REMARKS ABOUT THE EXISTENCE OF NON-LOCAL CHARGES IN TWO-DIMENSIONAL MODELS

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A simple derivation of the classical non-local conservation laws in two dimensions discovered by Lüscher and Pohlmeyer is given. Several classes of models are shown to possess the same structure.

A few years ago Lüscher and Pohlmeyer [1] discovered conserved non-local charges in the two dimensional non-linear σ-model. If the lagrangian is written as

$$\mathcal{L} = (2\lambda)^{-1}\partial_\mu \boldsymbol{S}\partial_\mu \boldsymbol{S}\,, \qquad \boldsymbol{S} = (S^1(x), ..., S^n(x))\,, \quad \boldsymbol{S}^2 = 1\,, \tag{1}$$

the simplest nontrivial conserved charge is

$$Q^{\alpha\beta}(t) = \int_{-\infty}^{+\infty} \mathrm{d}x \int_{-\infty}^{+\infty} \mathrm{d}x'\, j_0^{\alpha\gamma}(t,x) j_0^{\gamma\beta}(t,x')\,\theta(x-x') - \int_{-\infty}^{+\infty} j_1^{\alpha\beta}(t,x)\,\mathrm{d}x\,, \tag{2}$$

$$j_\mu^{\alpha\beta} = 2\,S_\alpha \overleftrightarrow{\partial}_\mu S_\beta\,. \tag{3}$$

These charges have been obtained by inverse scattering methods without use of conformal invariance and thus, as shown later by Lüscher [2], they are also quantum-mechanically conserved, implying the absence of particle production and factorization of the multiparticle S-matrix elements. These were the main hypotheses made by the Zamalodchikovs [3] in their construction of the S-matrix of the O(n) non-linear σ-model in two dimensions.

In this note we want to give a brief derivation of the existence of these conserved charges and to show that many generalized two-dimensional σ-models possess this same structure. Assume that we have found a set of matrices $A_\mu^{\alpha\beta}(x)$ with the following properties:

(i) $A_\mu(x)$ is a pure gauge, i.e. there exists a non-singular matrix g such that

$$A_\mu(x) = g^{-1}(x)\partial_\mu g(x)\,; \tag{I}$$

(ii) $A_\mu(x)$ is conserved as a consequence of the equations of motion of the model:

$$\partial_\mu A_\mu(x) = 0\,. \tag{II}$$

We can then construct an infinite set of (non-local) conserved currents by the following inductive procedure. Define the covariant derivative

$$D_\mu^{\alpha\beta} = \delta^{\alpha\beta}\partial_\mu + A_\mu^{\alpha\beta}\,. \tag{4}$$

It follows from (I) that

$$[D_\mu, D_\nu] = 0\,, \tag{5}$$

and from (II) that

$$\partial_\mu D_\mu = D_\mu \partial_\mu\,. \tag{6}$$

Let us assume that we have constructed the nth conserved current $J_\mu^{(n)}$. Therefore, there exists a function $\chi^{(n)}(x)$ such that:

$$J_\mu^{(n)} = \epsilon_{\mu\nu}\partial_\nu \chi^{(n)}\,, \quad n \geqslant 1\,. \tag{7}$$

The (n+1)th current is then defined as

$$J^{(n+1)} = D_\mu \chi^{(n)}\,, \quad n \geqslant 0\,. \tag{8}$$

The induction starts with $\chi^{(0)} = 1$, and thus $J_\mu^{(1)} = A_\mu$, which is conserved. To show that $J_\mu^{(n+1)}$ is conserved, we note that from eq. (6)

$$\partial_\mu J_\mu^{(n+1)} = D_\mu \partial_\mu \chi^{(n)}\,, \quad n \geqslant 1\,,$$

and from eq. (7)

$$\partial_\mu J_\mu^{(n+1)} = -\epsilon_{\mu\nu} D_\mu J_\nu^{(n)}\ , \quad n \geqslant 1\ . \tag{9}$$

We then obtain

$$\partial_\mu J_\mu^{(n+1)} = -\epsilon_{\mu\nu} D_\mu D_\nu \chi^{(n-1)}$$

$$= -[D_0, D_1]\chi^{(n-1)} = 0\ , \quad n \geqslant 1\ ,$$

which completes the induction. We thus have an infinite number of conserved charges

$$Q^{(n)}(t) = \int_{-\infty}^{+\infty} \mathrm{d}x\, J_0^{(n)}(t,x)\ . \tag{10}$$

It is easy to verify that $Q^{(2)}(t)$ has precisely the structure (2) in terms of $j^{(1)}$ since

$$\begin{aligned} Q^{(2)}(t) &= \int_{-\infty}^{+\infty} \mathrm{d}x\, (\partial_0 + j_0^{(1)})\chi^{(1)} \\ &= -\int_{-\infty}^{+\infty} \mathrm{d}x\, j_1^{(1)}(t,x) + \int_{-\infty}^{+\infty} \mathrm{d}x\, j_0^{(1)}(t,x)\chi^{(1)}(t,x)\ . \end{aligned} \tag{11}$$

We can now integrate (7) for $n = 1$,

$$j_0^{(1)}(t,x) = \partial_1\chi^{(1)} \Rightarrow \chi^{(1)}(t,x) = \int_{-\infty}^{x} \mathrm{d}x'\, j_0^{(1)}(t,x')$$

(apart from an irrelevant integration constant which would add to $Q^{(2)}$ a piece proportional to $Q^{(1)}$), and eq. (11) takes precisely the form (2). It is thus sufficient to find models which possess properties (I) and (II) in order to ensure the existence of these charges.

Models which fulfill the properties (I) and (II).

(i) The simplest examples are given by the lagrangian

$$\mathcal{L} = (2\alpha)^{-1}\,\mathrm{Tr}(\partial_\mu g(x)\,\partial_\mu g^{-1}(x))\ , \tag{12}$$

in which $g(x)$ varies in a group of matrices G. This lagrangian is invariant by a global transformation of $G \times G$: $g(x) \to g_1 g(x) g_2$. The most common examples correspond to $G = \mathrm{U}(n)$ ($gg^+ = 1$), or $G = \mathrm{O}(n)$ ($gg^\mathrm{T} = 1$). The Euler–Lagrange equations of motion are simply [‡1]

$$\partial_\mu(g^{-1}(x)\,\partial_\mu g(x)) = 0\ ,$$

and of course (I) and (II) are satisfied with $A_\mu = g^{-1}\partial_\mu g$.

(ii) Other examples are provided by the same lagrangian (12) in which $g(x)$ varies over a restricted set of invertible matrices of the group G. Consider, for instance, the set of orthogonal matrices

$$g(x) = \mathrm{e}^{\mathrm{i}\pi P(x)}\ , \tag{13}$$

in which $P(x)$ is a projector onto a one-dimensional subspace, namely $P^{\alpha\beta}(x) = S^\alpha(x)S^\beta(x)$ with $S^\alpha \cdot S^\alpha = 1$. It is elementary to verify that $g^{-1}(x) = g(x) = 1 - 2P(x)$ (i.e. $g(x)$ is a symmetry with respect to the plane orthogonal to the subspace onto which P is a projector) and that

$$\mathrm{Tr}\,\partial_\mu g\,\partial_\mu g^{-1} = 8\,\partial_\mu \boldsymbol{S}\,\partial_\mu \boldsymbol{S}\ , \quad g^{-1}\partial_\mu g = 2S_\alpha \overleftrightarrow{\partial_\mu} S_\beta\ .$$

The equations of motion of the model, $\partial_\mu S_\alpha \overleftrightarrow{\partial_\mu} S_\beta = 0$, complete the identification of the $\mathrm{O}(n)$-model with the required structure.

(iii) If we have again the structure (13) with P projecting onto a complex one-dimensional subspace,

$$P^{\alpha\beta}(x) = z^{\alpha *}(x) z^\beta(x) \quad \text{with} \quad z^{\alpha *} z^\alpha = 1\ .$$

We now have

$$\mathrm{Tr}\,\partial_\mu g \partial_\mu g^{-1} = 4\,\mathrm{Tr}\,\partial_\mu P \partial_\mu P\ ,$$

and the resulting lagrangian is that of the CP^n σ-model:

$$\mathcal{L} = (8/2\alpha)(\partial_\mu z^{*\alpha}\partial_\mu z^\alpha + (z^{*\alpha}\partial_\mu z^\alpha)(z^{*\beta}\partial_\mu z^\beta))\ .$$

The equations of motion have indeed the form

$$\partial_\mu A_\mu = 0 \quad \text{with} \quad A_\mu = -2[P, \partial_\mu P]\ .$$

(iv) More generally we can take g matrices of the form (13) which satisfy thus $g^2 = 1$, but with a projector P on a subspace of arbitrary dimension p [5], as for instance in

$$P^{\alpha\beta}(x) = \sum_{i=1}^{p} S_i^\alpha(x) S_i^\beta(x)\ , \quad \text{with} \quad S_i^\alpha \cdot S_j^\alpha = \delta_{ij}\ , \quad \alpha, \beta = 1, ..., n\ . \tag{14}$$

Again the lagrangian reads

$$\mathcal{L} = (2\alpha)^{-1}\,\mathrm{Tr}\,\partial_\mu g \partial_\mu g^{-1} = (2/\alpha)\,\mathrm{Tr}\,\partial_\mu P \partial_\mu P\ ,$$

‡1 This is obviously true for the full linear group $\mathrm{GL}(n)$. It is also true for other compact Lie groups since $g^{-1}\partial_\mu g$ is an element of the Lie algebra of the group. In the lattice formulation, recently discussed by A. Polyakov (unpublished) one must distinguish at this stage between $\mathrm{GL}(n)$ and for instance $\mathrm{O}(n)$.

with P given by eq. (14). This lagrangian is invariant by global $O(n)$-rotations, and local $O(p)$ rotations since we can obviously choose at any point of space–time an arbitrary basis in the subspace onto which P projects. This is a non-abelian generalization of CP^n which possesses the same conserved charges since the equations of motion are again

$$\partial_\mu A_\mu = 0 \,, \quad A_\mu = -2[P, \partial_\mu P] = e^{-i\pi P(x)} \partial_\mu e^{i\pi P(x)} \,.$$

As a side remark we note that these models are symmetric in the interchange $p \leftrightarrow n-p$ by replacing P by the complementary projector Q such that $PQ = QP = 0$, $P + Q = 1$.

Conclusion: all the known models which generalize the $O(n)$ σ-model and possess only one coupling constant have the same infinite set of non-local conserved charges with presumably a quantum-mechanical equivalent. The consequences concerning the classical complete integrability of the models or on the quantum theory have not yet been explored. The formalism used here can presumably be generalized to gauge models in three dimensions along the lines suggested by Polyakov (see footnote 1).

References

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