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Quantum Field Theory Techniques in Graphical Enumeration

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We present a method for counting closed graphs on a compact Riemannian surface, based on techniques suggested by quantum field theory.

1. INTRODUCTION

In this paper our aim is to present as simply, but as completely as possible, some results on the counting of graphs (to be defined precisely below) drawn on a Riemann surface. These were obtained using integral representations suggested by quantum field theory. It is sufficient to say here that the latter deals with integrals that are computed in power series, each term of which can be put in correspondence with a set of graphs, as originally suggested by Feynman. This is called a perturbative series.

For our purpose we prove this correspondence in Section 2 for a simple example that will be used to define our rules and serves to demonstrate the method. In Section 3 we reorganize the series for connected graphs according to their topological properties characterized by the smallest genus of a compact Riemannian surface on which they can be faithfully drawn.

We reformulate the expressions of interest in terms of a family of orthogonal polynomials in Section 4. In Section 5 we obtain the expression for the generating function $e_0(g)$ counting planar graphs (i.e., of genus zero). A further analysis in Section 6 enables us to derive explicitly the expressions of $e_1(g)$ and $e_2(g)$, the generating functions on a simple and double-torus for graphs with quartic vertices (Section 7).

Apart from formulating the methods, our explicit results are therefore contained in formula (5.11), giving the general result for $e_0(g)$, and formulas (7.19), (7.21), and (7.33), for $e_0(g)$, $e_1(g)$ and $e_2(g)$ in the quartic

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case. Of course the technique described here can be used to compute the higher $e_H(g)$ as well as other quantities of interest.

In Section 8 we briefly mention various other methods dealing with the same problem, as well as related topics. Most noteworthy among them is the subject of random matrices, to which we devote a lengthy appendix (Appendix 6). The remarks at the end of Appendix 6 were worked out in collaboration with E. Brézin. The other five appendixes collect technical results. The last of them is based on an idea due to J. M. Drouffe.

The algebraic topology used in this paper essentially reduces to Euler's characteristic formula, which serves to define the genus H of a surface. Similarly, the required results from the theory of the unitary group are derived in Appendix 3. Consequently, our presentation is largely self-contained.

2. Wick's Lemma and Perturbation Series

We first introduce and compute the following elementary integrals

$$\langle x_{\mu_1} x_{\mu_2} \cdots x_{\mu_n} \rangle = \frac{\int d^p x x_{\mu_1} x_{\mu_2} \cdots x_{\mu_n} \exp\left(-\frac{1}{2} \sum_{\mu, \nu} x_\mu A_{\mu\nu} x_\nu \right)}{\int d^p x \exp\left(-\frac{1}{2} \sum_{\mu, \nu} x_\mu A_{\mu\nu} x_\nu \right)}. \tag{2.1}$$

Here x is a point in the real Euclidean space $\mathbb{R}_p$; hence, each index μ takes p distinct values. The real $p \times p$ matrix A is symmetric and positive definite to ensure the convergence of the integral. Therefore, it admits an inverse A^{-1}. Note that the above mean value is invariant under any permutation P of the indices $\mu_1, \mu_2, \ldots, \mu_n$.

Lemma (Wick).

$$\langle x_{\mu_1} x_{\mu_2} \cdots x_{\mu_{2n+1}} \rangle = 0, \tag{2.2}$$

$$\langle x_{\mu_1} x_{\mu_2} \rangle = (A^{-1})_{\mu_1 \mu_2}, \tag{2.3}$$

$$\langle x_{\mu_1} x_{\mu_2} \cdots x_{\mu_{2n}} \rangle = \frac{1}{2^n n!} \sum_P \langle x_{\mu_{P_1}} x_{\mu_{P_2}} \rangle \cdots \langle x_{\mu_{P_{2n-1}}} x_{\mu_{P_{2n}}} \rangle. \tag{2.4}$$

To obtain (2.2) change x into $-x$ in the integrand of (2.1). For the last formula we observe that there are $2^n n!$ permutations which leave invariant a given pairing of the $2n$ indices $\mu_1, \mu_2, \ldots, \mu_{2n}$. Therefore, the right-hand side of Eq. (2.4) can be rewritten without any prefactor as a sum over all distinct pairings of the indices.

Proof. From Appendix 1,

$$\left\langle \exp\left(t \sum_{\mu=1}^{p} j_\mu x_\mu \right) \right\rangle = \exp\left(\frac{t^2}{2} \sum_{\mu,\nu=1}^{p} j_\mu (A^{-1})_{\mu\nu} j_\nu \right). \tag{2.5}$$

Expanding both sides in powers of t, we see that odd powers have a vanishing coefficient. The term of order t^2 yields Eq. (2.3), while the general term of order $2n$ reads

$$\frac{1}{(2n)!} \sum_{\mu_1, \mu_2, \ldots, \mu_{2n}} j_{\mu_1} j_{\mu_2} \ldots j_{\mu_{2n}} \langle x_{\mu_1} x_{\mu_2} \ldots x_{\mu_{2n}} \rangle = \frac{1}{2^n n!} \left(\sum_{\mu,\nu} j_\mu (A^{-1})_{\mu\nu} j_\nu \right)^n$$

$$= \frac{1}{2^n n!} \sum_{\mu_1, \mu_2, \ldots, \mu_{2n}} j_{\mu_1} j_{\mu_2} \ldots j_{\mu_{2n}} \frac{1}{(2n)!} \sum_{P} (A^{-1})_{\mu_{p_1} \mu_{p_2}} \ldots (A^{-1})_{\mu_{p_{2n-1}} \mu_{p_{2n}}}. \tag{2.6}$$

Identification of the symmetric coefficient of $j_{\mu_1} j_{\mu_2} \ldots j_{\mu_{2n}}$ yields the result (2.4), which completes the proof.

Now consider the slightly more intricate integral, typical of those with which we shall deal in what follows,

$$Z(\lambda) = \int d^p x \exp\left\{ -\frac{1}{2} \sum_{\mu,\nu} x_\mu A_{\mu\nu} x_\nu - \frac{\lambda}{4!} \sum_{\mu_1, \mu_2, \mu_3, \mu_4} g_{\mu_1 \mu_2 \mu_3 \mu_4} x_{\mu_1} x_{\mu_2} x_{\mu_3} x_{\mu_4} \right\}. \tag{2.7}$$

The quartic form $\sum g_{\mu_1 \mu_2 \mu_3 \mu_4} x_{\mu_1} x_{\mu_2} x_{\mu_3} x_{\mu_4}$ will be assumed to be positive definite to ensure the convergence of the integral for λ real and positive. Without loss of generality $g_{\mu_1 \mu_2 \mu_3 \mu_4}$ will be considered as symmetric in the permutation of its indices. Clearly $Z(\lambda)$ is holomorphic in the complex λ plane in the region $\mathrm{Re}\,\lambda > 0$. By making a change of variable $x = \lambda^{-1/4} y$ in (2.7) we see that

$$Z(\lambda) = \lambda^{-p/4} Y(\lambda^{-1/2}), \tag{2.8}$$

where $Y(z)$ is an entire function in z. Hence $Z(\lambda)$ is in fact analytic in the complex λ plane except for a cut on the negative real axis $(-\infty, 0]$. Furthermore, it is infinitely differentiable on the closed interval $[0, \infty)$ and the same property holds along any semi-axis $\lambda = \rho e^{i\theta}$, $-\pi < \theta < \pi$, for fixed θ and $\rho > 0$.

Therefore, it makes sense to discuss the asymptotic series

$$Z(\lambda)/Z(0) = 1 - z_1 \lambda + z_2 \lambda^2 + \ldots + (-1)^k z_k \lambda^k + \ldots, \tag{2.9}$$

which is the perturbative series alluded to in the introduction. Obviously,

$$z_k = \frac{1}{(4!)^k k!} g_{\mu_1^1 \mu_2^1 \mu_3^1 \mu_4^1} \ldots g_{\mu_1^k \mu_2^k \mu_3^k \mu_4^k} \langle x_{\mu_1^1} x_{\mu_2^1} \ldots x_{\mu_3^k} x_{\mu_4^k} \rangle. \tag{2.10}$$

Here we use the convention that repeated indices have to be summed over. The above mean value can be computed using Eqs. (2.3) and (2.4). Thus, a graphical representation appears natural since the effect of Wick's lemma is to replace the mean value of the $4k$ x's by products of elementary "contractions" $\langle x_\mu x_\nu \rangle$ of pairs of variables in all possible distinct manners. To represent the various terms obtained in this way, we introduce the following rules:

(i) Draw k distinct points called vertices, labeled $v_1, v_2, \ldots, v_k$ to which we attach, respectively, the factors

$$g_{\mu_1^1 \mu_2^1 \mu_3^1 \mu_4^1} \cdots g_{\mu_1^k \mu_2^k \mu_3^k \mu_4^k}.$$

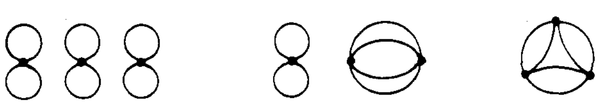
(ii) Pair in every possible way the $4k$ indices by drawing links between pairs of vertices. Four lines are incident at each vertex. The lines may connect two distinct vertices or one and the same vertex. To each line connecting indices μ and ν attach a "propagator," i.e., a factor $(A^{-1})_{\mu\nu}$. Recall that A^{-1} is a symmetric real matrix.

(iii) For each such pairing sum over all dummy indices.

(iv) Collect all terms (i.e., the graph contributions) obtained in this way, sum them, and divide by $(4!)^k k!$. This gives z_k. A term will be said to be of order k if the corresponding graph involves k vertices.

In the above rules the labeling of the vertices is immaterial; consequently, two graphs that differ only in the indices attached to the vertices yield the same contribution. It would therefore seem that the $k!$ labeled graphs correspond to the same unlabeled graph. This is, however, not quite true. It may happen that by labeling the vertices one obtains two topological equivalent graphs. More generally, let S_v be the order of the group of permutations of vertices that yield topologically equivalent labeled graphs. The operation of omitting the labels of the vertices will therefore incompletely cancel the factor $k!$, leaving a division factor S_v. We have drawn in Fig. 1 some terms of order 3 and indicated the corresponding value of S_v.

The graphs in Fig. 1 have not yet been labeled with the indices pertaining to factors $g_{\mu_1 \mu_2 \mu_3 \mu_4}$. To do this we want to take advantage of the symmetry of these factors in any permutation of their indices. When



$S_v = 3!$ $S_v = 2!$ $S_v = 3!$

FIG. 1. Some terms of third order, and the value of S_v giving the number of elements of the symmetry group of vertices.

(a) (b)

FIG. 2. (a) The simplest graph of order one. (b) Labeling of the first-order graph.

contracting two indices through the propagator $(A^{-1})_{\mu\nu}$ representing a given line we have to make a choice of the pair $\mu\nu$. Because of the symmetry of g it is immaterial which of its indices we call μ for the first vertex; similarly for ν.

In the generic case the four indices of a given vertex v will be contracted with four other ones pertaining to four distinct vertices of the graph, and there will be 4! equivalent choices. This will cancel the factor 4! attached to the vertex v due to our initial convention. Again this cancellation might be incomplete due to the symmetries of the graph. Take, for instance, the simplest graph of order one (Fig. 2a). In trying to contract two pairs of indices at the same vertex we have three possibilities giving the same contribution $g_{\mu\nu\rho\sigma}(A^{-1})_{\mu\nu}(A^{-1})_{\rho\sigma}$. Upon division by 4! this leaves a value $(1/2^3)g_{\mu\nu\rho\sigma}(A^{-1})_{\mu\nu}(A^{-1})_{\rho\sigma}$ corresponding to the labeled graph of Fig. 2b. The factor 2^3 may be analyzed as follows. A factor 2^2 corresponds to the permutations $\mu \leftrightarrow \nu$, $\rho \leftrightarrow \sigma$ under which g and A^{-1} are invariant. The remaining factor 2 is the order of the permutation group of the equivalent lines of the graph for fixed vertices. This factor we call S_l in the general case.

It is seen that the previous considerations extend to an arbitrary graph leading to the following Feynman rules. To a given order k:

(i) First draw all topologically inequivalent unlabeled graphs with k vertices. A graph is composed of vertices v and lines l joining pairs of vertices. (We also say that they are incident on the vertices at their end points.) For the computation of the integral (2.7) four lines are incident at each vertex, counting twice those which join a vertex to itself.

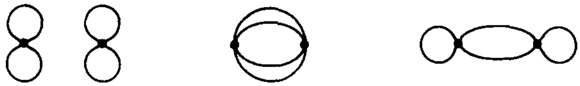
(ii) Label the end points of each line. Assign to the line with labels $\mu\nu$ a factor $(A^{-1})_{\mu\nu}$ and to each vertex with labels $\mu\nu\rho\sigma$ a factor $g_{\mu\nu\rho\sigma}$. The labeling is just a pictorial way to represent the manner in which the dummy indices are to be contracted and summed over.

(iii) For each graph divide the previous contribution by a symmetry factor S written as a product of three terms,

$$S = 2^c S_v S_l. \tag{2.11}$$

(iv) Sum over all possibilities to a given order. This is z_k.

The factors S_v and S_l have been previously defined and correspond, respectively, to the order of the group of permutations of vertices and lines



(a) (b) (c)

FIG. 3. The topologically inequivalent graphs of second order.

leaving the graph invariant. Finally, σ is the number of lines that start and end at the same vertex. In the language of graph theory they are sometimes called loops. We shall not adhere to this convention here, since we want to use the term loop for a different concept (to be defined below).

As an application we compute z_1 and z_2 using the above rules. For z_1 we have only one graph, shown in Fig. 2. Here $\sigma = 2$, $S_l = 2!$, $S_v = 1$. Therefore $S = 8$ and the contribution will be

$$z_1 = \tfrac{1}{8} g_{\mu\nu\rho\sigma}(A^{-1})_{\mu\nu}(A^{-1})_{\rho\sigma}. \tag{2.12}$$

For z_2 the three topologically inequivalent graphs are reproduced in Fig. 3. For the graph of Fig. 3a we have $\sigma = 4$, $S_l = 2^2$, $S_v = 2$. Therefore $S = 2^7 = 2 \times 8^2$. The corresponding contribution is

$$z_2^a = \tfrac{1}{2}\left[\tfrac{1}{8} g_{\mu\nu\rho\sigma}(A^{-1})_{\mu\nu}(A^{-1})_{\rho\sigma}\right]^2. \tag{2.13}$$

For the graph of Fig. 3b, $\sigma = 0$, $S_l = 4!$, $S_v = 2$. Therefore $S = 2 \times 4!$ and

$$z_2^b = \frac{1}{2\,4!} g_{\mu\nu\rho\sigma} g_{\bar\mu\bar\nu\bar\rho\bar\sigma}(A^{-1})_{\mu\bar\mu}(A^{-1})_{\nu\bar\nu}(A^{-1})_{\rho\bar\rho}(A^{-1})_{\sigma\bar\sigma}. \tag{2.14}$$

Finally, for the graph of Fig. 3c, $\sigma = 2$, $S_l = 2$, $S_v = 2$. Hence $S = 2^4$ and

$$z_2^c = \frac{1}{2^4} g_{\mu\nu\rho\sigma} g_{\bar\mu\bar\nu\bar\rho\bar\sigma}(A^{-1})_{\mu\nu}(A^{-1})_{\bar\mu\bar\nu}(A^{-1})_{\rho\bar\rho}(A^{-1})_{\sigma\bar\sigma}. \tag{2.15}$$

To check these results, let us specialize to the case $p = 1$ and $g = 1$. From the previous rules, we have, A being now a pure (positive) number,

$$\frac{Z(\lambda)}{Z(0)} = 1 - \frac{\lambda}{8}A^{-2} + \left[\frac{\lambda A^{-2}}{8}\right]^2 \frac{35}{6} + \cdots. \tag{2.16}$$

On the other hand, a direct calculation yields

$$\frac{Z(\lambda)}{Z(0)} = \frac{\int_{-\infty}^{+\infty} dx \exp\left(-\frac{A}{2}x^2 - \frac{\lambda}{4!}x^4\right)}{\int_{-\infty}^{+\infty} dx \exp\left(-\frac{A}{2}x^2\right)} = \sum_{k=0}^{\infty} (-1)^k z_k \lambda^k \tag{2.17}$$

with

$$z_k = \frac{1}{(4!)^k k!} \frac{\int_{-\infty}^{+\infty} dx \exp\left(-A\frac{x^2}{2}\right) x^{4k}}{\int_{-\infty}^{+\infty} dx \exp\left(-A\frac{x^2}{2}\right)} = \frac{2^{2k} A^{-2k} \Gamma\left(2k+\frac{1}{2}\right)}{(4!)^k k! \Gamma\left(\frac{1}{2}\right)}. \tag{2.18}$$

In particular,

$$z_1 = \frac{1}{8}A^{-2},$$

$$z_2 = \frac{35}{6}\left(\frac{A^{-2}}{8}\right)^2, \tag{2.19}$$

in agreement with the previous calculation.

Now consider the contribution of a graph composed of several disconnected pieces. Among them, some may be topologically isomorphic. Let G be the union of ν_1 copies of G_1, ν_2 copies of G_2, and so on, where $G_1, G_2, \ldots$ are connected. According to the above rules the contribution of G, apart from S, will factorize in ν_1 factors corresponding to G_1, ν_2 factors corresponding to G_2, and so on. The symmetry factor 2^σ will be equal to $(2^{\sigma_1})^{\nu_1}(2^{\sigma_2})^{\nu_2}\cdots$. Similarly, $S_l = (S_{l_1})^{\nu_1}(S_{l_2})^{\nu_2}\cdots$. In addition to the factors $(S_{v_i})^{\nu_i}$ the vertex symmetry factor will include extra factors $\nu_1!\nu_2!\ldots$. Thus $S_v = \nu_1!\nu_2!\ldots(S_{v_1})^{\nu_1}(S_{v_2})^{\nu_2}\ldots$.

This means that we can write

$$Z(\lambda)/Z(0) = \exp -E(\lambda) \tag{2.20}$$

with

$$E(\lambda) = \lambda E_1 - \lambda^2 E_2 + \ldots + (-1)^{k+1}\lambda^k E_k + \ldots, \tag{2.21}$$

where the E_k's are given by the same rules as above using only connected graphs. For instance, in the case $p = 1$, $g = 1$,

$$E(\lambda) = \frac{\lambda A^{-2}}{8} - \lambda^2 A^{-4}\left[\frac{1}{2\,4!} + \frac{1}{2^4}\right] + \ldots$$

$$= \frac{\lambda A^{-2}}{8} - \left[\frac{\lambda A^{-2}}{8}\right]^2 \frac{16}{3} + \ldots, \tag{2.22}$$

which can be checked directly using (2.16).

In addition to the quantities $Z(\lambda)/Z(0)$ or $E(\lambda)$, we can compute, using similar rules, averages of the form

$$G^{(2S)}_{\mu_1,\ldots,\mu_{2S}} = Z^{-1}(0)\int d^px x_{\mu_1}\ldots x_{\mu_{2S}} \times \exp\left(-\frac{1}{2}\sum_{\mu,\nu} x_\mu A_{\mu\nu} x_\nu - \frac{\lambda}{4!}\sum_{\mu_1,\ldots,\mu_4} g_{\mu_1\ldots\mu_4} x_{\mu_1\ldots\mu_4}\right), \tag{2.23}$$

which in a physical context are called Green functions. They are related to graphs having $2S$ external lines for which the previous rules are easily extended. Finally, we note that the graphical analysis generalizes to the case of an integration measure of the form $d^px \exp(-\frac{1}{2}xAx - V(x))$, where $V(x)$ is any polynomial in the components of x: each homogeneous monomial of degree r will introduce vertices with r incident lines.

3. The Topological Expansion

It is possible to rearrange the terms in the perturbative expansion according to the topology of the corresponding graphs. This will be demonstrated in the following example. Let us consider the N^2-dimensional real vector space of hermitean $N \times N$ matrices denoted generically M. This real vector space carries a representation of the group of unitary $N \times N$ matrices U (the adjoint representation) according to

$$M \xrightarrow{U} {}^UM = UMU^{-1}. \tag{3.1}$$

Clearly, we can restrict ourselves to the special unitary group, i.e., of unitary matrices with unit determinant, since in the adjoint representation (3.1) any U can be multiplied by a phase without modifying its action on hermitean matrices. The Lebesgue measure

$$dM = \prod_{i=1}^{N} dM_{ii} \prod_{1\le i<j\le N} d(\mathrm{Re}\, M_{ij})d(\mathrm{Im}\, M_{ij}) \tag{3.2}$$

is invariant under the unitary transformations (3.1). We shall now introduce and compute the integral

$$Z_N(g) = \int dM \exp\left\{-\frac{1}{2}\mathrm{tr}\, M^2 - \frac{g}{N}\mathrm{tr}\, M^4\right\} \tag{3.3}$$

using the methods of the previous section. This integral is of the type of (2.7) except for minor changes in the notation. Clearly the integrand is also invariant under the unitary transformations (3.1).

We have to specify the graphical rules pertaining to the integral (3.3) taking into account obvious modifications required in this case. In particular, the choice of a factor g/N in front of the quartic term will have important consequences.

Let us look first at the quadratic form $\frac{1}{2}\mathrm{tr}\, M^2$. We rewrite it as

$$\tfrac{1}{2}\mathrm{tr}\, M^2 = \tfrac{1}{2}\left\{ \sum_i M_{ii}^2 + 2\sum_{i<j} (\mathrm{Re}\, M_{ij})^2 + (\mathrm{Im}\, M_{ij})^2 \right\}. \tag{3.4}$$

It will appear to be more convenient to use the quantities M_{ij} and M_{ij}^* rather than $\mathrm{Re}\, M_{ij}$ and $\mathrm{Im}\, M_{ij}$. The restriction $i<j$ can also be lifted if we recall that $M_{ij} = M_{ji}^*$. Our elementary variables (field variables in physical language) will then be M_{ij}, i and j unrestricted. For the application of Wick's lemma, we need the elementary integral

$$\langle M_{ij} M_{kl}^* \rangle = \frac{\int dM \exp\left(-\frac{1}{2}\mathrm{tr}\, M^2\right) M_{ij} M_{kl}^*}{\int dM \exp\left(-\frac{1}{2}\mathrm{tr}\, M^2\right)} = \delta_{ik}\delta_{jl}. \tag{3.5}$$

It is of course possible, by linear combinations, to check that this is equivalent to the set of elementary contractions between the quantities M_{ii}, $\mathrm{Re}\, M_{ij}$, $\mathrm{Im}\, M_{ij}$, $i<j$, dictated by (3.4).

We represent this contraction by a double line connecting two pairs of indices. Under the action of the adjoint group, a hermitean $N \times N$ matrix has the same transformation law as the tensor product of an N vector z times its adjoint z^*, i.e., $z \otimes z^*$. By this we mean that we can distinguish the two indices of the matrix according to the transformation law under the unitary matrix U. We shall therefore put opposite arrows on the two lines of the contraction

$$\langle M_{ij} M_{kl}^* \rangle = \delta_{ik}\delta_{jl}, \tag{3.6}$$

which will be represented as indicated in Fig. 4.

The same rule extends to vertices which are drawn as a crossing of four double lines, each pair carrying two arrows, with the corresponding indices identified according to the scheme implied by the trace operation, as shown in Fig. 5:

$$\mathrm{tr}\, M^4 = \sum_{i,j,k,l=1}^{N} M_{ij} M_{jk} M_{kl} M_{li}. \tag{3.7}$$

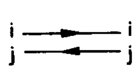


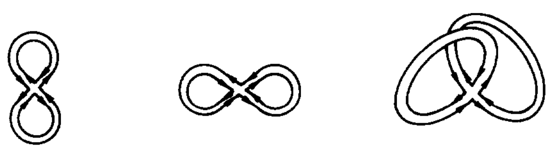
FIG. 4. The elementary contraction for hermitean matrices.

FIG. 5. The quartic vertex representing tr M^4.

Using Wick's lemma we now look at the term $\langle \text{tr } M^4 \rangle$. We have

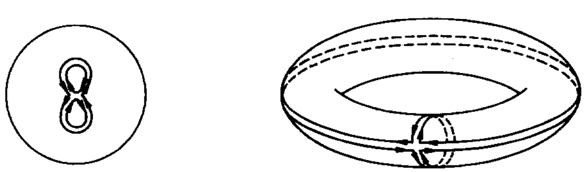
$$\begin{aligned}
\frac{g}{N}\langle \text{tr } M^4 \rangle &= \frac{g}{N} \sum_{i,j,k,l=1}^{N} \{\langle M_{ij} M_{kj}^* \rangle \langle M_{kl} M_{il}^* \rangle + \langle M_{ij} M_{lk}^* \rangle \langle M_{jk} M_{il}^* \rangle \\
&\quad + \langle M_{ij} M_{il}^* \rangle \langle M_{jk} M_{lk}^* \rangle\} \\
&= \frac{g}{N} \sum_{i,j,k,l=1}^{N} \{\delta_{ik}\delta_{jj}\delta_{ki}\delta_{ll} + \delta_{il}\delta_{jk}\delta_{ji}\delta_{kl} + \delta_{ii}\delta_{jl}\delta_{jl}\delta_{kk}\} \\
&= \frac{g}{N}\{N^3 + N + N^3\} = g\{2N^2 + 1\}. \qquad (3.8)
\end{aligned}$$

Identifying the independent indices according to the above rules leads to the graphs depicted in Fig. 6. Each closed loop corresponds to a dummy index running from 1 to N, and therefore, to a factor N. The isomorphic graphs, Figs. 6a and 6b, have three independent index loops, and therefore, a factor N^3, which upon multiplication by g/N yields a term gN^2. The last graph has only one index loop and contributes a term g. It may be noted that reversing the sign of all arrows is not considered as generating a new graph since the initial choice was arbitrary.



(a) (b) (c)

FIG. 6. The three contributions to $\langle \text{tr } M^4 \rangle$ obtained using Wick's lemma.



(a) (b)

FIG. 7. The two types of graphs of order 1. Type a is planar, type b is nonplanar. The corresponding Euler characteristics are, respectively, 0 and 1.

The distinction between the behavior for large N according to the topological structure of the graph is already apparent in this simple example. Indeed Figs. 6a and 6b can be drawn on a plane, or a sphere, while Fig. 6c can only be drawn on a torus (i.e., a surface with at least one handle; see Fig. 7) if we insist that lines only meet at vertices of the graph. Our aim is now to generalize this correspondence to an arbitrary order (number of vertices) according to the Euler characteristic of the graphs. As before, we write

$$\frac{Z_N(g)}{Z_N(0)} = \sum_{k=0}^{\infty} (-g)^k z_{N,k} \tag{3.9}$$

$$E_N(g) = \sum_{k=1}^{\infty} (-1)^{k+1} g^k E_{N,k} = -\ln Z_N(g)/Z_N(0) \tag{3.10}$$

with $z_{N,k}$ given by all graphs of order k, while in $E_{N,k}$ we keep only the connected ones. We have

$$z_{N,k} = \frac{1}{N^k k!}\langle (\mathrm{tr}\, M^4)^k \rangle, \tag{3.11}$$

which we expand using Wick's lemma. Similarly,

$$E_{N,k} = \frac{1}{N^k k!}\langle (\mathrm{tr}\, M^4)^k \rangle_c, \tag{3.12}$$

the lower index c meaning that we keep only the connected graphs.

We now state the rules for the computation of $E_{N,k}$. First, we draw all connected graphs with k vertices. Each vertex has a representation analogous to the one given in Fig. 5, except that we omit the labels, but keep the arrows. Vertices are connected by double lines respecting the flows dictated by the arrows. Each double line connects a double entry at one vertex with another double entry at another vertex, or possibly the same vertex. The same argument as given in Section 2 shows that the original factor $1/k!$ included in (3.12) is to be replaced for unlabeled graphs by $1/S_v$, where S_v is the order of the symmetry group of vertices.

Each factor carries, of course, a factor N^{-1}. Note that we did not divide g by 4! as was the case in Section 2, so we have to watch carefully the role of lines when applying Wick's lemma. Indeed, we cannot perform an arbitrary permutation of lines since at each vertex the connection of indices prescribed by the expression tr M^4 and illustrated in Fig. 5 forbids us to do so.

A given topological configuration will be obtained a certain number of times called $\rho(G)$. Furthermore, each loop carrying a dummy index will contribute a factor N upon summation. Call $L(G)$ the number of these loops of indices. A given connected graph of order k will therefore yield a term

$$\rho \frac{N^{L-k}}{S_v} \tag{3.13}$$

and $E_{N,k}$ will be the sum over all possible connected graphs of order k. In Fig. 8, we give the resulting analysis up to order 2. For each graph we also show a schematic form, omitting the structure of double lines (skeleton graph).

Because of topological obstructions certain graphs when drawn on a plane have additional and unwanted self-crossings. Imagine instead that we draw them on an orientable compact surface, topologically equivalent to a sphere with H handles; H is also called the genus of the surface. Let us choose for each graph the smallest possible H in such a way that no such self-crossing of lines occurs. On this surface the graph will be represented by a polygon with k vertices, P sides (when we identify both sides of a double line), and L faces bounded by the closed loops along which run the dummy indices. Indeed it is this abstract complex which is homeomorphic to the surface of genus H on which the graph is drawn.

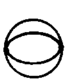
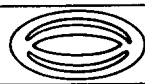
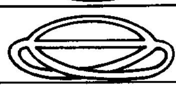
order k	skeleton	graphs	S_v	$\rho(G)$	$L(G)$	Contribut-ion	$E_{N,k}$
1			1	2	3	$2N^2$	$2N^2+1$
			1	1	1	1	
2			2	4	4	$2N^2$	$18N^2+30$
			2	20	2	10	
			2	32	4	$16N^2$	
			2	8	2	4	
			2	32	2	16	

FIG. 8. Analysis of first- and second-order graphs in the topological expansion.

Since each propagator (i.e., double line) connects two double entries on vertices and since each vertex has four double entries, we have $2P = 4k$, i.e.,

$$P = 2k. \tag{3.14}$$

Furthermore, each oriented loop of internal index is, as mentioned above, the unique boundary of an oriented face. We can now use Euler's formula, which states

$$k - P + L = 2 - 2H, \tag{3.15}$$

which, on account of Eq. (3.14), yields

$$L - k = 2 - 2H. \tag{3.16}$$

This is precisely the power of N associated with the graph. This means that we can rearrange the perturbative series (3.10) for $E_N(g)$ according to the topology of the graphs in the form

$$E_N(g) = \sum_{k=1}^{\infty} (-1)^{k+1} g^k E_{N,k} = N^2 \sum_{H=0}^{\infty} e_H(g) N^{-2H}, \tag{3.17}$$

where $e_H(g)$ is given as a sum of contributions relative to graphs without self-intersection that can be drawn on a compact oriented surface with at least H handles. Call this set of graphs g_H,

$$e_H(g) = - \sum_{G \in g_H} (-g)^{k(G)} \frac{\rho(G)}{S_v(G)}. \tag{3.18}$$

Equation (3.17) is the topological expansion looked for. The leading term

$$e_0(g) = \lim_{N \to \infty} \frac{1}{N^2} E_N(g) \tag{3.19}$$

is called, by abuse of language, the planar approximation.

The preceding development, carried on the example of an integral with (g/N) tr M^4 in the exponential term of the measure, can be generalized to a finite sum of terms of the form $(1/N^{(p/2)-1})$ tr M^p. It is also possible to replace the set of hermitean matrices by real symmetric or complex matrices, with slight modifications.

In the remainder of this paper we present a technique to compute in closed form the quantities $e_H(g)$. We also mention various alternatives, generalizations, and connections with related problems.

4. The Moment Problem Formulation

We generalize the results obtained in the previous sections to include an arbitrary polynomial $V(M)$, which we assume even for simplicity,

$$\operatorname{tr} V(M) = \frac{1}{2} \operatorname{tr} M^2 + \sum_{p \geq 2} \frac{g_p}{N^{p-1}} \operatorname{tr} M^{2p}. \tag{4.1}$$

As before, M is an $N \times N$ hermitean matrix and $g_p \geq 0$. With dM the unitary invariant measure introduced in (3.2), we define

$$Z_N(g) = \int dM e^{-\operatorname{tr} V(M)}. \tag{4.2}$$

This integral defines an analytic function of the parameters g_p at least in the region $\operatorname{Re} g_p > 0$. In Eq. (4.1) the coefficients of higher-order terms are weighted with inverse powers of N in such a way that the perturbative expansion of

$$E_N(g) = -\frac{1}{N^2} \ln \frac{Z_N(g)}{Z_N(0)} \tag{4.3}$$

will have the asymptotic expansion

$$E_N(g) = \sum_{H \geq 0} \frac{1}{N^{2H}} e_H(g). \tag{4.4}$$

As above, $e_H(g)$ is the generating function for counting graphs drawn on a surface with H handles: e_0 is associated to the spherical (or planar) topology, e_1 to the torus, and so on. According to (4.1) vertices are of order four, six, . . . , depending on the nonvanishing coefficients g_p.

Since $\operatorname{tr} V(M)$ as well as the measure dM are invariant under the N-dimensional unitary group, we can first perform an integral over the adjoint action of the group. This leaves us with an integral over the N eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$ of the matrix M (see Appendix 2). We can then rewrite the integral (4.2) as

$$Z_N(g) = \Omega_N (2\pi)^{-N} \int_{-\infty}^{+\infty} \prod_{1 \leq i \leq N} d\lambda_i \, \Delta^2(\lambda) \exp\left(-\sum_{i=1}^{N} V(\lambda_i) \right), \tag{4.5}$$

where

$$V(\lambda) = \frac{\lambda^2}{2} + \sum_{p \geq 2} \bar{g}_p \lambda^{2p}, \tag{4.6}$$

$$\bar{g}_p = \frac{g_p}{N^{p-1}}, \tag{4.7}$$

and

$$\Delta(\lambda) = \prod_{1 \le i < j \le N} (\lambda_i - \lambda_j) = \det \|\lambda_i^{j-1}\|. \tag{4.8}$$

The quantity Ω_N is related to the volume of the unitary group

$$\Omega_N = \frac{2^N \pi^{N(N+1)/2}}{\prod_{p=1}^{N} p!}. \tag{4.9}$$

Only the ratio $Z_N(g)/Z_N(0)$ enters in the definition (4.3). We can therefore use, instead of $Z_N(g)$, the following expression

$$\bar{Z}_N(g) = \int_{-\infty}^{+\infty} \prod_{i=1}^{N} d\mu(\lambda_i) \Delta(\lambda)^2, \tag{4.10}$$

where

$$d\mu(\lambda) = e^{-V(\lambda)}\, d\lambda. \tag{4.11}$$

To compute the integral (4.10) we introduce the set of orthogonal polynomials $P_n(\lambda)$ with respect to the measure $d\mu(\lambda)$

$$h_n \delta_{nm} = \int_{-\infty}^{+\infty} d\mu(\lambda) P_n(\lambda) P_m(\lambda), \tag{4.12}$$

where $P_n(\lambda)$ is normalized by the condition that the coefficient of its term of highest degree is equal to unity

$$P_n(\lambda) = \lambda^n + \ldots \tag{4.13}$$

and h_n is the norm squared of $P_n(\lambda)$. We remark that

$$\Delta(\lambda) = \det \|\lambda_i^{j-1}\| = \det \|P_{j-1}(\lambda_i)\|, \qquad 1 \le i,j \le N. \tag{4.14}$$

In (4.10) we then expand the determinant squared built up with the $P_k(\lambda_j)$. If $(-1)^{P(i_1, \ldots, i_N)}$ stands for the parity of the permutation $\{1, \ldots, N\} \to \{i_1, \ldots, i_N\}$ we obtain

$$\begin{aligned} \bar{Z}_N = & \sum_{\substack{i_1, \ldots, i_N \\ j_1, \ldots, j_N}} (-1)^{P(i_1, \ldots, i_N)} (-1)^{P(j_1, \ldots, j_N)} \\ & \times \prod_{k=1}^{N} \int_{-\infty}^{+\infty} d\mu(\lambda_k) P_{i_k - 1}(\lambda_k) P_{j_k - 1}(\lambda_k), \\ = & \sum_{\substack{i_1, \ldots, i_N \\ j_1, \ldots, j_N}} \delta_{i_1 j_1} \ldots \delta_{i_N j_N} h_{i_1 - 1} h_{i_2 - 1} \ldots h_{i_N - 1} = N! h_0 h_1 \ldots h_{N-1}. \end{aligned} \tag{4.15}$$

Let us now obtain an alternative form of this result. The polynomial $\lambda P_n(\lambda)$ of degree $n+1$ admits an expansion of the form

$$\lambda P_n(\lambda) = P_{n+1}(\lambda) - A_n P_n(\lambda) + R_n P_{n-1}(\lambda) + Q_{n-2}(\lambda), \tag{4.16}$$

where Q_{n-2} is a polynomial of degree at most equal to $n-2$. From the orthogonality properties (4.12) it follows that Q_{n-2} vanishes. Therefore, we get the well-known three-term recursion relation,

$$(\lambda + A_n)P_n(\lambda) = P_{n+1}(\lambda) + R_n P_{n-1}(\lambda). \tag{4.17}$$

In the case of the measure (4.11) which is even in λ, the polynomials P_n have parity $(-1)^n$ and A_n vanishes. Since, however, the present discussion could be extended to a situation where $V(\lambda)$ has no definite parity, and hence, where A_n is not vanishing, we keep it for the time being.

Since

$$\begin{aligned} h_{n+1} &= \int_{-\infty}^{+\infty} d\mu(\lambda) P_{n+1}(\lambda)\lambda P_n(\lambda) \\ &= \int_{-\infty}^{+\infty} d\mu(\lambda)[P_{n+2}(\lambda) - A_{n+1}P_{n+1}(\lambda) + R_{n+1}P_n(\lambda)]P_n(\lambda) \\ &= R_{n+1}h_n, \end{aligned} \tag{4.18}$$

we can rewrite (4.15) as

$$\bar{Z}_N = N!\, h_0^N R_1^{N-1} \ldots R_{N-2}^2 R_{N-1}, \tag{4.19}$$

where

$$h_0 = \int_{-\infty}^{+\infty} d\mu(\lambda). \tag{4.20}$$

Incidentally formula (4.19) provides an easy way to compute the factor Ω_N (see Appendix 2).

Finally, the function of interest $E_N(g)$ is given by

$$E_N(g) = -\frac{1}{N^2}\ln\frac{\bar{Z}_N(g)}{\bar{Z}_N(0)} = -\frac{1}{N}\sum_{k=1}^{N}\left(1-\frac{k}{N}\right)\ln\frac{R_k(g)}{R_k(0)} - \frac{1}{N}\ln\frac{h_0(g)}{h_0(0)}, \tag{4.21}$$

where $R_k(g)$ is the coefficient entering the three-term recursion relation for the set of orthogonal polynomials relative to the measure $d\lambda \exp -V(\lambda, g)$. Therefore the problem has been reduced to one of computing these coefficients R_k.

We could proceed by introducing the moments μ_k of the measure

$$\mu_k = \int_{-\infty}^{+\infty} d\mu(\lambda)\lambda^k \tag{4.22}$$

and their generating function

$$G(z) = \int_{-\infty}^{+\infty} \frac{d\mu(\lambda)}{z-\lambda} = \frac{\mu_0}{z} + \frac{\mu_1}{z^2} + \dots + \frac{\mu_k}{z^{k+1}} + \dots . \tag{4.23}$$

An expansion of $G(z)$ as a continued fraction

$$G(z) = \cfrac{h_0}{z + A_0 - \cfrac{R_1}{z + A_1 - \cfrac{R_2}{z + A_2 - \cfrac{R_3}{z + A_3 - \cdots}}}} \tag{4.24}$$

would answer our problem. We will not develop this subject here [4].

In the case of the measure (4.11) there is a simple method to obtain a nonlinear recursive relation among the R_n's. This will now be expounded. We start from the relations

$$\begin{aligned} nh_n &= \int_{-\infty}^{+\infty} d\lambda e^{-V(\lambda)} \lambda P_n'(\lambda) P_n(\lambda) \\ &= \int_{-\infty}^{+\infty} d\lambda e^{-V(\lambda)} P_n'(\lambda) [P_{n+1}(\lambda) + R_n P_{n-1}(\lambda) - A_n P_n(\lambda)] \\ &= R_n \int_{-\infty}^{+\infty} d\lambda e^{-V(\lambda)} P_n'(\lambda) P_{n-1}(\lambda) \\ &= R_n \int_{-\infty}^{+\infty} d\lambda e^{-V(\lambda)} V'(\lambda) P_n(\lambda) P_{n-1}(\lambda), \end{aligned} \tag{4.25}$$

where an integration by parts has been used to obtain the last equality. With $V(\lambda)$ even we know that $P_k(\lambda)$ has parity $(-1)^k$ and the A_k vanish. Use of the relation (4.17) to compute the last expression in (4.25) will therefore only involve the coefficients R_k. From (4.6),

$$V'(\lambda) = \lambda + \sum_{p \geq 2} \bar{g}_p 2p \lambda^{2p-1} \tag{4.26}$$

and

$$\int_{-\infty}^{+\infty} d\lambda e^{-V(\lambda)} P_n(\lambda) V'(\lambda) P_{n-1}(\lambda) = h_n \left\{ 1 + \sum_{p \geq 1} 2(p+1) \bar{g}_{p+1} \alpha_n^{[2p+1]} \right\}, \tag{4.27}$$

where we have set

$$h_n \alpha_n^{[2p+1]} = \int_{-\infty}^{+\infty} d\lambda e^{-V(\lambda)} P_n(\lambda) \lambda^{2p+1} P_{n-1}(\lambda). \tag{4.28}$$

Inserting these results back in (4.25) we find

$$n = R_n \left\{ 1 + \sum_{p \geq 1} 2(p+1) \bar{g}_{p+1} \alpha_n^{[2p+1]} \right\}. \tag{4.29}$$

In Appendix 3 we develop the simple arguments that enable one to express $\alpha_n^{[2p+1]}$ in terms of the R_k's using the recursion formula (4.17) once again. The result is the following:

$\alpha_n^{[2p+1]}$ is a sum over the $\binom{2p+1}{p}$ paths along a staircase, from the stair at height $n-1$ to the one at height n, in $2p+1$ steps, $p+1$ up, p down. A factor R_s occurs when descending from the stair at height s down to the one at height $s-1$. That is,

$$\alpha_n^{[2p+1]} = \sum_{\text{paths}} R_{s_1} R_{s_2} \ldots R_{s_p}. \tag{4.30}$$

To conclude this section we summarize what remains to be done. We have to expand for large N the expression (4.21) for $E_N(g)$ knowing that the R_k's satisfy the recursion relation

$$k = R_k \left\{ 1 + \sum_{p \geq 1} 2(p+1) \bar{g}_{p+1} \sum_{\text{paths}} R_{s_1} R_{s_2} \ldots R_{s_p} \right\}. \tag{4.31}$$

In the simplest case studied in Section 3, which we call the quartic case,

$$V(\lambda) = \frac{\lambda^2}{2} + \bar{g}\lambda^4 = \frac{\lambda^2}{2} + \frac{g}{N}\lambda^4, \tag{4.32}$$

the relation (4.31) reduces to

$$k = R_k(g) \left\{ 1 + \frac{4g}{N} \left[R_{k-1}(g) + R_k(g) + R_{k+1}(g) \right] \right\}. \tag{4.33}$$

5. Computation of $e_0(g)$ the Generating Function for Planar Graphs

By planar graphs we mean also, according to our terminology, those which are faithfully drawn on a sphere. The above material enables us to compute their generating function $e_0(g)$ in the case of an arbitrary even polynomial of the form (4.6).

From (4.31) we see that

$$R_k(g)|_{g=0} = k; \tag{5.1}$$

hence,

$$\frac{R_k(g)}{R_k(0)} = \frac{R_k(g)}{N}\left(\frac{k}{N}\right)^{-1}. \tag{5.2}$$

Taking into account (4.7), $R_k(g)/N$ fulfills

$$\frac{k}{N} = \frac{R_k(g)}{N}\left\{1 + \sum_{p\geq 1} 2(p+1)g_{p+1} \sum_{\text{paths}} \frac{R_{s_1}}{N}\frac{R_{s_2}}{N}\cdots\frac{R_{s_p}}{N}\right\}. \tag{5.3}$$

Clearly,

$$\frac{R_k(g)}{N} = r_0\left(\frac{k}{N}, g\right) + O\left(\frac{1}{N^2}\right), \tag{5.4}$$

where the function $r_0(x, g)$ is the unique positive solution, in the range $0 \leq x \leq 1$, of

$$x = r_0\left\{1 + \sum_{p\geq 2} g_p \frac{(2p)!}{p!\,(p-1)!} r_0^{p-1}\right\}. \tag{5.5}$$

We recall here that all g_p are considered positive or zero. The inverse function of $x(r_0)$, namely, $r_0(x)$, is holomorphic in a neighborhood of $[0, \infty)$, in particular on the closed interval $[0, 1]$.

According to Eqs. (4.4) and (4.21), $e_0(g)$ is given by

$$\begin{aligned} e_0(g) &= -\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \left(1 - \frac{k}{N}\right) \ln\left\{\frac{r_0(k/N) + O(1/N^2)}{(k/N)}\right\}, \\ &= -\int_0^1 dx(1-x) \ln\left\{\frac{r_0(x)}{x}\right\}. \end{aligned} \tag{5.6}$$

If we define

$$w(r) = r\left\{1 + \sum_{p\geq 2} g_p \frac{(2p)!}{p!\,(p-1)!} r^{p-1}\right\} \tag{5.7}$$

and a^2 as the unique solution of

$$1 = w(a^2), \qquad a^2 > 0, \tag{5.8}$$

we can rewrite (5.6) as

$$e_0(g) = \int_0^{a^2} dr\, w'(r)[1 - w(r)] \ln\left\{\frac{w(r)}{r}\right\}. \tag{5.9}$$

Instead of $w(r)$ we can use the function

$$\bar{w}(r) = \frac{w(r)}{r} = 1 + \sum_{p \geq 2} g_p \frac{(2p)!}{p!\,(p-1)!} r^{p-1}, \tag{5.10}$$

leading to the expression

$$e_0(g) = -\tfrac{1}{2}\ln a^2 - \tfrac{1}{2}\int_0^{a^2} dr\, r[\bar{w}(r)]'(2 - r\bar{w}(r)). \tag{5.11}$$

As the simplest example, in the quartic case (4.22), dropping the index 2 on the parameter g, we find

$$\bar{w}(r) = 1 + 12gr \tag{5.12}$$

and a^2 is the positive root of

$$12ga^4 + a^2 - 1 = 0, \tag{5.13}$$

i.e., the one regular at $g = 0$. This gives

$$e_0(g) = -\tfrac{1}{2}\ln a^2 + \tfrac{1}{24}(a^2 - 1)(9 - a^2), \tag{5.14}$$

which, upon expansion in powers of g, yields

$$e_0(g) = -\sum_{p=1}^{\infty} (-12g)^p \frac{(2p-1)!}{p!\,(p+2)!}. \tag{5.15}$$

This expression solves the counting problem for the planar graphs with no external lines and quartic vertices, while (5.9) or (5.11) gives the solution in the general case.

We refer the reader to Appendix 4 for an interpretation of these results in terms of the asymptotic distribution of eigenvalues of random matrices with nongaussian weight.

6. Analysis of the Recursion Formula

In Section 4 we have obtained a recursive relation connecting $(v - 1)$ R_k's, if v is the degree of the even polynomial $V(\lambda)$. To analyze the content of this relation somewhat further, we stick from now on to the simplest nontrivial case, i.e., the quartic case with $v = 4$. We shall also drop the index 2 on g_2 since no confusion is possible. We have

$$k = R_k\left\{1 + \frac{4g}{N}[R_{k-1} + R_k + R_{k+1}]\right\}. \tag{6.1}$$

In fact we are interested in the coefficient

$$\bar{R}_k\left(\frac{g}{N}\right) = \frac{R_k(g/N)}{R_k(0)} = \frac{R_k(g/N)}{k}, \tag{6.2}$$

which satisfies the relation

$$1 = \bar{R}_k + \frac{4g}{N}\bar{R}_k\left[(k-1)\bar{R}_{k-1} + k\bar{R}_k + (k+1)\bar{R}_{k+1}\right]. \tag{6.3}$$

Making the change of variable

$$\epsilon = 1/N \qquad x = k/N \tag{6.4}$$

and of function

$$r_\epsilon(x, g) = \frac{k}{N}\bar{R}_k\left(\frac{g}{N}\right), \tag{6.5}$$

we get

$$x = r_\epsilon(x, g) + 4g r_\epsilon(x, g)\{r_\epsilon(x-\epsilon, g) + r_\epsilon(x, g) + r_\epsilon(x+\epsilon, g)\} \tag{6.6}$$

with the condition

$$r_\epsilon(0, g) = 0. \tag{6.7}$$

In this relation it is seen that $r_\epsilon(x, g)$ is symmetric in ϵ. Thus it admits an expansion in even powers of ϵ

$$r_\epsilon(x, g) = r_0(x, g) + \epsilon^2 r_2(x, g) + \ldots + \epsilon^{2k} r_{2k}(x, g) + \ldots. \tag{6.8}$$

Consequently,

$$r_\epsilon(x-\epsilon) + r_\epsilon(x+\epsilon) = 2\sum_{n=0}^{\infty} \epsilon^{2n} \sum_{k+p=n} \frac{r_{2k}^{(2p)}(x)}{(2p)!}, \tag{6.9}$$

where $r_{2k}^{(2p)}(x)$ stands for $(d^{2p}/dx^{2p})r_{2k}(x)$.

We then identify the coefficient of ϵ^{2s} on both sides of (6.6), with the result

$$x\delta_{s,0} = r_{2s}(x) + 4g \sum_{m+n=s} r_{2m}(x)\left\{r_{2n}(x) + 2\sum_{k+p=n} \frac{r_{2k}^{(2p)}(x)}{(2p)!}\right\}. \tag{6.10}$$

For $s = 0$ we recover

$$x = r_0(x) + 12g r_0^2(x), \qquad r_0(x)|_{x=0} = 0 \tag{6.11}$$

i.e.,

$$r_0(x) = \frac{-1 + \sqrt{1 + 48gx}}{24g}. \tag{6.12}$$

For $s = 1$,

$$0 = r_2[1 + 24gr_0] + 4gr_0 r_0^{(2)}, \tag{6.13}$$

which gives

$$r_2(x) = \frac{96g^2 r_0(x)}{[1 + 24gr_0(x)]^4} = \frac{96g^2 r_0(x)}{[1 + 48gx]^2}. \tag{6.14}$$

Similarly, when $s = 2$,

$$0 = r_4[1 + 24gr_0] + 4g\left[3r_2^2 + r_2 r_0^{(2)} + r_0 r_2^{(2)} + \frac{r_0 r_0^{(4)}}{12}\right], \tag{6.15}$$

leading to

$$\begin{aligned} r_4(x) &= \tfrac{7}{72}(24g)^4 r_0(x)[5 - 48gr_0(x)][1 + 24gr_0(x)]^{-9}, \\ &= \tfrac{7}{72}(24g)^4 r_0(x)[5 - 48gr_0(x)][1 + 48gx]^{-9/2}. \end{aligned} \tag{6.16}$$

More generally, if we observe that $(d/dx)r_0(x) = [1 + 24gr_0(x)]^{-1}$, the set of equations (6.10) leads to a structure

$$r_{2s}(x) = \frac{r_0 p_{s-1}(gr_0)}{[1 + 24gr_0(x)]^{5s-1}}, \qquad s \geq 1, \tag{6.17}$$

where $p_{s-1}(gr_0)$ is a polynomial of degree $s - 1$ in $gr_0(x)$.

Remark. The relations (6.3) giving $\bar{R}_k$ are valid for $k \geq 1$, with the initial condition

$$\bar{R}_1 = R_1 = \frac{\mu_2(g/N)}{\mu_0(g/N)}. \tag{6.18}$$

Here $\mu_{2k}(g/N)$ stand for the moments

$$\begin{aligned} \mu_{2k}(g/N) &= \int_{-\infty}^{+\infty} \lambda^{2k} \exp\left(-\frac{\lambda^2}{2} - \frac{g}{N}\lambda^4\right) d\lambda \\ &= \sqrt{2\pi} \sum_{p=0}^{\infty} \frac{(2k + 4p - 1)!!}{p!}\left(-\frac{g}{N}\right)^p. \end{aligned} \tag{6.19}$$

In principle, the solution of (6.3) depends on the explicit knowledge of R_1. This fact seems in contradiction with formula (6.8) which states that

$$\frac{k}{N}\bar{R}_k\left(\frac{g}{N}\right) = \sum_{n=0}^{\infty} \frac{1}{N^{2n}} r_{2n}\left(\frac{k}{N}, g\right), \tag{6.20}$$

where the quantities $r_{2n}(x, g)$ are unambiguously obtained recursively from (6.10) as shown above and do not seem to depend on the initial condition giving R_1.

For instance, we can compare the values of $(k/N)\bar{R}_k(g/N)$ for $k = 1$ using (6.18) or (6.20). From the former

$$\frac{1}{N}\bar{R}_1\left(\frac{g}{N}\right) = \frac{1}{N}\frac{\mu_2\left(\frac{g}{N}\right)}{\mu_2(0)}\left[\frac{\mu_0\left(\frac{g}{N}\right)}{\mu_0(0)}\right]^{-1}$$

$$= \frac{1}{N}\frac{1 - \frac{15g}{N} + \frac{945}{2}\left(\frac{g}{N}\right)^2 + \dots + \frac{(4p+3)!!}{p!}\left(-\frac{g}{N}\right)^p + \dots}{1 - \frac{3g}{N} + \frac{105}{2}\left(\frac{g}{N}\right)^2 + \dots + \frac{(4p-1)!!}{p!}\left(-\frac{g}{N}\right)^p + \dots}. \tag{6.21}$$

On the other hand, from (6.20),

$$\frac{1}{N}\bar{R}_1\left(\frac{g}{N}\right) = r_0\left(\frac{1}{N}, g\right) + \frac{1}{N^2}r_2\left(\frac{1}{N}, g\right) + \frac{1}{N^4}r_4\left(\frac{1}{N}, g\right) + \dots. \tag{6.22}$$

It is not difficult to check using (6.12), (6.14), and (6.16) that the expansions of Eqs. (6.21) and (6.22) in inverse powers of N agree up to order N^{-4}. In fact (6.21) is an asymptotic series in g/N with zero radius of convergence. Therefore, our statement is slightly less stringent, in the sense that only the asymptotic expansion of $\bar{R}_k$ for N large does not depend on the initial condition. This is not surprising because we are selecting the neighborhood of the fixed point of a nonlinear mapping. This fixed point, as well as its neighborhood, is insensitive to the initial conditions. Of course, Eq. (6.3) admits two fixed points given by the two solutions of (6.11)

$$r_0^{\pm}(x) = \frac{-1 \pm \sqrt{1 + 48gx}}{24g}, \qquad x = \frac{k}{N}, \tag{6.23}$$

but the condition $r_0(x)|_{x=0} = 0$ selects out the solution $r_0^{+}(x)$.

Otherwise stated, the requirement $\bar{R}_k(g/N) \to 1$ for $N \to \infty$ selects out a unique initial condition, in the sense of asymptotic series, which is precisely (6.18). We have checked this statement to order N^{-4}. It would be interesting to have a proof to all orders.

7. Asymptotic Formula and Generating Functions for Higher Genus

We are now in a position to obtain explicit expressions for the generating functions $e_1(g)$ and $e_2(g)$ in the quartic case. Our starting point is formula (4.21), which reads

$$E_N(g) = -\frac{1}{N}\sum_{k=1}^{N}\left(1-\frac{k}{N}\right)\ln \bar{R}_k\left(\frac{g}{N}\right) - \frac{1}{N}\ln\frac{h_0(g)}{h_0(0)}. \tag{7.1}$$

To expand both sides of this relation in inverse powers of N not only do we need the expansion of $\bar{R}_k$, but also the Euler–Maclaurin formula. The latter states that if $f(x)$ admits continuous derivatives up to, and including, order $2p$ on the closed interval [0, 1], then

$$\begin{aligned}\frac{1}{N}\sum_{k=1}^{N} f\left(\frac{k}{N}\right) &= \int_0^1 dx\, f(x) + \frac{1}{2N}[f(1)-f(0)] \\ &+ \frac{B_1}{2!}\frac{1}{N^2}[f^{(1)}(1)-f^{(1)}(0)] - \frac{B_2}{4!}\frac{1}{N^4}[f^{(3)}(1)-f^{(3)}(0)] \\ &+ \dots + (-1)^p\frac{B_{p-1}}{(2p-2)!}\frac{1}{N^{2p-2}}[f^{(2p-3)}(1)-f^{(2p-3)}(0)] \\ &+ (-1)^{p+1}\frac{B_p}{(2p)!}\frac{1}{N^{2p+1}}[f^{(2p)}(x_1)+f^{(2p)}(x_2)+\dots+f^{(2p)}(x_N)],\end{aligned} \tag{7.2}$$

where

$$\frac{s-1}{N} < x_s < \frac{s}{N} \tag{7.3}$$

and B_j are the Bernouilli numbers

$$B_1 = \tfrac{1}{6}, \qquad B_2 = \tfrac{1}{30}, \dots \tag{7.4}$$

If we set

$$\phi_N\left(\frac{k}{N}, g\right) = \bar{R}\left(\frac{g}{N}\right), \tag{7.5}$$

applying the Euler–Maclaurin formula to

$$f\left(\frac{k}{N}\right) = \left(1 - \frac{k}{N}\right) \ln \phi_N\left(\frac{k}{N}, g\right)$$

yields

$$\begin{aligned} E_N(g) = &-\int_0^1 dx(1-x)\ln\phi_N(x,g) - \frac{1}{2N}\left\{2\ln\frac{h_0\left(\frac{g}{N}\right)}{h_0(0)} - \ln\phi_N(0,g)\right\} \\ &+\Bigg[-\frac{1}{12N^2}\{(1-x)\ln\phi_N(x,g)\}^{(1)} \\ &+\frac{1}{6!N^4}\{(1-x)\ln\phi_N(x,g)\}^{(3)} \\ &+\ldots+(-1)^p\frac{B_p}{(2p)!}\frac{1}{N^{2p}} \\ &\times\{(1-x)\ln\phi_N(x,g)\}^{(2p-1)}+\ldots\Bigg]\Bigg|_0^1 . \end{aligned} \tag{7.6}$$

The last symbol $|_0^1$ means of course that we have to take the difference between the values of the function at $x = 1$ and $x = 0$.

According to Eq. (6.20),

$$\phi_N(x, g) = \sum_{s=0}^{\infty} \frac{1}{N^{2s}} \frac{r_{2s}(x, g)}{x} . \tag{7.7}$$

Consequently, only even powers of $1/N$ can occur in (7.6), as we expect from the graphical analysis of Section 3, with the possible exception of the terms arising from the even part in $1/N$ of the expansion of

$$\left\{2 \ln \frac{h_0\left(\frac{g}{N}\right)}{h_0(0)} - \ln \phi_N(0, g)\right\}.$$

However, this even part vanishes identically. Indeed, it is shown in Appendix 5 that

$$\phi_N(0, g) = \frac{h_0(g/N)}{h_0(0)} \cdot \frac{h_0(-g/N)}{h_0(0)} . \tag{7.8}$$

Of course, this has to be interpreted as a formal identity among asymptotic series. Therefore in Eq. (7.6) the coefficient of $1/2N$ can be rewritten

$$\frac{1}{2N}\left\{2\ln\frac{h_0(g/N)}{h_0(0)}-\ln\phi_N(0,g)\right\}=\frac{1}{2N}\left\{\ln\frac{h_0(g/N)}{h_0(0)}-\ln\frac{h_0(-g/N)}{h_0(0)}\right\}, \tag{7.9}$$

which clearly only gives even contributions.

Let us now collect in (7.6) successive coefficients of inverse powers of N up to N^{-4}. We have first

$$\begin{aligned}-\int_0^1 dx(1-x)\ln\phi_N(x,g) &= -\int_0^1 dx(1-x)\ln\frac{r_0(x,g)}{x}\\ &\quad -\frac{1}{N^2}\int_0^1 dx(1-x)\frac{r_2(x,g)}{r_0(x,g)}\\ &\quad -\frac{1}{N^4}\int_0^1 dx(1-x)\\ &\quad \times\left\{\frac{r_4(x,g)}{r_0(x,g)}-\frac{1}{2}\left[\frac{r_2(x,g)}{r_0(x,g)}\right]^2\right\}+\cdots.\end{aligned} \tag{7.10}$$

Equation (6.19) for $k=0$ yields $\mu_0(g/N)=h_0(g/N)$. Thus

$$\frac{h_0(g/N)}{h_0(0)}=1-3\frac{g}{N}+\frac{105}{2}\left(\frac{g}{N}\right)^2-\frac{3465}{2}\left(\frac{g}{N}\right)^3+\cdots, \tag{7.11}$$

giving

$$-\frac{1}{2N}\left\{\ln\frac{h_0(g/N)}{h_0(0)}-\ln\frac{h_0(-g/N)}{h_0(0)}\right\}=3\frac{g}{N^2}+1584\frac{g^3}{N^4}+\cdots. \tag{7.12}$$

The next term in (7.6) gives rise to two contributions

$$-\frac{1}{12N^2}\left\{(1-x)\ln\frac{r_0(x,g)}{x}\right\}^{(1)}\Bigg|_0^1-\frac{1}{12N^4}\left\{(1-x)\frac{r_2(x,g)}{r_0(x,g)}\right\}^{(1)}\Bigg|_0^1+\cdots \tag{7.13}$$

and the following one reads

$$\frac{1}{6!N^4}\left\{(1-x)\ln\frac{r_0(x,g)}{x}\right\}^{(3)}\Bigg|_0^1+\cdots. \tag{7.14}$$

By identification with Eq. (4.4) we get

$$e_0(g) = -\int_0^1 dx(1-x)\ln\frac{r_0(x,g)}{x}, \tag{7.15}$$

$$e_1(g) = -\int_0^1 dx(1-x)\frac{r_2(x,g)}{r_0(x,g)} + 3g$$
$$-\frac{1}{12}\left\{(1-x)\ln\frac{r_0(x,g)}{x}\right\}^{(1)}\Bigg|_0^1, \tag{7.16}$$

$$e_2(g) = -\int_0^1 dx(1-x)\left\{\frac{r_4(x,g)}{r_0(x,g)} - \frac{1}{2}\left[\frac{r_2(x,g)}{r_0(x,g)}\right]^2\right\} + 1584g^3$$
$$-\frac{1}{12}\left\{(1-x)\frac{r_2(x,g)}{r_0(x,g)}\right\}^{(1)}\Bigg|_0^1 + \frac{1}{6!}\left\{(1-x)\ln\frac{r_0(x,g)}{x}\right\}^{(3)}\Bigg|_0^1. \tag{7.17}$$

Therefore,

$$\begin{aligned} e_0(g) &= -\frac{1}{2}\ln a^2 + \int_0^1 dx\left(x-\frac{x^2}{2}\right)\left\{\ln\frac{r_0(x,g)}{x}\right\}^{(1)}, \\ &= -\frac{1}{2}\ln a^2 - \frac{3}{8} + \frac{1}{2}\int_0^1 dx\left[1-\frac{x}{2}\right](1+48gx)^{-1/2}, \\ &= -\frac{1}{2}\ln a^2 - \frac{3}{8} - \frac{1}{48g} - \frac{1}{3(48g)^2} + \frac{(1+120g)(1+48g)^{1/2}}{3(48g)^2}, \end{aligned} \tag{7.18}$$

which is nothing but the result obtained previously in (5.14) and (5.15), i.e.,

$$e_0(g) = -\frac{1}{2}\ln a^2 + \frac{1}{24}(a^2-1)(9-a^2) = -\sum_{p=1}^{\infty}(-12g)^p\frac{(2p-1)!}{p!\,(p+2)!}. \tag{7.19}$$

In the same way, we have

$$e_1(g) = 3g + \tfrac{1}{12}\left[\ln a^2 - 12g\right] - 96g^2\int_0^1 dx(1-x)(1+48gx)^{-2}, \tag{7.20}$$

which yields

$$e_1(g) = \frac{1}{12}\ln(2-a^2) = -\frac{1}{24}\sum_{p=1}^{\infty}\frac{(-12g)^p}{p}\left[4^p - \frac{(2p)!}{(p!)^2}\right]. \tag{7.21}$$

To compute $e_2(g)$ we need some intermediate steps. Define

$$\rho(x) = \frac{r_0(x)}{x} = \frac{-1 + \sqrt{1 + 48gx}}{24gx}$$
$$= 1 - 12g\frac{x}{1!} + 4(12g)^2\frac{x^2}{2!} - 30(12g)^3\frac{x^3}{3!} + \cdots. \quad (7.22)$$

Not to superimpose notations we shall denote derivatives in x of ρ with primes, double primes, ..., instead of $\rho^{(1)}, \rho^{(2)}, \ldots,$ as we did up to now. Also an index zero or one will mean the value at $x = 0$ or 1. With these notations the last term occurring in (7.17) takes the following form, if we take into account that $\rho(x)$ is analytic on the closed interval [0, 1],

$$\frac{1}{6!}\{(1 - x)\ln\rho(x)\}'''|_0^1$$
$$= \frac{1}{6!}\left\{-3\left[\frac{\rho''}{\rho} - \left(\frac{\rho'}{\rho}\right)^2\right]\Bigg|_0^1 - \left[\rho_0''' - 3\rho_0''\rho_0' + 2\rho_0'^3\right]\right\}$$
$$= \frac{1}{6!}\left\{-3\left[\frac{\rho''}{\rho} - \left(\frac{\rho'}{\rho}\right)^2\right]_1 + 3\left[\rho_0'' + \rho_0'\rho_0'' - \rho_0'^2\right] - \rho_0''' - 2\rho_0'^3\right\}. \quad (7.23)$$

Using (7.22) and the fact that

$$\left(\frac{\rho_1'}{\rho_1}\right)^2 - \frac{\rho_1''}{\rho_1} = -3\frac{(12g)^2\rho_1^2}{(1 + 24g\rho_1)^3}(1 + 16g\rho_1)$$
$$= -2g[1 + 48g]^{-3/2}[(1 + 48g)\rho_1 - 1], \quad (7.24)$$

where the second equality uses the property that $(1 + 24g\rho_1)^2 = 1 + 48g$, we obtain finally

$$\frac{1}{6!}\{(1 - x)\ln\rho(x)\}'''|_0^1 = \frac{1}{6!}\{9(12g)^2 + 20(12g)^3$$
$$- 3(12g)[1 + 48g]^{-3/2}[(1 + 48g)\rho_1 - 1]\} \quad (7.25)$$

We now have to compute

$$\frac{-1}{12}\left\{(1 - x)\frac{r_2(x, g)}{r_0(x, g)}\right\}'\Bigg|_0^1 = -8g^2\{(1 - x)[1 + 48gx]^{-2}\}'|_0^1$$
$$= -8g^2\{1 + 96g - [1 + 48g]^{-2}\}. \quad (7.26)$$

The sum of the last two integrals

$$\frac{1}{2}\int_0^1 dx(1-x)\left[\frac{r_2(x,g)}{r_0(x,g)}\right]^2 = \frac{(96g^2)^2}{2}\int_0^1 dx(1-x)[1+48gx]^{-4} \tag{7.27}$$

$$-\int_0^1 dx(1-x)\frac{r_4(x,g)}{r_0(x,g)} = \tfrac{7}{72}(24g)^4\int_0^1 dx(1-x)[1+48gx]^{-9/2} \times\left[2\sqrt{1+48gx}-7\right] \tag{7.28}$$

is easily found equal to

$$\frac{(48g)^2}{1152}\left\{-9(48g)-\frac{28}{5}(1+48g)^{-5/2}+\frac{15}{6}(1+48g)^{-2}+\frac{31}{10}\right\}. \tag{7.29}$$

Adding all terms and setting

$$\xi = 48g, \tag{7.30}$$

we find

$$e_2(g) = \frac{1}{6!}\left\{-\frac{7}{2}\xi^2(1+\xi)^{-5/2}-\frac{3}{4}(1+\xi)^{-3/2}+\frac{9}{4}(1+\xi)^{-1/2}-\frac{3}{2} + \frac{65}{16}\xi^2(1+\xi)^{-2}\right\}. \tag{7.31}$$

Using the relation

$$\sqrt{1+\xi} = (2-a^2)/a^2, \tag{7.32}$$

we obtain

$$e_2(g) = \frac{1}{6!}\frac{(1-a^2)^3}{(2-a^2)^5}(82+21a^2-3a^4)$$
$$= \frac{1}{5.3^3.2^5}\sum_{p=3}^{\infty}(-12g)^p(p-1)\left[195\times 2^{2p-3}-(28p+9)\frac{(2p)!}{(p!)^2}\right]. \tag{7.33}$$

As expected, $e_2(g)$ starts at order g^3. Explicitly,

$$e_2(g) = 240g^3 - 32\,112g^4 + \dots\,. \tag{7.34}$$

Cross-checks

We can check our calculations in various ways:

(i) In general, the function $e_H(g)$, for $H \geq 1$, has an expansion in g, with a first term equal to

$$e_H(g) = \frac{2^{2H-2}(4H-3)!}{H!\,(H-1)!} g^{2H-1} + \ldots . \tag{7.35}$$

The leading power in g, namely $2H - 1$, is given by Euler's relation. Using the notation of Section 3, we have

$$\begin{aligned} k - P + L &= 2 - 2H, \\ P &= 2k, \end{aligned} \tag{7.36}$$

giving

$$k = (L-1) + (2H-1). \tag{7.37}$$

Since L is larger or equal to one, we conclude that the lowest possible value of k, i.e., the lowest power of g in $e_H(g)$ is indeed $2H - 1$. In (7.35) we have furthermore included the coefficient of this lowest power in g, derived in Appendix 6. Note that this coefficient counts the number of graphs drawn on a torus with at least H handles, having one loop only, i.e., only one face.

We readily see that (7.35) is in agreement with the expressions for e_1 and e_2.

(ii) The generating function for the sum of all connected graphs, pertaining to the case $N = 1$, is equal to

$$-\ln \frac{h_0(g)}{h_0(0)} = 3g - 48g^2 + 1584g^3 - 78\,336g^4 + \ldots . \tag{7.38}$$

We must have therefore

$$-\ln \frac{h_0(g)}{h_0(g)} = \sum_{H=0}^{\infty} e_H(g). \tag{7.39}$$

It is easy to see that our functions fulfill this identity up to order g^4 included, since

$$\begin{aligned} e_0(g) &= 2g - 18g^2 + 288g^3 - 6048g^4 + \ldots, \\ e_1(g) &= g - 30g^2 + 1056g^3 - 40{,}176g^4 + \ldots, \\ e_2(g) &= \qquad\qquad\quad 240g^3 - 32{,}112g^4 + \ldots . \end{aligned} \tag{7.40}$$

TABLE I

g	1	10
E_3 exact	0.424945	0.886812
e_0	0.419709	0.880709
$e_0 + (e_1/9)$	0.424891	0.886715
$e_0 + (e_1/9) + (e_2/81)$	0.424930	0.886757

To appreciate the accuracy of the asymptotic series given by the topological expansion, let us observe the following variations.

As g varies from 0 to ∞, a^2 decreases from 1 to 0, and $e_0(g)$, $e_1(g)$, $e_2(g)$ increase from 0 to, respectively,

$$+\infty\left(\sim\frac{1}{4}\ln 12g - \frac{3}{8}\right), \qquad \frac{1}{12}\ln 2, \qquad \frac{41}{11520}.$$

Typically if we set $g = 1$, which means $a^2 = \frac{1}{4}$, and $N = 3$ we find

$$e_0 \sim 4.196\ 10^{-1},$$

$$\frac{1}{9}e_1 \sim 5.181\ 10^{-3},$$

$$\frac{1}{81}e_2 \sim 3.837\ 10^{-5}.$$

We see that the asymptotic series $\Sigma_{H=0}^{\infty} e_H(g)/N^{2H}$ already gives an excellent estimate to $-(1/N^2)\ln(Z_N(g)/Z_N(0))$: the first terms decrease very rapidly (see Table I).

8. Conclusion

It would of course be very interesting to obtain $e_H(g)$ in closed form for any value of H. The method of this paper enabled us to do so up to $H = 2$, but works in the general case, although it requires an increasing amount of work. We conjecture a general expression of the form

$$e_H = \frac{(1-a^2)^{2H-1}}{(2-a^2)^{5(H-1)}} P_H(a^2), \qquad H \geq 2, \tag{8.1}$$

with P_H a polynomial in a^2, the degree of which could be obtained by a careful analysis of the above procedure. From (7.35) its value for $a^2 = 1$

would be equal to

$$P_H(1) = \frac{1}{2 \times 6^{2H-1}} \frac{(4H-3)!}{H!\,(H-1)!}. \tag{8.2}$$

Various methods have been proposed to compute $e_0(g)$ and similar quantities to leading order. Most of these methods do not require the sophisticated apparatus developed here.

In the physics literature, the topological expansion was first proposed by Veneziano and 't Hooft. For the Green functions, the counting problem was first solved using combinatorial methods by Koplik, Neveu, and Nussinov [1]. Their techniques seem closely related to the work of Tutte [2]. The relation to integral representations was presented in Ref. [3], where use was made of a saddle point method to obtain all planar quantities, including $e_0(g)$. Reference [4] considers for the first time the nonplanar topology by introducing the method of orthogonal polynomials which was further simplified by an unpublished remark due to G. Parisi.

At the end of Appendix 4 we give yet another derivation of the planar approximation, worked out in collaboration with E. Brézin.

However, we do not know of any method, except the one presented here, which enables one to study systematically the higher topologies.

Related problems have recently been studied in the physical literature. We mention here two of them. The first deals with unitary instead of hermitean matrices, and is the work of Gross and Witten [5] and Goldschmidt [6]. The other one considers coupled hermitean matrices. It was studied by two of the present authors [7], and finally solved by Mehta [8].

Our poor knowledge of the mathematical literature does not enable us to quote adequately related work done by mathematicians.

Appendix 1: Gaussian Integrals

We want to compute

$$\left\langle \exp\left(t \sum_{\mu=1}^{p} j_\mu x_\mu \right) \right\rangle = \frac{\int d^p x \exp\left(-\frac{1}{2} \sum_{\mu,\nu=1}^{p} x_\mu A_{\mu\nu} x_\nu + t \sum_{\mu=1}^{p} j_\mu x_\mu \right)}{\int d^p x \exp\left(-\frac{1}{2} \sum_{\mu,\nu=1}^{p} x_\mu A_{\mu\nu} x_\nu \right)} \tag{A.1.1}$$

with $A_{\mu\nu}$ a $p \times p$ real symmetric matrix, assumed to be positive definite.

Using the convention that repeated indices are to be summed over, we rewrite the numerator as

$$N(j,t) = \exp\left(\frac{t^2}{2} j_\mu (A^{-1})_{\mu\nu} j_\nu\right)$$
$$\times \int d^p x \exp\left\{-\frac{1}{2}\left(x_\mu + t j_\nu (A^{-1})_{\nu\mu}\right) A_{\mu\rho}\left(t(A^{-1})_{\rho\sigma} j_\sigma + x_\rho\right)\right\}$$
$$= \exp\left(\frac{t^2}{2} j_\mu (A^{-1})_{\mu\nu} j_\nu\right) \int d^p y \exp\left(-\frac{1}{2} y_\mu A_{\mu\nu} y_\nu\right). \quad \text{(A.1.2)}$$

By diagonalizing the quadratic form $y_\mu A_{\mu\nu} y_\nu$, we get

$$N(j,t) = \exp\left(\frac{t^2}{2} j_\mu (A^{-1})_{\mu\nu} j_\nu\right) (2\pi)^{p/2} (\det A)^{-1/2}. \quad \text{(A.1.3)}$$

Finally,

$$\langle e^{t j_\mu x_\mu} \rangle = \exp\left(\frac{t^2}{2} j_\mu (A^{-1})_{\mu\nu} j_\nu\right). \quad \text{(A.1.4)}$$

Appendix 2: Integration of an Invariant Function

Consider the integral

$$I_N = \int dM f(M), \quad \text{(A.2.1)}$$

where M is an $N \times N$ hermitean matrix, the measure dM stands for

$$dM = \prod_{1 \le i \le N} dM_{ii} \prod_{1 \le i < j \le N} d(\mathrm{Re}\, M_{ij}) d(\mathrm{Im}\, M_{ij}), \quad \text{(A.2.2)}$$

and $f(M)$ is invariant, under the adjoint action of the N-dimensional unitary group,

$$f({}^U M) = f(M) \qquad {}^U M = U M U^\dagger. \quad \text{(A.2.3)}$$

It can also be checked that dM is also invariant. Equation (A.2.3) means that $f(M)$ is a symmetric function of the eigenvalues of M. We introduce the unitary transformation V which diagonalizes M,

$$M = V \Lambda V^\dagger \qquad \Lambda_{ij} = \lambda_i \delta_{ij}. \quad \text{(A.2.4)}$$

Then the integral I_N can be written

$$I_N = \int_{-\infty}^{+\infty} \prod_{i=1}^{N} d\lambda_i \, f(\Lambda) J(\Lambda), \quad \text{(A.2.5)}$$

where we have to compute $J(\Lambda)$. The unitary transformation V^{-1} is such that, applied to M, it leads to a matrix with vanishing off-diagonal elements. Let us then introduce

$$\Delta^{-1}(M) = \int dU \prod_{1 \leq i < j \leq N} \delta^{(2)}\left[({}^{U}M)_{ij} \right]. \tag{A.2.6}$$

Here dU is the Haar invariant measure on the unitary group. For the time being, we leave its normalization arbitrary. Furthermore,

$$\delta^{(2)}\left[({}^{U}M)_{ij} \right] = \delta\left[\operatorname{Re}(UMU^{\dagger})_{ij} \right] \delta\left[\operatorname{Im}(UMU^{\dagger})_{ij} \right]. \tag{A.2.7}$$

In (A.2.6) there are $N(N-1)$ δ-distributions and N^2 integration variables, N of them being trivial, corresponding to a diagonal U. Clearly the invariance of the measure dU entails the invariance of $\Delta(M)$,

$$\Delta({}^{U}M) = \Delta(M). \tag{A.2.8}$$

Introducing (A.2.6) in the expression for I_N, we get

$$I_N = \int dM f(M) \Delta(M) \int dU \prod_{1 \leq i < j \leq N} \delta^{2}\left[({}^{U}M)_{ij} \right]. \tag{A.2.9}$$

Changing M into ${}^{U^{-1}}M$, and using the invariance of the measure dM, of $f(M)$ and of $\Delta(M)$, this reads

$$I_N = \int dU \int dM f(M) \Delta(M) \prod_{1 \leq i \leq j \leq N} \delta^{(2)}(M_{ij}). \tag{A.2.10}$$

The integral over the unitary group factorizes. If we set

$$\Omega_N = \int dU, \tag{A.2.11}$$

we get

$$I_N = \Omega_N \int \prod_{i=1}^{N} d\lambda_i \, f(\Lambda) \Delta(\Lambda). \tag{A.2.12}$$

Comparing with (A.2.5), we find

$$J(\Lambda) = \Omega_N \Delta(\Lambda). \tag{A.2.13}$$

The computation of $J(\Lambda)$ is thus reduced to the one of $\Delta(\Lambda)$.

To obtain $\Delta(\Lambda)$, we can take in (A.2.6) the matrix M diagonal; we call it Λ. Hence U is close to the identity, up to a permutation matrix. Therefore with A infinitesimal anti-hermitean,

$$U = e^{A} = I + A + \dots, \qquad A = -A^{\dagger}. \tag{A.2.14}$$

The unique solution of

$$(e^{A}\Lambda e^{-A})_{ij} = 0 \quad \text{for } i \neq j \qquad \text{is} \qquad A_{ij} = 0 \quad (i \neq j). \tag{A.2.15}$$

Recall that

$$\delta[F(x)] = \sum_{i} \frac{\delta(x - x_i)}{|F'(x_i)|}, \tag{A.2.16}$$

where the x_i are the zeros of $F(x)$. We can first integrate over the diagonal elements of the unitary group which commute with Λ and the infinitesimal off-diagonal ones. We choose to normalize the remaining part of the measure in the vicinity of the identity as

$$dU = d\mathcal{U}(2\pi)^{N} \prod_{1 \le i < j \le N} d(\operatorname{Re} A_{ij})d(\operatorname{Im} A_{ij}), \tag{A.2.17}$$

where $d\mathcal{U}$ is the measure over the diagonal part of the group, the integral of which is chosen to be unitary. Furthermore for $i \neq j$,

$$[U\Lambda U^{\dagger}]_{ij} = [A, \Lambda]_{ij} + \dots = A_{ij}(\lambda_j - \lambda_i) + \dots . \tag{A.2.18}$$

Here the omitted terms are of higher order in A. As a result

$$\begin{aligned} \Delta^{-1}(\Lambda) &= (2\pi)^{N} \int \prod_{1 \le i < j \le N} d(\operatorname{Re} A_{ij})d(\operatorname{Im} A_{ij}) \\ &\quad \times \prod_{1 \le i < j \le N} \delta^{(2)}[A_{ij}(\lambda_j - \lambda_i) + \dots] \\ &= \frac{(2\pi)^{N}}{\prod_{1 \le i < j \le N} (\lambda_i - \lambda_j)^2}. \end{aligned} \tag{A.2.19}$$

Therefore,

$$\Delta(\Lambda) = (2\pi)^{-N} \prod_{1 \le i < j \le N} (\lambda_i - \lambda_j)^2 \tag{A.2.20}$$

and

$$I_N = \frac{\Omega_N}{(2\pi)^{N}} \int_{-\infty}^{+\infty} \prod_{i=1}^{N} d\lambda_i \prod_{1 \le i < j \le N} (\lambda_i - \lambda_j)^2 f(\Lambda). \tag{A.2.21}$$

Finally, to obtain Ω_N we use an explicit evaluation of (A.2.1) for a gaussian

$f(M) = \exp - \frac{1}{2}\mathrm{tr}\, M^2$. Then

$$I_N^{(0)} = \int dM \exp\left(-\tfrac{1}{2}\mathrm{tr}\, M^2\right) = 2^{N/2}\pi^{N^2/2}. \tag{A.2.22}$$

On the other hand, from (4.5) and (4.15) in the text

$$I_N^{(0)} = \frac{\Omega_N}{(2N)^N} N!\, h_0 h_1 \ldots h_{N-1}, \tag{A.2.23}$$

where h_p corresponds to the normalization factor for Hermite polynomials (the case $g = 0$)

$$h_p = (2\pi)^{1/2} p!. \tag{A.2.24}$$

Hence,

$$I_N^{(0)} = \frac{\Omega_N}{(2\pi)^{N/2}} \prod_{p=1}^{N} p! = 2^{N/2}\pi^{N^2/2}. \tag{A.2.25}$$

This means

$$\Omega_N = \frac{2^N \pi^{N(N+1)/2}}{\prod_{p=1}^{N} p!}. \tag{A.2.26}$$

Appendix 3

In this appendix we evaluate the integral

$$\alpha_n^{[2p+1]} h_n = \int_{-\infty}^{+\infty} d\lambda\, e^{-V(\lambda)} P_n(\lambda) \lambda^{2p+1} P_{n-1}(\lambda). \tag{A.3.1}$$

The notation is that of Section 4.

For fixed n we write

$$\lambda^q P_{n-1}(\lambda) = \sum_i \alpha_i^{[q]} P_i(\lambda), \tag{A.3.2}$$

$$\lambda^{q+1} P_{n-1}(\lambda) = \sum_j \alpha_j^{[q+1]} P_j(\lambda) = \sum_i \alpha_i^{[q]} \left[P_{i+1}(\lambda) + R_i P_{i-1}(\lambda) \right].$$

Therefore,

$$\alpha_j^{[q+1]} = \alpha_{j-1}^{[q]} + R_{j+1} \alpha_{j+1}^{[q]}, \quad \alpha_j^{[0]} = \delta_{j,\, n-1}. \tag{A.3.3}$$

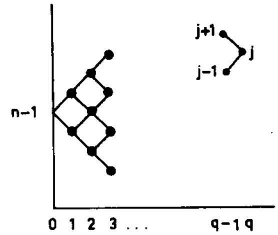


FIG. 9. The various paths connecting point $[n-1, 0]$ to point $[j, q]$.

To analyze this relation, we use a pictorial representation (Fig. 9), where we introduce all possible paths, made of q steps joining level $(n-1)$ to level j. Each step is up or down by one unit. When the step is up we attach a weighting factor one; when we leave s down to level $s-1$ the factor is R_s. The proof follows obviously by recursion from (A.3.3) if we observe that it is true for $q = 1$.

As a consequence, using the orthogonality of the polynomials $\{P_n\}$ we obtain the following statement:

$\alpha_n^{[2p+1]}$ is a sum over the $\binom{2p+1}{p}$ paths along a staircase, from the stair at height $n-1$, to the one at height n, in $2p+1$ steps, $p+1$ up and p down. For each path, a factor R_s occurs when descending from stair s down to stair $s-1$.

For instance when $p = 1$, we have three paths, depicted in Fig. 10. In this case

$$\alpha_n^{[3]} = R_{n-1} + R_n + R_{n+1}. \tag{A.3.4}$$

Similarly, for $p = 2$ we have 10 paths and we find

$$\alpha_n^{[5]} = R_{n-2}R_{n-1} + R_{n-1}^2 + 2R_{n-1}R_n + R_{n-1}R_{n+1} + R_n^2 + 2R_nR_{n+1} + R_{n+1}R_{n+2}. \tag{A.3.5}$$

We conclude this appendix with a remark concerning the relation with Jacobi matrices. Consider the orthonormal polynomials,

$$\mathcal{P}_n(\lambda) = h_n^{-1/2}P_n(\lambda), \tag{A.3.6}$$

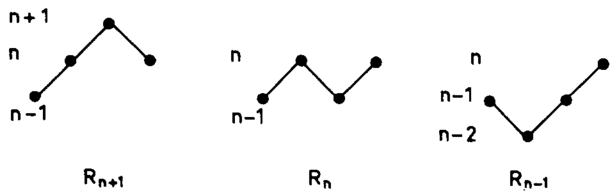


FIG. 10. The three paths corresponding to $p = 1$ and their contributions.

which verify the recursion relations

$$\begin{aligned}\lambda\mathcal{P}_n(\lambda) &= \left(\frac{h_{n+1}}{h_n}\right)^{1/2}\mathcal{P}_{n+1}(\lambda) + R_n\left(\frac{h_{n-1}}{h_n}\right)^{1/2}\mathcal{P}_{n-1}(\lambda)\\ &= R_{n+1}^{1/2}\mathcal{P}_{n+1}(\lambda) + R_n^{1/2}\mathcal{P}_{n-1}(\lambda).\end{aligned}\tag{A.3.7}$$

We represent these relations using the Jacobi matrix,

$$J = \begin{bmatrix} 0 & R_1^{1/2} & 0 & 0 & 0 & \cdots \\ R_1^{1/2} & 0 & R_2^{1/2} & 0 & 0 & \cdots \\ 0 & R_2^{1/2} & 0 & R_3^{1/2} & 0 & \cdots \\ 0 & 0 & R_3^{1/2} & 0 & R_4^{1/2} & \cdots \\ - & - & - & - & - & - \end{bmatrix}\tag{A.3.8}$$

and the (infinite) vector $|\mathcal{P}\rangle$ with components

$$\langle n|\mathcal{P}\rangle = \mathcal{P}_n.\tag{A.3.9}$$

Then (A.3.7) reads

$$J|\mathcal{P}\rangle = \lambda|\mathcal{P}\rangle.\tag{A.3.10}$$

Clearly,

$$J^q|\mathcal{P}\rangle = \lambda^q|\mathcal{P}\rangle,\tag{A.3.11}$$

or

$$\lambda^q\mathcal{P}_{n-1} = \sum_i \langle n-1|J^q|i\rangle\mathcal{P}_i,\tag{A.3.12}$$

which shows that $\alpha_i^{[q]}$ is the $(n-1, i)$ matrix element of the Jacobi matrix raised to the power q.

Appendix 4: Random Matrices

The formalism of this paper can be translated in terms of properties of random matrices [9]. Consider the set of $N \times N$ hermitean matrices with random matrix elements M_{ij}. Define a probability law

$$p_N(M)dM = Z_N^{-1}e^{-\operatorname{tr}V(M)}dM.\tag{A.4.1}$$

The joint density of probability to have the set of eigenvalues in the range $(\lambda_i, \lambda_i + d\lambda_i)$ is obtained by integrating over the unitary group, as in

Appendix 2, with the result that

$$p_N(\lambda_1, \ldots, \lambda_N) = \bar{Z}_N^{-1}\Delta^2(\Lambda_1, \ldots, \lambda_N)\exp\left(-\sum_{i=1}^{N} V(\lambda_i)\right). \quad \text{(A.4.2)}$$

If we are only interested by the probability $\bar{p}(\lambda)d\lambda$ to have one eigenvalue in the range $\{\lambda, \lambda + d\lambda\}$ we have to integrate over the remaining $(N-1)$ eigenvalues. Thus,

$$\bar{p}_N(\lambda) = \bar{Z}_N^{-1}e^{-V(\lambda)}\int_{-\infty}^{+\infty}\prod_{i=2}^{N} d\lambda_i e^{-V(\lambda_i)}\Delta^2(\lambda, \lambda_2, \ldots, \lambda_N). \quad \text{(A.4.3)}$$

Up to a factor the Vandermonde determinant Δ can be replaced by the determinant over the orthonormalized polynomials $\mathcal{P}_k(\lambda)$ introduced in the previous appendix,

$$\Delta(\lambda_1, \lambda_2, \ldots, \lambda_N) = \prod_{j=0}^{N-1} h_j^{1/2}\det\|\mathcal{P}_{k-1}(\lambda_l)\|, \qquad 1 \leq k, l \leq N. \quad \text{(A.4.4)}$$

Inserting this result in (A.4.3) and integrating over $\lambda_2, \ldots, \lambda_N$ we find

$$\bar{p}_N(\lambda) = \frac{1}{N}\sum_{n=0}^{N-1}\mathcal{P}_n^2(\lambda)e^{-V(\lambda)}. \quad \text{(A.4.5)}$$

The normalization constant has been adjusted in such a way that

$$\int d\lambda\, \bar{p}_N(\lambda) = 1. \quad \text{(A.4.6)}$$

Let us remark that $\bar{p}_N(\lambda)$ also defines the density distribution of zeros of the orthogonal polynomial $\mathcal{P}_N(\lambda)$ (or $P_N(\lambda)$) for N large. To see this let us average over the probability (A.4.1) the characteristic polynomial $C_N(\lambda)$ of the matrix M. We have

$$C_N(\lambda) = \prod_{i=1}^{N}(\lambda - \lambda_i)$$

$$\begin{aligned}\langle C_N(\lambda)\rangle &= \bar{Z}_N^{-1}\int_{-\infty}^{+\infty}\prod_{i=1}^{N}\{d\lambda_i e^{-V(\lambda_i)}(\lambda - \lambda_i)\}\Delta^2(\lambda_1, \ldots, \lambda_N)\\ &= P_N(\lambda). \end{aligned} \quad \text{(A.4.7)}$$

Clearly, the right-hand side is a polynomial of degree N in λ^N, with a coefficient of the term of highest degree equal to 1. To prove the stated equality we only need to show that for $0 \leq s \leq N-1$ we have

$$\int_{-\infty}^{+\infty} d\mu(\lambda_{N+1}) P_N(\lambda_{N+1}) \lambda_{N+1}^s = 0, \tag{A.4.8}$$

where for short we write $d\mu(\lambda) = e^{-V(\lambda)} d\lambda$. Now

$$\begin{aligned}
\bar{Z}_N \int d\mu(\lambda_{N+1}) P_N(\lambda_{N+1}) \lambda_{N+1}^s &= \int \prod_{k=1}^{N+1} d\mu(\lambda_k) \Delta(\lambda_1, \ldots, \lambda_{N+1}) \\
&\quad \times \lambda_{N+1}^s \Delta(\lambda_1, \ldots, \lambda_N) \\
&= \frac{1}{N+1} \int \prod_{k=1}^{N+1} d\mu(\lambda_k) \Delta(\lambda_1, \ldots, \lambda_{N+1}) \\
&\quad \times \sum_{l=1}^{N+1} (-1)^{N+1-l} \lambda_l^s \Delta(\lambda_1, \ldots, \hat{\lambda}_l, \ldots, \lambda_{N+1}).
\end{aligned} \tag{A.4.9}$$

The sum in the last integrand is the expansion of the determinant

$$\begin{vmatrix}
1 & \lambda_1 & \cdots & \lambda_1^{N-1} & \lambda_1^s \\
1 & \lambda_2 & \cdots & \lambda_2^{N-1} & \lambda_2^s \\
\vdots & & & & \\
1 & \lambda_{N+1} & \cdots & \lambda_{N+1}^{N-1} & \lambda_{N+1}^s
\end{vmatrix} \tag{A.4.10}$$

with respect to its last column. It vanishes for $0 \leq s \leq N-1$, which proves the assertion. Since $P_N(\lambda)$ is therefore the average of $\Pi_{i=1}^N(\lambda - \lambda_i)$, we may think of $\bar{p}_N(\lambda)$ for N large as giving the density of zeros of $P_N(\lambda)$.

Let us now compute the limiting even distribution $u(\mu)$

$$\begin{aligned}
u(\mu) &= \lim_{N\to\infty} N^{1/2} \bar{p}_N(N^{1/2}\mu), \\
\int d\mu u(\mu) &= 1.
\end{aligned} \tag{A.4.11}$$

The choice of the scaling factor $\lambda = N^{1/2}\mu$ is made, as we shall see, to ensure the existence of the limit. The reader will not confuse the Lebesgue measure $d\mu$ used in the remainder of this appendix with the measure $d\mu(\lambda) = d\lambda e^{-V(\lambda)}$ introduced before. Consider the moments

$$\begin{aligned}
v_{2p} = \int d\mu u(\mu) \mu^{2p} &= \lim_{N\to\infty} \int d\mu(\lambda) \frac{1}{N^{p+1}} \sum_{n=0}^{N-1} \mathscr{P}_n(\lambda) \lambda^{2p} \mathscr{P}_n(\lambda) \\
&= \lim_{N\to\infty} \frac{1}{N^{p+1}} \sum_{n=0}^{N-1} \langle n | J^{2p} | n \rangle,
\end{aligned} \tag{A.4.12}$$

where we use the Jacobi matrix of Appendix 3.

For n large we have approximately

$$\langle n|J^{2p}|n\rangle \sim R_n^p \frac{(2p)!}{(p!)^2}, \tag{A.4.13}$$

the last coefficient being the number of paths of length $2p$ connecting the nth stair to itself.

To leading order in N we have therefore

$$\nu_{2p} = \frac{(2p)!}{(p!)^2} \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \left[r_0\!\left(\frac{n}{N}, g\right)\right]^p. \tag{A.4.14}$$

To obtain the limit we replace the sum by an integral and use the integral representation

$$\frac{(2p)!}{(p!)^2} = \int_{-1}^{+1} \frac{dy}{\pi} \frac{(2y)^{2p}}{\sqrt{1-y^2}}. \tag{A.4.15}$$

In this way

$$\nu_{2p} = \int d\mu u(\mu)\mu^{2p} = \int_0^1 dx [r_0(x,g)]^p \int_{-1}^{+1} \frac{dy}{\pi} \frac{(2y)^{2p}}{\sqrt{1-y^2}}. \tag{A.4.16}$$

Incidentally, we have justified the scaling factor $N^{1/2}$ as claimed above. Comparison of the two expressions for ν_{2p} yields

$$\begin{aligned} u(\mu) &= \int_0^1 dx \int_{-1}^{+1} \frac{dy}{\pi} \frac{1}{\sqrt{1-y^2}} \delta\!\left(\mu - 2y\sqrt{r_0(x,g)}\right) \\ &= \int_0^1 \frac{dx}{\pi} \frac{\theta[4r_0(x,g) - \mu^2]}{\sqrt{4r_0(x,g) - \mu^2}}. \end{aligned} \tag{A.4.17}$$

From Eqs. (5.5), (5.7), and (5.8) of the text we see that

$$w[r_0(x,g)] = x \qquad w(a^2) = 1. \tag{A.4.18}$$

Therefore,

$$u(\mu) = \frac{1}{\pi} \int_{\mu^2/4}^{a^2} \frac{w'(r)\,dr}{\sqrt{4r - \mu^2}}. \tag{A.4.19}$$

This result gives the limiting distribution of eigenvalues in closed form. Note that it is concentrated in the range $-2a \le \mu \le 2a$. In the quartic case, for instance, where

$$w(r) = r + 12gr^2, \tag{A.4.20}$$

$$\begin{aligned} u(\mu) &= \frac{1}{2\pi}\left[1 + 4g(\mu^2 + 2a^2)\right]\sqrt{4a^2 - \mu^2}\,, \\ &= \frac{1}{6\pi a^4}\left[\mu^2(1 - a^2) + 2a^2 + a^4\right]\sqrt{4a^2 - \mu^2}\,. \end{aligned} \tag{A.4.21}$$

This result is called Wigner's semicircle law in the case $g = 0$, $a^2 = 1$, and is then relative to the distribution of zeros of Hermite polynomials. In general, we have a polynomial in μ^2 multiplying $\sqrt{4a^2 - \mu^2}$.

From the knowledge of $u(\mu)$ it is easy to derive the leading (planar) term of the topological expansion $e_0(g)$. Indeed, from Section 4, and sticking to the quartic case

$$\begin{aligned} e_0(g) = \int_{-2a}^{+2a} d\mu u(\mu)\left[\tfrac{1}{2}\mu^2 + g\mu^4\right] - \int\int_{-2a}^{+2a} d\mu_1 d\mu_2 u(\mu_1) u(\mu_2) \ln|\mu_1 - \mu_2| \\ - \{g \to 0\}. \end{aligned} \tag{A.4.22}$$

Using the expression (A.4.22) we rewrite this

$$\begin{aligned} e_0(g) &= \int_0^{2a} d\mu u(\mu)\left(\tfrac{1}{2}\mu^2 + g\mu^4 - 2\ln\mu\right) - \{g \to 0\} \\ &= -\tfrac{1}{2}\ln a^2 + \tfrac{1}{24}(a^2 - 1)(9 - a^2) \end{aligned} \tag{A.4.23}$$

in agreement with (7.19).

The method presented in this appendix could be used, in principle, to compute the $1/N$ correction to $u(\mu)$. If we are only interested in the dominant term (A.4.2), and therefore in $e_0(g)$, there exists a much simpler and elegant technique which we shall now briefly review.

To do so we first rescale M as

$$\begin{aligned} M &= N^{1/2} m \\ p_N(m)\,dm &= N^{N^2/2} Z_N^{-1} e^{-N \operatorname{tr} v(m)}\,dm \\ v(m) &= \tfrac{1}{2}m^2 + \sum_{p \geq 2} g_p m^{2p} \end{aligned} \tag{A.4.24}$$

in accordance with the fact that a limiting distribution $u(\mu)d\mu$ will exist for the density of eigenvalues of m.

$$F_N(z) = \left\langle \frac{1}{N}\operatorname{tr}\left(\frac{1}{z - m}\right)\right\rangle = \int dm\, p_N(m)\frac{1}{N}\operatorname{tr}\left(\frac{1}{z - m}\right); \tag{A.4.25}$$

then

$$u(\mu) = \lim_{N\to\infty} -\frac{\operatorname{Im} F(\mu + i\epsilon)}{\pi}. \tag{A.4.26}$$

Using the invariance of the measure dm under a translation δm we derive the analog of the nonlinear relation among the coefficients R_n,

$$\left\langle \left(\frac{1}{z-m}\right)_{ac} \delta m_{cd} \left(\frac{1}{z-m}\right)_{db} \right\rangle = N \left\langle \left(\frac{1}{z-m}\right)_{ab} \delta m_{cd} v'(m)_{dc} \right\rangle, \tag{A.4.27}$$

where repeated indices are of course summed over. As a consequence

$$\left\langle \left[\frac{1}{N}\operatorname{tr}\left(\frac{1}{z-m}\right)\right]^2 \right\rangle = \left\langle \frac{1}{N}\operatorname{tr}\left(\frac{v'(m)}{z-m}\right)\right\rangle. \tag{A.4.28}$$

Now we have

$$\operatorname{tr}\left(\frac{v'(m)}{z-m}\right) = v'(z)\operatorname{tr}\left(\frac{1}{z-m}\right) - \operatorname{tr}\frac{v'(z)-v'(m)}{z-m}. \tag{A.4.29}$$

To proceed we restrict ourselves to the simple quartic case, although the extension to the general case is quite easy. Then

$$v(m) = \tfrac{1}{2}m^2 + gm^4. \tag{A.4.30}$$

Define

$$t_2^{[N]} = \left\langle \frac{1}{N}\operatorname{tr} m^2 \right\rangle \tag{A.4.31}$$

in such a way that from (A.4.29)

$$\left\langle \frac{1}{N}\operatorname{tr}\left(\frac{v'(m)}{z-m}\right)\right\rangle = (z + 4gz^3)F_N(z) - \left[1 + 4g\left(t_2^{[N]} + z^2\right)\right] \tag{A.4.32}$$

and

$$\left\langle \left[\frac{1}{N}\operatorname{tr}\left(\frac{1}{z-m}\right)\right]^2 \right\rangle = (z + 4gz^3)F_N(z) - \left[1 + 4g\left(t_2^{[N]} + z^2\right)\right]. \tag{A.4.33}$$

To use this equation we observe that to leading order the mean value of a product of unitary invariants (such as traces) factorizes

$$\langle AB \rangle = \langle A \rangle \langle B \rangle + O(1/N). \tag{A.4.34}$$

This is the crucial observation, which enables us to obtain, in the limiting case $N \to \infty$, an algebraic equation for the function $F(z)$, with $t_2 = \lim_{N\to\infty} t_2^{[N]}$. This equation reads

$$F^2(z) - (z + 4gz^3)F(z) + \left[1 + 4g(t_2 + z^2)\right] = 0. \qquad \text{(A.4.35)}$$

The unknown coefficient t_2 is obtained by requiring that $F(z)$ admits on the real z axis a positive discontinuity. The discriminant δ of the above equation

$$\delta = z^2(1 + 4gz^2)^2 - 4(1 + 4gz^2 + 4gt_2) \qquad \text{(A.4.36)}$$

is cubic in z^2. We therefore require that it have a negative double root and call $4a^2$ its remaining positive root, i.e., we identify it with

$$\delta = (4gz^2 + \xi)^2(z^2 - 4a^2). \qquad \text{(A.4.37)}$$

This yields

$$\begin{gathered} \xi = 1 + 8ga^2, \\ t_2 = \frac{a^2(4 - a^2)}{3}, \\ 12ga^4 + a^2 - 1 = 0. \end{gathered} \qquad \text{(A.4.38)}$$

We recognize Eq. (5.13) which determines $a^2(g)$, and with these values we obtain the expressions of $F(z)$ and $u(\mu)$ as

$$F(z) = \tfrac{1}{2}\left\{(z + 4gz^3) - (1 + 8ga^2 + 4gz^2)\sqrt{z^2 - 4a^2}\right\}, \qquad \text{(A.4.39)}$$

$$u(\mu) = (1/2\pi)(1 + 8ga^2 + 4g\mu^2)\sqrt{4a^2 - \mu^2}, \qquad -2a \le \mu \le 2a, \qquad \text{(A.4.40)}$$

in agreement with (A.4.21).

Appendix 5

In this appendix we derive formula (7.8) of the text. In the sense of formal power series, we show that

$$\frac{h_0\left(\frac{g}{N}\right)}{h_0(0)} \cdot \frac{h_0\left(-\frac{g}{N}\right)}{h_0(0)} = \phi_N(0, g) = \left.\frac{r_\epsilon(x, g)}{x}\right|_{x=0}. \qquad \text{(A.5.1)}$$

From Eq. (6.6) we have

$$1 = \phi_N(0, g) + 4g\phi_N(0, g)\{r_\epsilon(-\epsilon, g) + r_\epsilon(\epsilon, g)\}. \tag{A.5.2}$$

Taking into account that $r(x, g)$ is even in ϵ, and the fact that from (6.18),

$$r_\epsilon(\epsilon, g) = \frac{1}{N}\bar{R}_1\left(\frac{g}{N}\right) = \frac{1}{N}R_1\left(\frac{g}{N}\right) = \frac{1}{N}\frac{\mu_2(g/N)}{\mu_0(g/N)}, \tag{A.5.3}$$

the identity to be proven reduces to

$$1 + 4\frac{g}{N}\left\{\frac{\mu_2^+}{\mu_0^+} - \frac{\mu_2^-}{\mu_0^-}\right\} = \frac{h_0^2(0)}{h_0^+h_0^-}, \tag{A.5.4}$$

where we use the notation

$$f^{\pm} = f\left(\pm\frac{g}{N}\right). \tag{A.5.5}$$

Introducing

$$\beta = g/N$$
$$\bar{\mu}_{2k}(\beta) = \left(1/\sqrt{2\pi}\right)\int_{-\infty}^{+\infty} dy y^{2k} \exp\left(-\tfrac{1}{2}y^2 - \beta y^4\right) \tag{A.5.6}$$

and recalling that $h_0 = \mu_0$, we rewrite (A.5.4)

$$\bar{\mu}_0^+\bar{\mu}_0^- + 4\beta\left[\bar{\mu}_2^+\bar{\mu}_0^- - \bar{\mu}_2^-\bar{\mu}_0^+\right] = 1. \tag{A.5.7}$$

Of course, this can only be meant in the sense of formal series since the functions $\bar{\mu}$ are singular at $\beta = 0$. To cope with this singularity we use the following device. Let $A(\beta)$ be an infinitely differentiable function in an interval $0 \leq \beta \leq b$. Then $[A(\beta)]_k$ will stand for the polynomial of degree k in β which coincides up to the term β^k with the formal Taylor expansion of $A(\beta)$. Clearly we have

$$[A(\beta)B(\beta)]_k = \left[[A(\beta)]_k[B(\beta)]_k\right]_k. \tag{A.5.8}$$

With this notation we want to prove the meaningful statement that for any integer k, (A.5.7) is true with every function replaced by its expansion to order k. Using the integral representation (A.5.6) this means

$$[1]_k = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{dxdy}{2\pi}\exp\left(-\frac{1}{2}(x^2 + y^2)\right)$$
$$\times\left[\left[e^{-\beta x^4}\right]_k\left[e^{\beta y^4}\right]_k\left[1 + 4\beta(x^2 - y^2)\right]_k\right]_k. \tag{A.5.9}$$

For $k = 0$ this is trivially verified, while for $k \geq 1$ it reduces to

$$0 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{dxdy}{2\pi} \exp\left(-\frac{1}{2}(x^2+y^2)\right)$$
$$\times (y^4 - x^4)^{k-1}\{y^4 - x^4 + 4k(x^2 - y^2)\}. \qquad \text{(A.5.10)}$$

Using polar coordinates, the angular integral factorizes

$$0 = (-1)^k \int_0^{2\pi} \frac{d\theta}{2\pi}(\cos 2\theta)^k \int_0^{\infty} d\rho\rho e^{-\rho^2/2}\rho^{4k-2}(\rho^2 - 4k). \qquad \text{(A.5.11)}$$

But the last integral over ρ vanishes identically as it is equal to $2^{2(1-k)}[(2k)! - 2k(2k-1)!]$. Therefore (A.5.1) is proved.

Appendix 6: Number of Graphs of Genus H with One Loop

In this appendix we compute the number n_H of quartic graphs of genus H with $2H - 1$ vertices and only one loop. This coefficient appears in the first nonvanishing term of the expansion of $e_H(g)$ given in (7.35)

$$e_H(g) = n_H g^{2H-1} + O(g^{2H}), \qquad H \geq 1. \qquad \text{(A.6.1)}$$

The line of reasoning, due to J. M. Drouffe, entirely different from the previous method, would enable one in fact to compute various similar quantities.

Let us return to the analysis of Section 3 using graphs with double lines as in Figs. 5 through 8. At each of the $k = 2H - 1$ vertices we label arbitrarily the double lines with an integer i running from 1 to $4k$. At each vertex a line with an entering arrow is linked to one with an outgoing arrow. This defines an element σ of the permutation group on $4k$ indices Σ_{4k}. Clearly this permutation has k cycles of length four. We represent σ according to its decomposition in cycles, using the conventional notation for the classes of the symmetric group,

$$\sigma \in \{4^k\}. \qquad \text{(A.6.2)}$$

For instance, in the case depicted in Fig. 11, $k = 3$, $H = 2$, and for the labeling indicated in the figure

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 2 & 3 & 4 & 1 & 6 & 7 & 8 & 5 & 10 & 11 & 12 & 9 \end{pmatrix}. \qquad \text{(A.6.3)}$$

Note that the class of σ is independent of the arbitrary labeling of double

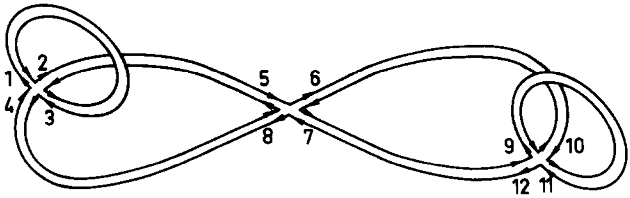


FIG. 11. Example of labeling the double lines at each vertex.

lines as well as of the possibility of reversing all arrows, which would change σ into σ^{-1}.

The pairing of double lines emerging from each vertex, implied by Wick's lemma, introduces a second permutation τ, where i and $\tau(i)$ are the labels of the two ends of each double line. Since τ^2 is the identity, it is of course immaterial which end we call i and which we call $\tau(i)$. This permutation τ obviously has $2k$ cycles ($2k$ is the number of propagators) of length 2

$$\tau \in \{2^{2k}\}. \tag{A.6.4}$$

In the case depicted in Fig. 11,

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 3 & 5 & 1 & 8 & 2 & 10 & 12 & 4 & 11 & 6 & 9 & 7 \end{pmatrix} \tag{A.6.5}$$

Finally, the condition that the graph has only one loop, or one face ($L = 1$ in Eq. (3.16)), amounts to saying that the product $\sigma\tau$ has only one cycle

$$\sigma\tau \in \{4k\}. \tag{A.6.6}$$

We can check this for our example

$$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 4 & 6 & 2 & 5 & 3 & 11 & 9 & 1 & 12 & 7 & 10 & 8 \end{pmatrix}. \tag{A.6.7}$$

Up to the supplementary factor $1/k!$ appearing in Eq. (3.12), n_H is the number of permutations τ fulfilling (A.6.4) and (A.6.6) for a fixed σ belonging to the class $\{4^k\}$,

$$n_H = \frac{1}{k!} \sum_{\tau \in \Sigma_{4k}} \delta_{\{\tau\},\,\{2^{2k}\}} \delta_{\{\sigma\tau\},\,\{4k\}}. \tag{A.6.8}$$

We now introduce the characters $\chi^{(r)}$ of the symmetric group Σ_{4k} [10].

For each $\sigma \in \Sigma_{4k}$, $\chi^{(r)}(\sigma)$ only depends on the class $\{\sigma\}$ of σ. The characters satisfy the completeness relation

$$\sum_r \chi^{(r)}(\sigma)\chi^{(r)}(\sigma') = \frac{(4k)!}{\nu_{\{\sigma\}}} \delta_{\{\sigma\},\,\{\sigma'\}}, \tag{A.6.9}$$

where $\nu_{\{\sigma\}}$ is the number of elements in the class $\{\sigma\}$. If $\sigma = \{1^{\alpha_1}2^{\alpha_2}3^{\alpha_3}\ldots\}$,

$$\nu_{\{\sigma\}} = \frac{(4k)!}{1^{\alpha_1}\alpha_1!2^{\alpha_2}\alpha_2!3^{\alpha_3}\alpha_3!\ldots}. \tag{A.6.10}$$

Using the representation (A.6.9) of $\delta_{\{\sigma\},\{\sigma'\}}$ in Eq. (A.6.8), we obtain

$$n_H = \frac{\nu_{\{2^{2k}\}}\nu_{\{4k\}}}{k![(4k)!]^2}\sum_{r,s}\chi^{(r)}_{\{2^{2k}\}}\chi^{(s)}_{\{4k\}}\sum_{\tau}\chi^{(r)}_{\{\tau\}}\chi^{(s)}_{\{\sigma\tau\}}. \tag{A.6.11}$$

Owing to the orthogonality properties of the characters, the unconstrained summation over τ can be performed. If d_r denotes the dimension of the irreducible representation indexed by r, we have

$$\sum_{\tau}\chi^{(r)}_{\{\tau\}}\chi^{(s)}_{\{\sigma\tau\}} = \delta_{r,s}\frac{(4k)!}{d_r}\chi^{(r)}_{\{\sigma\}}. \tag{A.6.12}$$

As a result

$$n_H = \frac{\nu_{\{2^{2k}\}}\nu_{\{4k\}}}{k!\,(4k)!}\sum_{r} d_r^{-1}\chi^{(r)}_{\{2^{2k}\}}\chi^{(r)}_{\{4k\}}\chi^{(r)}_{\{4^k\}} \tag{A.6.13}$$

since σ belongs to the class $4k$.

Similar expressions can be written for the number of graphs with arbitrary number of vertices and loops; however, their analysis looks quite difficult. The nice feature of the case $k = 2H - 1$ is that the sum on the right-hand side of (A.6.13) reduces to one over a limited set of representations corresponding to a Young tableau with only a first row of length possibly exceeding one. In the conventional notation these representations are written

$$(r) = (4k - p, 1^p), \qquad p = 0, 1, \ldots, 4k - 1. \tag{A.6.14}$$

Indeed those are the only representations for which the character χ yields a nonvanishing value for the class $\{4k\}$, namely,

$$\chi^{(p)}_{\{4k\}} = (-1)^p. \tag{A.6.15}$$

Here we have used the integer p as an index for the representation (r) defined by (A.6.14). Moreover, for these representations the other quantities occurring in (A.6.13) take the form

$$d_p = \binom{4k-1}{p}$$

$$\chi^{(p)}_{\{2^{2k}\}} = \delta^{[2]}_{p,0}(-1)^{p/2}\binom{2k-1}{p/2} - \delta^{[2]}_{p,1}(-1)^{(p-1)/2}\binom{2k-1}{(p-1)/2}$$

$$\chi^{(p)}_{\{4^k\}} = \delta^{[4]}_{p,0}(-1)^{p/4}\binom{k-1}{p/4} - \delta^{[4]}_{p,1}(-1)^{(p-1)/4}\binom{k-1}{(p-1)/4} \quad \text{(A.6.16)}$$

$$+ \delta^{[4]}_{p,2}(-1)^{(p-2)/4}\binom{k-1}{(p-2)/4} - \delta^{[4]}_{p,3}(-1)^{(p-3)/4}\binom{k-1}{(p-3)/4},$$

where $\delta^{[s]}_{p,q}$ means zero unless $p = q$ modulo s.

It is then a matter of patience to compute

$$n_H = \frac{1}{k^2 2^{2k+1}} \sum_{q=0}^{k-1} \frac{(-1)^q (4q)!\,(4k-4q-1)!}{q!\,(k-1-q)!\,(2q)!\,(2k-2q-1)!}$$

$$\times \left[1 - 2\frac{4q+1}{4k-4q-1} + \frac{(4q+1)(4q+3)}{(4k-4q-1)(4k-4q-3)}\right]. \quad \text{(A.6.17)}$$

This can be rewritten

$$n_H = \frac{2^{2k-2}}{kk!}(2k)!\frac{1}{\pi}\int_0^1 dt\,t^{-1/2}(1-t)^{-1/2}(1-2t)^{k+1}$$

$$= 2^{k-3}\frac{k+1}{k}\frac{(2k)!}{\left[\Gamma\left(\frac{k+3}{2}\right)\right]^2} = 2^{2H-2}\frac{(4H-3)!}{H!\,(H-1)!} \quad \text{(A.6.18)}$$

and this is the result quoted in (7.35). The first few values of n_H are

$$n_1 = 1, \qquad n_2 = 240, \qquad n_3 = 483{,}840, \qquad n_4 = 2{,}767{,}564{,}800 \ldots . \quad \text{(A.6.19)}$$

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