_{i} \equiv \check{R}_{i, i+1}(z, 0)$. Via these projections, we may relate the vector $\Psi$ at generic $q$ to solutions of a modified $q \mathrm{KZ}$ equation for size one more, but with the extra spectral parameter sent to zero or infinity. In the case of size $2 n+1$, we have found that $P_{0}$ creates a solution of this modified $q \mathrm{KZ}$ equation in size $2 n+2$ with $z_{2 n+2} \rightarrow 0$, whose base component $\Theta_{\pi_{0}}$, corresponding to the fully nested link pattern $\pi_{0}$ relating $i$ to $2 n+3-i, i=1,2, \ldots, n+1$, simply reads

$$
\begin{equation*}
\Theta_{\pi_{0}}\left(z_{1}, \ldots, z_{2 n+1}\right)=\Phi_{\pi_{0}}\left(z_{1}, \ldots, z_{2 n+1}, 0\right) G\left(z_{1}, \ldots, z_{2 n+1}, \infty\right) \tag{5.2}
\end{equation*}
$$

where $\Phi_{\pi_{0}}$ stands as before for the (completely factorized) base component of the loop model on an unpunctured disc of size $2 n+2$, while $G$ stands for the so-called Gaudin determinant

$$
\begin{align*}
G\left(z_{1}, \ldots, z_{2 n+2}\right) & =\frac{\prod_{1 \leq i, j \leq n+1}\left(q z_{i}-q^{-1} z_{n+j+1}\right)\left(q^{1 / 2} z_{i}-q^{-1 / 2} z_{n+j+1}\right)}{\prod_{1 \leq i<j \leq n+1}\left(z_{i}-z_{j}\right)\left(z_{n+i+1}-z_{n+j+1}\right)} \\
& \times \operatorname{det}_{1 \leq i, j \leq n+1}\left(\frac{1}{q z_{i}-q^{-1} z_{n+j+1}} \frac{1}{q^{1 / 2} z_{i}-q^{-1 / 2} z_{n+j+1}}\right) \tag{5.3}
\end{align*}
$$

and the symbol $\infty$ in place of $z_{2 n+2}$ stands as usual for the suitably normalized large $z_{2 n+2} \rightarrow \infty$ limit. Note that $P_{\infty}$ does just the opposite, namely interchanges 0 and $\infty$ in equation (5.2). This is remarkably reminiscent of the higher degree solution to the $q \mathrm{KZ}$ equation found by Pasquier [9] on the even-size unpunctured discs, for which the cyclicity condition analogous to that of the second line of equation (2.14) has the same shift $s=q^{3}$ (as opposed to $s=q^{6}$ for the minimal degree solution).

## Acknowledgments

We thank M Kasatani and V Pasquier, who are currently working on related issues, for interesting discussions. We acknowledge the support of the Geocomp project (ACI Masse de Données), the European networks 'ENIGMA', grant MRT-CT-2004-5652 and 'ENRAGE', grant MRTN-CT-2004-5616, and the ANR program 'GIMP', ANR-05-BLAN-0029-01.

## References

[1] Razumov A V and Stroganov Yu G, Spin chains and combinatorics, 2001 J. Phys. A: Math. Gen. 345335 [cond-mat/0012141]
[2] Batchelor M T, de Gier J and Nienhuis B, The quantum symmetric XXZ chain at $\Delta=-1 / 2$, alternating sign matrices and plane partitions, 2001 J. Phys. A: Math. Gen. 34 L265 [cond-mat/0101385]
[3] Razumov A V and Stroganov Yu G, 2004 Teor. Mat. Fiz. 138395
Razumov A V and Stroganov Yu G, Combinatorial nature of ground state vector of $O(1)$ loop model, 2004 Theor. Math. Phys. 138333 [math.CO/0104216] (translation)
[4] Pearce P A, Rittenberg V and de Gier J, Critical $Q=1$ Potts model and Temperley-Lieb stochastic processes, 2001 Preprint cond-mat/0108051
[5] Razumov A V and Stroganov Yu G, 2005 Teor. Mat. Fiz. 142284
Razumov A V and Stroganov Yu G, $O(1)$ loop model with different boundary conditions and symmetry classes of alternating-sign matrices, 2005 Theor. Math. Phys. 142237 [cond-mat/0108103] (translation)
[6] Pearce P A, Rittenberg V, de Gier J and Nienhuis B, Temperley-Lieb stochastic processes, 2002 J. Phys. A: Math. Gen. 35 L661 [math-ph/0209017]
[7] Di Francesco P, A refined Razumov-Stroganov conjecture, 2004 J. Stat. Mech. P08009 [cond-mat/0407477]
Di Francesco P, A refined Razumov-Stroganov conjecture: II, 2004 J. Stat. Mech. P11004 [cond-mat/0409576]
[8] Di Francesco P and Zinn-Justin P, Around the Razumov-Stroganov conjecture: proof of a multi-parameter sum rule, 2005 Eur. J. Combin. 12 R6 [math-ph/0410061]
[9] Pasquier V, Quantum incompressibility and Razumov Stroganov type conjectures, 2005 Preprint cond-mat/0506075
Pasquier V, Incompressible representations of the Birman-Wenzl-Murakami algebra, 2005 Preprint math.QA/0507364
[10] Di Francesco P and Zinn-Justin P, The quantum Knizhnik-Zamolodchikov equation, generalized Razumov-Stroganov sum rules and extended Joseph polynomials, 2005 J. Phys. A: Math. Gen. 38 L815 [math-ph/0508059]
[11] de Gier J and Nienhuis B, Brauer loops and the commuting variety, 2004 Preprint math.AG/0410392
[12] Di Francesco P and Zinn-Justin P, Inhomogeneous model of crossing loops and multidegrees of some algebraic varieties, 2006 Commun. Math. Phys. 262459 [math-ph/0412031]
[13] Knutson A and Zinn-Justin P, A scheme related to the Brauer loop model, 2005 Preprint math.AG/0503224
[14] Di Francesco P, Inhomogeneous loop models with open boundaries, 2005 J. Phys. A: Math. Gen. 386091 [math-ph/0504032]
[15] Di Francesco P, Boundary qKZ equation and generalized Razumov-Stroganov sum rules for open IRF models, 2005 J. Stat. Mech. P11003 [math-ph/0509011]
[16] Mitra S, Nienhuis B, de Gier J and Batchelor M T, Exact expressions for correlations in the ground state of the dense O(1) loop model, 2004 J. Stat. Mech. P09010 [cond-mat/0401245]
[17] Bressoud D, 1999 Proofs and Confirmations: The Story of the Alternating Sign Matrix Conjecture (Cambridge: Cambridge University Press)
[18] Robbins D P, Symmetry classes of alternating sign matrices, 2000 Preprint math.CO/0008045
[19] Kuperberg G, Symmetry classes of alternating-sign matrices under one roof, 2002 Ann. Math. 156 835 [math.CO/0008184]
[20] Razumov A V and Stroganov Yu G, Enumerations of half-turn symmetric alternating-sign matrices of odd order, 2005 Preprint math-ph/0504022
[21] Okada S, Enumeration of symmetry classes of alternating sign matrices and characters of classical groups, 2004 Preprint math.CO/0408234
[22] Razumov A V and Stroganov Yu G, Spin chains and combinatorics: twisted boundary conditions, 2001 J. Phys. A: Math. Gen. 345335 [cond-mat/0102247]
[23] Di Francesco P and Zinn-Justin P, From orbital varieties to alternating sign matrices, 2006 Proc. FPSAC'06 at press [math-ph/0512047]


[^0]:    ${ }^{4}$ Note that the following choice of normalization for $t_{i}$ is ad hoc to ensure rotational invariance of the crossing move, and coincides with the standard crossing operators of knot theory. However, the prefactors $-q^{ \pm 3 / 2}$ will be irrelevant at the particular point $q^{1 / 2}=\mathrm{e}^{-\mathrm{i} \pi / 3}$ to which we will restrict later on.

