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Graph rings and integrable perturbations of $N = 2$ superconformal theories

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We show that the connection between certain integrable perturbations of $N = 2$ superconformal theories and graphs found by Lerche and Warner extends to a broader class. These perturbations are such that the generators of the perturbed chiral ring may be diagonalized in an orthonormal basis. This allows one to define a dual ring, whose generators are labelled by the ground states of the theory and are encoded in a graph or set of graphs, that reproduce the pattern of the ground states and interpolating solitons. All known perturbations of the ADE potentials and some others are shown to satisfy this criterion. This suggests a test of integrability.

1. Introduction

$N = 2$ superconformal theories have been under active investigation during recent years. Of particular interest is to understand their perturbations by relevant operators, that preserve the $N = 2$ supersymmetry and make them massive field theories, and among them, those that are integrable ([1], for a review and a list of references see ref. [2]). Integrability seems to imply nice features on the pattern of solitons that interpolate between the ground states of the theory.

Two years ago, Lerche and Warner have studied the perturbations of the $N = 2$ theories described by an ADE Landau–Ginzburg potential [3,4] and perturbed by the least relevant operator [5]. They made the intriguing observation that the pattern of minima of the potential in field space then reproduces the shape of the corresponding Dynkin diagram, and that the values of the polynomial representatives of the chiral ring at the minima of the potential are proportional to the various eigenvectors of the Cartan matrix. In the A-case, the chiral algebra satisfied by these polynomials is nothing else than the Verlinde fusion algebra of the corresponding SU(2) theory, whereas in the other D- or E-cases, its interpretation remained more elusive.

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In this paper, we want to point out that this feature extends to a much broader class of integrable perturbations, that the association between the corresponding ring (or algebra) and a graph is something that has already been encountered some time ago in the slightly different context of relations between lattice integrable models attached to graphs and conformal field theories [6] and that conversely this connection might suggest a simple criterion of integrability. In sect. 2, we review shortly some facts on $N=2$ superconformal field theories, their perturbations and the structure of their "chiral ring". We focus on cases where the multiplication matrices C_i in a natural basis of the (perturbed) chiral ring are *normal*, i.e. commute with their adjoint $[C_i, C_i^\dagger]=0$. This condition is equivalent to the property of C_i to be diagonalizable in an orthonormal basis *. More generally, we assume that the matrices may be made normal after a change of basis that respects the natural gradation of the problem (U(1) charge). This we call the *normalizability* property. We then recall in sect. 3 some definitions of what we call a *graph* ring, of its dual ring, and how in many cases a subring of it enables one to identify a structure of blocks with nice modular properties. Applied to the ADE perturbations mentionned above, it tells us how to group the polynomials into blocks in one-to-one correspondence with the blocks of the corresponding SU(2) modular invariant. A survey of the ADE cases (sect. 4) and of cases related to SU(N) (sect. 5) shows that the property of normalizability is quite restrictive and allows only sparse solutions. Strangely, all the known integrable cases of ADE potentials are normalizable, and the dual ring has integral coefficients that may thus be encoded in a graph. In so far as the normal matrices are some natural generalization of the fusion matrices, this connection between the property of normalizability and integrability represents an extension of a conjecture of Gepner [7] that there is an underlying conformal field theory behind each integrable deformation of $N=2$ theory. Moreover, this condition of normalizability enables us to identify some plausible candidates to integrability. One of them has been checked to possess indeed conserved quantities [8]. We speculate in sect. 6 on the issues raised in this paper. Four appendices gather some technical material on the form of the D-potential and free energy (Appendix A), on the exceptional cases (Appendices B and D) and on SU(3) at level 2 (Appendix C).

2. Perturbations of $N=2$ Landau–Ginzburg superconformal theories

Consider a $N=2$ superconformal theory that admits a description by a Landau–Ginzburg superpotential. The latter is a quasi-homogeneous polynomial in

* A simple proof of this fact is provided by the decomposition of C into its hermitian and antihermitian parts, $C=A+iB$, $A=A^\dagger$, $B=B^\dagger$; the commutation of C and $C^\dagger$ translates into the commutation of A and B that may thus be diagonalized simultaneously in an orthonormal basis.

the superfields, $W_0(x, y, \dots)$. The case of the "minimal" $N = 2$ theories (with $c < 3$) is particularly striking since this superpotential is given by one of the well-known ADE singularities [9], thus matching the classification of modular invariants [3,4]. This description is substantiated by the comparison of the chiral ring of the $N = 2$ theory with the local ring of the singularity. The former describes the non-singular pointwise product of fields that satisfy the constraint $h = \frac{1}{2}q$, with h their conformal weight and q their U(1) charge, and the latter describes the multiplication in the ring of polynomials mod $\partial_x W_0, \partial_y W_0, \dots$.

The important question of the $N = 2$ supersymmetry preserving perturbations [10] may then be investigated using this potential description.

Let $W(x, \dots, t_.)$ be the perturbed potential in terms of the flat coordinates t_i, (see ref. [2] for a definition and references), let $p_l(x, \dots, t_.) = -\partial W/\partial t_l$ be the corresponding basis of the (deformed) chiral ring with $p_0 = 1$ and $p_{\bar{0}}$ the unique basis element with maximal U(1) charge. The structure constants of the ring

$$p_i p_j = C_{ij}{}^k p_k \mod \partial W \tag{2.1}$$

are functions of the coordinates $t_.$ and have been proved to satisfy two kinds of constraints (in addition to the associativity and commutativity which are obvious from that definition) [11],

(i) the metric tensor defined as $\eta_{ij} = C_{ij}{}^{\bar{0}}$ is independent of the t's;

(ii) $C_{ij}{}^k$ satisfy the integrability condition that enable one to write $C_{ijk} = \eta_{kl} C_{ij}{}^l (\partial^3/\partial t_i \partial t_j \partial t_k) F(t_.)$, where $F(t_.)$ is some function, the free energy of the theory.

We should mention that Dubrovin has undertaken the classification of the solutions to these constraints, independently of the existence of a potential and polynomial representation of the chiral ring [12]. For more on these topics, see ref. [13].

For any given perturbation, let us consider the chiral ring. Consider first for simplicity the case where the potential depends on a single variable x. Let C_1 be the matrix of structure constants encoding the multiplication by $p_1(x) = x$. We thus have a representation of the chiral algebra by a set of polynomials in x and according to an argument given for example in ref. [14], this implies that the constraint $W'(x) = 0$ is the characteristic equation satisfied by C_1:

$$W'(x) = \det(x\mathbf{1} - C_1). \tag{2.2}$$

In particular, if the perturbed potential W is a good "resolution" of the singular $W_0 = x^{n+1}/(n+1)$, namely if the zeros of W' are distinct, then C_1 has distinct eigenvalues and is diagonalizable. (An example of a non-diagonalizable case is provided by $W = x^6/6 - x^3$; C_1 has a double eigenvalue at 0 and is not diagonalizable. The singularity has not been resolved.)

Suppose now that C_1 is "normalizable". By definition, this means that a *diagonal* * change of basis in the polynomial representation makes C_1 normal

$$C_{1i}^{\ j} = \frac{\rho_1 \rho_i}{\rho_j} M_{1i}^{\ j} \tag{2.3}$$

with M_1 normal, hence diagonalizable in an orthonormal basis $\psi_a^{(i)}$; in fact as all C_i commute, they are all made normal by the same change

$$C_{ij}^{\ k} = \frac{\rho_i \rho_j}{\rho_k} M_{ij}^{\ k}, \tag{2.4a}$$

$$(M_i)_j^{\ k} = \sum_a \lambda_i^{(a)} \psi_a^{(j)} \psi_a^{(k)*} \tag{2.4b}$$

and because of the symmetry $i \leftrightarrow j$, the condition $p_0 = 1$ hence $M_0 = \mathbf{1}$ and the orthonormality of the ψ's, one finds that the eigenvalues have the form $\lambda_i^{(a)} = \psi_a^{(i)} / \psi_a^{(0)}$ hence

$$(M_i)_j^{\ k} = \sum_a \frac{\psi_a^{(i)} \psi_a^{(j)} \psi_a^{(k)*}}{\psi_a^{(0)}}. \tag{2.5}$$

Thus the eigenvalues of C_1, i.e. the zeros of W', or the extrema of W, are

$$x_a = \rho_1 \frac{\psi_a^{(1)}}{\psi_a^{(0)}}. \tag{2.6}$$

The case where the potential involves more than one variable is easy to deal with, but at one point less explicit. For definiteness we consider the case of two variables but the extension to any larger number is straightforward. The assumption that the theory is described by a potential $W(x, y)$ amounts to saying that the chiral ring is represented by polynomials in the variables x and y, i.e. that all the matrices C_i are polynomials in the matrices $C_0 = \mathbf{1}$, C_x and C_y

$$x p_i(x, y) = C_{xi}^{\ j} p_j(x, y) \mod \partial_x W, \partial_y W,$$

$$y p_i(x, y) = C_{yi}^{\ j} p_j(x, y) \mod \partial_x W, \partial_y W. \tag{2.7}$$

* Our insistence on this diagonal change of basis is due to the fact that it must preserve the gradation by the U(1) charge; when several fields have the same gradation, a wider set of redefinitions would be conceivable; we have not studied this systematically.

Thus for any extremum of the potential x_a, y_a, $\partial_x W(x_a, y_a) = \partial_y W(x_a, y_a) = 0$

$$x_a p_i(x_a, y_a) = C_{xi}{}^j p_j(x_a, y_a),$$

$$y_a p_i(x_a, y_a) = C_{yi}{}^j p_j(x_a, y_a) \tag{2.8}$$

and hence x_a is an eigenvalue of C_x and y_a one of C_y for the same eigenvector. If moreover C_x and C_y are normalizable, then all the C's may be diagonalized in the orthonormal basis $\psi_a^{(i)}$ according to eq. (24a) and as before,

$$x_a = \rho_x \frac{\psi_a^{(x)}}{\psi_a^{(0)}},$$

$$y_a = \rho_y \frac{\psi_a^{(y)}}{\psi_a^{(0)}}. \tag{2.9}$$

In contrast with the one-variable case, the reconstruction of W from C_x, C_y is not obvious.

The appearance of algebras of the form (2.5), generalizing the Verlinde formula for fusion algebras is something that has already been encountered in association with graphs. We shall devote the next section to recall some facts on what we call graph algebras.

3. Graph algebras

3.1. THE TWO DUAL ALGEBRAS ATTACHED TO A GRAPH

We present here some concepts on rings (or algebras) attached to graphs [15,6]. Let us consider a graph defined by its adjacency matrix G, whose non-negative entries G_{ab} count the number of edges connecting the vertex a to the vertex b. The graph may possibly be oriented, and thus the matrix G be non-symmetric, but we request it to be normal. (Here, G is real and thus $G^\dagger = G^t$). Clearly any symmetric matrix (hence any adjacency matrix of an unoriented graph) is normal. Let us denote $\psi_a^{(l)}$ the components of the orthonormal eigenvectors, where the index l labels the eigenvector. Note that in general, l and a take an equal number of values (equal to the size n of the G-matrix) but belong to different sets. For convenience, we shall label the vertices a by integers running from 0 to $n-1$, whereas l will also for simplicity be taken as an integer taking in particular the value 0 *. By convention, $l = 0$ will denote the Perron–Frobenius eigenvector. We

* We depart from our previous conventions of ref. [6], where both a and l were taking values starting from 1 rather than 0; this change is motivated by the mismatch with the degrees of polynomials or other gradings (U(1) charges...) that are natural in the problem at hand.

also *assume* that G_{ab} possesses a "unit" vertex a_0 such that

$$\text{(i)}\ \forall l,\ \psi_{a_0}^{(l)} \neq 0. \tag{3.1a}$$

$$\text{(ii)}\ \exists!\ \text{vertex } f \text{ such that } G_{a_0,a} = \delta_{a,f}. \tag{3.1b}$$

At the possible price of a relabelling of the vertices of the graph we shall take $a_0 = 0$. The role of this unit vertex 0 is clear: the graph associated to G encodes some sort of "fusion" by the vertex f, in the sense that we can write $f \times 0 = f$, $f \times a = G_{ab} b$. In the particular case of ADE Dynkin diagrams, l is a Coxeter exponent minus 1 taking n values between 0 and $h-2$, with h the Coxeter number. More generally, in a variety of cases related to a Lie algebra, it is more natural to regard it as taking its values in a bounded domain of the weight lattice of this Lie algebra [6]. The archetypical case is provided by the A_n Dynkin diagram, where the $\psi_a^{(l)}$, l, $a = 0, \ldots, n-1$ are also the matrix elements of the modular S-matrix for the $SU(2)_{n-1}$ current algebra. This is readily seen on the celebrated Verlinde formula [16] written as

$$M_{lm}{}^{p} = \sum_{a=0}^{n-1} \frac{\psi_a^{(l)} \psi_a^{(m)} \psi_a^{(p)*}}{\psi_a^{(0)}}, \tag{3.2}$$

expressing the fusion coefficients in terms of ψ's. The A Dynkin diagram has the uncommon property of being self-dual in the sense that the two sets of a-indices and l-labels may be identified: this is due to the symmetry of the $S = \psi$ matrix. Now, the Verlinde formula suggests to form similar sums for the other D or E Dynkin diagrams, or more generally for a generic graph with a normal adjacency matrix. Then the two sets $\{a\}$ and $\{l\}$ are no longer equivalent and there are two possible summations. The one carried out in (3.2), and the dual one

$$N_{ab}{}^{c} = \sum_{l=0}^{n-1} \frac{\psi_a^{(l)} \psi_b^{(l)} \psi_c^{(l)*}}{\psi_0^{(l)}}. \tag{3.3}$$

Contrary to the case of the A Dynkin diagram, the M's are not in general integers. In contrast, for the D and E Dynkin diagrams as well as a larger class of graphs studied in ref. [6], the N's are! Moreover, for a subset of these graphs – the A, D_{even} and $E_{6,8}$ cases among the Dynkin diagrams – both M's and N's turn out to be non-negative. We stress that these are empirical observations and that we know no sufficient condition on the graph that ensures the integrality of the N's. Note that the matrices M_l and N_a defined, respectively, by

$$(M_l)_m{}^{p} = M_{lm}{}^{p},$$

$$(N_a)_b{}^{c} = N_{ab}{}^{c}, \tag{3.4}$$

satisfy an associative and commutative algebra

$$M_l M_m = M_{lm}{}^p M_p,$$

$$N_a N_b = N_{ab}{}^c N_c \tag{3.5}$$

and the orthonormality condition ensures that $M_0 = \mathbf{1}$ and $N_0 = \mathbf{1}$ are the units of these algebras.

In writing (3.2) and (3.3), we have implicitly assumed that these summations make sense, i.e. that no vanishing denominator occurs. Although the Perron–Frobenius theorem tells us that the components of the eigenvector with the largest eigenvalue are non-negative, it does not forbid the vanishing of some of these. We do know cases where either one of the $\psi_0^{(l)}$ or one of the $\psi_a^{(0)}$ vanishes. To avoid the possibility of a vanishing $\psi_0^{(l)}$, we included the condition (i) (3.1a) in the definition of the unit vertex 0 of the graph. Moreover, a sufficient condition to avoid the vanishing of $\psi_a^{(0)}$'s is to suppose that the graph is connected, i.e. there exists a path $a_1 = a, a_2, \ldots, a_p = b$ between any couple of vertices (a, b) of the graph, $G_{a_1 a_2} G_{a_2 a_3} \ldots G_{a_{p-1} a_p} \neq 0$.

We should also note that there are cases where the matrix G has degenerate eigenvalues and there is a problem of choosing the appropriate combination of the corresponding eigenvectors. Such is the case of the D_{2l} Dynkin diagram: the middle exponent $2l-1$ is twice degenerate and the coefficients $N_{ab}{}^c$ are integers only for a specific choice of the eigenvectors $\psi^{(2l-2)\pm}$; for l even, this choice involves complex combinations of the real eigenvectors, whence the relevance of the complex conjugation in eqs. (3.2)–(3.3).

Except in the simplest case of A-type, no physical interpretation of these algebras, and in particular of the integral N's as some multiplicities is known (see ref. [17], however).

3.2. SUBALGEBRAS AND MODULAR INVARIANCE

In this section, we review the connections between (some of) these graph algebras and fusion algebras of rational conformal field theories. This is not in the main stream of our paper, but we include it here to illustrate some cross relations between these topics and to show that some of the considerations of ref. [5] may be extended. Whenever all the M's and N's are non-negative (case referred as "type I" in the ref. [6]), one may find a subalgebra of the graph algebra (3.3), i.e. a stable subset T of vertices of the graph, which encodes the fusion rules of the underlying WZW theory. Namely, in the cases considered in ref. [6], each of the graphs forms the target space for an integrable lattice model whose continuum limit is described by a coset c.f.t. $\mathrm{G}_{k-1} \times \mathrm{G}_1 / \mathrm{G}_k$ and is in correspondence with a modular invariant of the relevant G_k WZW theory. The l labels indexing the algebra M are in correspondence with integrable weights of the G_k WZW model at some fixed level k. On the other hand the modular invariants of the WZW theories at a given level can be of two forms: the "block-diagonal" ones formed by a sum of absolute squares of sums of WZW characters χ_l (the so-called "extended" characters), and the "twisted" ones, obtained by combining left and right blocks of the preceding class in a non-diagonal way. Concentrating on the block-diagonal invariants (also called type I in ref. [6]) *, we see that they are characterized by a partition I_1, $I_2, \ldots, I_p$ of the set of labels of representations which form the theory, the modular invariant reading

$$Z = \sum_{i=1}^{p} \left| \sum_{l \in I_i} \chi_l \right|^2 .$$

We found that the graph subalgebra coincided in all known cases with the fusion rules for the primary states of the "extended" symmetry, whose characters are the blocks

$$\chi_{I_k} = \sum_{l \in I_k} \chi_l . \tag{3.6}$$

More precisely, from the data of the graph ring and of its subring associated with the stable subset T of vertices, we can define the equivalence relation $\simeq$ on the eigenvalue labels [19]

$$l \simeq m \quad \text{iff} \sum_{a \in T} \psi_a^{(l)} \psi_a^{(m)*} \neq 0. \tag{3.7}$$

The equivalence classes form the desired partition of the set of eigenvalue labels into blocks $I_1, \ldots, I_p$. The latter are in one-to-one correspondence with the elements of T, and we relabel them I_a, $a \in T$. The corresponding S-matrix of modular transformation is given by

$$S_a{}^b = \frac{\psi_a^{(l)}}{\left(\sum_{c \in T} |\psi_c^{(l)}|^2\right)^{1/2}}, \quad \text{independent on } l \in I_b . \tag{3.8}$$

Like any fusion algebra of a rational conformal field theory, the subalgebra with fusion coefficients $N_{ab}{}^c$, a, b, $c \in T$ is self-dual, due to the symmetry of S. The dual $M_{ab}{}^c$ can also be expressed in terms of the original dual algebra $M_{lm}{}^n$ but in a less straightforward way. In particular its one-dimensional representations have a

* This distinction between type I and non-type I graphs and/or modular invariants seems to have multiple aspects, as testified by the existence or non-existence of a flat connection on the space of paths on the graph [18]; see also the end of appendix A.

TABLE 1

A, D, E perturbations. Normalizable perturbations of the A, D, E potentials are displayed as follows. First column: name of the potential; second column: name of the perturbation (list of the non-zero t's); third column: the dual ring of the normalized ring, through the graph of one of its generators (all the cycles have to be understood as oriented anti-clockwise); fourth column: locus of the extrema of the perturbed potential (x in the complex plane in the one variable case, x- and y-planes otherwise); fifth column: values taken by the perturbed potential at the various extrema, in the complex plane (the links correspond to the minimal solitons interpolating between the extrema); sixth column: a check-mark in case of known integrability, a question-mark otherwise.

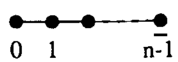
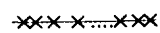
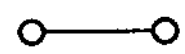
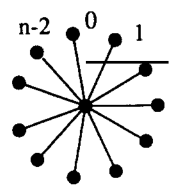
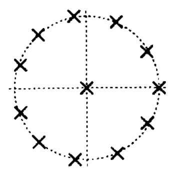
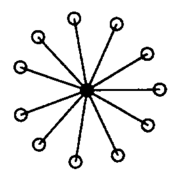
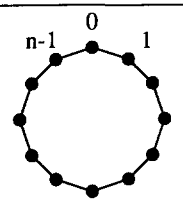
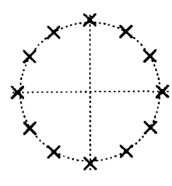
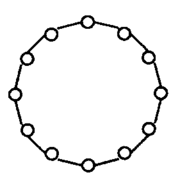
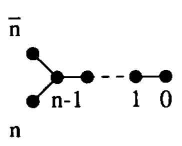
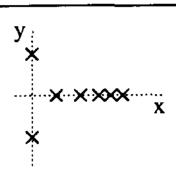
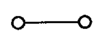
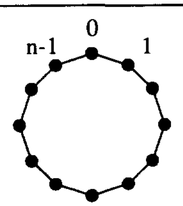
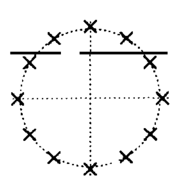
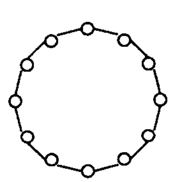
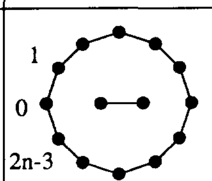
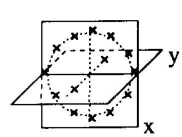
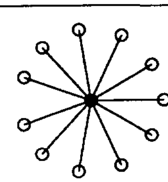
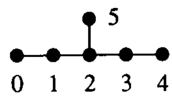
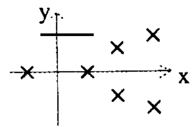
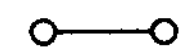
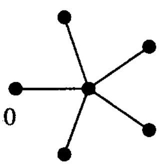
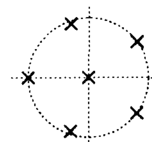
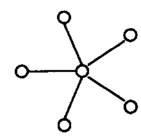
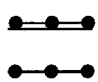
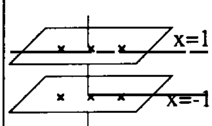
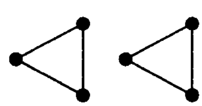
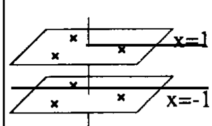
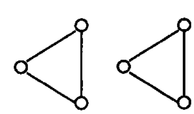
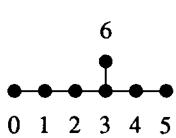
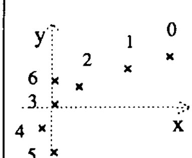
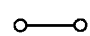
A_n	t_{n-1}				✓
	t_2				✓
	t_1				✓
D_{n+2}	t_{2n}				✓
	τ				?
D_{2n}	t_2				✓

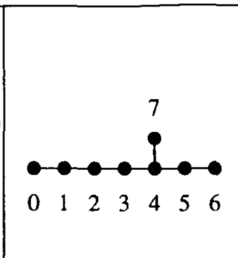
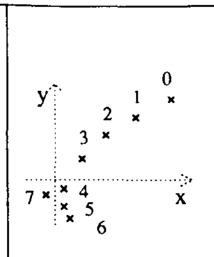
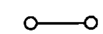
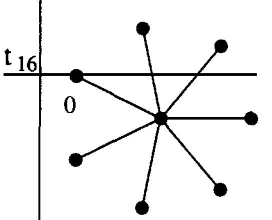
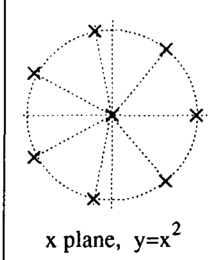
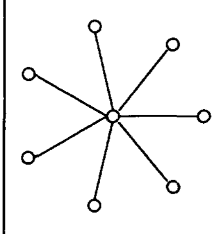
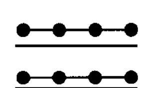
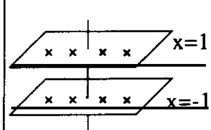
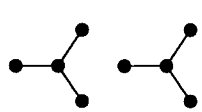
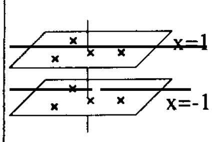
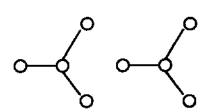
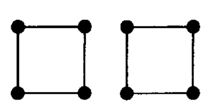
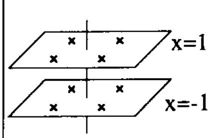
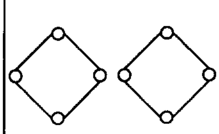
TABLE 1 (continued)

E_6	t_{10}				✓
	t_7		 x plane, y=x 2		?
	t_4 t_6				✓
	t_3 t_4				✓
E_7	t_{16}				✓

simple realization in terms of the one-dimensional representations of $M_{lm}{}^n$. The latter are of the form

$$\tilde{p}_l^a = \frac{\psi_a^{(l)}}{\psi_a^{(0)}} \tag{3.9}$$

TABLE 1 (continued)

E_8	t_{29}				✓
	t_{16}				?
	t_{10} t_{18}				✓
	t_{10} t_{12}				✓
	t_6 t_{10}				✓

and satisfy $\tilde{p}^a_l \tilde{p}^a_m = M_{lm}{}^n \tilde{p}^a_n$, for any vertex a. The corresponding one-dimensional representations of the self-dual subalgebra read

$$\Pi^{(b)}_a = \frac{S_a{}^b}{S_a{}^0}, \quad a, b \in \mathrm{T}, \tag{3.10}$$

and one finds, using the various relations between the S's and the ψ's

$$\Pi_a^{(b)} = \sum_{l \in I_b} \left[\sum_{c \in T} |\psi_c^{(0)}|^2 \sum_{c \in T} |\psi_c^{(l)}|^2 \right]^{1/2} \tilde{p}_l^a, \tag{3.11}$$

for any a, $b \in T$.

We should stress that all these considerations are empirical and based on a case by case examination of all the type I cases pertaining to G = SU(2) or SU(3) (the classification of modular invariants for the latter case has been completed in ref. [20]).

Let us illustrate this on an example. We consider the E_6 diagram of table 1, and the subalgebra of the graph algebra formed by the end-point vertices, $T = \{0, 4, 5\}$. It is isomorphic to the fusion algebra of the Ising model, (known to be that of the blocks of the E_6 theory), upon identification of $0 \equiv \mathrm{Id}$, $4 \equiv \epsilon$ and $5 \equiv \sigma$, respectively the identity, energy and spin conformal blocks of the Ising model. The eigenvalues of the adjacency matrix $[E_6]_{ab}$ are labelled by the Coxeter exponents shifted by -1

$$\beta^{(l)} = 2 \cos \pi \frac{l+1}{12}, \qquad l = 0, 3, 4, 6, 7, 10.$$

Applying eq. (3.7), we find the corresponding equivalence classes of the set of exponents

$$I_0 = \{0, 6\}, \quad I_4 = \{4, 10\}, \quad I_5 = \{3, 7\},$$

and the associated modular invariant

$$Z_{E_6} = |\chi_0 + \chi_6|^2 + |\chi_4 + \chi_{10}|^2 + |\chi_3 + \chi_7|^2. \tag{3.12}$$

Moreover we get the one-dimensional representations of the self dual subalgebra

$$\Pi_a^{(0)} = \frac{1}{3+\sqrt{3}} \tilde{p}_0^a + \frac{1}{\sqrt{6}} \tilde{p}_6^a,$$

$$\Pi_a^{(4)} = \frac{1}{\sqrt{6}} \tilde{p}_4^a + \frac{1}{3+\sqrt{3}} \tilde{p}_{10}^a,$$

$$\Pi_a^{(5)} = \frac{1}{\sqrt{2(3+\sqrt{3})}} \left[\tilde{p}_3^a + \tilde{p}_7^a \right]. \tag{3.13}$$

3.3. CHEBISHEV RESOLUTION OF THE ADE SINGULARITIES

Let us show how the considerations of sect. 2 apply to the case considered by Lerche and Warner, namely the perturbations of the ADE potentials by their least relevant operator. In the A_n case, the corresponding potential is nothing else than the Chebishev polynomial of the first kind ($T_n(2\cos\theta) = 2\cos n\theta$): $W(x) = (n+1)^{-1}T_{n+1}(x)$; the basis of the chiral ring derived from the flat coordinates is provided by the Chebishev polynomials of the second kind ($U_l(2\cos\theta) = \sin(l+1)\theta/\sin\theta$), $p_l(x) = U_{l-1}(x)$, $l = 1,\dots,n$, and the chiral algebra that they satisfy is just the $\mathrm{SU}(2)_{k=n-1}$ fusion algebra. The potential $W(x) = (n+1)^{-1}T_{n+1}(x)$ is the fusion potential that encodes these fusion rules [21,14].

The other cases D and E have also been discussed [5]. By inspection, one finds that

(i) the corresponding matrices C_x and C_y can be made normal by a diagonal redefinition of the basis;

(ii) the dual algebra has among its generators the incidence matrix of the Dynkin diagram, or equivalently the $\psi_a^{(l)}$ are the eigenvectors of the Cartan matrix of the D or E Lie algebra and according to the discussion of sect. 2, $p_l(x_a, y_a) \propto \psi_a^{(l)}/\psi_a^{(0)}$;

(iii) the pattern of extrema of the potential in the x–y plane reproduces the shape of the Dynkin diagram.

Then the previous discussion applies: in the "good" cases D_{even}, E_6 and E_8, one can find linear combinations of the polynomials $p_l(x, y)$ that generate a *subring* of the chiral ring isomorphic to the fusion ring of the corresponding SU(2) modular invariant. Let us illustrate this again on the case of E_6. We start from the deformed E_6 potential W given in ref. [11] and recalled for convenience in appendix B. The polynomials $p_i(x, y, t) = -(\partial/\partial t_i)W$ form a ring (modulo $\partial_x W$, $\partial_y W$) with structure constants derived from the free energy given in ref. [22]. The potential W and the p's are quasihomogeneous polynomials of x (of degree 4), y (degree 3) and t_i (degree $12-i$).

If only $t_{10} = t$, the coupling to the least relevant operator, is non-vanishing, the polynomials p_i reduce to $p_0 = 1$, $p_3 = y$, $p_4 = x - \frac{1}{2}t^2$, $p_6 = y^2 - tx + \frac{1}{6}t^3$, $p_7 = xy - t^2 y$ and $p_{10} = xy^2 - \frac{3}{2}t^2y^2 + \frac{1}{3}t^3 x$. After a change of scale

$$\tilde{p}_0 = p_0, \qquad \tilde{p}_3 = \frac{\sqrt{3}}{t^{3/2}} p_3,$$

$$\tilde{p}_4 = \frac{\sqrt{6}}{t^2} p_4, \qquad \tilde{p}_6 = \frac{3\sqrt{2}}{t^3} p_6,$$

$$\tilde{p}_7 = \frac{6}{t^{7/2}} p_7, \qquad \tilde{p}_{10} = \frac{6\sqrt{3}}{t^5} p_{10},$$

the dimensionless $\tilde{p}$'s have structure constants given by the algebra dual to the one generated by the E_6 Dynkin diagram, in the sense of sect. 3.1. In other words,

$$\tilde{p}_i \tilde{p}_j = M_{ij}{}^k \tilde{p}_k .$$

To make contact with the fusion algebra of the underlying SU(2) model we finally form linear combinations of the $\tilde{p}$'s according to eq. (3.13), and we find that the polynomials Π_0, Π_4 and Π_5 form a subring isomorphic to the fusion ring of the theory (3.12).

The property of normalizability enjoyed by the chiral ring of the Chebishev resolution of the ADE singularities has prompted us to systematically examine what are the normalizable deformations of these cases. This will be our endeavour in the next section.

4. Normalizable deformations of the ADE singularities

This section is a catalog of the normalizable deformations of the ADE potentials by a single non-vanishing parameter t_i (see sect. 6 for a discussion of this restriction). Because the potential is a quasihomogeneous polynomial of the variable(s) x (and possibly y) and of this parameter, t_i may be rescaled to the value 1. This will be assumed in the following.

4.1. A_n

We first examine the A_n deformed potential $W(x, t_0, \dots, t_{n-1})$ of ref. [11]. The matrix C_1 in that case reads

$$C_1 = \begin{pmatrix} 0 & 1 & 0 & \cdots & \\ t_{n-1} & 0 & 1 & 0 & \\ t_{n-2} & t_{n-1} & 0 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \\ t_1 & \cdots & t_{n-2} & t_{n-1} & 0 \end{pmatrix} \tag{4.1}$$

and the potential W is reconstructed from (2.2) by one quadrature. We assume that only $t_{p+1} = t$ is non-vanishing. Then C_1 reads

$$C_1 = \begin{pmatrix} 0 & 1 & 0 & \cdots & & \cdots & 0 \\ & 0 & 1 & 0 & & & 0 \\ & & 0 & 1 & \ddots & & \\ 0 & & & \ddots & \ddots & \ddots & \\ t & 0 & \cdots & & & \ddots & 0 \\ & \ddots & \ddots & & & 0 & 1 \\ 0 & \cdots & t & 0 & & \cdots & 0 \end{pmatrix}, \tag{4.2}$$

where the diagonal of t's starts in position $(n-p,\,1)$ on the matrix. We look for a diagonal change of basis $P=\operatorname{diag}(\rho_1,\dots,\rho_n)$, such that $\tilde{C}_1=P^{-1}C_1P$ is normal. The normality condition imposes constraints on the ρ's, which are all expressed in terms of $\sigma_i=\rho_i^{-2}$. For all values of p, we get

$$\frac{\sigma_{n-p}}{\sigma_1}=\frac{\sigma_{n-p+1}}{\sigma_2}=\dots=\frac{\sigma_n}{\sigma_{p+1}}. \tag{4.3}$$

According to the respective positions of p and $\frac{1}{2}n$, we find for $p\geqslant 1$

(i) $p<\frac{1}{2}n$:

$$\begin{aligned}
\frac{\sigma_j}{\sigma_{j+1}}&=j\frac{\sigma_1}{\sigma_2} && 1\leqslant j\leqslant p+1\\
\frac{\sigma_j}{\sigma_{j+1}}&=(p+1)\frac{\sigma_1}{\sigma_2} && p+2\leqslant j\leqslant n-p-1\\
\frac{\sigma_j}{\sigma_{j+1}}&=(n-j)\frac{\sigma_1}{\sigma_2} && n-p\leqslant j\leqslant n-1.
\end{aligned}$$

(ii) $p\geqslant\frac{1}{2}n$:

$$\begin{aligned}
\frac{\sigma_j}{\sigma_{j+1}}&=j\frac{\sigma_1}{\sigma_2} && 1\leqslant j\leqslant n-p-1\\
\frac{\sigma_j}{\sigma_{j+1}}&=(n-p-1)\frac{\sigma_1}{\sigma_2} && n-p\leqslant j\leqslant p+1\\
\frac{\sigma_j}{\sigma_{j+1}}&=(n-j)\frac{\sigma_1}{\sigma_2} && p+2\leqslant j\leqslant n-1.
\end{aligned}$$

From (i) and (4.3) we get

$$\frac{\sigma_{n-p}}{\sigma_{n-p+1}}=\frac{\sigma_1}{\sigma_2}=p\frac{\sigma_1}{\sigma_2},$$

possible only if $p=1$. Analogously, from (ii) and (4.3) we get

$$\frac{\sigma_{n-p}}{\sigma_{n-p+1}}=\frac{\sigma_1}{\sigma_2}=(n-p-1)\frac{\sigma_1}{\sigma_2},$$

possible only if $p=n-2$. This leaves us with the three cases

(1) $p=0$: $\quad\sigma_k=t^{-2(k-1)/(n-1)},\quad k=1,2,\dots,n.$

(2) $p=1$: $\quad\sigma_k=2(2t)^{-2(k-1)/(n-1)},\quad k=2,3,\dots,n-1,$
$\sigma_1=1,\quad\sigma_n=t^{-2},$

(3) $p=n-2$: $\quad\sigma_k=t^{-(k-1)},\quad k=1,2,\dots,n.$

We conclude that the only normalizable cases with a single non-vanishing t-parameter are the three cases $t_1\neq 0$, $t_2\neq 0$ and $t_{n-1}\neq 0$. What is most striking is that these three cases have been identified as integrable deformations [23]: the perturbation by the most relevant operator has long been recognized as integrable ([4,24–27]), the one by t_2 is discussed in refs. [1,24] while the case of t_{n-1} is treated in refs. [5,28].

There are several things that can be done on these normalizable cases:

(a) From the diagonalization of the matrix $\tilde{C}_1 = \rho_1 M_1$ in an orthonormal basis, we can construct the dual algebra $N_{ab}{}^c$. In all these cases, it exists (because all $\psi_0^{(l)} \neq 0$) and it leads to *non-negative* integers! Each of these can be regarded as the adjacency matrix of a (possibly disconnected) graph.

(b) We can also determine the extrema of the potential $W(x, t)$, i.e. both the location of the extrema x_a (on the real line or in the complex plane) and the value of W at this x_a. We find that the location of the x_a follows the pattern of vertices of one of the graphs of the dual algebra, call it $N_{fb}{}^c$; consequently, it seems natural to link the extrema x_a by edges of the graph of N_f. As for the extremal values of W, they are such that for two extrema x_a and x_b linked as just explained, $|W(x_a) - W(x_b)|$ takes only one value $|\Delta W|$. The interpretation [1] is that the link exists between the ground states a and b of the potential if and only if there is a "fundamental" soliton interpolating between them, and the mass of this soliton is just given by $|\Delta W|$. These features are apparent on the graphs tabulated in table 1. We comment briefly the results.

(i) For the perturbation by t_{n-1}, the Chebishev resolution discussed before, the extrema lie at $x_a = 2\cos \pi(a+1)/(n+1)$, $a = 0, \dots, n-1$, and the value of W at these points is $W_a = 2(-1)^{a+1}/(n+1)$. The graph encoding C_x as well as one of its dual is the A_n Dynkin diagram.

(ii) For the perturbation by x^2, $W = x^{n+1}/(n+1) - \frac{1}{2}x^2$, one finds extrema at $x_0 = 0$ and $x_a = \exp(2i\pi(a-1)/(n-1))$, $a = 1, \dots, n-1$; they form a centered $(n-1)$-gon in the complex plane, like the corresponding values of W: $0, [(n-1)/2(n+1)] \exp(4i\pi(a-1)/(n-1))$. The graph of the dual $N_{1a}{}^b$ has the daisy shape depicted in the third column of table 1.

(iii) For the perturbation by x, the results are similar, with a non-centered *oriented* polygon: $x_a = \exp(2i\pi a/(n-1))$, $W_a = [-n/(n+1)] \exp(2i\pi a/(n-1))$, $a = 0, \dots, n-1$. On table 1, only the graph associated with N_1 has been drawn, but the other N's would connect other pairs of ground states, corresponding to the other, non-fundamental, solitons.

4.2. D_{n+2}

It may be useful to first recall that the D_{n+2} perturbed potential may be obtained from the A_{2n+1} one by an orbifold procedure. We devote appendix A to a review of this construction and of various properties of the D-potential and free energy, including a curious positivity property of the coefficients of F for D_{even}.

The bottom line is that the D_{n+2} potential involves a new variable y, and after inserting an extra deformation parameter τ coupled to y, it reads

$$W_{D_{n+2}}(x, y; t_0, t_2, \dots, t_{2n}, \tau) = W_{A_{2n+1}}\left(x' = \sqrt{x}, t_0, 0, t_2, \dots, t_{2n}\right) + \tfrac{1}{2}xy^2 + \tau y$$
$$= \frac{x^{n+1}}{2(n+1)} + \dots \tag{4.4}$$

The free energy F of the D-models may also be determined fairly explicitly in terms of the A one. One finds that

$$F_{D_{n+2}}(t_0, t_2, \dots, t_{2n}, \tau) = F_{A_{2n+1}}(t_0, 0, t_2, \dots, t_{2n}) - \tfrac{1}{2}\tau^2\Phi(t_0, \dots, t_{2n}), \tag{4.5}$$

where the expression of Φ is given in appendix A.

From the orbifold connection between the D_{n+2} and the A_{2n+1} cases, it seems reasonable to expect the perturbations by the least relevant and the most relevant operator to be (i) integrable, (ii) normalizable; the former observation has been made in refs. [24,29], for the most relevant (and n even), and in ref. [5], for the least relevant; as for normalizability, it is readily checked for these two perturbations and one also finds that the perturbation by τ is normalizable. The proof goes as follows.

The matrix encoding the multiplication by $p_1(x)$ is given by

$$(C_1)_i^j = \frac{\partial}{\partial t_2}\frac{\partial}{\partial t_i}\frac{\partial}{\partial t_k}\left(F_{A_{2n+1}} - \tfrac{1}{2}\tau^2\Phi(t.)\right)\eta^{j,k},$$

$$C_1 = \begin{pmatrix} & & & 0 \\ & A & & \vdots \\ & & & -\tau \\ \tau & 0 & \dots & -t_{2n} \end{pmatrix}, \tag{4.6}$$

where A is an $(n+1)\times(n+1)$ matrix, which can be expressed in terms of the matrix $\hat{C}_1$ encoding multiplication by x in the A_{2n+1} model. The relation is as follows:

$$(A)_i^j = \left(\hat{C}_1^2\right)_{2i}^{\ 2j} - t_{2n}\delta_{ij}, \quad i = 0, \dots, n. \tag{4.7}$$

Looking at specific perturbations we now get:

(i): τ-perturbation: The matrix C_1 simplifies dramatically and is

$$C_1 = \begin{pmatrix} 0 & 1 & 0 & \dots & & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & & & \ddots & \vdots \\ 0 & \dots & & 0 & 1 & 0 \\ 0 & \dots & \dots & 0 & 0 & -\tau \\ \tau & \dots & & & 0 & 0 \end{pmatrix}. \tag{4.8}$$

We see that this is a cycle and it is clearly normalizable.

(ii): t_i-perturbation. The matrix C_1 has the following form:

$$C_1 = \begin{pmatrix} & & 0 \\ & A & \vdots \\ 0 & \dots & -t_{2n} \end{pmatrix}. \tag{4.9}$$

Obviously if in this case A is normalizable, the same property holds for C_1, therefore the normalizable cases in A_{2n+1} with $t_{\mathrm{odd}} = 0$ will also be in this case. It is likely that these are the only normalizable perturbations with a single non-vanishing t_i, although we have no complete proof. There is, however, the possibility to mix the two operators of same degree, and for D_6, for example, one finds that the operators coupled to $t_n \pm i\tau$ are normalizable (see below sect. 5.2).

We thus discuss in turn the perturbations

(i) by t_{2n}. This is the Chebishev perturbation discussed in refs. [5,28]. The minima in the x–y plane build up the shape of the Dynkin diagram, while W takes only two values. It is still true that the locations of the extrema are related to the eigenvectors of the adjacency matrix of the Dynkin diagram D_{n+2} as in (2.9). If we want to reconstruct the whole $M_{ab}{}^c$ dual algebra, however, we have to distinguish the cases of even and odd n. For even n, one chooses for the vertex 0 the end of the longest leg of the diagram, since all the components $\psi_0^{(l)}$ are non-vanishing (after some judicious choice of linear combinations of the eigenvectors pertaining to the same eigenvalue). For n odd, in contrast, one has to take rather the end of a short leg to have a well-defined expression and then the $N_{ab}{}^c$'s are not all non-negative, as recalled in sect. 3.1.

(ii) by τ. The potential $W = x^{n+1}/2(n+1) + \frac{1}{2}xy^2 - y$ has minima at $x_a \propto \exp 2i\pi a/(n+2)$, $a = 0, \dots, n-1$ and takes there the values $W_a \propto x_a^{-1}$ (the overall factors or phases have been discarded in table 1). The multiplication by x yields a matrix of cyclic permutation, and the dual N_1 has the same form: in table 1, the links of the graph should be oriented. The integrability of that case is, to the best of our knowledge, not established.

(iii) by t_2. The potential $W = [x^{n+1}/2(n+1)] + \frac{1}{2}xy^2 - x$ has minima at $x_a = 2^{1/n} \exp 2i\pi a/n$, $y_a = 0$ for $a = 0, 1, \dots, n-1$ and $x = 0$, $y = \pm\sqrt{2}$, and the values of the potential are $W(x_a, y_a) = -(n2^{1/n}/n+1)\exp(2i\pi a/n)$, $W(0, \pm\sqrt{2}) = 0$. For even n, the ring generators are normal in the basis

$$1,\ x,\ x^2,\dots,x^{n/2-1},\frac{x^{n/2}\pm\omega y}{\sqrt{2}},\ x^{n/2+1},\dots,x^{n-1},\ x^n-1,$$

for any ω such that $\omega^4 = 1$. The dual-algebra generator is a cyclic permutation of n vertices together with the exchange of the two remaining ones. For n odd, the ring generators are normal in the basis $1, x,\dots, x^{n-1}, x^n - 1, y$. The dual ring cannot be constructed, due to a failure of condition (3.1a). For even n, the perturbation has been argued to be integrable, as the most relevant one [29].

4.3. E_6

The only normalizable perturbations with a single non-vanishing t_i are:

(i) by t_{10}. This is the Chebishev perturbation, with the extrema of the potential at either $y=0$, $x=\pm 1/2\sqrt{3}$, $W=\mp 1/36\sqrt{3}$, or $x = 1\pm 1/2\sqrt{3}$, $y=\epsilon\sqrt{1\pm 1/\sqrt{3}}$, ($\epsilon=\pm 1$), $W=\mp 1/36\sqrt{3}$. The matrix N_1 is the adjacency matrix of the Dynkin diagram.

(ii) by t_7. This is an interesting case where the perturbation couples the two variables x and y: $W=\frac{1}{3}x^3+\frac{1}{4}y^4-xy$. The extrema occur either at $x_0=y_0=0$, $W_0=0$ or at $x=\exp[6i\pi(a-1)/5]$, $y=\exp[2i\pi(a-1)/5]$, $W_a = -\frac{5}{12}\exp[-2i\pi(a-1)/5]$, $a=1,\dots,5$. The dual N_5 is the adjacency matrix of a daisy graph (like in the case of A_6 perturbed by t_2).

(iii) by t_4, t_6. In this case and the next, we allow two different t's to be non-vanishing, in apparent contradiction to our previous assumption. This is because the two variables x and y are in fact uncoupled, and we are dealing with the tensor product of a A_2 x^3-potential perturbed by x and a A_3 y^4-potential perturbed by y^2. As before the perturbation parameters t_4, t_6 may be absorbed into a redefinition of x and y and we choose them equal to 1. The extrema lie at $x=0$, 1 and $y=0$, $\pm\sqrt{2}$, $W=\pm\frac{2}{3}$ and twice $\pm\frac{5}{3}$.

(iv) by t_3, t_4. The extrema are at $x=\pm 1$, $y=\exp(2ia\pi/3)$, $a=0, 1, 2$ with W taking six values in the plane.

The integrability of the case t_{10} has been discussed in refs. [5,28]. It would be quite interesting to find a conserved quantity or any other evidence of integrability in the case of the t_7 perturbation.

4.4. E_7

The expression of the perturbed potential and free energy may be found in ref. [22] (with a little misprint corrected in our appendix B) *. The only normalizable perturbations by a single non-vanishing flat coordinate are

* We are grateful to A. Klemm for a communication on this subject.

(i) by t_{16}. This Chebishev resolution, already discussed in ref. [5], leads to a dual algebra that involves signs but is well defined. The generator N_1 is the adjacency matrix of the E_7 Dynkin diagram. The extrema of W take place at points that also reproduce that diagram, (see table 1) and W takes only two values (with the conventions of ref. [22]), $W = \pm(2 \times 3^{10})^{-1}$ (4 times +, 3 times −).

(ii) by t_{10}: The potential $W = \frac{1}{3}x^3 + xy^3 - xy$ has extrema that lie in the x and y complex planes, making the picture more difficult to read. Also, all the ψ_a have some vanishing component, making the N algebra ill-defined. Accordingly, there is no corresponding entry in table 1. The integrability of that case is not known.

4.5. E_8

The parametrization of the perturbed potential by flat coordinates may be found in ref. [22]. The normalizable cases are perturbations by

(i) by t_{28}: Extrema are at $y_a = 1 + (1/\sqrt{15})\psi_a^{(1)}/\psi_a^{(0)}$, $x_a = y_a - \frac{4}{5} + \frac{1}{15}\psi_a^{(2)}/\psi_a^{(0)}$, where $\psi^{(0,1,2)}$ are the three eigenvectors of the E_8 adjacency matrix pertaining to the eigenvalues $2\cos\frac{1}{30}\pi$ (1, 7, 11). The corresponding critical values of W take only two values, $W = \pm 1/20\,250\sqrt{5}$. Once again, the extrema display nicely in the x–y plane the shape of the E_8 Dynkin diagram. The adjacency matrix of the latter is reproduced by the N_1 matrix.

(ii) by t_{16}: The extrema lie at the origin, with $W_0 = 0$ and at the seventh roots of unity, $x_a = \exp(2i\pi(a-1)/7)$, $y_a = x_a^2$, $W_a = -\frac{7}{15}x_a^3$, $a = 1,\dots,7$. These extrema and the resulting N_7 graph are again like in the case of A_8 perturbed by t_2.

(iii) by t_{18}, t_{10};

(iv) by t_{12}, t_{10};

(v) by t_{10}, t_6: these last three cases corresponds to decoupled cases $A_4 \otimes A_6$. Their extrema and graphs are thus obtained as tensor products of the A-cases discussed above.

5. Non-ADE cases

5.1. THE SU(N) CASES

The study of effective Landau–Ginzburg theories beyond the ADE potentials becomes more delicate. The main difficulty is the appearance of *modules* in the singularities, i.e. dimensionless parameters decorating the potentials. The simplest example of a module is provided by the P_8 singularity of ref [9], with a potential

$$x^3 + y^3 + z^3 + axyz,$$

where the dimensionless parameter a is the module of the singularity [30]. However, Gepner [21] found some geometrical potentials for the fusion rings of the SU(N) WZW theories at level k, best expressed through their generating function

$$\sum_{m\geqslant 0} t^m W_m^{(N)}(x_1, x_2, \dots, x_{N-1})$$
$$= -\log\left(1 - tx_1 + t^2x_2 - t^3x_3 + \dots + (-1)^{N-1}t^{N-1}x_{N-1} + (-1)^N t^N\right). \quad (5.1)$$

The fusion ring is then the polynomial ring $\mathbb{C}[x_1, \dots, x_{N-1}]/\{\partial_{x_i} W_m^{(N)}\}$, with $m = k+N$. The ring basis corresponding to the integrable weights of $\widehat{\mathrm{SU}(N)}_k$, $(\lambda_1, \dots, \lambda_{N-1})$, $\lambda_i \geqslant 0$, $\Sigma\lambda_i \leqslant k$ is formed by generalized Chebishev polynomials. It was argued that this is just a particular perturbation of the chiral ring of an $N=2$ Landau–Ginzburg theory of $N-1$ superfields $\Phi_1, \dots, \Phi_{N-1}$, with a quasi-homogeneous potential $w_m^{(N)}$, generated by

$$\sum_{m\geqslant 0} t^m w_m^{(N)}(\Phi_1, \dots, \Phi_{N-1}) = -\log\left(1 - t\Phi_1 + t^2\Phi_2 - \dots + (-1)^{N-1}t^{N-1}\Phi_{N-1}\right). \quad (5.2)$$

There is now a bulk of evidence [3,4,27,31,32] that these Landau–Ginzburg theories describe the $N=2$ superconformal Kazama–Suzuki models [33] based on the cosets $(\mathrm{SU}(N)_k \times \mathrm{SO}(2(N-1))_1/(\mathrm{SU}(N-1)_{k+1} \times \mathrm{U}(1))$. The potential $w_{k+N}^{(N)}$ is a quasi-homogeneous function of degree $k+N$, if we assign the degree j to the field Φ_j. The "Chebishev" perturbation reproducing the fusion ring of $\widehat{\mathrm{SU}(N)}_k$ is therefore a perturbation by the degree k operator corresponding to the weight $(k, 0, \dots, 0)$ (see table 1, where $t_{k,0,\dots,0}$ is denoted by t_k). The task of computing flat coordinates for generic perturbations of these potentials $w_m^{(N)}$ is formidable. However, it was carried out in one special case, corresponding to $\mathrm{SU}(3)_3$ [22]. The appearance of modules, i.e. of coupling parameters with negative or zero dimension is clear from inspection of the possible degrees of operators in a generic perturbed theory. Let $\mathscr{U}_{\lambda_1,\dots,\lambda_{N-1}}(\Phi_1, \dots, \Phi_{N-1})$ denote the ring basis element (generalized Chebishev polynomial) with weight $(\lambda_1, \dots, \lambda_{N-1})$. It behaves like

$$\mathscr{U}_{\lambda_1,\dots,\lambda_{N-1}}(\Phi_1, \dots, \Phi_{N-1}) = \Phi_1^{\lambda_1}\Phi_2^{\lambda_2}\dots\Phi_{N-1}^{\lambda_{N-1}} + \dots,$$

hence its degree is $\lambda_1 + 2\lambda_2 + \dots + (N-1)\lambda_{N-1}$, and can go up to $(N-1)k$: it can become larger than the degree of the attached potential, $k+N$, as soon as $k \geqslant 2$ for $N>3$, or $k \geqslant 3$ for $N=3$. Hence the perturbations by such operators will have zero- (marginal operators) or negative- (irrelevant operators) dimension coupling constants to preserve the quasi-homogeneity of the perturbed potential. In all cases, these will enable one to construct dimensionless couplings, whence modules. This explains also how the cases $\mathrm{SU}(3)_1$ and $\mathrm{SU}(3)_2$ avoid the problem, being just part of the A, D, E classification of singularities without modules (resp. A_3 and D_6, see next subsection for a detailed study), as well as $SU(N)_1$ (A_N).

Although the complete expression for perturbed potentials is not known, we can consider some special perturbations which are relevant enough to avoid the problem of modules, such as the "Chebishev" perturbation for instance. In the remainder of this section we will concentrate on the SU(3) case at levels $k \geqslant 3$. The generating function for the potentials $w_{k+3}^{(3)}$ is easily recognized as a special form of the SU(2) Chebishev potential generating function, i.e.

$$\sum_{m \geqslant 0} t^m w_m^{(3)}(x, y) = \sum_{m \geqslant 0} t^m y^{m/2} w_m^{(2)}\left(x/\sqrt{y}\right),$$

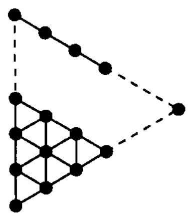
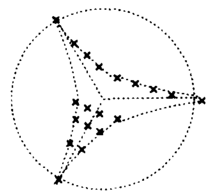
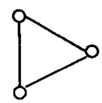
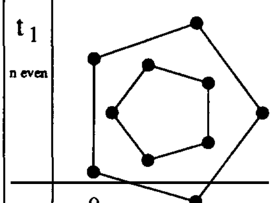
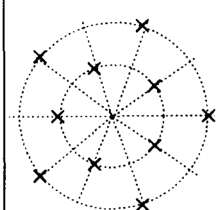
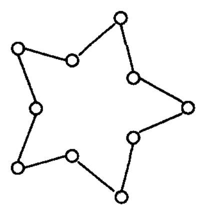
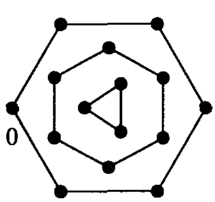
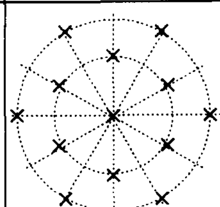
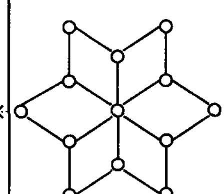
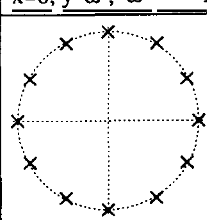
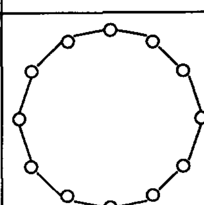
where $w_m^{(2)}(x) = T_m(x)/m$, T_m the Chebishev polynomial of the first kind. This enables us to study the most relevant perturbations of the conformal theory, by the operators $\mathscr{U}_{1,0} = x + \dots$, $\mathscr{U}_{2,0} = x^2 - y + \dots$, $\mathscr{U}_{0,1} = y + \dots$, with respective degrees 1, 2, 2, and for which the perturbed potentials read

$$\begin{aligned}
x: &\qquad w = y^{n/2} \frac{T_n\left(x/\sqrt{y}\right)}{n} - t_{1,0} x, \\
y: &\qquad w = y^{n/2} \frac{T_n\left(x/\sqrt{y}\right)}{n} - t_{0,1} y, \\
x^2 - y: &\qquad w = y^{n/2} \frac{T_n\left(x/\sqrt{y}\right)}{n} - t_{2,0}\left(x^2 - y\right).
\end{aligned}$$

For these perturbations, we worked out the perturbed ring and found that only $\mathscr{U}_{1,0} = x$ and $\mathscr{U}_{0,1} = y$ were normalizable, together with the Chebishev perturbation by $\mathscr{U}_{k,0}$. On the other hand, we computed the extrema of the potential, and found striking similarities between the dual of the normalized ring and the positions of the extrema and values taken by the potential at those. The results are collected in pictorial form on table 2. To comment briefly, the "Chebishev" perturbation by t_{n-3} leads to a set of extrema in the x-plane inside a three-cusp hypocycloid, a deformed version of the Weyl chamber of level $n-3$. The potential takes three possible values according to the triality of the ground state, on the vertices of an equilateral triangle. The perturbation by t_1 coupled to x leads to two possible pictures depending on the parity of n, because of parity properties of Chebishev polynomials. For n even, the extrema in the x-plane lie at the vertices of $\frac{1}{2}(n-2)$ concentric regular $(n-1)$-gons. For n odd, they are on $\frac{1}{2}(n-3)$ concentric $(n-1)$-gons while the origin is $\frac{1}{2}(n-1)$ times degenerate (the latter degeneracy is lifted by the y-coordinate). In either case, the dual generators reproduce these features. Finally, another case that may be discussed easily is the t_2' perturbation by y (see table 2).

TABLE 2

Normalizable perturbations of the diagonal $\mathscr{A}^{(n=k+3)}$ series of the Kazama–Suzuki cosets $SU(3)_k \times SO(4)_1/(SU(2)_{k+1} \times U(1))$. The columns are organized as on table 1. The first perturbation $t_k = t_{n-3} \equiv t_{k,0,\dots,0}$ is the "Chebishev" one, the corresponding third column displays the weight diagram of $SU(3)_k$ (generalization of the A Dynkin diagram) and should be understood as oriented in order for each elementary triangle to be itself oriented anti-clockwise. The other cycles of the third column have to be understood as oriented anti-clockwise.

$A^{(n)}$	t_{n-3}	 0	 x plane, $y=\bar{x}$		✓
	t_1 n even	 0	 x plane, $y=\bar{x}$		✓
	t_1 n odd	 0	 x plane, $y=\bar{x}$, $x\neq 0$, $\underline{x=0}$, $\underline{y=\omega}$, $\underline{\omega^{n-1/2}=1}$		✓
	t'_2	same as t_1	 y plane		?

Note again that, like in the SU(2) case, the values taken by the perturbed potential in the complex plane can be all linked in a connected graph with only straight segments of the same length, corresponding to the mass of the unique minimal soliton interpolating between the nearest-neighbouring vacua.

Beyond the SU(3) case, it might be possible to investigate some (relevant enough) perturbations in the general SU(N), based on the following natural conjecture. Rearranging the SU(2) type-A perturbed potentials of ref. [11], one can derive the following generating function:

$$\sum_{m\geqslant 0} u^m W_m^{(2)}(x,\tau_2,\tau_3,\ldots) = -\log\left(1-ux+u^2\tau_2+u^3\tau_3+\ldots\right), \qquad (5.3)$$

where for a given level $k=m-2$, we retain only the couplings $\tau_2=t_k$, $\tau_3=t_{k-1},\ldots,\tau_{k+2}=t_0$. The Chebishev potentials are obtained by taking $\tau_2=1$, and $\tau_p=0$ for $p>2$. Based on the Chebishev and $(1,0,\ldots,0)$ perturbation cases, we conjecture that this can serve also as a generating function for certain perturbations of the SU(N) potentials, by substituting in (5.3) $x\equiv x_1$, $\tau_2\equiv x_2$, $\tau_3\equiv -x_3,\ldots,\tau_{N-1}\equiv(-1)^{N-1}x_{N-1}$, and identifying the remaining couplings as perturbations ($\tau_p=t_{k+N-p,\,0,\ldots,0}$ couples to the $\mathscr{U}_{k+N-p,\,0,\ldots,0}$ operator).

5.2. FAKE NON-ADE CASES

Let us return to the few SU(N) cases which avoid the appearance of modules.

SU(N)$_1$ *case*. It is easy to see that the general perturbed potential takes the form

$$W_{N+1}^{(N)}(x_1,\ldots,x_{N-1}) = w_{N+1}^{(N)}(x_1,\ldots,x_{N-1}) - \sum_{j=1}^{N-1} s_j x_j - s_0.$$

Working out the perturbed ring, we find that it is isomorphic to the perturbed ring of the SU(2)$_{N-1}$ A$_N$ theory for some special coordinates s_j, the identification of the basis elements being $x_j\equiv(0,\ldots,0,\,1,\,0,\ldots,0)\to(j)\equiv x^j$ (if the 1 is in jth position in the SU(N)$_1$ weight).

SU(3)$_2$ *case*. As mentioned in the previous section, the perturbed potential, of degree 5, involves only couplings of positive dimensions 1, 2, 3, 3, 4, 5, to operators with respective dimensions 4, 3, 2, 2, 1, 0 (see appendix C for the complete expression.). The latter match exactly those of the D$_6$ model of SU(2) at level 8. It is actually straightforward to find the isomorphism between the corresponding perturbed rings. It involves a change of basis of the ring, preserving the initial grading (hence allowing for rotations in the two-dimensional space of dimension-two ring elements); accordingly the parameters $t_{0,0}$, $t_{1,0}$, $t_{1,1}$, $t_{0,2}$ are proportional to the D$_6$ parameters t_0, t_2, t_6, t_8, whereas $t_{2,0}$, $t_{0,1}$ are proportional to $t_4\pm i\tau_4$. In this sense, SU(3)$_2$ is within the A, D, E classification.

It is then an easy matter to examine what are the normalizable perturbations of that case, involving only one non-zero flat coordinate *in the* SU(3) *language*. One finds that the solutions are $t_{1,0} \neq 0$, or $t_{0,1} \neq 0$, or $t_{2,0} \neq 0$ or $t_{0,2} \neq 0$, thus excluding $t_{1,1}$. The first two are just particular cases of the discussion above, the $t_{2,0}$ perturbation is the "Chebishev" one leading to the $SU(3)_2$ fusion potential, and $t_{0,2}$ is the least relevant perturbation. The first and the last have been already found in the discussion of D_6. The perturbation by $t_{2,0}$ that gives a chiral ring isomorphic to the fusion ring of $SU(3)_2$ is also known to be integrable [26]. Only the $t_{0,1}$ perturbation had not previously been recognized as integrable. In a recent calculation to first order, it has been checked that this perturbation admits indeed a spin-three conserved quantity [8].

6. Discussion

In this paper, we have explored a special class of perturbations of $N=2$ superconformal theories, in which the basis of the chiral ring is made of what we called normalizable matrices. We showed that this property is fairly restrictive, and that for the ADE potentials, it allows only a finite and small number of perturbations, *if we insist on perturbations in which only one flat coordinate is non-vanishing*. We have then shown that this normalizability property leads naturally to the consideration of the algebra dual to the original chiral one; except in a few cases, this dual algebra is well defined and admits a basis made of matrices with non-negative entries (type I) or in which at least one matrix has this property (non-type I). In all those cases, such a matrix may be regarded as the adjacency matrix of a graph. The surprising empirical fact, that generalizes an observation by Lerche and Warner, is that this graph resembles the pattern of the extrema of the potential in coordinate space. This implies that there is a natural action of the dual algebra on these extrema, namely on the ground states of the theory. Finally, a last empirical observation is that there seems to be a connection between the integrability of the theory and this normalizability condition: more precisely, *all* the known integrable perturbations of $N=2$ theories with an ADE Landau–Ginzburg potential and a few others have been found among the normalizable perturbations. It is tempting to conjecture that there is an identity between the two classes. In other words, normalizability could be a criterion of integrability.

We now want to discuss this and related questions raised by the previous findings.

(i) What is the meaning of the normalizability condition? This condition has been introduced on a technical ground, namely to allow the diagonalization of the chiral ring in an orthogonal basis and the construction of the dual ring. Clearly a more physical interpretation would be desirable. Let us point out that this condition is *stronger* than the condition that the singularity has been fully resolved (in a physical language, that all the degeneracy of the extrema has been lifted, and the theory describes only massive excitations). Indeed, the normalizability condition implies that the coordinates of the extrema are expressible in terms of the eigenvectors according to (2.9); the independence of the latter implies the non-degeneracy of the former. Conversely, it is easy to see that the perturbation by $t_4 = t \neq 0$ of the A_6 potential, viz. $W = \frac{1}{7}x^7 - tx^4 + t^2x$ is a full resolution of the x^7 singularity, but by the theorem of sect. 4.1, the matrix C_1 of (4.2) is not normalizable.

(ii) What is the good justification of keeping only a single $t \neq 0$? Certainly the introduction of a single non-zero parameter t_i, hence of a single perturbing operator, is the simplest and most natural thing to do. The situation is confused, however, by the existence of some normalizable cases involving *several* non-vanishing parameters $t_\cdot$, presumably non-integrable. For example, all the matrices of the chiral ring of the perturbed A_3 theory are normalizable for arbitrary t_1 and t_2. It is very unlikely that all these perturbations are integrable! Yet another example is provided by a class of fusion potentials. Whenever a potential is known to be the fusion potential of a rational conformal field theory, it certainly satisfies the normalizability condition in a suitable basis. Such is the case of the A_6 potential perturbed by $t_4 = 1$ and $t_1 = 2$. In ref. [14], it has been showed that this potential $W = \frac{1}{7}x^7 - x^4 - x$ provides a one-variable representation of the fusion ring $SU(3)_2$. Although the matrix C_1 of (4.2) that encodes the multiplication by x in the ordinary basis $1, x, x^2, x^3 - 1, x^4 - 2x, x^5 - 3x^2$ of the A_6 case is *not* normalizable, after a change of basis to the basis

$$1,\ x \tfrac{1}{2}(x^5 - 3x^2),\ \tfrac{1}{2}(5x^2 - x^5),\ \tfrac{1}{2}(x^3 - 1),\ \tfrac{1}{2}(x^4 - 3x)$$

it reads

$$C_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \tag{6.1}$$

and is normal. Clearly then the M and N algebras are isomorphic (we are in a fusion case, hence self-dual). It is amusing to see again that the location of extrema in the x-plane reproduces the pattern of the integrable weights of $SU(3)_2$. However, it is doubtful that this corresponds to an integrable perturbation of the A_6 theory. Note that this instance illustrates the possibility of attaching several consistent gradings to the chiral rings of a given potential.

Another example is provided by the $Sp(2)_2$ case. The potentials for the $Sp(N)_k$

fusion algebra have been worked out [34,35]. We choose to present now the case of $N=k=2$, due to its relation to the $SU(3)_2$ case. The potential reads

$$W=\tfrac{1}{5}x^5-x^3y+xy^2+xy-x.$$

Comparing with the $SU(3)_2$ general perturbations of appendix C, we find that this is a special perturbation by $\mathscr{U}_{1,1}=xy+\dots$ *and* $\mathscr{U}_{1,0}=x$ *simultaneously*, corresponding to $t_{1,1}=-1$ and $t_{1,0}=\frac{1}{2}$, the other t's being zero. As the fusion ring of a WZW model, this point is normalizable, but corresponds again to a perturbation mixing two directions. In this case too, the integrability of the $N=2$ theory described by this potential has not been established, to the best of our knowledge.

(i) It seems therefore that a refined version of our conjecture should be: the normalizability of a perturbation by a single flat coordinate is equivalent to integrability.

(ii) Although we have used the language of potentials and polynomial representation, it must be clear that the issue of normalizability depends only on the structure constants and may therefore also be addressed in cases where no potential is available. We hope to return to such instances in a near future.

(iii) What may be the origin of such an alleged connection between integrability and normalizability? The form of the C_1 matrix in the simplest cases (see (4.2)) suggests a possible connection with generalized Toda theories and/or hamiltonian reduction. This too will be left to future investigation.

(iv) What is the physical meaning of the graph and/or of the dual algebra? The existence of the dual ring, with a basis labelled in the same way as the ground states, means that one may define a ring structure on these ground states. What is the meaning of this ring? The whole discussion has some features reminiscent of a recent discussion by Cecotti and Vafa [36]. These authors have been able to relate the counting of solitons (weighted by their fermionic number) interpolating between pairs of ground states with the intersection numbers of homology cycles of the (perturbed) potential. Their discussion, contrary to ours, is not limited to the integrable or normalizable perturbations. With this additional assumption, we are able to obtain quite explicit formulae and new results on the pattern of ground states. It would be quite interesting to understand if our results have any bearing on that more general and systematic approach.

(v) Is there a conformal field theory associated with the integrable cases, in the sense that there is a subring of the chiral ring isomorphic to the fusion ring of that conformal theory as in the Chebishev cases? In all the other ADE cases that we have encountered, there was always a cyclic $\mathbb{Z}_N$ subring. In that sense, one may say that there was an underlying $SU(N)_1$ conformal theory, but it is not clear what is gained from that.

Special thanks are due to Nick Warner for his indefatigable patience in explaining us the beauties and intricacies of $N=2$ theories. Also, we have benefited from discussions with D. Gepner, C. Itzykson, W. Lerche, D. Nemeschansky and N. Sochen, F. Lesage is supported by a Canadian NSERC 67 scholarship.

Appendix A

THE D_{n+2} POTENTIAL AND FREE ENERGY

The heuristic idea to connect the D_{n+2} and the A_{2n+1} is to take an orbifold of the latter. We follow here a route slightly different from ref. [11]. Suppose that only the t_{even} parameters are non-vanishing in the A_{2n+1} potential $W_A(x', t.)$, which is thus an even function of x', $W_A(x', t.) = V(x'^2, t.)$. We imagine that this potential is used as an action, i.e. in an exponentiated form as a weight in integrals.

$$\langle f \rangle = \int \mathrm{d}x' \exp(-W_A(x', t.)) f(x'). \tag{A.1}$$

If one restricts oneself to even functions of x', $f(x') = F(x'^2)$, one may perform the $x'^2 \to x$ change of variables, and up to irrelevant factors and discarding all problems of convergence

$$\langle f \rangle = \int \frac{\mathrm{d}x}{\sqrt{x}} \exp(-V(x)) F(x)$$

$$= \int \mathrm{d}x \, \mathrm{d}y \exp(-V(x) - \tfrac{1}{2}y^2 x) F(x). \tag{A.2}$$

The orbifold D_{n+2} potential is thus identified as the term in the exponential; the jacobian of the transformation has forced us to introduce a new variable y, and after inserting an extra deformation parameter τ coupled to y, the D_{n+2} potential reads

$$W_{D_{n+2}}(x, y; t_0, t_2, \dots, t_{2n}, \tau) = W_{A_{2n+1}}(x' = \sqrt{x}, t_0, 0, t_2, \dots, t_{2n}) + \tfrac{1}{2}xy^2 + \tau y.$$

$$= \frac{x^{n+1}}{2(n+1)} + \dots. \tag{A.3}$$

It is a quasihomogeneous polynomial of x (degree 2), y (degree n) and the t's. The free energy F of these D-models may also be determined explicitly in terms of the

A-one. Expressing the multiplication of the polynomials $p_i(x, t_.)$ and $p_n' = y$ in terms of that of the $p_{2i}^{(A)}(x', t_.)$ modulo $\partial_x W$, $\partial_y W$, one finds that

$$F_{D_{n+2}}(t_0, t_2, \dots, t_{2n}, \tau) = F_{A_{2n+1}}(t_0, 0, t_2, \dots, t_{2n}) - \tfrac{1}{2}\tau^2 \Phi(t_0, \dots, t_{2n}). \quad (A.4)$$

We recall that expressions for F_A or its partial derivatives with respect to the t's have been given in refs. [11,37]; the function Φ is a polynomial in $t_0, \dots, t_{2n}$ that is determined by its partial derivatives: $-\partial/\partial t_i \Phi$ is the coefficient of p_{n-i} in the expansion of $2V'(x)$. Using the relation $W'_{A_{2n+1}}(x') = p_{2n+1}(x')$ and recursion formulae between the p's, one finds

$$-\frac{\partial}{\partial t_i}\Phi = \sum_{\substack{r \geqslant 0,\, j_p \geqslant 0 \\ r + \Sigma j_p = i}} (-1)^r \prod_{p=1}^{r} t_{2n-2j_p}. \quad (A.5)$$

As a side remark, we want to comment the positivity properties of the coefficients of the resulting F. While all the monomials of F_A have positive coefficients, at first sight the monomials of the polynomial Φ seem to have either sign. We have checked in the cases D_4 and D_6, and it is very likely to be true for general D_{n+2}, n *even*, that one may rewrite F with positive signs only in terms of $t_0, t_2, \dots, t_{n-2}$, $t_{n+2}, \dots, t_{2n}$, and $t_{\pm} = \sqrt{\tfrac{1}{2}}(t_n \pm i^{n/2-1}\tau)$. This is of interest in view of our earlier observation that the chiral ring is a generalization of the M algebra associated with the graph, and that the latter has non-negative structure constants only for D_{even}. It tells us the change of basis to be performed to deal with an algebra with positive structure constants. In contrast for the D_{odd} cases, the signs in F are irreducible. One can see that the same positivity properties of the coefficients of F holds for the A_n, E_6 and E_8 cases [22], but not for E_7. Thus this is one more manifestation of the type I–non-type I distinction alluded to above.

Appendix B

THE PERTURBED POTENTIALS OF THE $E_{6,7,8}$ CASES

The E_6 potential reads

$$\begin{aligned} W = {} & \tfrac{1}{3}x^3 + \tfrac{1}{4}y^4 - t_{10}xy^2 - t_7 xy - \left(t_6 - \tfrac{1}{2}t_{10}^3\right)y^2 \\ & - \left(\tfrac{1}{12}t_{10}^4 - t_6 t_{10} + t_4\right)x - \left(t_3 - t_7 t_{10}^2\right)y \\ & - \tfrac{1}{6}t_6 t_{10}^3 + \tfrac{1}{2}t_4 t_{10}^2 + \tfrac{1}{2}t_7^2 t_{10} + \tfrac{1}{2}t_6^2 - t_0. \end{aligned} \quad (B.1)$$

The E_7 potential is

$$W = \tfrac{1}{3}x^3 + xy^3 - t_{16}x^2y - \left(t_{12} - \tfrac{4}{27}t_{16}^3\right)x^2 - \left(t_{10} + \tfrac{1}{27}t_{16}^4 - \tfrac{4}{3}t_{12}t_{16}\right)xy$$
$$- t_8 y^2 - \left(t_6 - \tfrac{1}{729}t_{16}^6 + \tfrac{5}{27}t_{12}t_{16}^3 - \tfrac{5}{18}t_{10}t_{16}^2 - \tfrac{1}{3}t_8t_{16} - \tfrac{5}{6}t_{12}^2\right)x$$
$$- \left(t_4 - \tfrac{1}{162}t_{12}t_{16}^4 + \tfrac{1}{54}t_{10}t_{16}^3 + \tfrac{1}{6}t_{12}^2t_{16} - \tfrac{1}{3}t_6t_{16} - \tfrac{1}{3}t_{10}t_{12}\right)y$$
$$- t_0 + \tfrac{1}{118098}t_{16}^9 - \tfrac{1}{1458}t_{12}t_{16}^6 + \tfrac{1}{81}t_8t_{16}^4 + \tfrac{1}{27}t_{12}^2t_{16}^3 - \tfrac{1}{9}t_{10}t_{12}t_{16}^2 + \tfrac{1}{9}t_4t_{16}^2$$
$$- \tfrac{4}{9}t_8t_{12}t_{16} + \tfrac{1}{9}t_{10}^2t_{16} - \tfrac{4}{27}t_{12}^3 + \tfrac{2}{3}t_6t_{12} + \tfrac{1}{3}t_8t_{10}. \qquad \text{(B.2)}$$

Finally, the E_8 potential reads

$$W(x, y, t.)$$
$$= \tfrac{1}{3}x^3 + \tfrac{1}{5}y^5 - s_{28}xy^3 - s_{22}xy^2 - s_{18}y^3 - s_{16}xy - s_{12}y^2 - s_{10}x - s_6y - s_0, \qquad \text{(B.3)}$$

where

$$s_0 = t_0 - t_{28}^3t_6 - \tfrac{103}{450}t_{28}^{15} + \tfrac{421}{450}t_{22}t_{28}^{11} + \tfrac{14}{45}t_{18}t_{28}^9 - \tfrac{29}{30}t_{16}t_{28}^8 - \tfrac{43}{45}t_{22}^2t_{28}^7$$
$$+ \tfrac{7}{15}t_{12}t_{28}^6 + \tfrac{4}{5}t_{10}t_{28}^5 + \tfrac{5}{3}t_{16}t_{22}t_{28}^4 + \tfrac{1}{3}t_{22}^3t_{28}^3 + \tfrac{1}{2}t_{18}^2t_{28}^3 + \tfrac{1}{2}t_{16}t_{18}t_{28}^2$$
$$+ \tfrac{1}{2}t_{18}t_{22}^2t_{28} - t_{10}t_{22}t_{28} - \tfrac{1}{2}t_{16}^2t_{28} - \tfrac{43}{30}t_{18}t_{22}^6 - \tfrac{1}{2}t_{16}t_{22}^2 - t_{12}t_{18},$$
$$s_6 = t_6 + \tfrac{82}{75}t_{28}^{12} - \tfrac{107}{30}t_{22}t_{28}^8 - \tfrac{7}{30}t_{18}t_{28}^6 + \tfrac{12}{5}t_{16}t_{28}^5 + \tfrac{7}{3}t_{22}^2t_{28}^4 - \tfrac{4}{3}t_{12}t_{28}^3$$
$$+ 2t_{18}t_{22}t_{28}^2 - t_{10}t_{28}^2 - 2t_{16}t_{22}t_{28} - \tfrac{1}{3}t_{22}^3 - t_{18}^2,$$
$$s_{10} = t_{10} - \tfrac{11}{45}t_{28}^{10} - \tfrac{2}{15}t_{22}t_{28}^6 + \tfrac{3}{2}t_{18}t_{28}^4 - \tfrac{1}{2}t_{16}t_{28}^3 + \tfrac{1}{2}t_{22}^2t_{28}^2 - t_{12}t_{28} - t_{18}t_{22},$$
$$s_{12} = t_{12} - \tfrac{28}{15}t_{28}^9 + \tfrac{23}{5}t_{22}t_{28}^5 - t_{18}t_{28}^3 - \tfrac{3}{2}t_{16}t_{28}^2 - \tfrac{3}{2}t_{22}^2t_{28},$$
$$s_{16} = t_{16} + \tfrac{19}{15}t_{28}^7 - \tfrac{2}{3}t_{22}t_{28}^3 - 2t_{18}t_{28},$$
$$s_{18} = t_{18} + \tfrac{6}{5}t_{28}^6 - 2t_{22}t_{28}^2,$$
$$s_{22} = t_{22} - 2t_{28}^4,$$
$$s_{28} = t_{28}. \qquad \text{(B.4)}$$

Appendix C

THE $SU(3)_2$ POTENTIAL AND FREE ENERGY

The $SU(3)_2$ perturbed potential is parametrized as follows:

$$\begin{aligned}W(x_1, x_2, t.) = &\left[\tfrac{1}{5}x_1^5 - x_1^3x_2 + x_1x_2^2\right]\\ &- t_{02}x_2^2 - t_{11}x_1x_2 - (t_{11}t_{02} + t_{20})x_1^2 - \left(t_{01} - t_{20} - t_{02}{}^3 - t_{11}t_{02}\right)x_2\\ &- \left(-\tfrac{1}{2}t_{11}t_{02}{}^2 + t_{20}t_{02} + \tfrac{1}{2}t_{11}{}^2 + t_{10}\right)x_1\\ &+ t_{02}{}^2t_{01} - \tfrac{3}{10}t_{02}{}^5 - t_{20}t_{11} - t_{00}.\end{aligned}$$

The polynomials $p_i(x_1, x_2, t.) = -\partial W/\partial t_i$ form a basis of the chiral ring with structure constants derived from the free energy

$$\begin{aligned}F = &\tfrac{1}{2}t_{00}{}^2t_{02} + t_{00}t_{10}t_{11} + \tfrac{1}{2}t_{00}\left(t_{20}{}^2 + t_{01}{}^2\right) + \tfrac{1}{2}t_{10}{}^2(t_{20} + t_{01}) - \tfrac{1}{2}t_{10}{}^2t_{11}t_{02}\\ &+ \tfrac{1}{6}t_{10}{}^2t_{02}{}^3 - t_{10}t_{20}t_{01}t_{02} - \tfrac{1}{2}t_{10}(t_{20} + t_{01})t_{11}{}^2 + \tfrac{1}{2}t_{10}(t_{20} + t_{01})t_{11}t_{02}{}^2\\ &+ \tfrac{1}{6}t_{10}t_{11}{}^3t_{02} + \tfrac{1}{6}\left(t_{20}^3 + t_{01}^3\right)t_{02}{}^2 - \tfrac{1}{2}t_{20}t_{01}(t_{20} + t_{01})t_{11} + \tfrac{1}{2}\left(t_{20}{}^2 + t_{20}t_{01} + t_{01}^2\right)t_{11}{}^2t_{02}\\ &+ \tfrac{1}{20}\left(t_{20}{}^2 + t_{01}^2\right)t_{02}{}^5 - \tfrac{1}{2}t_{20}t_{01}t_{11}t_{02}{}^3 + \tfrac{1}{8}(t_{20} + t_{01})t_{11}{}^4 - \tfrac{1}{4}(t_{20} + t_{01})t_{11}{}^3t_{02}{}^2\\ &+ \tfrac{1}{8}(t_{20} + t_{01})t_{11}{}^2t_{02}{}^4 - \tfrac{3}{40}t_{11}{}^5t_{02} + \tfrac{1}{8}t_{11}{}^4t_{02}{}^3 - \tfrac{1}{24}t_{11}{}^3t_{02}{}^5 + \tfrac{1}{56}t_{11}{}^2t_{02}{}^7 + \tfrac{1}{3960}t_{02}{}^{11},\end{aligned}$$

and the only non-vanishing $\eta_{ij} = \eta_{ji}$ are given by

$$\eta_{(00)(02)} = \eta_{(10)(11)} = 1,$$

$$\eta_{(20)(20)} = \eta_{(01)(01)} = 1.$$

Note that F is symmetric under the interchange $t_{20} \leftrightarrow t_{01}$.

Upon restriction to $t_{00} = t_{10} = t_{11} = t_{02} = 0$, $t_{20} = -1$, $t_{01} = 0$, the potential reduces to

$$W = \tfrac{1}{5}x_1^5 - x_1^3x_2 + x_1x_2^2 + x_1^2 - x_2$$

and the polynomials to

$$p_{00} = 1, \quad p_{10} = x_1,$$

$$p_{20} = x_1^2 - x_2, \quad p_{01} = x_2,$$

$$p_{11} = x_1x_2 - 1, \quad p_{02} = x_2^2 - x_1,$$

that are the polynomials that represent the fusion ring of $SU(3)_2$.

There is a simple change of variables that maps these expressions to those pertaining to D_6. Let

$$x_1 = a(x - t_4), \qquad x_2 = y + \tfrac{1}{2}a^2\left(x^2 - 3xt_4 + 3t_4^2 - t_3\right),$$

$$t_{02} = -at_4, \qquad t_{10} = -a^2 t_3,$$

$$t_{20} = -\tfrac{1}{2}a^3 t_2 - a\tau, \qquad t_{01} = -\tfrac{1}{2}a^3 t_2 + a\tau,$$

$$t_{11} = 2t_1, \qquad t_{02} = 2at_0$$

with $a^4 = -4$. Then $W_{SU(3)_2}(x_1, x_2, t_{02}, \dots, t_{00}) = W_{D_6}(x, y, t_4, \dots, t_0)$.

Appendix D

THE FREE ENERGIES OF THE $E_{6,7,8}$ CASES

1. The E_6 free energy. The free energy reads

$$\begin{aligned} F = {} & \tfrac{1}{185\,328}t_{10}^{13} + \tfrac{1}{576}t_7^2 t_{10}^8 + \tfrac{1}{252}t_6^2 t_{10}^7 + \tfrac{1}{24}t_6 t_7^2 t_{10}^5 + \tfrac{1}{60}t_4^2 t_{10}^5 + \tfrac{1}{24}t_4 t_7^2 t_{10}^4 \\ & + \tfrac{1}{24}t_3^2 t_{10}^4 + \tfrac{1}{24}t_7^4 t_{10}^3 + \tfrac{1}{6}t_3 t_6 t_7 t_{10}^3 + \tfrac{1}{6}t_4 t_6^2 t_{10}^3 + \tfrac{1}{4}t_6^2 t_7^2 t_{10}^2 + \tfrac{1}{2}t_3 t_4 t_7 t_{10}^2 \\ & + \tfrac{1}{6}t_3 t_7^3 t_{10} + \tfrac{1}{2}t_4 t_6 t_7^2 t_{10} + \tfrac{1}{12}t_6^4 t_{10} + \tfrac{1}{2}t_3^2 t_6 t_{10} + \tfrac{1}{6}t_4^3 t_{10} + \tfrac{1}{2}t_0^2 t_{10} \\ & + \tfrac{1}{12}t_6 t_7^4 + \tfrac{1}{4}t_4^2 t_7^2 + \tfrac{1}{2}t_3 t_6^2 t_7 + t_0 t_3 t_7 + t_0 t_4 t_6 + \tfrac{1}{2}t_3^2 t_4 . \end{aligned}$$

2. The E_7 free energy

$$\begin{aligned} F = {} & \tfrac{1}{1\,001\,094\,543\,576}t_{16}^{19} + \tfrac{1}{55\,269\,864}t_{12}^2 t_{16}^{13} + \tfrac{1}{5\,196\,312}t_{10}^2 t_{16}^{11} - \tfrac{1}{1\,417\,176}t_{12}^3 t_{16}^{10} + \tfrac{1}{157\,464}t_{10} t_{12}^2 t_{16}^9 \\ & + \tfrac{1}{236\,196}t_8^2 t_{16}^9 - \tfrac{1}{52\,488}t_8 t_{12}^2 t_{16}^8 + \tfrac{1}{104\,976}t_{10}^2 t_{12} t_{16}^8 + \tfrac{7}{157\,464}t_{12}^4 t_{16}^7 + \tfrac{1}{8\,748}t_8 t_{10} t_{12} t_{16}^7 \\ & + \tfrac{1}{52\,488}t_{10}^3 t_{16}^7 + \tfrac{1}{10\,206}t_6^2 t_{16}^7 - \tfrac{7}{52\,488}t_{10} t_{12}^3 t_{16}^6 + \tfrac{1}{2\,916}t_6 t_{10} t_{12} t_{16}^6 - \tfrac{1}{2\,916}t_8^2 t_{12} t_{16}^6 \\ & - \tfrac{1}{5\,832}t_8 t_{10}^2 t_{16}^6 + \tfrac{1}{972}t_8 t_{12}^3 t_{16}^5 + \tfrac{1}{972}t_{10}^2 t_{12}^2 t_{16}^5 - \tfrac{1}{486}t_6 t_8 t_{12} t_{16}^5 + \tfrac{1}{972}t_6 t_{10}^2 t_{16}^5 + \tfrac{1}{810}t_4^2 t_{16}^5 \\ & - \tfrac{7}{11\,664}t_{12}^5 t_{16}^4 + \tfrac{5}{2\,916}t_6 t_{12}^3 t_{16}^4 - \tfrac{5}{972}t_8 t_{10} t_{12}^2 t_{16}^4 + \tfrac{5}{5\,832}t_{10}^3 t_{12} t_{16}^4 + \tfrac{1}{162}t_4 t_8 t_{12} t_{16}^4 \\ & + \tfrac{1}{324}t_6^2 t_{12} t_{16}^4 + \tfrac{1}{648}t_4 t_{10}^2 t_6^4 + \tfrac{1}{162}t_6 t_8 t_{10} t_{16}^4 - \tfrac{1}{486}t_8^3 t_{16}^4 + \tfrac{25}{5\,832}t_{10} t_{12}^4 t_{16}^3 \\ & - \tfrac{1}{486}t_4 t_{12}^3 t_{16}^3 + \tfrac{1}{54}t_8^2 t_{12}^2 t_{16}^3 + \tfrac{1}{162}t_8 t_{10}^2 t_{12} t_{16}^3 + \tfrac{1}{27}t_4 t_6 t_{12} t_{16}^3 + \tfrac{1}{486}t_{10}^4 t_{16}^3 \end{aligned}$$

$$
\begin{aligned}
&-\tfrac{1}{54}t_4t_8t_{10}t_{16}^3+\tfrac{1}{54}t_6^2t_{10}t_{16}^3-\tfrac{1}{72}t_8t_{12}^4t_{16}^2-\tfrac{1}{324}t_{10}^2t_{12}^3t_{16}^2+\tfrac{1}{36}t_4t_{10}t_{12}^2t_{16}^2\\
&+\tfrac{1}{18}t_6t_8t_{12}^2t_{16}^2+\tfrac{1}{18}t_6t_{10}^2t_{12}t_{16}^2-\tfrac{1}{18}t_8^2t_{10}t_{12}t_{16}^2-\tfrac{1}{18}t_4^2t_{12}t_{16}^2+\tfrac{1}{18}t_4t_6t_{10}t_{16}^2\\
&+\tfrac{1}{18}t_4t_8^2t_{16}^2-\tfrac{1}{18}t_6^2t_8t_{16}^2+\tfrac{1}{486}t_{12}^6t_{16}-\tfrac{1}{108}t_6t_{12}^4t_{16}+\tfrac{1}{18}t_8t_{10}t_{12}^3t_{16}\\
&+\tfrac{1}{54}t_{10}^3t_{12}^2t_{16}-\tfrac{1}{6}t_4t_8t_{12}^2t_{16}+\tfrac{1}{9}t_6^2t_{12}^2t_{16}-\tfrac{1}{9}t_6t_8t_{10}t_{12}t_{16}+\tfrac{2}{27}t_8^3t_{12}t_{16}\\
&+\tfrac{1}{27}t_6t_{10}^3t_{16}+\tfrac{1}{18}t_8^2t_{10}^2t_{16}+\tfrac{1}{6}t_4^2t_{10}t_{16}+\tfrac{1}{3}t_4t_6t_8t_{16}+\tfrac{1}{9}t_6^3t_{16}\\
&+\tfrac{1}{2}t_0^2t_{16}-\tfrac{1}{216}t_{10}t_{12}^5+\tfrac{1}{72}t_4t_{12}^4+\tfrac{1}{18}t_6t_{10}t_{12}^3-\tfrac{2}{27}t_8^2t_{12}^3\\
&-\tfrac{1}{18}t_8t_{10}^2t_{12}^2-\tfrac{1}{6}t_4t_6t_{12}^2+\tfrac{1}{108}t_{10}^4t_{12}+\tfrac{1}{3}t_4t_8t_{10}t_{12}+\tfrac{1}{6}t_6^2t_{10}t_{12}+\tfrac{1}{3}t_6t_8^2t_{12}\\
&+t_0t_4t_{12}+\tfrac{1}{18}t_4t_{10}^3-\tfrac{1}{18}t_8^3t_{10}+t_0t_6t_{10}-\tfrac{1}{2}t_0t_8^2-\tfrac{1}{2}t_4^2t_8+\tfrac{1}{2}t_4t_6^2
\end{aligned}
$$

3. The E_8 free energy. The function F has the following simple expression:

$$
\begin{aligned}
F=&\tfrac{1}{210}t_{28}^7t_6^2+\tfrac{1}{6}t_{22}t_{28}^3t_6^2+\tfrac{1}{2}t_{18}t_{28}t_6^2+\tfrac{1}{2}t_{16}t_6^2+t_0\left(t_{22}t_6+t_{10}t_{18}+t_{12}t_{16}\right)\\
&+t_{16}\left(\tfrac{1}{60}t_{18}t_{28}^6+\tfrac{1}{2}t_{18}t_{22}t_{28}^2+\tfrac{1}{2}t_{18}^2\right)t_6+\tfrac{1}{45}t_{12}t_{22}t_{28}^6t_6+\tfrac{1}{60}t_{18}t_{22}^2t_{28}^5t_6\\
&+\tfrac{1}{16}t_{10}t_{22}t_{28}^5t_6+\tfrac{1}{2}t_{16}^2\left(\tfrac{1}{20}t_{28}^5+t_{22}t_{28}\right)t_6+t_{16}\left(\tfrac{1}{12}t_{22}^2t_{28}^4+\tfrac{1}{6}t_{22}^3\right)t_6\\
&+\tfrac{1}{6}t_{12}t_{18}t_{28}^4t_6+\tfrac{1}{72}t_{22}^4t_{28}^3t_6+\tfrac{1}{6}t_{18}^2t_{22}t_{28}^3t_6+\tfrac{1}{6}t_{12}t_{16}t_{28}^3t_6+\tfrac{1}{2}t_{12}t_2^2t_{28}^2t_6\\
&+\tfrac{1}{2}t_{10}t_{16}t_{28}^2t_6+\tfrac{1}{6}t_{18}t_{22}^3t_{28}t_6+\tfrac{1}{2}t_{10}t_{22}^2t_{28}t_6+\tfrac{1}{2}t_{12}^2t_{28}t_6+t_{12}t_{18}t_{22}t_6\\
&+t_{10}t_{12}t_6+\tfrac{1}{245\,764\,125\,000}t_{128}^3+\tfrac{1}{27\,945\,000}t_{22}^2t_{28}^{23}\\
&+t_{22}^3\left(\tfrac{1}{729\,000}t_{28}^{19}+\tfrac{2}{30\,375}t_{22}t_{28}^{15}+\tfrac{13}{10\,800}t_{22}^2t_{28}^{11}+\tfrac{11}{1\,080}t_{22}^2t_{28}^{11}+\tfrac{1}{54}t_{22}^4t_{28}^3\right)\\
&+t_{18}^2\left(\tfrac{1}{1\,539\,000}t_{28}^{19}+\tfrac{1}{360}t_{22}^2t_{28}^{11}+\tfrac{7}{180}t_{22}^3t_{28}^7+\tfrac{17}{72}t_{22}^4t_{28}^3\right)+\tfrac{1}{459\,000}t_{16}^2t_{28}^{17}\\
&+t_{18}t_{22}^2\left(\tfrac{1}{81\,000}t_{28}^{17}+\tfrac{7}{16\,200}t_{22}t_{28}^{13}+\tfrac{77}{6480}t_{22}^2t_{28}^9+\tfrac{7}{90}t_{22}^3t_{28}^5+\tfrac{1}{18}t_{22}^4t_{28}\right)\\
&+\tfrac{1}{2}t_{16}t_{22}^2\left(\tfrac{1}{40\,500}t_{28}^{16}+\tfrac{13}{8\,100}t_{22}t_{28}^{12}+\tfrac{1}{30}t_{22}^2t_{28}^8+\tfrac{7}{36}t_{22}^3t_{28}^4+\tfrac{2}{45}t_{22}^4\right)\\
&+t_{16}t_{18}t_{22}\left(\tfrac{1}{5\,400}t_{28}^{14}+\tfrac{11}{1\,800}t_{22}t_{28}^{10}+\tfrac{7}{45}t_{22}^2t_{28}^6+\tfrac{11}{24}t_{22}^3t_{28}^2\right)\\
&+\tfrac{1}{6}t_{18}^3\left(\tfrac{1}{1\,800}t_{28}^{13}+\tfrac{1}{20}t_{18}t_{28}^7+\tfrac{7}{10}t_{22}^2t_{28}^5+t_{22}^3t_{28}+\tfrac{3}{10}t_{18}^2t_{28}\right)\\
&+\tfrac{1}{2}t_{16}^2t_{22}\left(\tfrac{1}{5\,400}t_{28}^{13}+\tfrac{1}{60}t_{22}t_{28}^9+\tfrac{7}{30}t_{22}^2t_{28}^5+\tfrac{5}{12}t_{22}^3t_{28}\right)\\
&+\tfrac{1}{2}t_{12}^2\left(\tfrac{1}{8\,775}t_{28}^{13}+\tfrac{1}{270}t_{22}t_{28}^9+\tfrac{1}{5}t_{22}^2t_{28}^5+\tfrac{2}{3}t_{22}^3t_{28}\right)
\end{aligned}
$$

$$+ t_{12}t_{18}t_{22}\left(\tfrac{1}{2700}t_{28}^{12} + \tfrac{1}{60}t_{22}t_{28}^{8} + \tfrac{5}{18}t_{22}^{2}t_{28}^{4} + \tfrac{1}{6}t_{22}^{3}\right)$$

$$+ t_{12}t_{16}t_{22}\left(\tfrac{1}{900}t_{28}^{11} + \tfrac{2}{45}t_{22}t_{28}^{7} + \tfrac{4}{9}t_{22}^{2}t_{28}^{3}\right)$$

$$+ \tfrac{1}{2}t_{16}^{2}t_{18}\left(\tfrac{1}{1800}t_{28}^{11} + \tfrac{1}{15}t_{22}t_{28}^{7} + \tfrac{3}{20}t_{18}t_{28}^{5} + \tfrac{7}{6}t_{22}^{2}t_{28}^{3} + t_{18}t_{22}t_{28}\right)$$

$$+ t_{10}^{2}\left(\tfrac{1}{4950}t_{28}^{11} + \tfrac{1}{20}t_{18}t_{28}^{5} + \tfrac{1}{6}t_{22}^{2}t_{28}^{3} + \tfrac{1}{6}t_{10}t_{28} + \tfrac{1}{2}t_{16}t_{22}\right)$$

$$+ t_{10}t_{16}\left(\tfrac{1}{900}t_{22}t_{28}^{10} + \tfrac{7}{180}t_{22}^{2}t_{28}^{6} + \tfrac{1}{3}t_{22}^{3}t_{28}^{2}\right)$$

$$+ t_{12}t_{22}^{3}\left(\tfrac{11}{8100}t_{29}^{10} + \tfrac{7}{270}t_{22}t_{28}^{6} + \tfrac{1}{12}t_{22}^{2}t_{28}^{2}\right)$$

$$+ t_{16}^{3}\left(\tfrac{1}{2700}t_{28}^{10} + \tfrac{7}{360}t_{22}t_{28}^{6} + \tfrac{1}{24}t_{18}t_{28}^{4} + \tfrac{1}{24}t_{16}t_{28}^{3} + \tfrac{1}{4}t_{22}^{2}t_{28}^{2} + \tfrac{1}{3}t_{18}t_{22}\right)$$

$$+ t_{12}t_{16}t_{18}\left(\tfrac{1}{180}t_{28}^{9} + \tfrac{1}{6}t_{22}t_{28}^{5} + \tfrac{1}{3}t_{18}t_{28}^{3} + \tfrac{3}{2}t_{22}^{2}t_{28}\right)$$

$$+ t_{10}t_{18}^{2}\left(\tfrac{1}{360}t_{28}^{9} + \tfrac{1}{6}t_{18}t_{28}^{3} + \tfrac{1}{2}t_{22}^{2}t_{28}\right) + t_{10}t_{22}^{3}\left(\tfrac{1}{1620}t_{28}^{9} + \tfrac{7}{360}t_{22}t_{28}^{5} + \tfrac{1}{30}t_{22}^{2}t_{28}\right)$$

$$+ \tfrac{1}{2}t_{16}t_{18}^{2}\left(\tfrac{1}{20}t_{22}t_{28}^{8} + \tfrac{2}{3}t_{22}^{2}t_{28}^{4} + t_{18}t_{22}t_{28}^{2} + \tfrac{2}{3}t_{22}^{3}\right)$$

$$+ t_{10}t_{12}t_{22}\left(\tfrac{1}{90}t_{28}^{8} + \tfrac{1}{6}t_{22}t_{28}^{4} + \tfrac{1}{3}t_{22}^{2}\right) + t_{10}t_{18}\left(\tfrac{1}{45}t_{22}^{2}t_{28}^{7} + \tfrac{1}{6}t_{22}^{3}t_{28}^{3}\right)$$

$$+ \tfrac{1}{2}t_{12}^{2}t_{18}\left(\tfrac{1}{90}t_{28}^{7} + \tfrac{2}{3}t_{22}t_{28}^{3} + t_{18}t_{28}\right) + t_{10}t_{16}^{2}\left(\tfrac{1}{120}t_{28}^{7} + \tfrac{1}{6}t_{22}t_{28}^{3} + \tfrac{1}{2}t_{18}t_{28} + \tfrac{1}{6}t_{16}\right)$$

$$+ \tfrac{1}{2}t_{12}t_{18}^{2}\left(\tfrac{7}{45}t_{22}t_{28}^{6} + t_{22}^{2}t_{28}^{2} + \tfrac{2}{3}t_{18}t_{22}\right) + t_{12}^{2}t_{16}\left(\tfrac{1}{45}t_{28}^{6} + \tfrac{1}{2}t_{22}t_{28}^{2} + \tfrac{1}{2}t_{18}\right)$$

$$+ t_{10}t_{12}\left(t_{16}\left(\tfrac{1}{30}t_{28}^{5} + t_{22}t_{28}\right) + \tfrac{1}{6}t_{12}t_{28}^{3} + t_{18}t_{22}t_{28}^{2}\right) + t_{10}t_{16}\left(\tfrac{1}{3}t_{18}t_{22}t_{28}^{4} + \tfrac{1}{2}t_{18}t_{22}^{2}\right)$$

$$+ t_{12}t_{16}^{2}\left(\tfrac{1}{4}t_{22}t_{28}^{4} + \tfrac{1}{2}t_{18}t_{28}^{2} + \tfrac{1}{6}t_{16}t_{28} + \tfrac{1}{2}t_{22}^{2}\right) + \tfrac{1}{2}t_{0}^{2}t_{28} + \tfrac{1}{3}t_{12}^{3}t_{22}.$$

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