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A Classification Programme of Generalized Dynkin Diagrams

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The purpose of this note is to present a problem of classification of graphs according to their spectral properties. This problem is encountered in several issues of current interest in mathematical physics. The graphs which appear are generalizations both of the simply laced Dynkin diagrams (i.e. of ADE type) and of fusion graphs drawn on the weight lattices of the $sl(N)$ Lie algebras.

Cette note vise à présenter un problème de théorie des graphes d'intérêt dans plusieurs domaines de physique mathématique contemporaine. Il s'agit de classifier des graphes dont les matrices d'adjacence obéissent certaines conditions, en particulier de spectre. Les graphes qui apparaissent généralisent d'une part les diagrammes de Dynkin ADE, d'autre part les graphes de fusion, tracés sur les réseaux de poids des algèbres $sl(N)$.

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This note aims at attracting the attention of graph theorists to a problem of classification of *oriented* graphs according to their spectral properties. This problem arises in current problems of theoretical physics, and seems to have a number of interesting implications. I now briefly describe the graphs and the context in which they appear.

I first introduce some group theoretic notations. Let $\Lambda_1, \cdots, \Lambda_{N-1}$ be the fundamental weights of $sl(N)$ and let ρ denote their sum. The set of integrable weights of level k (shifted by ρ) of the affine algebra $\widehat{sl}(N)_k$ is [1]

$$\mathcal{P}_{++}^{(k+N)} = \{\lambda = \lambda_1\Lambda_1 + \cdots + \lambda_{N-1}\Lambda_{N-1} | \lambda_i \in \mathbb{N}\,,\ \lambda_i \geq 1,\ \lambda_1 + \cdots \lambda_{N-1} \leq k+N-1\}\ . \quad (1)$$

We shall also make use of the N (linearly dependent) vectors e_i

$$e_1 = \Lambda_1, \qquad e_i = \Lambda_i - \Lambda_{i-1}, \quad i = 2, \cdots, N-1, \qquad e_n = -\Lambda_{N-1}\ , \quad (2)$$

endowed with the scalar product

$$(e_i, e_j) = \delta_{ij} - \frac{1}{N}\ . \quad (3)$$

Let $\mathcal{G}$ be a graph satisfying the following properties

i) $\mathcal{G}$ is connected.

ii) $\mathcal{G}$ is unoriented. To each vertex a is attached a $\mathbb{Z}/N\mathbb{Z}$ grading $\tau(a)$ and the only non-vanishing entries of the adjacency matrix G are between vertices with different values of τ. One may thus split the matrix G into a sum of $N-1$ matrices

$$G = G_1 + G_2 + \cdots + G_{N-1} \quad (4)$$

where G_p is the adjacency matrix describing the edges that connect vertices with τ differing by p, namely

$$(G_p)_{ab} \neq 0 \qquad \text{only if} \qquad \tau(b) = \tau(a) + p \bmod N\ , \quad (5)$$

$$(G_p)_{ab} = (G_{N-p})_{ba} \qquad \text{i.e.} \quad {}^tG_p = G_{N-p}\ . \quad (6)$$

Accordingly, the graph may be regarded as the superposition of $N-1$ (oriented and non necessarily connected) graphs $\mathcal{G}_p$ with adjacency matrices G_p, $p = 1, \cdots, N-1$ on the same set of vertices.

iii) There exists an involution $a \mapsto \bar{a}$ such that $\tau(\bar{a}) = -\tau(a)$ and

$$G_{ab} = G_{\bar{b}\bar{a}} \ . \tag{7}$$

iv) The matrices G_p commute among themselves

$$[G_p, G_q] = 0 \qquad p, q = 1, \cdots N-1 \ ; \tag{8}$$

the real matrices G_p are thus "normal", i.e. they commute with their transposed and are diagonalizable in a common orthonormal basis;

v) The common eigenvectors are labelled by integrable weights $\lambda \in \mathcal{P}_{++}^{(k+N)}$ for some level k, and the corresponding eigenvalues of G_1, G_2, ..., G_{N-1} are given by the following formulae

$$\begin{aligned}
\gamma_1^{(\lambda)} &= \sum_{i=1}^{N} \exp -\frac{2i\pi}{h}(e_i, \lambda) \\
\gamma_2^{(\lambda)} &= \sum_{1\le i<j\le N} \exp -\frac{2i\pi}{h}((e_i + e_j), \lambda) \\
\vdots \qquad & \qquad \vdots \qquad\qquad \vdots \\
\gamma_{N-1}^{(\lambda)} &= \sum_{1\le i_1<\cdots i_{N-1}\le N} \exp -\frac{2i\pi}{h}((e_{i_1} + \cdots + e_{i_{N-1}}), \lambda)
\end{aligned} \tag{9}$$

where $h = k + N$. Some of these λ may occur with multiplicities larger than one. Note that one has $\gamma_{N-p}^{(\lambda)} = \left(\gamma_p^{(\lambda)}\right)^*$ as a consequence of $\sum e_i = 0$.

vi) The weight $\rho = \Lambda_1 + \cdots + \Lambda_{N-1}$ is among these λ, with multiplicity 1: it corresponds to the eigenvector of largest eigenvalue, the so–called Perron–Frobenius eigenvector;

Problem: Classify all graphs satisfying conditions i)–vi).

These conditions are fulfilled by the "fusion" graphs of $\widehat{sl}(N)$. Let us recall that fusion is an associative and commutative multiplication defined on the set $\mathcal{P}_{++}^{(k+N)}$ of eq. (1), analogous to the tensor product of representations for finite dimensional algebras [2]. The fusion coefficients are the analogues of the multiplicities in the decomposition of tensor products into irreducible representations, and hence are non negative integers which may be used as entries of adjacency matrices of graphs. More precisely, we get a solution of the previous problem as follows:

* the set of vertices is the set of weights $\mathcal{P}_{++}^{(k+N)}$, for some k;

* τ is the natural grading of representations of $sl(N)$, i.e. counts the number of boxes of the Young tableau modulo N;

* $\bar{a}$ is the complex conjugate of the representation a;

* the matrix G_p, $p = 1 \cdots, N-1$, encodes the fusion by the representation of weight Λ_p;

* finally the spectral properties follow from the formula of Verlinde [3], that expresses the fusion coefficients in terms of the modular matrix written by Kac and Peterson [4].

For $N = 2$ the problem involves *symmetric* adjacency matrices, hence non-oriented graphs, with two-colourable vertices, a trivial involution and a spectrum of eigenvalues of the form $2\cos\frac{\pi\ell}{h}$, hence between -2 and $+2$. This is a well known problem, whose solution is provided by the simply laced Dynkin diagrams of ADE type [5]. In this case, the values of ℓ are the Coxeter exponents, and h is the Coxeter number of the ADE algebra.

For $N = 3$, the two matrices G_1 and G_2 are transposed of one another and it is thus sufficient to draw the graph $\mathcal{G}_1$. A certain number of graphs have been found satisfying conditions i)–vi), as in Fig. 1 and [6], but already in this case, a complete classification is still missing.

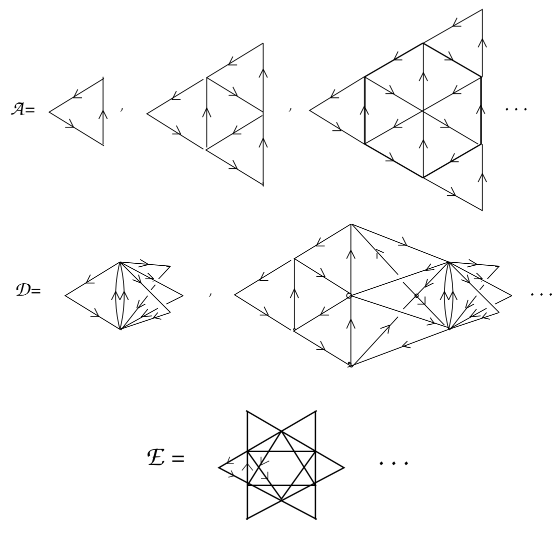

Fig. 1: Some of the graphs $\mathcal{G}_1$ for $N = 3$. (Not all orientations of edges have been shown on the last one; the missing ones are obtained by rotations of $\frac{2\pi p}{6}$.)

Indications on related issues

The classification problem first arose in two distinct but related problems, the classification of two-dimensional conformal field theories based on $sl(N)$ on the one hand, and the construction and classification of integrable lattice models on the other. In the first context, one deals with the characters $\chi_\lambda(q)$ of the $\widehat{sl}(N)$ affine algebra [4] that are labelled by weights in (1), and one wants to classify all modular invariant sesquilinear forms in these characters $\sum N_{\lambda\bar\lambda}\chi_\lambda(q)\chi_{\bar\lambda}(\bar q)$ with non negative integer coefficients $N_{\lambda\bar\lambda}$ and $N_{\rho\rho}=1$. One proves in the simplest case of $\widehat{sl}(2)$ [7], and one suspects in the more general case of $N>2$, that this problem is tightly connected with the search of graphs satisfying conditions i)–vi). Typically the *diagonal* terms of the sesquilinear form are generalized "exponents", i.e. those weights λ that index the spectrum of one of the graphs. The second context has to do with the construction of integrable lattice models (solutions to the Yang-Baxter equation), again based on $sl(N)$. There the degrees of freedom, or "heights", attached to the lattice sites, are vertices of a graph, whose adjacency matrix G_1 specifies which pairs of heights may occur on neighbouring sites [8]. In other words, the configurations that one reads along a line drawn on the lattice are paths on the graph $\mathcal{G}_1$. Moreover the Boltzmann weights of the model are constructed out of representations of the Hecke algebra on the space of paths on the graph $\mathcal{G}_1$. It is again known for $N=2$ and believed for $N>2$ that a necessary condition for the construction of a solution of the Yang-Baxter in such a model is that the graph fulfills conditions i)–vi).

In both problems, however, these conditions were not stringent enough and some graphs found in [6] had to be rejected.

Note also the recent work of Ocneanu who has made use of these graphs to construct invariants of three-manifolds, à la Turaev-Viro [9].

One other context in which these graphs have been encountered is the construction of so-called $\mathcal{N}=2$ superconformal field theories and/or topological field theories in two dimensions (see [10] for a review). Once more, families of such theories are known to be based on the algebra $sl(N)$. In some cases, a connection is made with the theory of singularities [11], and the generalized Dynkin diagram of the singularity plays a central role. For example, for $N=2$, the so-called simple singularities [12], known to be classified by the ADE Dynkin diagrams, are the relevant ones. For $N>2$, the connection with singularity theory is less systematic and misses some instances. In all cases, however, it has been argued by Cecotti and Vafa [13] that if one perturbs these quantum field theories in a way that breaks conformal invariance but preserves the $\mathcal{N}=2$ supersymmetry, they develop a set of degenerate ground states; it is then appropriate to consider the graphs that describe the pattern of solitons interpolating between these ground states. It seems that the graphs discussed in this note are the graphs of solitons pertaining to a specific perturbation, namely the "less relevant" one.

Finally from a yet different standpoint, Dubrovin has been studying the so-called topological (or cohomological) field theories, that are known to be related to the foregoing $\mathcal{N} = 2$ theories by a "twisting" procedure. He showed that the constitutive equations of these theories in genus zero, the so-called Witten-Dijkgraaf-Verlinde-Verlinde equations [14], may be rephrased in a geometric language: there appear two flat metrics on the space of moduli of the theories, and the differential system that connects the two systems of flat coordinates has a monodromy group generated by reflections [15]. In the simplest cases associated with $N = 2$ theories, this group is a finite Coxeter group (encompassing the ADE cases). On the other hand, it has been shown recently that each of the graphs discussed above may be regarded as encoding the geometry of a root system, thereby generalizing the common interpretation of Dynkin diagrams, and thus allows the construction of a group generated by the reflections in the hyperplanes orthogonal to these roots [16]. There is some evidence that for those theories that are based on $sl(N)$ these reflection groups are explicit realizations of the monodromy groups of Dubrovin [17].

For a short review of these topics and a list of references, see for example [16].

Thus for the purpose of classifications of conformal field theories, lattice integrable height models, invariants of three-manifolds, $\mathcal{N} = 2$ superconformal field theories, and topological field theories, all based on the algebra $sl(N)$, it would be very useful to have a systematic way to classify the graphs satisfying the above properties.

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