# FACTORIZATION AND SELECTION RULES OF OPERATOR PRODUCT ALGEBRAS IN CONFORMAL FIELD THEORIES 

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Factorization of the operator product algebra in conformal field theory into independent left and right components is investigated. For those theories in which factorization holds we propose an ansatz for the number of independent amplitudes which appear in the fusion rules, in terms of the crossing matrices of conformal blocks in the plane. This is proved to be equivalent to a recent conjecture by Verlinde. The monodromy properties of the conformal blocks of 2-point functions on the torus are investigated. The analysis of their short-distance singularitities leads to a precise definition of Verlinde's operations.

## 1. Introduction

Conformal field theories are fully characterized by the central charge $c$ of the left and right Virasoro algebras, by the set of primary fields and their conformal weights ( $h_{i}, h_{i}$ ), and by the structure constants $C_{I J K}$ of the operator product algebra [1]. The problem of classifying possible consistent choices of these data is actively pursued. Much progress has been done on the classification of the possible operator contents, following Cardy's observation [2] that they are encoded in the genus-one (torus) partition function. The partition function can be written in terms of the Virasoro characters

$$
\begin{equation*}
X=\sum N_{i i} \chi_{i}(q) \chi_{i}(\bar{q}), \tag{1.1}
\end{equation*}
$$

and must be modular invariant. (See, for example, ref. [3] for a review and a rather extensive list of references.) On the other hand, little is known about the possible choices of structure constants $[1,4,5]$.

[^0]Recently, Verlinde [6] has made the remarkable observation that the operator product algebra (O.P.A.) may be determined simply in terms of the matrix $S$, which carries out the modular transformation $\tau \rightarrow-1 / \tau$ on the characters. Conversely, from this algebra, some a priori restrictions on the possible values of $c$ and $h$ may be derived [6] (see also ref. [7] for analogous results).

Verlinde's arguments left many questions unanswered. The operations he suggested on characters were not defined precisely. These operations form an algebra with multiplicities, the interpretation of which was not clear (at least to us). Finally, the relationship of this algebra with the O.P.A. of the theory is only tentative.

These questions motivated the present work. We have achieved a precise realization of Verlinde's operations on characters. The analysis of 2-point functions on the torus enables us to express Verlinde's algebra in terms of the crossing matrices of the conformal blocks of 4-point functions in the plane. This means that Verlinde's ansatz concerning the O.P.A. may equally be expressed in terms of these crossing matrices. We discuss several theories in which non-trivial multiplicities appear and try to elucidate their general meaning. Our analysis is done only for conformal field theories in which the O.P.A. factorizes into independent left and right algebras. It seems that the analysis of the O.P.A. in unfactorized theories remains an open problem.

Throughout this paper, we focus on the O.P.A. of fields which are mutually local and belong to the untwisted sector of the theory and hence appear in the partition function. Extension to other situations should not present particular difficulties.

Our discussion is far from being complete. Many conjectures remain to be (hopefully) proved... or invalidated. Still, we believe that our partial results clarify enough the situation and raise interesting questions.

The paper is organized as follows. Sect. 2 is a review of standard lore in conformal field theory. It is mainly intended to establish notations and possibly refresh the reader's memory. In sect. 3 , we discuss the issue of left $\times$ right factorization of the O.P.A. In cases where it does factorize we propose an ansatz for its selection rules, in terms of the crossing matrices of conformal blocks in the plane. In sect. 4 , we discuss the monodromy properties of the conformal blocks of the 2-point function on the torus. The analysis of their short-distance singularities leads to a precise definition of Verlinde's operations. The ansatz of sect. 3 is then directly related to Verlinde's conjecture. Sect. 5 contains examples and a discussion, based on these examples, of how degeneracies may occur, whilst some additional comments and a recapitulation of all our conjectures appear in sect. 6. Some technical details are gathered in appendix $A$.

## 2. Generalities on the Operator Product Algebra

We consider a conformal field theory, the primary fields of which take their conformal weights in a certain set $\left\{h_{i}\right\}$, finite or infinite, discrete or continuous:
$h, \bar{h} \in\left\{h_{i}\right\}$. Accordingly, the primary fields are denoted $\varphi_{I}(z, \bar{z})$, labeled by the pair $I=(i, \bar{i})$ such that $h=h_{i}, \bar{h}=h_{i}$. The spin $h_{i}-h_{i}$ is integer. It may happen that there is no one-to-one correspondence between pairs $(h, \bar{h})$ and primary fields of the theory. In such cases, the label $I$ has to be supplemented by some additional index, usually in the representation of some group ( $\mathrm{Z}_{3}$ for the three state Potts model, a Lie group for the WZW theories, etc.). This will be implicitly contained in the index $I$ or $i$. We have, however, to introduce a separate notion for the conjugate field $\varphi_{\hat{I}}$ such that

$$
\begin{equation*}
\left\langle\varphi_{I}(z, \bar{z}) \varphi_{\hat{l}}(0,0)\right\rangle=\frac{1}{z^{2 h_{i}}} \times " \text { c.c" } \tag{2.1}
\end{equation*}
$$

Here, and in what follows, "c.c" is an abuse of notation to denote the analogous expression with bar variables.

We also make use of the descendants of the primary field $\varphi_{I}$,

$$
\begin{equation*}
\varphi_{I}^{\{k\},(\bar{k}\}}=\varphi_{I}^{\left(--k_{1}, \ldots,-k_{p},-\bar{k}_{1}, \ldots,-\bar{k}_{q}\right)}=L_{-k_{1}} \ldots \bar{L}_{-\bar{k}_{q}} \varphi_{I}(z, \bar{z}), \tag{2.2}
\end{equation*}
$$

of level $|k|=\sum k_{i},|\bar{k}|=\sum \bar{k}_{j}$, and of the corresponding in and out states

$$
\begin{align*}
& |I,\{k\}\{\bar{k}\}\rangle=\lim _{z, \bar{z} \rightarrow 0} \varphi_{I}^{\{k\},\{\bar{k}\}}(z, \bar{z})|0\rangle \\
& \langle I,\{k\}\{\bar{k}\}|=\lim _{z, \bar{z} \rightarrow \infty} z^{2\left(h_{i}+|k|\right) \bar{z}^{2\left(h_{t}+|\bar{k}|\right)}\langle 0| \varphi_{\bar{I}}^{\{k) \cdot\{\bar{k}\}}(z, \bar{z})} \tag{2.3}
\end{align*}
$$

It is assumed that an orthonormal basis of such descendants has been constructed ${ }^{\star}$, denoted by

$$
\begin{align*}
|I, N\rangle & =\sum_{\{k\}\{\bar{k}\}} n_{\{k\}} \bar{n}_{\{\bar{k}\}}|I,\{k\}\{\bar{k}\}\rangle,  \tag{2.4}\\
\langle I N \mid J M\rangle & =\delta_{I J} \delta_{N M}, \tag{2.5}
\end{align*}
$$

where $N$ stands for the set of coefficients $\left(n_{\{k\}} \bar{n}_{\{\bar{k}\}}\right)$. The corresponding linear combination of descendant fields is denoted $\varphi_{I}^{(N)}$. The assumption of positivity of the norm (2.5), consistent with the hermiticity properties of the Virasoro generators ( $L_{n}^{\dagger}=L_{-n}, \bar{L}_{n}^{\dagger}=\bar{L}_{-n}$ ) means that we are dealing with a unitary theory [10].

[^1]The O.P.A. is usually characterized by the so called "structure constants" which are defined through one of the following equivalent formulae

$$
\begin{align*}
& \left\langle\varphi_{I}\left(z_{1}, \bar{z}_{1}\right) \varphi_{J}\left(z_{2}, \bar{z}_{2}\right) \varphi_{K}\left(z_{3}, \bar{z}_{3}\right)\right\rangle \\
& \quad=C_{I J K}\left(\frac{1}{\left(z_{1}-z_{2}\right)^{n_{i}+h_{j}-h_{k}}\left(\bar{z}_{1}-\bar{z}_{2}\right)^{h_{i}+h_{j}-h_{\bar{k}}}} \times \text { perm }\right),  \tag{2.6}\\
& \langle\hat{I}| \varphi_{J}(1,1)|K\rangle=C_{I J K} \tag{2.7}
\end{align*}
$$

or any permutation of the l.h.s.

By orthonormality

$$
\begin{equation*}
\beta_{i j, k n} \cdot \beta_{i j, \bar{k} \bar{n}}=\frac{\langle\hat{K} N| \varphi_{I}(1,1)|J\rangle}{\langle\hat{K}| \varphi_{I}(1,1)|J\rangle} \tag{2.9}
\end{equation*}
$$

and $\beta_{i j, k 0}=1$. The left $\times$ right factorization of these coefficients follows from the factorized holomorphic $\times$ antiholomorphic form of the 3-point function $\left\langle\varphi_{K}^{(N)}\left(z_{1}, \bar{z}_{1}\right) \varphi_{I}\left(z_{2}, \bar{z}_{2}\right) \varphi_{J}\left(z_{3}, \bar{z}_{3}\right)\right\rangle$ (see ref. [1] and appendix A).

The compact notation $I$, that we use to label a state, may at this stage be slightly misleading. In the general case where $I$ incorporates extra indices, such as group indices, the structure constants $C_{I J K}$ may also be a tensor in these indices. For example, in the $\mathrm{SU}(3)$ WZW model and in particular when dealing with the fields transforming as octets of the left and right algebras, we write $I=(i, \bar{i}, \alpha, \bar{\alpha})$ where $\alpha, \bar{\alpha}=1, \ldots, 8$, and

$$
\begin{equation*}
C_{I J K}=d_{\alpha \beta \gamma} d_{\bar{\alpha} \bar{\beta} \bar{\gamma}} C_{i \bar{i}, j \bar{j}, k \bar{k}}^{(d)}+f_{\alpha \beta \gamma} f_{\bar{\alpha} \bar{\beta} \bar{\gamma}} C_{i \bar{i}, j j, k \bar{k}}^{(f)} \tag{2.10}
\end{equation*}
$$

How are the structure constants determined? The idea [1] is to consider the 4-point function and use the associativity of the O.P.A. to derive a system of relations between the $C$ 's. For our present purpose, it is sufficient to consider a special 4-point function, namely, $\left\langle\varphi_{\hat{J}} \varphi_{\hat{I}} \varphi_{I} \varphi_{J}\right\rangle$. Using $S L(2, \mathrm{C})$ invariance, one writes

$$
\begin{align*}
& \left\langle\varphi_{j}\left(z_{1}, \bar{z}_{1}\right) \varphi_{\hat{I}}\left(z_{2}, \bar{z}_{2}\right) \varphi_{I}\left(z_{3}, \bar{z}_{3}\right) \varphi_{J}\left(z_{4}, \bar{z}_{4}\right)\right\rangle \\
& \quad=\frac{1}{\left(z_{1}-z_{4}\right)^{2 h_{j}}}\left(\frac{1-z}{z_{2}-z_{3}}\right)^{2 h_{i}} \times \text { "c.c" } \mathscr{F}(z, \bar{z}), \tag{2.11}
\end{align*}
$$

in terms of the cross-ratio $z=\left(z_{1}-z_{2}\right)\left(z_{3}-z_{4}\right) /\left(z_{1}-z_{3}\right)\left(z_{2}-z_{4}\right)$. Multiplying by $z_{1}^{2 h_{j}} \bar{z}_{1}^{2 h_{j}}$ and considering the limit $z_{1}, \bar{z}_{1} \rightarrow \infty, z_{2} \rightarrow 1$ (respectively 0 ), $z_{4} \rightarrow 0$ (respectively 1 ), we see that $z$ approaches $z_{3}$ (respectively $1-z_{3}$ ) and

$$
\begin{equation*}
\mathscr{F}(z, \bar{z})=\langle J| \varphi_{\tilde{I}}(1,1) \varphi_{I}(z, \bar{z})|J\rangle=\langle J| \varphi_{J}(1,1) \varphi_{I}(1-z, 1-\bar{z})|\hat{I}\rangle \tag{2.12}
\end{equation*}
$$

The operator product expansions of $\varphi_{I}(z, \bar{z}) \varphi_{J}(0)$ (respectively $\varphi_{I}(1-z$, $1-\bar{z}) \varphi_{\hat{I}}(0)$ ), are then inserted into these two amplitudes

$$
\begin{align*}
\langle J| \varphi_{\hat{I}}(1,1) \varphi_{I}(z, \bar{z})|J\rangle & =\sum_{K}\left|C_{I J K}\right|^{2} I\left[\begin{array}{l}
i i \\
j j
\end{array}\right]_{k}(z) I\left[\begin{array}{l}
\bar{i} \bar{i} \\
\bar{j} \bar{j}
\end{array}\right]_{\bar{k}}(\bar{z}) \\
& =\sum_{L} C_{I \hat{I} L} C_{J \hat{J}_{L}}^{*} I\left[\begin{array}{c}
i j \\
\hat{i} \hat{j}
\end{array}\right]_{I}(1-z) I\left[\begin{array}{c}
\bar{i} \bar{j} \\
\overline{\hat{j}} \hat{j}
\end{array}\right]_{i}(1-\bar{z}) . \tag{2.13}
\end{align*}
$$

The homomorphic "conformal blocks", $I$, are built up by the summation over descendants

$$
I\left[\begin{array}{l}
i i  \tag{2.14}\\
j j
\end{array}\right]_{k}(z)=\sum_{n}\left|\beta_{i j, k n}\right|^{2} z^{-h_{i}-h_{j}+h_{k}+|n|}
$$

(Remember that the leading term is normalized to $\beta_{i j, k 0}=1$.) A pictorial representation may be favorably substituted for those cumbersome notations (fig. 1). The


Fig. 1. Pictorial representation of the crossing transformation of conformal blocks.
conformal blocks in the two channels are related by a linear transformation

$$
I\left[\begin{array}{l}
i i  \tag{2.15}\\
j j
\end{array}\right]_{k}(z)=\sum_{l} X_{(i j) k l} I\left[\begin{array}{l}
i j \\
\hat{i j}
\end{array}\right]_{l}(1-z)
$$

In particular, in minimal theories, Dotsenko and Fateev have computed these "crossing matrices" $X$ by using integral representations of the conformal blocks. (As any rational theory is likely to be the minimal theory of some adequately extended algebra [11], their method is presumably generic.)

From eqs. (2.13) and (2.15), we see that the $C$ 's must satisfy the following system of equations

$$
\begin{equation*}
\sum_{k}\left|C_{I J K}\right|^{2} X_{(i j) k l} X_{(i j j \overline{k l}}^{*}=C_{I \hat{I L L}} C_{J J L}^{*} \tag{2.16}
\end{equation*}
$$

Therefore, whenever the pair ( $l, \bar{l}$ ) does not correspond to an operator present in the theory, the 1.h.s. of eq. (2.16) must vanish. In particular

$$
\sum_{k}\left|C_{I J K}\right|^{2} X_{(i j) k l} X_{(i j) \bar{k} l}^{*}=0 \quad \text { if } \quad(l, \bar{l}) \quad \text { corresponds to non-integer spin. }
$$

In the case of minimal theories with only spinless operators - those classified as $(A, A)$ in refs. [12,13] - these equations have been solved in ref. [4]. In this case

$$
\begin{equation*}
\sum_{k}\left|C_{I J K}\right|^{2} X_{(i j) k l} X_{(i j) k i}^{*}=C_{I \hat{L} L} C_{J \hat{L} L}^{*} \delta_{i \bar{l}} \tag{2.18}
\end{equation*}
$$

Take $l=1$, for which we know that $C_{I I 1}=1$. This determines

$$
\begin{equation*}
\left|C_{I J K}\right|^{2}=\frac{\left(X_{(i j)}^{-1}\right)_{1 k}}{X_{(i j) k 1}^{*}} \tag{2.19}
\end{equation*}
$$

and likewise

$$
\begin{equation*}
\left|C_{I \hat{I} L}\right|^{2}=\frac{\left(X_{(i \hat{i})}^{-1}\right)_{1 l}}{X_{(i i) l 1}^{*}} \tag{2.20}
\end{equation*}
$$

So far, the reality properties of the structure constants $C$ and the crossing matrices $X$ were not specified. In unitary theories, using charge conjugation symmetry, $C_{I J K}^{*}=C_{\hat{I} \hat{J} \hat{K}}=C_{I J K}$, so that the structure constants are real. So presumably are the crossing matrices. In what follows we assume that the crossing matrices in unitary theories are real. Moreover, in unitary minimal (A, A) theories, it has been observed [4] that the structure constants may be consistently chosen to be positive. Note that the structure constants are generically transcendental numbers.

Instead of insisting on the associativity of the O.P.A. and on the crossing relations (2.16) that it entails, one may alternatively put the emphasis on the monodromy properties of the conformal blocks $I(z)$. The latter have singularities at 0,1 , and $\infty$ which give rise to non-trivial monodromy properties as $z$ circles around these special points. The monodromy around 0 is read off from eq. (2.14) as

$$
z \rightarrow z \mathrm{e}^{2 i \pi} \Rightarrow I\left[\begin{array}{l}
i i  \tag{2.21}\\
j j
\end{array}\right]_{k}(z) \rightarrow \mathrm{e}^{2 i \pi\left(h_{k}-h_{i}-h_{j}\right)} I\left[\begin{array}{l}
i i \\
j j
\end{array}\right]_{k}(z)
$$

The monodromy properties around 1 may be obtained by using eq. (2.15) and the simple monodromy properties of $I\left[\begin{array}{l}i j \\ \hat{j}\end{array}\right](1-z)$ around $1-z=0$; finally, the monodromy properties around $\infty$ follow from the composition of those around 0 and 1. The requirement that the physical quantity $\langle J| \varphi_{\hat{I}}(1) \varphi_{I}(z)|J\rangle$ be a uniform function of joint variables $(z, \bar{z})$ is then fulfilled if only integer-spin operators appear in either channel. Hence, uniformity around 0 results from the summation over integer-spin intermediate states $K$ in eq. (2.13), while uniformity around 1 follows from eq. (2.17). It is then seen that the monodromy group is tightly connected to the crossing property.

## 3. Selection rules on factorized Operator Product Algebras

The data ( $c,\{I\},\left\{C_{I J K}\right\}$ ) characterize fully the conformal field theory. As the previous discussion suggests, the determination of the $C$ 's in a given theory is a laborious task and their a priori classification seems, for the time being, out of reach. Selection rules on the $C$ 's, on the other hand, are better understood $[1,4,5,14,15]$. It seems that a general formula exists for the indicator of $C_{I J K}$. The indicator is zero or integer depending on whether $C_{I J K}$ vanishes or not. More precisely, define $N_{I J K}$ as the number of independent terms appearing in the fusion of $I \otimes J \rightarrow \hat{K}$ (see, e.g. eq. (2.10)). Clearly, these numbers are fully symmetric in $I, J, K$. The ansatz of Verlinde [6] amounts to a closed expression for $N_{I J K}$ (in a certain class of theories).

Notice that determining the operator content (the set $\{I\}$ ), and the selection rules $N_{I J K}$ do not suffice to fix uniquely the structure constants, hence not the whole theory. A standard counterexample is provided by the $\mathrm{E}_{8} \times \mathrm{E}_{8}$ and $\mathrm{SO}(32) / \mathrm{Z}_{2}$ level one WZW theories, which have the same spectrum of $\left\{h_{i}, h_{i}\right\}$, the same genus-one partition function, the same $N_{I J K}$ (see ref. [6] and below) and still are inequivalent theories.

To proceed, we have to address the issue of left $\times$ right factorization of the O.P.A. The reader may have noticed that while the holomorphic $\times$ antiholomorphic structure of the 2-point function and the 3-point function was emphasised (eqs. (2.1) and (2.6)) nothing was assumed on the $C$ 's. All the preceding discussion was held
irrespective of whether the structure constants factorize, i.e.

$$
\begin{equation*}
C_{I J K}=c_{i j k} c_{i j \bar{k}} \tag{3.1}
\end{equation*}
$$

There is a class of theories in which this factorization is obviously satisfied, namely theories with only spinless operators. For these theories the distinction between $I$ and $i$ is immaterial, and we may write

$$
\begin{equation*}
C_{I J K}=C_{i j k}=\left(c_{i j k}\right)^{2}, \tag{3.2}
\end{equation*}
$$

and the genus-one partition function is a sum of squared moduli of characters

$$
\begin{equation*}
Z=\sum N_{i}\left|\chi_{i}\right|^{2} \tag{3.3}
\end{equation*}
$$

More generally, consider a theory with an extended algebra whose partition function takes the diagonal form (3.3) in terms of supercharacters of this extended algebra. With respect to the extended algebra, all the primary fields are diagonal, i.e. $\{I\}=\{i, i\}$ where $i$ is an index of the extended algebra. Hence, factorization of the structure constants for these fields is obvious. When the representations of the extended algebra are decomposed into representations of the Virasoro algebra, the partition function reads

$$
\begin{equation*}
Z=\sum_{[i]} N_{[i]}\left|\sum_{j \in[i]} \chi_{j}\right|^{2} \tag{3.4}
\end{equation*}
$$

where the index [ $i$ ] in eq. (3.4) is an index of the extended algebra. The partition function (3.4) exhibits Virasoro primary fields of non-zero spin but the left $\times$ right factorization is preserved. Therefore, it is very tempting to conjecture that, in general, for theories whose partition function has the general form (3.4), the O.P.A. factorizes (even if the underlying extended algebra has not been identified). We refer to such theories as of class I. For example the minimal theories of class I are those classified by $(A, A),\left(A, D_{\text {even }}\right),\left(A, E_{6}\right)$ or $\left(A, E_{8}\right)$ in refs. [12,13]. Theories of class II are those theories whose partition function cannot be written in the form of eq. (3.4).

For theories of class II, there may exist operators of non zero integer spin $h_{i}-h_{i} \in Z$ such that either of the two spinless operators $\left(h_{i}, h_{i}\right)$ or ( $h_{i}, h_{i}$ ) does not exist. Again referring to minimal theories, this is what happens in the theories ${ }^{\star}$ $\left(\mathrm{A}, \mathrm{D}_{\text {odd }}\right)$ or $\left(\mathrm{A}, \mathrm{E}_{7}\right)$. In such a case, it would clearly be inconsistent to write eq. (3.1). Consider for example taking $I=J=(i, \bar{i})$ and some appropriate $K=(k, \bar{k})$.

[^2]The factorization $C_{I J K}=c_{i i k} c_{i i i \bar{k}}$ would then be in contradiction with the absence of either $I^{\prime}=(i, i)$ or $I^{\prime \prime}=(\bar{i}, \bar{i})$ which implies the vanishing of $C_{I^{\prime} I^{\prime} K^{\prime}}=\left(c_{i i k}\right)^{2}$ or of $C_{I^{\prime \prime} I^{\prime \prime} K^{\prime \prime}}=\left(c_{i i \bar{k}}\right)^{2}$. Note that not all theories of class II may be disposed of by this simple argument. Such would be the case of a theory with a partition function of the form

$$
\begin{equation*}
Z=\left|\chi_{1}\right|^{2}+\left|\chi_{2}\right|^{2}+\left|\chi_{3}\right|^{2}+\left(\chi_{1}\left(\chi_{2}^{*}+\chi_{3}^{*}\right)+\text { c.c. }\right) . \tag{3.5}
\end{equation*}
$$

From now on we restrict the discussion to class I theories. Observe that belonging to the same cluster [ $i$ ] defines an equivalence relation between fields. We shall soon need, as a consistency relation, that

$$
\begin{equation*}
Y_{[i j[j] k]}=\sum_{k \in[k]} C_{i j k}^{2} X_{(i j) k 1} \text { depends only on the equivalence class of } i, j, k . \tag{3.6}
\end{equation*}
$$

The sum in eq. (3.6) runs over the $k$ 's which appear in $i \otimes j$ and belong to a certain class [ $k$ ]. The intuitive meaning of eq. (3.6) is that couplings between various representatives of the same equivalence class are related. This can be checked in the Potts model.

The fact that the coefficient $Y$ depends only on the class of $k$ results from its definition. At the end of appendix A, we prove the $i$ and $j$ part of this statement in any diagonal theory (cf. eq. (3.3)) with an extended algebra. The idea is to repeat, for the generators of this extended algebra, a discussion carried out for the Virasoro generators, showing that the residue of the leading $z \rightarrow 1$ singularity of $\sum_{N}\langle J| \varphi_{I}(1)|K N\rangle\langle K N| \varphi_{I}(z, \bar{z})|J\rangle$ does not depend on the descendant of $I$ or $J$. Again it is tempting to assume that this property holds for any "class I" theory.

Notice that some care has to be exercised when dealing with extended algebras; arguments familiar within the Virasoro algebra may turn out to be incorrect. For example, in the fusion of two extended primary fields, the leading term within a cluster may not be the primary field of that cluster, whereas the leading term in eq. (2.8) for a given $k$ is a Virasoro primary (see, e.g., claim 2 of ref. [15]). A counterexample within extended algebras is provided by the Potts model where the fusion rule of two energy operators $\epsilon\left(h=\frac{s 2}{5}\right)$ is $\epsilon \otimes \epsilon=\mathbb{1}+\epsilon^{\prime}$. The operator $\epsilon^{\prime}\left(h=\frac{7}{5}\right)$ is a Virasoro primary but a descendant of $\epsilon$ in Fateev-Zamolodchikov algebra [17]*.

Assuming eq. (3.6), the crossing relation (2.16) for $l=\bar{l}=1$ yields the sum rule

$$
\begin{equation*}
1=\sum_{k, \bar{k}} c_{i j k}^{2} c_{i j \bar{k}}^{2} X_{(i j) k 1} X_{(i j) \bar{k} 1}=\sum_{[k]}\left(\sum_{k \in[k]} c_{i j k}^{2} X_{(i k) k 1}\right)^{2}=\sum_{[k]} Y_{[i][j][k]}^{2} . \tag{3.7}
\end{equation*}
$$

It is hoped that for the theories of class I the general solution of the crossing relation, together with the additional information implicit in eq. (3.4), may be shown to satisfy factorization and eq. (3.6), and hence eq. (3.7).

[^3]The factorization of the $C$ 's implies that the $N$ 's factorize

$$
\begin{equation*}
N_{I J K}=N_{i j k} N_{i j \bar{k}} . \tag{3.8}
\end{equation*}
$$

Recall that $N_{i j k}$ counts the number of independent amplitudes in the fusion $i \otimes j \rightarrow \hat{k}$ (see, for example, eq. (2.10) and sect. 5). Actually, in theories of class I, it seems that the best we can do is to determine the number of independent amplitudes in the fusion to a given cluster $[\hat{k}]$

$$
N_{i j[k]}= \begin{cases}0, & \text { if } N_{i j k}=0 \text { for all } k \in[k],  \tag{3.9}\\ N_{i j k}, & \text { the common value of some non-zero } \quad N_{i j k} ; k \in[k] .\end{cases}
$$

The fact that $N_{i j[k]}$ does not depend on the choice of the non-vanishing $N_{i j k}$ follows from the interpretation of the $N$ 's as counting the number of independent amplitudes. The various $k$ 's in [ $k$ ] have "internal quantum numbers" of the same nature, and the number of amplitudes, if non-zero, should be the same for all $k \in[k]$. By the same considerations as for eq. (3.6), it is easy to see that $N_{i j[k]}$ depends only on [ $i$ ] and [ $j$ ] and should therefore be written as $N_{[i][j] k]}$.

We are ready now to propose our ansatz for $N_{[i[j][k]}$ in terms of the conformal blocks in the plane*

$$
\begin{equation*}
N_{[i][j] k]}=\frac{Y_{[i][j][k]} Y_{[i][j] k]}}{Y_{[i[i] i][1]}} . \tag{3.10}
\end{equation*}
$$

Note that this ansatz seems to require the knowledge of the $c$ 's to determine their indicator! It may, however, be recast in other forms. In particular as we shall see, it is equivalent to Verlinde's ansatz [6]. In this simple case where $[i],[j],[k]$ contain a single (Virasoro) primary field it reduces to

$$
\begin{equation*}
N_{i j k}=\frac{c_{i j k}^{4} X_{(i j) k 1} X_{(i k) j 1}}{X_{(i) 11}} \tag{3.11}
\end{equation*}
$$

Obviously, $N_{i j k}$ vanishes with $c_{i j k}$ and is symmetric under $j \leftrightarrow k$, but neither its integrality, nor even its full symmetry in ( $i, j, k$ ) is apparent. The main evidence we have in favor of this conjecture comes from explicit checks in many cases, either in this form, or in Verlinde's form (see below eq. (4.26)). In simple cases where only one channel is open to $i \otimes \hat{i}, i \otimes j$, and $i \otimes k$, namely $\mathbb{1}, k$, and $j$, respectively, eq. (3.6) fixes $N_{i j k}$ (up to a sign) to be 1 . The gaussian model provides a trivial example for such a case. For this model each factor in eq. (3.11) is actually a Kronecker delta function! Another example is provided by the Ising model for $I, J, K=\mathbb{1}$ or $\epsilon$. Here

[^4]$1, \epsilon$ are the identity and energy operators. In the case of the diagonal ( $\mathrm{A}, \mathrm{A}$ ), minimal theories, we may use the explicit expressions for $C_{I J K}$ given in eq. (2.19) to rewrite our ansatz as
\[

$$
\begin{equation*}
N_{i j k}^{2}=\frac{\left(X_{(i j)}^{-1}\right)_{1 k} X_{(i j) k 1}\left(X_{(i k)}^{-1}\right)_{1 j} X_{(i k) j 1}}{\left(X_{(i i)}^{-1}\right)_{11} X_{(i i) 11}} \tag{3.12}
\end{equation*}
$$

\]

It is easy to check, using the explicit expressions of ref. [4], that this agrees with the known fusion rules of minimal models [1] and their interpretation as addition of $\mathrm{SU}(2)$ spins.

The identification of the multiplicities with the r.h.s. of eq. (3.12) demands that the r.h.s. of these equations does not depend on a change of normalization of the conformal blocks. This is manifest in eq. (3.12). If $X_{(i j)} \rightarrow \Lambda X_{(i j)} \tilde{\Lambda}^{-1}$ here $\Lambda, \tilde{\Lambda}$ are diagonal matrices, $\left(X_{(i j)}^{-1}\right)_{l k} X_{(i j) k l}$ and therefore also $N_{i j k}$ are indeed invariant. Actually, Dotsenko and Fateev [4] have used a basis in which the crossing matrices are algebraic numbers. It may also be observed that $X_{(i j)} \mathrm{D} X_{(i j)}^{-1}$ where $\mathrm{D}=$ $\operatorname{diag}\left(\exp 2 i \pi\left(h_{l}-2 h_{i}\right)\right)$, is the monodromy matrix of the conformal block $I\left[\begin{array}{l}i j \\ i j\end{array}\right](z)$ around $z=1$.

All these remarks point to the possibility that eq. (3.12) is of more general validity and that the $N$ 's may be expressed in terms of monodromy matrices. In any case proving either eq. (3.10) or eq. (3.11) remains a challenge.

## 4. Verlinde's algebra

In order to define Verlinde's operations on characters, we consider the 2-point function $\left\langle\varphi_{I}(w, \bar{w}) \varphi_{i}(0)\right\rangle$ on a torus of periods $a=1$ and $b=\tau$. It may be written in operator language, using the $w$ and $\bar{w}$ translation operators on the cylinder of period $a=1$

$$
\begin{equation*}
Z\left\langle\varphi_{I}(w, \bar{w}) \varphi_{\hat{I}}(0)\right\rangle_{\mathrm{torus}}=\operatorname{tr}\left(\mathrm{e}^{[i(\tau-w) H+\mathrm{c.c.}]} \hat{\varphi}_{I} \mathrm{e}^{[i w H+\mathrm{c} . \mathrm{c} \cdot]} \hat{\varphi}_{\hat{I}}\right) \tag{4.1}
\end{equation*}
$$

with $0<\operatorname{Im} w<\operatorname{Im} \tau$. Mapping the cylinder on the plane by $z=\mathrm{e}^{2 i \pi w}$ and setting $q=\mathrm{e}^{2 i \pi \tau}$ one finds [2], for $|q|<|z|<1$

$$
\left.\begin{array}{rl}
Z\left\langle\varphi_{I}(w, \bar{w}) \varphi_{\hat{I}}(0)\right\rangle_{\text {torus }} \\
= & (2 i \pi)^{2 h_{i}}(-2 i \pi)^{2 h_{i}} \operatorname{tr}\left[\left(\frac{q}{z}\right)^{L_{0}-c / 24}\left(\frac{\bar{q}}{\bar{z}}\right)^{\bar{L}_{0}-c / 24} \hat{\varphi}_{I} z^{L_{0}-c / 24} \bar{z} \bar{L}_{0}-c / 24\right. \\
\hat{\varphi}_{\hat{I}}
\end{array}\right] .
$$

In order to exhibit the factorized structure of this 2-point function in terms of holomorphic and antiholomorphic blocks we insert a complete set of orthonormal intermediate states into eq. (4.2). With the notations of sect. 2

$$
\begin{align*}
& Z\left\langle\varphi_{I}(w, \bar{w}) \varphi_{\hat{I}}(0)\right\rangle_{\text {torus }}=(2 i \pi)^{2 h_{i}}(-2 i \pi)^{2 h_{i}} \sum_{J_{K, M}}\left(\frac{q}{z}\right)^{h_{j}+|m|-c / 24}\left(\frac{\bar{q}}{\bar{z}}\right)^{h_{j}+|\bar{m}|-c / 24} \\
& \times z^{h_{k}+|n|-c / 24} \bar{z} h_{\bar{K}}+|\bar{n}|-c / 24  \tag{4.3}\\
&\left.\langle\hat{K} N| \hat{\varphi}_{I}|J M\rangle\right|^{2}
\end{align*}
$$

The factorized form is now evident:

$$
\begin{equation*}
Z\left\langle\varphi_{I}(w, \bar{w}) \varphi_{\hat{i}}(0)\right\rangle_{\mathrm{torus}}=\sum_{J, K} C_{I J K}^{2} \Gamma_{i j k}(z, q) \Gamma_{i \bar{j} k}(\bar{z}, \bar{q}) . \tag{4.4}
\end{equation*}
$$

The holomorphic (in both $z$ and $q$ ) conformal block reads

$$
\begin{equation*}
\Gamma_{i j k}(z, q)=(2 i \pi)^{2 h_{i}}\left(\frac{q}{z}\right)^{h_{j}-c / 24} z^{n_{k}-c / 24} \sum_{\{m\},\{n\}}\left(\frac{q}{z}\right)^{|m|} z^{|n|}\left|\frac{\langle\hat{k} n| \hat{\varphi}_{i}|j m\rangle}{\langle\hat{k}| \hat{\varphi}_{i}|j\rangle}\right|^{2} \tag{4.5}
\end{equation*}
$$

where $|q|<|z|<1$. The last ratio in eq. (4.5) results from the factorization properties of the 3-point functions in the plane and depends on the "left" labels $h_{i}, h_{j}, h_{k}, m, n, \ldots$ only (see appendix A for details). A pictorial representation of $\Gamma_{i j k}$ is given in fig. 2.

These conformal blocks are singular at $z=1$, i.e. $w=0$, corresponding to the short-distance singularity of the 2-point function. They may have non-trivial monodromy properties when $w$ winds around 0 or is continued around the torus, i.e. $w \rightarrow w+1$ or $w \rightarrow w+\tau$. Only the physical amplitude (4.4) is a uniform, doubly periodic function on the torus.

The conformal blocks $\Gamma_{i j k}$ are relatives of those pertaining to the 4-point function in the plane, as the comparison between eqs. (2.14) and (4.5) shows. This similarity is an explicit manifestation of a general property emphasised in ref. [18].


Fig. 2. Pictorial representation of $\Gamma_{i j k}$ and its deformation into $\Gamma_{i k j}$.

In the singular limit $\tau \rightarrow i \infty, q \rightarrow 0$ where the torus degenerates ("pinches"), $\Gamma_{i j k}$ actually reduces to $I\left[\begin{array}{c}i i \\ j j\end{array}\right]_{k}(z)$

$$
\Gamma_{i j k}(z, q) \underset{q \rightarrow 0}{\sim}(2 i \pi)^{2 h_{i}} q^{h_{j}-c / 24^{h_{i}} I}\left[\begin{array}{l}
i i  \tag{4.6}\\
j j
\end{array}\right]_{k}(z)(\mathbb{1}+\mathrm{O}(q)) .
$$

Another interesting limit is $z \rightarrow 1$. It may be shown (appendix A) that in this limit

$$
\begin{equation*}
\sum_{\{n\}} z^{|n|-|m|}\left|\frac{\langle\hat{k} n| \hat{\varphi}_{i}|j m\rangle}{\langle\hat{k}| \hat{\varphi}_{i}|j\rangle}\right|^{2} \sim \frac{B_{i j k}}{(1-z)^{2 h_{i}}}, \tag{4.7}
\end{equation*}
$$

with a residue $B_{i j k}$ which is independent of the descendant $m$ of $j$. Inserting this expression into eq. (4.5), one finds

$$
\begin{align*}
\Gamma_{i j k}(z, q) & \underset{z \rightarrow 1}{\sim}\left(\frac{2 i \pi}{1-z}\right)^{2 h_{i}} B_{i j k} \sum_{\{m\}} q^{h_{j}+|m|-c / 24} \\
& \sim\left(\frac{2 i \pi}{1-z}\right)^{2 h_{i}} B_{i j k} \chi_{j}(q), \tag{4.8}
\end{align*}
$$

by definition of the character. The block $\Gamma_{i j k}(z, q)$ therefore interpolates between $I\left[\begin{array}{c}i i \\ j j\end{array}\right]_{k}(z)$ and $\chi_{j}(q)$. The residue $B_{i j k}$ is easily identified by considering the double limit $q \rightarrow 0, z \rightarrow 1$. The crossing relation (eq. (2.15)) and the fact that the leading singularity in the crossed channel (in a unitary theory) comes from the identity operator, gives

$$
\begin{equation*}
B_{i j k}=X_{(i j) k 1} . \tag{4.9}
\end{equation*}
$$

Note that eq. (4.8) expresses a natural decoupling of the short-distance singularity of $\Gamma_{i j k}$, depicted in fig. 3.

Let us now examine the monodromy properties of $\Gamma_{i j k}(z, q)$ as $w=(1 / 2 i \pi) \log z$ winds around the torus. The monodromy property of the block as $w \rightarrow w+1$, $z \rightarrow e^{2 i \pi_{z}}$ is easily obtained from eq. (4.5)

$$
\begin{equation*}
\Gamma_{i j k}(z, q) \rightarrow \mathrm{e}^{2 i \pi\left(h_{k}-h_{j}\right)} \Gamma_{i j k}(z, q) ; \tag{4.10}
\end{equation*}
$$

the effect of $w \rightarrow w+\tau, z \rightarrow z q$ cannot be read off directly from eq. (4.5), as it violates its condition of validity: $|q|<|z|<1$. For our purposes, however, it is sufficient to consider the change $w \rightarrow \tau-w, z \rightarrow q / z$ which does preserve it. One finds

$$
\begin{equation*}
\Gamma_{i j k}(q / z, q)=\Gamma_{i k j}(z, q) \tag{4.11}
\end{equation*}
$$

A pictorial interpretation of eq. (4.11) is given in fig. 2.


Fig. 3. Short-distance behaviour of $\Gamma_{i j k}$.

We are now in position to define Verlinde's operations on characters. We restrict ourselves to theories of class I (using the terminology of sect. 3). We shall make use of the normalized combinations of $\Gamma$ 's defined by

$$
\begin{equation*}
\Gamma_{i[j]}(z, q)=\sum_{j \in[j][k] \subset[i] \otimes[j]} Y_{[i] \mid j \| k]}\left[\sum_{k \in[k]} \Gamma_{i j k} c_{i j k}^{2}\right] \tag{4.12}
\end{equation*}
$$

By virtue of eqs. (3.6), (4.8) and the sum rule (3.7) we obtain for $z \sim 1$

$$
\begin{align*}
\Gamma_{i[j]}(z, q) & \sim\left(\frac{2 i \pi}{1-z}\right)^{2 h_{i}} \sum_{j \in[j]} \chi_{j}(q) \sum_{[k]} Y_{[i][j][k]}^{2} \\
& =\left(\frac{2 i \pi}{1-z}\right)^{2 h_{i}} \sum_{j \in[j]} \chi_{j}(q) \\
& =\left(\frac{2 i \pi}{1-z}\right)^{2 h_{i}} \chi_{[j]}(q), \tag{4.13}
\end{align*}
$$

where we have introduced the character of the cluster [ $j$ ], pertaining to the irreducible representation of the extended algebra.

Consider first the block $\Gamma_{i[1]}(z, q)$

$$
\begin{equation*}
\Gamma_{i[1]}(z, q)=\sum_{l \in[1]} \sum_{\hat{i^{\prime}} \in[\hat{i}]} c_{i l \hat{l}^{2}}^{2} \Gamma_{i l \hat{i}^{\prime}}(z, q) . \tag{4.14}
\end{equation*}
$$

This equation results from $Y_{[i j 11] \hat{i}]}=c_{i l \hat{i}}^{2} X_{(i) \hat{i} i}=1$. At small separation $w \rightarrow 0$,
$\operatorname{Im} z>0,|z| \rightarrow 1$

$$
\begin{equation*}
\Gamma_{i[1]} \sim\left(\frac{2 i \pi}{1-z}\right)^{2 h_{i}} \sum_{l \in[1]} \chi_{i}(q)=\left(\frac{2 i \pi}{1-z}\right)^{2 h_{i}} \chi_{[1]}(q) \tag{4.15}
\end{equation*}
$$

We then let $w$ wind around the torus and approach the point $\tau$, still satisfying $\operatorname{Im} w<\operatorname{Im} \tau$. This means that $z=q / z^{\prime}, z^{\prime} \rightarrow 1,\left|z^{\prime}\right|<1$

$$
\begin{align*}
& \Gamma_{i[1]}(z, q)=\sum_{l \in[1]} \sum_{\hat{i}^{\prime} \in[\hat{i}]} c_{i \hat{i} i^{\prime}}^{2} \Gamma_{i \hat{i}^{\prime} l}\left(z^{\prime}, q\right) \\
& \underset{z^{\prime} \rightarrow 1}{\sim}\left(\frac{2 i \pi}{1-z^{\prime}}\right)^{2 h_{i}} Y_{[i[i \hat{i}][1]} \sum_{i^{\prime} \in[i]} \chi_{i^{\prime}}(q)=\left(\frac{2 i \pi}{1-z^{\prime}}\right)^{2 h_{i}} Y_{[i]|\hat{i}|][1]} \chi[i](q) . \tag{4.16}
\end{align*}
$$

The comparison of eqs. (4.15) and (4.16) shows that we have defined a deformation of $\chi_{[1]}$ into $\chi_{[i]}$. The operator $\Phi_{i}(b)$ of Verlinde may be therefore identified with $\left(Y_{[i][i][1]}\right)^{-1}$ times this deformation.

Next we perform this deformation on some other character $\chi_{[j]}$. We thus start from $\Gamma_{i[j]}$ instead of $\Gamma_{i[1]}$. At short distance

$$
\begin{align*}
& \Gamma_{i[j]}(z, q)=\sum_{j \in[j][k]} \sum_{[i][j] k]} \sum_{k \in[k]} c_{i j k}^{2} \Gamma_{i k j}\left(z^{\prime}, q\right) \\
& \underset{z^{\prime} \rightarrow 1}{\sim}\left(\frac{2 i \pi}{1-z^{\prime}}\right)^{2 h_{i}} \sum_{[k]} Y_{[i[j][k]} \sum_{\substack{j \in[j] \\
k \in[k]}} c_{i j k}^{2} X_{(i k) j 1} \chi_{k}(q) \\
&=\left(\frac{2 i \pi}{1-z^{\prime}}\right)^{2 h_{i}} \sum_{[k]} Y_{[i][j][k]]} Y_{[i][k][j]} \sum_{k \in[k]} \chi_{k}(q) \\
&=\left(\frac{2 i \pi}{1-z^{\prime}}\right)^{2 n_{i}} \sum_{[k]} Y_{[i j[j][k]} Y_{[i][k][j]} \chi_{[k]}(q) \tag{4.17}
\end{align*}
$$

where we have used eq. (3.6). After multiplication by $\left(Y_{[i][\hat{i}[1]}\right)^{-1}$, this realizes Verlinde's operation

$$
\begin{equation*}
\Phi_{i}(b) \chi_{[j]}(q)=\sum_{[k]} N_{[i[j][k]} \chi_{[k]}(q) \tag{4.18}
\end{equation*}
$$

with $N_{[i][j][k]}$ given by eq. (3.10).
We can follow the same steps to define the deformation around the period 1 of the torus, i.e. $z \rightarrow z \mathrm{e}^{2 i \pi}$ with $(1-z) \rightarrow \mathrm{e}^{i \pi}(1-z)$ (see fig. 4). Using eq. (4.10), one


Fig. 4. The relation between the $w$ and $z$ planes. The two types of arrows correspond to the two periods of the torus.
finds:

$$
\begin{align*}
\Gamma_{i[j]}\left(z \mathrm{e}^{2 i \pi}, q\right) & \sim\left(\frac{2 i \pi}{1-z}\right)^{2 h_{i}} \sum_{j \in[j]} \chi_{j}(q) \sum_{[k]} Y_{[i \| j j][k]} \sum_{k \in[k]} \mathrm{e}^{2 i \pi\left(h_{k}-h_{i}-h_{j}\right)} c_{i j k}^{2} X_{(i j) k 1} \\
& =\left(\frac{2 i \pi}{1-z}\right)^{2 h_{i}} \sum_{j \in[j]} \chi_{j}(q) \sum_{[k]} Y_{[i]] j] k]}^{2} \mathrm{e}^{2 i \pi\left(h_{k}-h_{i}-h_{j}\right)} \\
& =\left(\frac{2 i \pi}{1-z}\right)^{2 h_{i}} \sum_{[k]} Y_{[i]] j][k]}^{2} \mathrm{e}^{2 i \pi\left(h_{k}-h_{i}-h_{j}\right)} \chi_{[j]}(q) \tag{4.19}
\end{align*}
$$

because the phase factor $\exp \left[2 i \pi\left(h_{k}-h_{i}-h_{j}\right)\right]$ depends only on the class [ $k$ ] (and $[i],[j])$. Multiplying again by $\left(Y_{[i l l \hat{i}[1]}\right)^{-1}$ allows us to define the operator $\Phi_{i}(a)$ of Verlinde. We find

$$
\begin{equation*}
\Phi_{i}(a) \chi_{[j]}(q)=\lambda_{i}^{(j)} \chi_{[j]}(q) \tag{4.20}
\end{equation*}
$$

with

$$
\begin{equation*}
\lambda_{i}^{(j)}=\frac{\sum_{[k]} Y_{[i l[j][k]}^{2} \mathrm{e}^{2 i \pi\left(h_{k}-h_{i}-h_{j}\right)}}{Y_{[i][\mathrm{i}][1]}} . \tag{4.21}
\end{equation*}
$$

The cases where either $i=1$ or $j=1$ are trivial

$$
\begin{array}{ll}
N_{[1][j][k]}=\delta_{[\hat{j} \mid k]}, & \lambda_{1}^{(j)}=1, \\
N_{[i j[1][k]}=\delta_{[\hat{i}] k]}, & \lambda_{i}^{(1)}=\left(Y_{[i \| i \hat{i}] 1]}\right)^{-1} . \tag{4.22}
\end{array}
$$

Consider now the effect of the modular transformation $S, S: \tau \mapsto-1 / \tau$ on the previously defined deformations. Recall that $S$ acts linearly on characters $\chi_{[i]}(-1 / \tau)=\sum_{[j]} S_{[i] j]} \chi_{[j]}(\tau)$. The matrix $S$ is a unitary matrix and its square exchanges conjugate fields

$$
\begin{equation*}
S S^{\dagger}=1, \quad\left(S^{2}\right)_{[i][j]}=\delta_{[j] i \hat{i}]} \tag{4.23}
\end{equation*}
$$

It is clear that the two deformations are related by $S: S \Phi_{i}(a) S^{-1}=\Phi_{i}(b)$. This suffices, together with eq. (4.22), to repeat Verlinde's steps and express $N_{[i \| j][k]}$ and $\lambda_{i}^{(j)}$ in terms of the matrix $S_{[i] j]}$

$$
\begin{gather*}
N_{[i \| j] I[k]}=\sum_{[n]} S_{[j][n]} \lambda_{i}^{(n)} S_{[n][k]}^{\dagger},  \tag{4.24}\\
\lambda_{i}^{(j)}=\frac{S_{[i][j]}}{S_{[11] j]}}, \tag{4.25}
\end{gather*}
$$

and hence

$$
\begin{equation*}
N_{[i l \| j][k]}=\sum_{[n]} \frac{S_{[i] n]]} S_{[j \| n]} S_{[n][k]}^{\dagger}}{S_{[1][n]}} \tag{4.26}
\end{equation*}
$$

From these expressions it follows that the operators $\Phi(a)$ and $\Phi(b)$ that we defined obey Verlinde's algebra

$$
\begin{equation*}
\Phi_{i}(c) \Phi_{j}(c)=\sum_{[k]} N_{[i][j][k]} \Phi_{k}(c), \quad \text { where } \quad c=a \quad \text { or } \quad b \tag{4.27}
\end{equation*}
$$

It is interesting to compare the two alternative expressions obtained for $N_{[i][j][k]}$ and $\lambda_{i}^{\left({ }^{\prime}\right)}$, i.e. eq. (3.11) versus eq. (4.26) and eq. (4.21) versus eq. (4.25). One expression is given by a sum while the other consists of a single term. They seem to be a sort of dual expressions.

The fact that it is possible to find expressions for the $N$ 's and $\lambda$ 's in terms of the matrix $S$ and in terms of the crossing matrices $X$ suggests that there is a hidden relationship betwaeen the $X$ 's and $S$. Actually, using the formulae of ref. [4], we obtain for minimal (A, A) theories that, for $k \in i \otimes j$,

$$
\begin{equation*}
X_{(i j) 1 k}^{-1} X_{(i j) k 1}=\frac{S_{11} S_{1 k}}{S_{1 i} S_{1 j}}, \tag{4.28}
\end{equation*}
$$

corresponding to $N_{i j k}=1$ (see eq. (3.12)).
As a last conjecture let us mention that

$$
\begin{equation*}
Y_{[i][j][k]}^{2}=N_{[i][j][k]} \frac{S_{[1][1]} S_{[1][k]}}{S_{[1][i]} S_{[1][j]}} \tag{4.29}
\end{equation*}
$$

is consistent with eqs. (3.7), (3.10) and (4.26) and is indeed satisfied in minimal ( $\mathrm{A}, \mathrm{A}$ ) theories. It might be the missing link between the crossing matrices and modular transformations.

## 5. Examples

All our discussion so far was quite formal. We have given an explicit realization of Verlinde's operations in terms of deformations on conformal blocks. We proposed an ansatz for the number of independent amplitudes appearing in the fusion rules of primary fields which is equivalent to Verlinde's ansatz. It would be, however, both useful and illuminating to see how these formal considerations apply to specific models. The simplest possible model is the gaussian model where everything is trivial and our ansatz is clearly satisfied (see also the discussion following eq. (3.11)). The only other model (leaving aside orbifolds of the gaussian model and low-level WZW models), where all the correlation functions are known for genus one, is the Ising model. In the untwisted sector of the Ising model the three primary fields $\mathbb{\downarrow}, \sigma, \epsilon$ satisfy the following O.P.A.:

$$
\begin{array}{lll}
\mathbb{1} \otimes \mathbb{1}=\mathbb{1}, & \mathbb{1} \otimes \epsilon=\boldsymbol{\epsilon}, & \mathbb{1} \otimes \sigma=\sigma \\
\boldsymbol{\epsilon} \otimes \boldsymbol{\epsilon}=\mathbb{1}, & \boldsymbol{\epsilon} \otimes \boldsymbol{\sigma}=\sigma, & \boldsymbol{\sigma} \otimes \sigma=\mathbb{1}+\boldsymbol{\epsilon} \tag{5.1}
\end{array}
$$

All the structure constants but $c_{\sigma \sigma \epsilon}=\frac{1}{2}$ are necessarily equal to one (cf. eq. (3.7)). The appropriate crossing-matrix elements may be evaluated [4] and through eq. (3.11), lead to selection rules in perfect agreement with eq. (5.1), with all the non-zero $N$ 's being equal to 1 . On the other hand, the analysis of sect. 4 may be applied, using the result of ref. [19] where the various correlation functions on the torus are given ${ }^{\star}$. The conformal blocks can be calculated and their monodromy properties may then be studied according to the discussion of sect. 4. In fact one can check and verify explicitly all the discussion carried out in sect. 4. In particular, one has to form the normalized combination (cf. eq. (4.12))

$$
\begin{align*}
& \Gamma_{\sigma \sigma}=\sqrt{\frac{1}{2}}\left(\Gamma_{\sigma \sigma l}+\Gamma_{\sigma \sigma \epsilon}\right)=\frac{\left[\theta_{2}\left(\frac{1}{2} w\right)\right]^{1 / 2}}{[2 \eta(q)]^{1 / 2}\left(-\theta_{1}(w) / \theta_{1}^{\prime}(0)\right)^{1 / 8}} \\
& \underset{w \rightarrow 0}{\sim}\left(\frac{2 i \pi}{1-z}\right)^{1 / 8} \chi_{\sigma}(q), \tag{5.2}
\end{align*}
$$

[^5]which verifies eq. (4.13). In short, the Ising model realizes our construction in a non-trivial way.

Let us now discuss the models with $\mathrm{SU}(2)$ current algebra of level $k$ [20,21]. We limit ourselves to those models which we expect to have left-right factorized $N_{I J K}$ (see our discussion of sect. 3); they are classified by $\mathrm{A}_{k+1}, \mathrm{D}_{2 \rho+2}$ if $k=4 \rho, \mathrm{E}_{6}$ if $k=10$, or $\mathrm{E}_{8}$ if $k=28$ [12,13]. It is convenient to label the primary fields by $\lambda=2 j+1$, where $j$ is their $\mathrm{SU}(2)$ spin, satisfying

$$
\begin{equation*}
0 \leqslant j \leqslant \frac{1}{2} k \quad \text { i.e. } \quad 1 \leqslant \lambda \leqslant k+1 \tag{5.3}
\end{equation*}
$$

The $N_{\lambda_{1} \lambda_{2} \lambda_{3}}$ coefficients ${ }^{\star}$ have been computed for the $\mathrm{A}_{k+1}$ series in ref. [6] and shown to agree with the selection rules expected for $\mathrm{SU}(2) \mathrm{WZW}$ model [21]

$$
\begin{equation*}
N_{\lambda_{1} \lambda_{2} \lambda_{3}}=1 \quad \text { if and only if }\left|j_{1}-j_{2}\right| \leqslant j_{3} \leqslant \min \left(j_{1}+j_{2}, k-j_{1}-j_{2}\right) \tag{5.4}
\end{equation*}
$$

The same selection rules for tensoring the representations $\{j\}$ of $\mathrm{SU}(2)$ are obtained by using the following rules:
(i) associativity;
(ii) in the multiplication of $\{j\}$ by $f=\left\{\frac{1}{2}\right\}$, the representations of spin $j^{\prime}>\frac{1}{2} k$ are discarded.

To see how it works take, for example, the case of level $k=4$

$$
\begin{align*}
\left\{\frac{1}{2}\right\} *\{2\} & =\left\{\frac{3}{2}\right\} \\
\left(\left\{\frac{1}{2}\right\} *\left\{\frac{1}{2}\right\}\right) *\{2\} & =(\{0\}+\{1\}) *\{2\} . \tag{5.5}
\end{align*}
$$

Using associativity

$$
\begin{equation*}
\left(\left\{\frac{1}{2}\right\} *\left\{\frac{1}{2}\right\}\right) *\{2\}=\left\{\frac{1}{2}\right\} *\left(\left\{\frac{1}{2}\right\} *\{2\}\right)=\left\{\frac{1}{2}\right\} *\left\{\frac{3}{2}\right\}=\{1\}+\{2\} \tag{5.6}
\end{equation*}
$$

Comparing eq. (5.5) with eq. (5.6) we obtain $\{1\} *\{2\}=\{1\}$, in agreement with eq. (5.4).

The models labeled $\mathrm{E}_{6}$ and $\mathrm{E}_{8}$ are easy to handle; they are actually interpreted as level-1 $\mathrm{C}_{2}$ and level-1 $\mathrm{G}_{2}$ WZW models respectively [22,23]. Their Verlinde algebra is the Ising one (5.1) for the $\mathrm{E}_{6}$ case and

$$
\begin{equation*}
\mathbb{1} \otimes \mathbb{1}=\mathbb{1}, \quad \mathbb{1} \otimes \varphi=\varphi, \quad \varphi \otimes \varphi=\mathbb{1}+\varphi \tag{5.7}
\end{equation*}
$$

for the $\mathrm{E}_{8}$ case.

[^6]The models labeled $D_{2 \rho+2}$ are more interesting. Their genus-one partition functions exhibit a factor 2

$$
\begin{equation*}
Z=\sum_{\lambda_{\text {odd }}=1}^{2 \rho-1}\left|\chi_{\lambda}+\chi_{4 \rho+2-\lambda}\right|^{2}+2\left|\chi_{2 \rho+1}\right|^{2} \tag{5.8}
\end{equation*}
$$

signaling that there are two primary fields of spin $j=\rho$ [21]. In the $\mathrm{D}_{4}$ case, this may be related to the interpretation of this model as a level-1 $\mathrm{SU}(3)$ WZW model [22], with two fields of degenerate conformal weight transforming as the 3 and the $\overline{3}$ representations. Note that these representations transform as the vector representation of the $\operatorname{SU}(2)$ which is conformally embedded in $\mathrm{SU}(3)$ level 1 [22]. In that case the matrix $S$ of modular transformation reads

$$
S=\frac{1}{\sqrt{3}}\left(\begin{array}{ccc}
1 & 1 & 1  \tag{5.9}\\
1 & \omega & \omega^{2} \\
1 & \omega^{2} & \omega
\end{array}\right), \quad \omega=\mathrm{e}^{2 i \pi / 3}, \quad S^{2}=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{array}\right)
$$

where the first row and column refer to the $(\lambda=1)+(\lambda=5)$ cluster in the language of $\mathrm{SU}(2)$, or to the representation 1 in the $\mathrm{SU}(3)$ interpretation. The corresponding algebra reads

$$
\begin{equation*}
\varphi_{3} \otimes \varphi_{3}=\varphi_{\overline{3}}, \quad \varphi_{3} \otimes \varphi_{\overline{3}}=\mathbb{1}, \tag{5.10}
\end{equation*}
$$

in agreement with the existence of a conserved $Z_{3}$ charge (triality). In this form the resulting non-zero $N$ 's are equal to 1 .

The $\mathrm{D}_{6}$ case may be treated in a similar fashion. However, the degeneracy between the two $\lambda=5$ representations has no clear interpretation. Nevertheless, there are, at first sight, several ways of splitting it. We use the basis $\chi_{1}+\chi_{9}, \chi_{3}+$ $\chi_{7}, \chi_{5}, \chi_{5}$, to write

$$
S+\left(\begin{array}{cccc}
2 s_{1} & 2 s_{3} & 1 & 1  \tag{5.11}\\
2 s_{3} & 2 s_{1} & -1 & -1 \\
1 & -1 & \alpha & \beta \\
1 & -1 & \beta & \alpha
\end{array}\right)
$$

where $s_{l}=\sin \left(\frac{1}{10} \pi l\right) ; \alpha$ and $\beta$ are such that $\alpha+\beta=1$, to recover the ordinary modular transformation when $\chi_{5} \equiv \chi_{5^{\prime}}$. One may consider two possibilities: either

$$
S^{2}=\left(\begin{array}{cccc}
1 & 0 & 0 & 0  \tag{5.12}\\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{array}\right)
$$

signaling that (5) and ( $5^{\prime}$ ) are conjugate, or $S^{2}=\mathbb{1}$. It turns out that only the second alternative, with $\alpha, \beta=\frac{1}{2}(1 \pm \sqrt{5})$ leads to integer coefficients $N_{i j k}$. The non-zero $N$ 's turn out, again, to be 1 . Denoting the four clusters by $1, \varphi, \psi_{1}$, and $\psi_{2}$,
respectively, the algebra reads

$$
\begin{gather*}
\varphi \otimes \varphi=\mathbb{1}+\varphi+\psi_{1}+\psi_{2}, \quad \varphi \otimes \psi_{1}=\varphi+\psi_{2} \\
\psi_{1} \otimes \psi_{1}=\mathbb{1}+\psi_{1}, \quad \psi_{1} \otimes \psi_{2}=\varphi \tag{5.13}
\end{gather*}
$$

and similar relations obtained by interchanging $1 \leftrightarrow 2$. This alleged O.P.A. should, of course, be tested against explicit calculations in the realization of this model as a $\mathrm{SO}(3)$ level 4 WZW model [21]. The case of higher $\mathrm{D}_{n}$ models will be discussed briefly at the end of this section.

We now want to discuss a feature which has not appeared in the simple cases discussed so far, i.e. the occurrence of multiplicities $N_{i j k}>1$. Since the previous discussion has shown the connection with suitably modified Clebsch-Gordan decompositions, it seems natural to consider higher rank algebras, and the corresponding conformal-current algebra. Take $\mathrm{SU}(3)$ for illustration, and label the fields by the dimension of their $\operatorname{SU(3)}$ representation. At level $k$ all the fields that may appear correspond to Young tableaux with at most $k$ columns. Working out the algebra using either eq. (4.26) or the calculus (5.5) adapted for $\mathrm{SU}(3)$, with $f=(3)$, one finds:
level 1 - see eq. (5.10);
level 2 - representations ( $1,3, \overline{3}, 6, \overline{6}, 8$ ), $N_{888}=1$ etc.;
level 3 -representations $(1,3, \overline{3}, 6, \overline{6}, 8,10, \overline{1} 0,15, \overline{1} 5), N_{888}=2$ etc.
The interpretation is clear. At low levels, the truncation is effective and one does not recover the well-known composition rules of representations $8 \times 8=1+8+8+10$ $+\overline{1} 0+27$. Instead we find $8 * 8=1+8+(8+10+\overline{1} 0)+(27)$ where the representations in the first brackets appear for $k \geqslant 3$ and the (27) appears for $k \geqslant 4$.

As we have discussed in sect. 2 the meaning of $N_{888}=2$ is that there are two possible couplings between three primary fields in the octet representation (see eq. (2.10)), for high enough level.

We finally return to the " $\mathrm{D}_{2 \rho+2}$ " $\mathrm{SU}(2)$ models, for $\rho \geqslant 3$. Surprisingly, non-trivial multiplicities arise there too! The matrix $S$ is written in a form generalizing eq. (5.11), with $\alpha+\beta=(-1)^{\rho}$, and $S^{2}=\mathbb{1}$ for $\rho$ even, $S^{2}$ of the form of eq. (5.12) for $\rho$ odd. This somewhat arbitrary prescription is shown to lead to integer $N_{i j k}$. Alternatively, one may reproduce this algebra by a calculus analogous to eq. (5.5); one uses this time only integer spins running between 0 and $\rho$, the latter being twice degenerate is denoted by $\rho$ and $\rho^{\prime}$. The role of $f$ is played by $j=1$, and one postulates that

$$
\begin{align*}
(1) *(\rho) & =(\rho-1)+\rho^{\prime}, \\
(1) *\left(\rho^{\prime}\right) & =(\rho-1)+\rho . \tag{5.14}
\end{align*}
$$

Somehow, the allowed range of $j$ at level $4 \rho: 0 \leqslant j \leqslant 2 \rho$ has been folded at the midpoint $j=\rho$, with an identification $j=2 \rho-j, 0 \leqslant j \leqslant \rho-1$. These rules, or this folding are then responsible for the occurrence of non-trivial multiplicities. For
example, in the $\mathrm{D}_{8}$ case: for $j=2, \lambda=5: N_{555}=2$. This must signal the existence of a two-dimensional space of couplings in the corresponding channel. We emphasize that it would be extremely interesting to test these mysterious rules, and this interpretation of $N_{555}=2$, by a direct study of the O.P.A. in the SO(3) WZW model, or in the corresponding " $\mathrm{A}, \mathrm{D}$ " minimal theory.

## 6. Conclusions

In a class of conformal theories in which the structure constants of the operator algebra decouple into a product of left and right contributions, an ansatz has been proposed for the multiplicities $N_{[i][j] k]}$ which count the dimension of the space of couplings in the fusion $i \otimes j \rightarrow \hat{k}$. This ansatz expresses $N_{[i[j j \| k]}$ in terms of elements of the crossing matrices of conformal blocks of four-point functions in the plane. This ansatz has been shown to be equivalent to that of Verlinde. We have constructed a realization of Verlinde's operations in terms of deformations of conformal blocks of the 2-point function on the torus.

As several points along our line of argument remain unproven, let us recapitulate our conjectures.
(1) We made the simplifying assumption that the crossing matrices of unitary theories are real. This is indeed what happens in minimal theories [4]. Relaxing it would replace the factor $Y_{[i][j] k]}$ in eq. (3.10) by it complex conjugate and thus spoil the symmetry under the interchange of $j$ and $k$. This is why we believe this reality to hold in general. Still, a general argument would be desirable.
(2) The precise characterization of the set of factorizable O.P.A. is missing. It is clear that it is not empty, and includes at least all the theories which are "diagonal" in terms of some extended algebra, i.e. contain only spinless primary fields of this algebra. We have conjectured that this class extends to all theories with a genus-one partition function of the form of eq. (3.4) (class I); tests of this assumption are underway [24]. Our subsequent analysis, however, does not depend on this conjecture.
(3) To achieve the connection between the ansatz and the deformation of characters, we need the consistency condition (3.6). We have proved it within theories endowed with an extended algebra, but a general (and more elegant) proof would be desirable. Whether it holds for all theories of class I remains to be investigated. Likewise, a general derivation of eq. (3.9) and its dependence on the classes $[i],[j],[k]$ is missing.
(4) The main challenge remains of course the justification of the ansatz, either in the form of eq. (3.10) or eq. (4.26). Since the $N$ 's are integers it would be gratifying if one could give some topological meaning to the r.h.s. of eq. (3.12), thus ensuring its integrality.

It is to be noted that these points all imply some knowledge of the crossing matrices, more precisely about relations between crossing matrices in various related
channels, e.g. $X_{(i j) k 1}, X_{(i k) j 1}$ and $X_{(i \hat{i}) 11}$. According to our remarks at the end of sect. 3, it may be that the ansatz can be expressed in terms of the monodromy matrices of conformal blocks, and that its proof relies on the study of monodromy properties and the corresponding braid group [25,26] of the theory.

It would also be very interesting to test directly the somewhat puzzling results on the O.P.A. of some models which follow from our ansatz (see sect. 5). Finally we have stressed that understanding of the O.P.A. in non-factorizable cases remain an open problem. Conformal field theory is still wrapped in a few mysteries... .

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## Note Added in proof

Soon after completion of this work, we received a paper by G. Moore and N. Seiberg (Phys. Lett. B212 (1988) 451) proving the main conjecture. The case of "class II" theories has been discussed in ref. [24] and by Dijkgraft and Verlinde (Utrecht preprint) and Moore and Seiberg (Princeton preprint).

## Appendix A

We discuss correlation functions of descendant fields, derive the explicit form of the coefficients $\beta_{i j, k n}$ in eq. (2.9) and prove the result announced in eqs. (3.6) and (4.7).

We use the conformal Ward identity for $N$ primary fields and $M+1$ stress energy-momentum tensors [1]

$$
\begin{align*}
\left\langle T(\xi) T\left(\xi_{1}\right) \ldots\right. & \left.T\left(\xi_{M}\right) \varphi_{1}\left(z_{1}\right) \ldots \varphi_{N}\left(z_{N}\right)\right\rangle=\left\{\sum_{i=1}^{N}\left[\frac{h_{i}}{\left(\xi-z_{i}\right)^{2}}+\frac{1}{\left(\xi-z_{i}\right)} \frac{\partial}{\partial z_{i}}\right]\right. \\
& \left.+\sum_{j=1}^{M}\left[\frac{2}{\left(\xi-\xi_{j}\right)^{2}}+\frac{1}{\left(\xi-\xi_{j}\right)} \frac{\partial}{\partial \xi_{j}}\right]\right\}\left\langle T\left(\xi_{1}\right) \ldots \varphi_{N}\left(z_{N}\right)\right\rangle \\
& +\sum_{j=1}^{M} \frac{c}{\left(\xi-\xi_{j}\right)^{4}}\left\langle T\left(\xi_{1}\right) \ldots \widehat{T\left(\xi_{j}\right)} \ldots T\left(\xi_{M}\right) \varphi_{1}\left(z_{1}\right) \ldots \varphi_{N}\left(z_{N}\right)\right\rangle . \tag{A.1}
\end{align*}
$$

The $\bar{z}$ variables of the $\varphi$ fields are implicit and the caret over $T$ denotes its omission. By integration of $T$ on a contour encircling one and only one of the $z$ 's, one generates the descendants of $\varphi(z)$.

$$
\begin{equation*}
\varphi^{(-k)}(z)=\left(L_{-k} \varphi\right)(z)=\frac{1}{2 i \pi} \oint_{z} \mathrm{~d} \xi(\xi-z)^{-k+1} T(\xi) \varphi(z) \tag{A.2}
\end{equation*}
$$

Arbitrary descendants are obtained by repeated such integrations.

By performing $M+1$ such integrations around $z_{1}$ in eq. (A.1), and using eq. (A.1) itself recursively, it is easy to see that

$$
\begin{equation*}
\left\langle\varphi_{1}^{\left(-k_{1}, \ldots,-k_{M+1}\right)}\left(z_{1}\right) \ldots \varphi_{N}\left(z_{N}\right)\right\rangle=\mathscr{P}\left(\mathscr{D}_{-l_{1}}, \mathscr{D}_{-l_{2}} \ldots\right)\left\langle\varphi_{1}\left(z_{1}\right) \ldots \varphi_{N}\left(z_{N}\right)\right\rangle, \tag{A.3}
\end{equation*}
$$

where $\mathscr{D}_{-l}$ is the differential operator

$$
\begin{equation*}
\mathscr{D}_{-l}\left(z_{1}\right)=\sum_{i=2}^{N} \frac{(l-1) h_{i}}{\left(z_{i}-z_{1}\right)^{\prime}}-\frac{1}{\left(z_{i}-z_{1}\right)^{l-1}} \frac{\partial}{\partial z_{i}} \tag{A.4}
\end{equation*}
$$

homogeneous of degree $-l$ in the $z$ 's, and $\mathscr{P}$ is a polynomial in the $\mathscr{D}$ 's, homogeneous in the $z$ 's of degree $-\sum_{1}^{M+1} k_{j}$. The interpretation is simple; the $\mathscr{D}$ 's satisfy Virasoro algebra

$$
\begin{equation*}
\left[\mathscr{D}_{-k}\left(z_{1}\right), \mathscr{D}_{-l}\left(z_{1}\right)\right]=(l-k) \mathscr{D}_{-k-l}\left(z_{1}\right), \tag{A.5}
\end{equation*}
$$

with no contribution from the central charge since $-k-l<0$. Therefore the $\mathscr{D}$ 's just realize Virasoro algebra on the field $\varphi_{1}\left(z_{1}\right)$ and eq. (A.3) may therefore be recast in the simple form

$$
\begin{align*}
\left\langle\varphi_{1}^{\left(-k_{1}, \ldots,-k_{M+1}\right)}\left(z_{1}\right) \ldots \varphi_{N}\left(z_{N}\right)\right\rangle & =\left\langle L_{-k_{M+1}} \ldots L_{-k_{1}} \varphi_{1}\left(z_{1}\right) \ldots \varphi_{N}\left(z_{N}\right)\right\rangle \\
& =\mathscr{D}_{-k_{M+1}} \ldots \mathscr{D}_{-k_{1}}\left\langle\varphi_{1}\left(z_{1}\right) \ldots \varphi_{N}\left(z_{N}\right)\right\rangle \tag{A.6}
\end{align*}
$$

As a simple application, this justifies the factorization of $\langle\hat{K} N| \varphi_{I}(1,1)|J\rangle$ displayed in eq. (2.9).

$$
\begin{align*}
\langle\hat{K}\{l\} & \left.\{\bar{l}\}\left|\varphi_{I}(1,1)\right| J\right\rangle \\
= & \lim _{z_{1}, \bar{z}_{1} \rightarrow \infty} z_{1}^{2\left(h_{k}+|l|\right)} \times \text { "c.c." }\left\langle\varphi_{K}^{\left(-l_{1}, \ldots,-i_{q}\right)}\left(z_{1}, \bar{z}_{1}\right) \varphi_{I}(1,1) \varphi_{J}(0,0)\right\rangle \\
= & \lim _{z_{1}, \bar{z}_{1} \rightarrow \infty} z_{1}^{2\left(h_{k}+|l|\right)} \times \text { "c.c." } \\
& \times \mathscr{D}_{-l_{p}}\left(z_{1}\right) \ldots \mathscr{D}_{-l_{1}}\left(z_{1}\right) \mathscr{D}_{-i_{q}}\left(\bar{z}_{1}\right) \ldots \mathscr{D}_{-i_{1}}\left(\bar{z}_{1}\right) \\
& \quad \times\left.\left\langle\varphi_{K}\left(z_{1}, \bar{z}_{1}\right) \varphi_{I}\left(z_{2}, \bar{z}_{2}\right) \varphi_{J}\left(z_{3}, \bar{z}_{3}\right)\right\rangle\right|_{x_{2}=1} \\
& x_{3}=0 \tag{A.7}
\end{align*}
$$

with

$$
\begin{align*}
\beta_{i j, k\{l\}}= & \lim _{\substack{z_{1} \rightarrow \infty \\
z_{2} \rightarrow 1 \\
z_{3} \rightarrow 0}} z_{1}^{2\left(h_{k}+|| |) \mathscr{D}_{-l_{p}}\left(z_{1}\right)\right.} \\
& \ldots \mathscr{D}_{-l_{1}}\left(z_{1}\right) \frac{1}{\left(z_{1}-z_{2}\right)^{h_{k}+h_{i}-h_{j}}\left(z_{2}-z_{3}\right)^{h_{i}+h_{j}-h_{k}}\left(z_{3}-z_{1}\right)^{h_{j}+h_{k}-h_{i}}} . \tag{A.8}
\end{align*}
$$

This is a mere repeat of results of ref. [1].

Let us now consider a correlation function with two descendant fields. For our purpose, it is sufficient to consider a 4-point function of the form

$$
\begin{equation*}
\left\langle\varphi_{\hat{J}}^{\left(-l_{1}, \ldots,-t_{p}\right)}\left(z_{1}\right) \varphi_{\hat{I}}\left(z_{2}\right) \varphi_{I}\left(z_{3}\right) \varphi_{J}^{\left(-k_{1}, \ldots,-k_{q}\right)}\left(z_{4}\right)\right\rangle \tag{A.9}
\end{equation*}
$$

It cannot be simply given by a homogeneous polynomial in the $\mathscr{D}\left(z_{1}\right)$ and $\mathscr{D}\left(z_{4}\right)$ acting on $\left\langle\varphi_{\hat{j}}\left(z_{1}\right) \ldots \varphi_{J}\left(z_{4}\right)\right\rangle$. The commutation of the two Virasoro algebras at $z_{1}$ and $z_{4}$ is still quite complicated and in general "feels" the central charge. By inspection, it is, however, easy to convince oneself that the differential operator is now a homogenous polynomial in the $\mathscr{D}_{-l}\left(z_{1}\right), \mathscr{D}_{-l}\left(z_{4}\right)$ and $\Delta_{-n}\left(z_{1}\right), \Delta_{-m}\left(z_{4}\right)$ where

$$
\begin{equation*}
\Delta_{-m}\left(z_{4}\right)=\frac{a_{m}}{\left(z_{4}-z_{1}\right)^{m}}-\frac{b_{m}}{\left(z_{4}-z_{1}\right)^{m-1}} \frac{\partial}{\partial_{z_{4}}} \tag{A.10}
\end{equation*}
$$

does not depend on $z_{2}, z_{3}$.
Let us finally apply this result to the study of the singularities of eq. (A.9) as $z_{2} \rightarrow z_{3}$. We start from eq. (2.11) and apply the differential operator $\mathscr{P}\left(\mathscr{D}_{-}\left(z_{1}\right), \mathscr{D}_{-}\left(z_{4}\right), \Delta_{-}\left(z_{1}\right), \Delta_{-.}\left(z_{4}\right)\right)$. We claim that whenever such an operator acts on $\left(z_{2}-z_{3}\right)^{-2 h_{i}}$ or on the cross ratio $z$, it produces a term proportional to $\left(z_{2}-z_{3}\right)$ which therefore does not contribute to the leading singularity

$$
\begin{align*}
{\left[\sum_{i^{\prime}=2}^{3} \frac{(l-1) h_{i}}{\left(z_{i^{\prime}}-z_{1}\right)^{l}}\right.} & \left.-\frac{1}{\left(z_{i^{\prime}}-z_{1}\right)^{l-1}} \frac{\partial}{\partial z_{i^{\prime}}}\right] \frac{1}{\left(z_{2}-z_{3}\right)^{2 h_{i}}} \\
& \sim \frac{l(l-1) h_{i}}{\left(z_{2}-z_{3}\right)^{2 h_{i^{\prime}-1}\left(z_{2}-z_{1}\right)^{l+1}}+\ldots,} \\
\frac{\partial}{\partial z_{4}}\left(\frac{z_{1}-z_{2}}{z_{1}-z_{3}} \frac{z_{3}-z_{4}}{z_{2}-z_{4}}\right) & =-\frac{z_{1}-z_{2}}{\left(z_{1}-z_{3}\right)\left(z_{2}-z_{4}\right)^{2}}\left(z_{2}-z_{3}\right) . \tag{A.11}
\end{align*}
$$

Therefore, the only contribution to the leading $z_{2} \rightarrow z_{3}$ singularity comes from the action of the polynomial $\mathscr{P}\left(\mathscr{D}_{-}^{\prime}\left(z_{1}\right), \mathscr{D}_{-}^{\prime}\left(z_{4}\right), \Delta_{-}\left(z_{1}\right), \Delta_{\ldots}\left(z_{4}\right)\right)$ on $\left(z_{1}-z_{4}\right)^{-2 h}$, where in $\mathscr{D}^{\prime}$ the terms in eq. (A.4) involving $z_{2}$ or $z_{3}$ have been removed. But the result is nothing but

$$
\begin{equation*}
\lim _{\substack{z_{1} \rightarrow \infty \\ z_{4} \rightarrow \infty}} z_{1}^{2\left(h_{k}+|/|\right)} \mathscr{P}\left(\mathscr{D}^{\prime}, \mathscr{D}^{\prime}, \Delta, \Delta\right) \frac{1}{\left(z_{1}-z_{4}\right)^{-2 h_{j}}}=\langle J\{-l\} \mid J\{-k\}\rangle \tag{A.12}
\end{equation*}
$$

and the same result applies to each term in eq. (2.13), i.e. for a given intermediate $\hat{K}$

$$
\begin{equation*}
\sum_{N}\langle J M| \varphi_{\hat{I}}(1)|\hat{K} N\rangle\langle\hat{K} N| \varphi_{I}(z, \bar{z})|J M\rangle \underset{z \rightarrow 1}{\sim} \sum_{N}\langle J| \varphi_{\hat{I}}(1)|\hat{K} N\rangle\left\langle\hat{K_{N}}\right| \varphi_{I}(z, \bar{z})|J\rangle \tag{A.13}
\end{equation*}
$$

This is the result announced in eq. (4.7).

By the same token, this justifies the consistency condition (3.6) in the class of factorizable theories with an extended algebra. In the latter, the various generators including $T$ are expected to satisfy with their primary fields Ward identities generalizing (A.1) (with poles of different orders) [27]. A repeat of the previous discussion leads to the conclusion that if $J^{\prime}$ is a descendant of the primary field $J$, then

$$
\begin{equation*}
\sum_{K N}^{\prime}\left\langle J^{\prime}\right| \varphi_{\hat{I}}(1)|\hat{K} N\rangle\langle\hat{K} N| \varphi_{I}(z, \hat{z})\left|J^{\prime}\right\rangle \underset{z \rightarrow 1}{\sim} \sum_{K N}^{\prime}\langle J| \varphi_{\hat{I}}(1)|\hat{K} N\rangle\langle\hat{K} N| \varphi_{I}(z, \bar{z})|J\rangle \tag{A.14}
\end{equation*}
$$

where the sum runs over $K$ in a certain cluster [ $K$ ]. This means that $\sum_{k \in[k]} c_{i j k}^{2} X_{(i j) k 1}$ depends only on the class [ $j$ ] in the terminology of sect. 3. Let us now examine the effect of changing the representative of [i], say from $i$, the primary, (of the extended algebra) to $i^{\prime}$, some descendant. The 4-point function $\left\langle J_{\hat{I}^{\prime}} \varphi_{I^{\prime}} J\right\rangle$ is related to $\left\langle J \varphi_{\hat{I}} \varphi_{I} J\right\rangle$ again by the same extended Ward identity, and the leading singularity as $z \rightarrow 1$ of the former comes entirely from the action of differential operators $\mathscr{D}_{1}\left(z_{2}, z_{3}\right)$ generalizing eq. (A.4) on the latter. This only affects the order of the pole which is changed from $2 h_{i}$ to $2 h_{i}^{\prime}$ but, for well-normalized descendants $\varphi_{i^{\prime}}$, not the residue, hence not $\sum_{k \in[k]} c_{i j k}^{2} X_{(i j) k 1}$.

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[^0]:    * Supported in part by the U.S.-Israel Science foundation and by the Israeli Academy of Science.

[^1]:    * In practice the actual construction of such a basis is a formidable task, requiring a diagonalization of the contravariant form $[8,9]$ and a consistent elimination of the null states and their descendants.

[^2]:    ${ }^{*}$ The fact that $\mathrm{D}_{\text {odd }}$ and $\mathrm{E}_{7}$ models have specific features has been also noticed in a similar context but a different language in ref. [16].

[^3]:    ${ }^{\star}$ Epsilon does not appear in $\epsilon \otimes \epsilon$ because it is odd under the self-duality of the Potts model whereas $\epsilon^{\prime}$ is even.

[^4]:    * The subscript 1 refers, as usual, to the identity operator.

[^5]:    * It now becomes clear why in that later work the $\langle\sigma \sigma\rangle$ correlation function was found to be a sum of four moduli of holomorphic contributions while the $\langle\epsilon \epsilon\rangle$ function was a sum of three. What was interpreted there in the fermionic language, specific to the Ising model, may now be recast in the general form (4.4) and reflects the various possibilities offered by the O.P.A. (5.1).

[^6]:    ${ }^{\star}$ In the rest of this section the indices refer to the extended algebra. For simplicity of notations we write them as $i, j, \ldots$ rather than $[i],[j], \ldots$.

