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COVARIANT DIFFERENTIAL EQUATIONS AND SINGULAR VECTORS IN VIRASORO REPRESENTATIONS

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We give explicit expressions for the singular vectors in highest weight representations of the Virasoro algebra using a precise definition of fusion.

1. Introduction

The rich representation theory of the Virasoro algebra has produced a vast amount of applications in statistical physics, conformal field theory and string theory. It is largely based on the fundamental work of Kac [1], Feigin and Fuchs [2] and Belavin et al. [3]. The Kac determinant formula yields the complex pattern of inclusions of Verma modules, leads to the consideration of rational theories and produces character formulas and differential equations for field correlations. The methods of proofs though improved by a number of authors [4–7], did not yet give very explicit expressions for these embeddings dictated by the existence of singular vectors in Verma modules. It is the aim of this work to produce such expressions.

A first important step – perhaps not sufficiently noticed – was accomplished by Benoit and Saint-Aubin [8] who gave a sub-series of these vectors. These formulas can be recast in a transparent matrix form in complete parallel to the formalism

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developed by Drinfeld and Sokolov [9] in the context of (generalized) KdV flows in a covariant discussion of differential equations [10, 11]. It was then natural to suspect that "fusion" [3] (or Wilson's short-distance expansions) would allow to obtain a completely general answer. Even though this has already been largely discussed in the literature, we could not pin-point a definite source to make the concept precise and operative. It is perhaps the most important part of this paper to achieve this goal (sect. 4). It turned out that (i) this can only be defined among irreducible modules; (ii) it involves three isomorphic but distinct copies of the same Virasoro algebra, by which we mean that the central charge is the same for the three; (iii) it introduces in a natural way the Kac determinant as the determinant of an inhomogeneous linear system; (iv) it leads to precise fusion rules, the latter being selection rules which dictate when fusion can occur and (v) last but not least, it allows to generate from singular vectors in the original modules, singular vectors in the target one. This is discussed in sect. 5 where we obtain the required expressions.

The paper has not been written in the spirit of a polished mathematical work. Rather we follow an inductive path, progressing through examples which illustrate the general properties. We did not wish to present here an axiomatic deduction which would hide the machinery, although at the present stage it is very likely that this can be done. Moreover, by turning the process upside down one can now presumably rederive the Kac formula as well as its consequences.

Our approach was suggested by the study of classical W-algebras. It was known that there are strong links between this subject and the representation theory of the Virasoro algebra. In a sense this is completely vindicated by the present work which could alternatively and provocatively be entitled a variation on the theme of second-order linear differential equations.

The essential result is summarized in proposition 5.2 (sect. 5) which gives a complete and straightforward algorithm for finding singular vectors – starting from a differential equation with coefficients in the Virasoro algebra. This is in disguise the Benoit–Saint-Aubin equation properly re-interpreted through fusion. Unfortunately to reach this stage we had to cope with a number of technicalities which explains the length and detours of the paper. A concise but slightly cumbersome version was given in a previous letter [12]. We hope to present in the future a more abstract version for a mathematically inclined reader. It is also possible that the fusion procedure can be applied fruitfully to other local algebras like Kac–Moody algebras, superconformal superconformal algebras and W-algebras themselves.

It is a pleasure to thank Raymond Stora who made us aware of the literature on covariant differential equations. In particular we are grateful to Pr. Peetre for providing us with copies of his articles on the subject. By revitalizing the "transvectant" formula of Gordan he has helped us to realize that this is the key to the construction of various bases of W-algebras and their interrelationship. We have borrowed from his work in the last paragraph of sect. 2.

2. Classical W-algebras

Classical W-algebras are the algebras of deformations of linear differential operators in one variable that preserve their transformations under diffeomorphisms. Alternatively, the W-algebras may be presented in terms of a Poisson (or hamiltonian) structure on the space of differential operators, with a finite set of k-differentials as generators. We summarize briefly the discussion given in ref. [10].

Let $\mathscr{F}_\lambda$ stand for the set of regular (i.e. infinitely differentiable or locally analytic) λ-differentials, i.e. such that the same element is represented in two coordinates x and $\tilde{x}$ by functions f and $\tilde{f}$ with the property

$$\tilde{f}(\tilde{x})\,\mathrm{d}\tilde{x}^\lambda = f(x)\,\mathrm{d}x^\lambda . \tag{2.1}$$

This definition makes sense for arbitrary λ in infinitesimal transformations by assuming the jacobian close to unity and can be extended to finite ones in regions where the jacobian does not vanish.

A normalized nth order differential operator Q_n is written in coordinate x as

$$Q_n = \mathrm{d}^n + a_2(x)\,\mathrm{d}^{n-2} + a_3(x)\,\mathrm{d}^{n-3} + \dots , \tag{2.2}$$

where $\mathrm{d} \equiv \mathrm{d}/\mathrm{d}x$ and the coefficient $a_1(x)$ of d^{n-1} has been set equal to zero. This is achieved if necessary by rescaling $Q_n \to g^{-1}(x) Q_n g(x)$ with $g(x) = \exp\{-(1/n)\int^x \mathrm{d}x'\, a_1(x')\}$. Then Q_n is a map

$$Q_n\colon\quad \mathscr{F}_{(1-n)/2} \to \mathscr{F}_{(1+n)/2} , \tag{2.3}$$

which dictates its covariance under diffeomorphisms. The coefficient of d^{n-1} vanishes in any coordinate and the wronskian of n linearly independent elements in $\ker Q_n$ is a constant.

Eq. (2.3) implies that

$$\tilde{a}_2(\tilde{x})\,\mathrm{d}\tilde{x}^2 = a_2(x)\,\mathrm{d}x^2 + \frac{n(n^2-1)}{12}\{x, \tilde{x}\}\,\mathrm{d}\tilde{x}^2 , \tag{2.4}$$

where the bracket denotes the schwarzian derivative

$$\{g(x), x\} = \frac{g'''(x)}{g'(x)} - \frac{3}{2}\left(\frac{g''(x)}{g'(x)}\right)^2 . \tag{2.5}$$

The other coefficients a_k have less transparent transformation laws. One can, however, perform an invertible transformation to differential polynomials $w_k \in \mathscr{F}_k$, $3 \leqslant k \leqslant n$. The choice made in ref. [10] was to require the w's to be linear in $a_3, a_4, \dots$ and their derivatives with a normalization given by $w_k = a_k + \dots$. The

w's were then shown to be generators for the graded ring R of those differential polynomials in the a's which are p-differentials, $p \geqslant 3$. An example is the antisymmetric combination

$$[[w_k, w_l]] = l w'_k w_l - k w_k w'_l$$

which also satisfies the Jacobi identity. This is however not the Poisson structure on functionals of w's discussed in refs. [9, 10], a subject which will not be pursued here.

The generators w_k were introduced in ref. [10] by a splitting $Q_n = \Delta_2(a_2) + \Delta_3^{(n)}(w_3, a_2) + \Delta_4^{(n)}(w_4, a_2) + \dots$ with each $\Delta_k^{(n)}$ linear in w_k and its derivatives ($k \geqslant 3$) mapping $\mathscr{F}_{(1-n)/2}$ into $\mathscr{F}_{(1+n)/2}$. This expression as well as the above formula are examples of a construction dating back to Gordan which we recall at the end of this section.

Let us introduce another set W_k of generators for the same ring R which appears more canonical and is useful for the sequel. It will be such that

$$w_k = \sigma_{k-1} W_k + \dots, \tag{2.6}$$

with σ_{k-1} a numerical constant. The additional terms in (2.6) are absent for $2 \leqslant k \leqslant 5$ (we set $a_2 = w_2$) and in general stand for a non-linear differential polynomial in the W's. The transformation is of course invertible.

To motivate, and define, this new choice, we return to Drinfeld and Sokolov's presentation [9] which substitutes for Q_n a $n \times n$ first-order differential operator,

$$\hat{Q}_n = \begin{pmatrix} \mathrm{d} & a_2 & a_3 & \dots & a_n \\ -1 & \mathrm{d} & 0 & & 0 \\ 0 & -1 & \mathrm{d} & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & -1 & \mathrm{d} \end{pmatrix}. \tag{2.7}$$

This is the so called A_{n-1} case, i.e. $\hat{Q}_n = \mathrm{d} + \mathscr{A}$, where the matrix $\mathscr{A}$ belongs to the Lie algebra A_{n-1}. The last component of a vector-valued function in $\ker \hat{Q}_n$ belongs to $\ker Q_n$ and conversely from an element in $\ker Q_n$ we can construct one in $\ker \hat{Q}_n$, so these two n-dimensional vector spaces are isomorphic. Take $f \in \ker \hat{Q}_n$, its last component may be chosen to belong to $\mathscr{F}_{(1-n)/2}$, but the next components are its successive derivatives. This entails the complicated behaviour of the a_k's under changes of coordinates.

It was one of the original ideas discussed in ref. [9] that the above form is by no means unique if one is to retain the isomorphism between $\ker Q_n$ and $\ker \hat{Q}_n$, for one can choose the components of the vectors on which $\hat{Q}_n$ acts as appropriate combinations of lower derivatives. This leads to a covariance under transformations of the type

$$\hat{Q}_n \to N^{-1} \hat{Q}_n N, \tag{2.8}$$

where N is an x-dependent element in $\mathscr{T}$, the group of upper triangular matrices with 1's on the diagonal (we call T its nilpotent Lie algebra of strictly upper triangular matrices). The above transformation does not affect the lowest component of vectors. To be more explicit let

$$J = \begin{pmatrix} 0 & 0 & \dots & & 0 \\ 1 & 0 & & & 0 \\ 0 & 1 & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & \dots & 1 & 0 \end{pmatrix}, \qquad a = \begin{pmatrix} 0 & a_2 & a_3 & \dots & a_n \\ 0 & 0 & & & 0 \\ 0 & 0 & & & 0 \\ \vdots & & & & \\ 0 & & & \dots & 0 \end{pmatrix},$$

$$\hat{Q}_n = -J + \mathrm{d} + a . \tag{2.9}$$

Under the action of N

$$\hat{Q}_n \to -J + \mathrm{d} + \left\{ N^{-1} a N + N^{-1} (\mathrm{d} N + [N, J]) \right\}, \tag{2.10}$$

where the matrix in curly brackets is again upper triangular, but not confined to the first row any more. The lowest component of $f \in \ker \hat{Q}_n$ being unaffected by this transformation leads to the same nth order operator Q_n. Therefore the new matrix operator which we call again $\hat{Q}_n = -J + d + a$, $a \in \mathrm{T}$, is as good for our purpose. We now use this gauge freedom to obtain the new canonical form.

The matrix $-J + a \equiv A$ belongs to the Lie algebra A_{n-1} of traceless matrices graded according to

$$\mathrm{grade}(A_{ij}) = j - i , \tag{2.11}$$

so that $\mathrm{grade}(J) = -1$. The Chevalley presentation of A_{n-1} in terms of generators h_i, e_i, f_i, $1 \leqslant i \leqslant n-1$, can be written in symbolic form

$$\begin{pmatrix} \frac{1}{2} h_1 & e_1 & [e_1, e_2] & [e_1, [e_2, e_3]] & \dots \\ f_1 & \frac{1}{2}(h_2 - h_1) & e_2 & [e_2, e_3] & \dots \\ [f_2, f_1] & f_2 & \frac{1}{2}(h_3 - h_2) & e_3 & \dots \\ \vdots & & & \ddots & \ddots \\ & & & f_{n-1} & -\frac{1}{2} h_{n-1} \end{pmatrix}, \tag{2.12}$$

where the entries stand for the non-vanishing elements of the corresponding matrices. In particular

$$J = \sum_{i=1}^{n-1} f_i , \tag{2.13}$$

while $e_1, \dots, e_{n-1}$ of grade 1 generate the nilpotent algebra T graded from 1 to $n-1$,

$$\mathrm{T} = \bigoplus_{k=1}^{n-1} \mathrm{T}^{(k)} . \tag{2.14}$$

The operator ad J induces an injective map

$$\operatorname{ad} J: \quad \mathrm{T}^{(k)} \to \mathrm{T}^{(k-1)} \qquad 2 \leqslant k \leqslant n-1 , \tag{2.15}$$

and maps $\mathrm{T}^{(1)}$ on the Cartan subalgebra. Since $\mathrm{T}^{(k)}$ has dimension $n-k$, we have

$$\dim\left(\mathrm{T}^{(k)} / \operatorname{ad} J(\mathrm{T}^{(k+1)})\right) = 1 . \tag{2.16}$$

Thus choosing in $\mathrm{T}^{(k)}$ a representative R_k of $\mathrm{T}^{(k)} \bmod \operatorname{ad} J(\mathrm{T}^{(k+1)})$

$$\mathrm{T}^{(k)} \sim \mathrm{C} R_k \oplus \operatorname{ad} J(\mathrm{T}^{(k+1)}) . \tag{2.17}$$

Drinfeld and Sokolov then show that using the gauge freedom on $\hat{Q}_n$ one can bring a in the form

$$a(x) = \sum_{k=1}^{n-1} r_k(x) R_k . \tag{2.18}$$

The initial choice was

$$R_k = \left[e_1, \left[e_2, \dots \left[e_{k-1}, e_k\right] \dots\right]\right] \tag{2.19}$$

which has a unique non-zero entry 1 in the first row, column $k+1$. Since an element in $\operatorname{ad} J(\mathrm{T}^{(k+1)}) \subset \mathrm{T}^{(k)}$ is characterized by the fact that the sum of its entries in the grade k principal diagonal vanishes, the above choice is perfectly justified.

However, the n-dimensional vector space carries also an irreducible representation of SL(2) of spin $j = (n-1)/2$. The Lie algebra is spanned by J_-, J_0, J_+ satisfying

$$[J_+, J_-] = 2J_0 , \qquad [J_0, J_\pm] = \pm J_\pm , \tag{2.20}$$

and in the spin-j representation the spectrum of J_0 is $(j, j-1, \dots, -j)$. One can embed the $n \times n$ matrices $J_0, J_\pm$ as elements of A_{n-1} (the so-called principal sl(2) embedding) as follows:

$$J_- = \sum_{i=1}^{n-1} f_i, \qquad J_+ = \sum_{i=1}^{n-1} i(n-i) e_i \in \mathrm{T}^{(i)},$$

$$J_0 = \sum_{i=1}^{n-1} i(n-i) h_i. \tag{2.21}$$

Thus

$$J_+^k \in \mathrm{T}^{(k)}. \tag{2.22}$$

The sum of entries of J_+^k,

$$\sigma_k = \sum_{i=1}^{n-k} i(n-i)(i+1)(n-i-1)\dots(i+k-1)(n-i-k+1)$$

$$= k!^2 \binom{n+k}{2k+1} = \frac{k!^2}{(2k+1)!} n(n^2-1)\dots(n^2-k^2), \tag{2.23}$$

is non-vanishing; we postpone a proof of this formula to the end of this section. Therefore we can choose J_+^k as the element R_k above. We then write, and this defines the W's [11]:

$$\hat{Q}_n = -J_- + \mathrm{d} + \sum_{k=1}^{n-1} W_{k+1} J_+^k. \tag{2.24}$$

Let Q_n be the corresponding nth-order operator.

Proposition 2.1.
(i) *For* $3 \leqslant k \leqslant n$, $W_k \in \mathscr{F}_k$, *i.e.*

$$\tilde{W}_k(\tilde{x})\, \mathrm{d}\tilde{x}^k = W_k(x)\, \mathrm{d}x^k.$$

(ii) $a_2(Q_n) = \sigma_1 W_2 = \frac{1}{6} n(n^2-1) W_2$, *hence*

$$\tilde{W}_2(\tilde{x})\, \mathrm{d}\tilde{x}^2 = W_2(x)\, \mathrm{d}x^2 + \tfrac{1}{2}\{x, \tilde{x}\}\, \mathrm{d}\tilde{x}^2.$$

(iii) *The* W_k, $3 \leqslant k \leqslant n$ *generate the graded ring R of r-differentials, r integral* $\geqslant 3$.
Since $w_k = a_k + \dots$ for our previous choice, we have indeed $W_k = (1/\sigma_{k-1}) w_k + \dots$. This property entails (ii) and (iii) from (i) using previous results in refs. [10]. So we need only prove (i).

Let $f = (f_j, f_{j-1}, \dots, f_{-j})^{\mathrm{T}}$ be the vector on which $\hat{Q}_n$ acts, with $n = 2j+1$. We want to find a consistent transformation law of f under diffeomorphisms such that on ker $\hat{Q}_n$ the last component f_{-j} transforms as

$$\delta f_{-j} = (\epsilon \mathrm{d} - j\epsilon')f_{-j}, \tag{2.25}$$

whereas

$$\delta W_k = (\epsilon \mathrm{d} + k\epsilon')W_k + \tfrac{1}{2}\delta_{k,2}\epsilon'''. \tag{2.26}$$

Let us write ($J \equiv J_-$)

$$\hat{Q}_n = -J_- + \mathrm{d} + a, \qquad a = \sum_{k=1}^{n-1} W_{k+1} J_+^k \tag{2.27}$$

and make the ansatz

$$\begin{aligned}\delta f &= \chi f, \\ \chi &= \epsilon(J_- - a) + \epsilon' J_0 - \tfrac{1}{2}\epsilon'' J_+ \\ &= \epsilon \mathrm{d} + \epsilon' J_0 - \tfrac{1}{2}\epsilon'' J_+ - \epsilon \hat{Q}_n.\end{aligned} \tag{2.28}$$

Observe that ϵ and $\hat{Q}_n$ do not commute ($[\hat{Q}_n, \epsilon] = \epsilon'$). Clearly on ker $\hat{Q}_n$ the last component transforms as indicated in eq. (2.25). It then follows that

$$\begin{aligned}\delta\hat{Q}_n &\equiv \sum_{k=1}^{n-1} \delta W_{k+1} J_+^k \\ &= \left[\chi, \hat{Q}_n\right] = \left[\epsilon \mathrm{d} + \epsilon' J_0 - \tfrac{1}{2}\epsilon'' J_+, -J_- + \mathrm{d} + \sum_{k=1}^{n-1} W_{k+1} J_+^k\right] - \left[\epsilon\hat{Q}_n, \hat{Q}_n\right] \\ &= \sum_{k=1}^{n-1} \{(\epsilon \mathrm{d} + \epsilon'(k+1))W_{k+1}\} J_+^k + \tfrac{1}{2}\epsilon''' J_+,\end{aligned} \tag{2.29}$$

which amounts to formula (2.26) and proves the proposition.

Remarks

(i) Another expression for χ is

$$\epsilon \mathrm{d} - \chi = \epsilon\hat{Q}_n + \tfrac{1}{2}\epsilon'\left[J_+, \hat{Q}_n\right] + \tfrac{1}{4}\epsilon''\left[J_+, \left[J_+, \hat{Q}_n\right]\right]. \tag{2.30}$$

(ii) For an alternative proof, see the end of this section.

(iii) As was said above, our former definition of w's was such that

$$Q_n = \Delta_2^{(n)}(w_2) + \Delta_3^{(n)}(w_3, w_2) + \Delta_4^{(n)}(w_4, w_2) + \dots, \tag{2.31}$$

where $w_2 \equiv a_2$, $\Delta_k^{(n)}$ maps $\mathcal{F}_{(1-n)/2} \to \mathcal{F}_{(1+n)/2}$ and $\Delta_k^{(n)}$, $k \geqslant 3$, is linear in w_k and its derivatives. In the coordinate where $w_2 \equiv a_2$ vanishes the explicit expressions become

$$\Delta_2^{(n)}(0) = \mathrm{d}^n ,$$

$$\Delta^{(n)}(w_k, 0) = \sum_{l=0}^{n-k} \frac{\binom{k+l-1}{l}\binom{n-k}{l}}{\binom{2k+l-1}{l}} w_k^{(l)} \mathrm{d}^{n-k-l}, \qquad k \geqslant 3 . \tag{2.32}$$

The relation between W's and w's (for $k \geqslant 3$) is obviously independent of $a_2 \equiv w_2$ (while $w_2 = \sigma_1 W_2$) so we can set $a_2 = 0$ to compare them. As we saw,

$$w_k = \sigma_{k-1} W_k \qquad 2 \leqslant k \leqslant 5 , \tag{2.33}$$

whereas for $k \geqslant 6$ there are additional terms on the r.h.s.. Direct computation shows for instance that for $n = 6$

$$w_6 = 5!^2\left(W_6 + \tfrac{1}{9} W_3^2\right) \tag{2.34}$$

while for $n = 7$

$$w_6 = 12 \times 5!^2\left(W_6 + \tfrac{5}{24} W_3^2\right),$$

$$w_7 = 6!^2\left(W_7 + \tfrac{1}{36} W_3 W_4\right), \tag{2.35}$$

and for $n = 8$

$$w_6 = 78 \times 5!^2\left(W_6 + \tfrac{191}{1950} W_3^2\right),$$

$$w_7 = 2 \times 10!\left(W_7 + \frac{19}{5!} W_3 W_4\right),$$

$$w_8 = 7!^2\left(W_8 + \left(\frac{2}{5!}\right)^2\left(6W_3 W_3'' - 7W_3'^2\right) + \tfrac{2}{15} W_3 W_5 + \left(\tfrac{1}{4} W_4\right)^2\right), \tag{2.36}$$

and so forth. These formulas are clearly invertible and it could well be that the Poisson structure expressed in this basis takes a more transparent form. The maximal embedding of sl(2) in other simple Lie algebras is relevant in the construction of the corresponding W-algebras (see ref. [11]).

Proof of the formula for σ_k. Observe that by construction

$$\sigma_k = \mathrm{tr}\left(J_-^k J_+^k\right), \tag{2.37}$$

in the representation of spin j. We set $n=2j+1$ and define $\sigma_0=n$. Introduce the generating function

$$\sigma(\beta)=\sum_{k=0}^{\infty}\frac{\beta^{2k}}{k!^2}\sigma_k=\operatorname{tr}\mathrm{e}^{\beta J_-}\mathrm{e}^{\beta J_+}=\operatorname{tr}\mathrm{e}^{2i\theta J_0}=\frac{\sin n\theta}{\sin\theta}, \tag{2.38}$$

where $2\cos\theta=2+\beta^2$. This last expression is obtained by using the structure of the Lie algebra sl(2) or equivalently by looking at the 2×2 representation

$$\mathrm{e}^{\beta J_-}\mathrm{e}^{\beta J_+}\to\begin{pmatrix}1 & \beta\\ \beta & 1+\beta^2\end{pmatrix}. \tag{2.39}$$

Now $\sin n\theta/\sin\theta$, the character of the spin-j representation, is a polynomial of degree $n-1$ in $\cos\theta$ and in view of the addition of angular momenta fulfills

$$2\cos\theta\frac{\sin n\theta}{\sin\theta}=\frac{\sin(n+1)\theta}{\sin\theta}+\frac{\sin(n-1)\theta}{\sin\theta}.$$

From this relation we construct again a generating function

$$\begin{aligned}\varphi\left(t,\cos\theta=1+\tfrac{1}{2}\beta^2\right)&=\sum_{n=1}^{\infty}t^{n-1}\frac{\sin n\theta}{\sin\theta}\\ &=\frac{1}{1-2t\cos\theta+t^2}=\frac{1}{(1-t)^2-\beta^2}\\ &=\sum_{n=1}^{\infty}t^{n-1}\oint\frac{\mathrm{d}u}{2i\pi u^n}\sum_{k=0}^{\infty}\frac{\beta^{2k}u^k}{(1-u)^{2(k+1)}},\end{aligned} \tag{2.40}$$

so that comparing with the definition of σ_k

$$\begin{aligned}\sigma_k&=k!^2\oint\frac{\mathrm{d}u}{2i\pi u^{n-k}(1-u)^{2(k+1)}}\\ &=k!^2\oint\frac{\mathrm{d}u}{2i\pi u^{n-k}}\sum_{r\geqslant 0}u^r\binom{r+2k+1}{2k+1}\\ &=k!^2\binom{n+k}{2k+1},\end{aligned} \tag{2.41}$$

as claimed.

An alternative proof is obtained by noting that the desired identity (2.23) is a particular case of a more general combinatorial identity

$$\sum_{l-p\leqslant i\leqslant n-k} \binom{n-i}{k}\binom{i+p}{l} = \binom{n+p+1}{k+l+1}, \tag{2.42}$$

which is readily proved by induction on p.

Projective connection and Gordan's "transvectant" formula★. From the work of Janson and Peetre [13] one realizes that the construction of our original w-basis for the ring R as well as the various differential polynomials in the w's giving again an element in R are parts of the same construction described as follows. Associate to $a_2(x)$ a projective connection b such that $a_2(x) = \frac{1}{12}n(n^2-1)\alpha(x)$ where the quantity $\alpha = b' - \frac{1}{2}b^2$ transforms as a 2-differential with an added term equal to the schwarzian derivative. This enables one to define a covariant differential

$$\mathrm{D}: \quad f \in \mathscr{F}_\lambda \to \mathrm{D}f \in \mathscr{F}_{\lambda+1}, \tag{2.43}$$

through $\mathrm{D}f = (\mathrm{d} - \lambda b)f$. For $\lambda = (1-n)/2$ the operator $\Delta_2(a_2)$ is then given by $\Delta_2(a_2)f = \mathrm{D}^n f$. With the notation

$$(r)_l = \Gamma(r+l)/\Gamma(r) = r(r+1)\dots(r+l-1), \tag{2.44}$$

for the Pochammer symbol, Gordan's "transvectant" [14] is then a map $\mathscr{F}_\lambda \otimes \mathscr{F}_\mu \to \mathscr{F}_{\lambda+\mu+N}$ such that if $f \in \mathscr{F}_\lambda$, $g \in \mathscr{F}_\mu$, then

$$F = \sum_{l=0}^{N} (-1)^l \binom{N}{l} \frac{\mathrm{D}^{N-l}f}{(2\lambda)_{N-l}} \frac{\mathrm{D}^l g}{(2\mu)_l}, \tag{2.45}$$

belongs to $\mathscr{F}_{\lambda+\mu+N}$ and depends on b only through the combination α. This is again proved by using the fact that under a variation δb which leaves α (hence $w_2 = a_2$) invariant, one has $(\mathrm{d} - \lambda b)\delta b = \delta b(\mathrm{d} - (\lambda - 1)b)$. Setting $\lambda = (1-n)/2$, $\mu = k$ a positive integer and $N = n-k$, we let $g \to w_k$ and normalize the leading derivative of f to unity to obtain the other terms of our decomposition of the operator Q_n in the form

$$\Delta_k(a_2, w_k)f \equiv \sum_{l=0}^{n-k} \frac{\binom{k+l-1}{l}\binom{n-k}{l}}{\binom{2k+l-1}{l}} (\mathrm{D}^l w_k)\mathrm{D}^{n-k-l}f, \tag{2.46}$$

as claimed in eq. (2.32). Similarly, given $w_k, w_l \in \mathrm{R}$ (k, l positive integers), we

★ The "transvectant" in Gordan's work is named "Überschiebung".

obtain for each $N \geqslant 0$ a map from $R \times R \to R$ giving a $k+l+N$-differential. For instance for $N=1$ it reduces to an example given in the text. It is interesting to note that in the context of automorphic forms Gordan's formula was rediscovered by Cohen [15] when restricted to PSL_2 transformations, a case where b vanishes.

Remark. As an alternative proof of proposition 2.1, one may consider the operator

$$\check{Q}_n = -J_- + (\mathrm{d} - bJ_0) + \sum_{k=2}^{n-1} W_{k+1} J_+^k \,, \tag{2.47}$$

where the W_k's are k-differentials. Define the n-dimensional vectors $\check{f} = (\check{f}_j, \ldots, \check{f}_{-j})^{\mathrm{T}}$ and $\check{F} = (\check{F}, 0, \ldots, 0)^{\mathrm{T}}$, where $\check{f}_{-j}$ is a $-j$-differential, and the other $\check{f}_k$ are determined by the relation $\check{F} = \check{Q}_n \check{f}$. Then $\check{f}_k$ is a k-differential, and $\mathrm{d} - bJ_0$ acts as covariant derivative on $\check{f}$, so that $\check{F}$ is a $j+1$-differential. The operator $\hat{Q}_n$ of eq. (2.24) is then obtained from $\check{Q}_n$ by a gauge transformation

$$\hat{Q}_n = \mathrm{e}^{-(b/2)J_+} \check{Q}_n{}^{(b/2)J_+} \,, \tag{2.48}$$

provided $W_2 = b'/2 - b^2/4$. This gauge transformation does not affect the components $\check{f}_{-j}$ and $\check{F}$ and makes thus Q_n a covariant operator from $\mathscr{F}_{(1-n)/2}$ to $\mathscr{F}_{(1+n)/2}$.

3. A subfamily of singular vectors in Virasoro modules

The graded Virasoro algebra is generated by L_m, $m \in \mathbb{Z}$, and a central element (grade 0), such that

$$[L_m, L_n] = (m-n) L_{m+n} + \tfrac{1}{12} c m (m^2 - 1) \delta_{m+n,0} \,, \tag{3.1}$$

and c is understood as multiplying the identity operator in any representation (or module) of interest.

Consider the subalgebra B generated by L_m, $m \geqslant 0$. Highest weight representations are characterized by a unique cyclic vector f (up to multiplication by a scalar) which carries a one-dimensional representation of B in the form (h is the conformal weight)

$$L_0 f = hf \,, \qquad L_m f = 0 \,, \qquad m > 0 \,. \tag{3.2}$$

Verma modules are graded highest weight representations $\mathrm{V}(c, h)$ freely generated by elements

$$f_{p_1, p_2, \ldots, p_k} = L_{-p_1} L_{-p_2} \ldots L_{-p_k} f \,, \qquad 1 \leqslant p_1 \leqslant p_2 \leqslant \ldots \leqslant p_k \,. \tag{3.3}$$

Finally singular vectors at level n arise from injective maps $V(c, h+n) \to V(c, h)$ where the image of the highest vector in $V(c, h+n)$ takes the form

$$F = \sum_{p_1+p_2+\ldots=n} C_{p_1, p_2, \ldots} f_{p_1, p_2, \ldots} = \phi(L_{-1}, L_{-2}, \ldots) f \tag{3.4}$$

satisfying therefore

$$L_0 F = (h+n) F, \qquad L_m F = 0 \qquad m > 0. \tag{3.5}$$

Parametrize c, h and the positive integer n as follows, with j and j' integers or half-integers

$$\begin{aligned} c &= 1 + 6(\theta + \theta^{-1})^2, \\ h &= -(j\theta + j'\theta^{-1})((j+1)\theta + (j'+1)\theta^{-1}), \\ n &= (2j+1)(2j'+1), \end{aligned} \tag{3.6}$$

and θ is any complex number, the values 0 and ∞ being included in a limiting sense (for $j' = 0$ and $j = 0$ respectively). Then a form of the Kac determinant formula is

Proposition 3.1. (Kac [1], Feigin and Fuchs [2, 16]). *For c, h, n as above, $V(c, h)$ admits a non-zero singular vector at level n and conversely, all singular vectors are obtained in this way.*

We emphasized above a notation involving spins j and j'. In terms of more familiar notations, with m a complex number

$$\begin{aligned} &r = 2j' + 1, \qquad s = 2j + 1, \\ &t = \theta^2 = -\frac{m}{m+1}, \qquad c = 1 - \frac{6}{m(m+1)}, \\ &h = \frac{(r(m+1) - sm)^2 - 1}{4m(m+1)}, \qquad n = rs. \end{aligned} \tag{3.7}$$

Since the early investigations on the representation theory of the Virasoro algebra it has been a problem to give an explicit expression of singular vectors, namely of the polynomial $\phi(L_{-1}, L_{-2}, \ldots)$, normalized by $\phi = L_{-1}^n + \ldots$. For small values of j, j', one can obtain such expressions by brute force. The goal is however to obtain universal formulas.

Let us first describe three general properties of singular vectors quoted by Feigin and Fuchs [16] with reference to work by Lutsyuk [17].

(i) The operator $\phi(L)$ is a finite Laurent series in the variable $t=\theta^2$. Then, as t tends to 0 or ∞,

$$t\to 0,\qquad \phi_{j',j}(L)\sim\left(L_{-(2j'+1)}\right)^{2j+1},$$
$$t\to\infty,\qquad \phi_{j',j}(L)\sim\left(L_{-(2j+1)}\right)^{2j'+1}. \tag{3.8}$$

A more precise form is $t\to 0$, $\phi_{j',j}(L)\sim(\phi_{j',0}(L))^{2j+1}$, $t\to\infty$, $\phi_{j',j}(L)\sim(\phi_{0,j}(L))^{2j'+1}$. Moreover (with the normalization $\phi=L_{-1}^{n}+\dots$) Feigin and Fuchs add the *conjecture* that under the transformation $t\to -t$, (hence $c\to 26-c$), $L_{-k}\to(-1)^{k-1}L_{-k}$, $\phi(L,t)$ remains invariant.

(ii) For each positive integer k the subalgebra generated by $L_{-k},L_{-k-1},\dots$ is locally nilpotent. Let U_{-k} denote its enveloping algebra. We have a chain of inclusions $\mathrm{U}_{-1}\supset\mathrm{U}_{-2}\supset\mathrm{U}_{-3}\dots$. It then makes sense to look at $\mathrm{U}_{-1}\bmod\mathrm{U}_{-2}$ freely generated by L_{-1}, $\mathrm{U}_{-1}\bmod\mathrm{U}_{-3}$ freely generated by L_{-1},L_{-2} considered as commuting, $\mathrm{U}_{-1}\bmod\mathrm{U}_{-4}$ isomorphic to the enveloping algebra of the Heisenberg Lie algebra, etc... (it would be interesting to analyze $\mathrm{U}_{-1}\bmod\mathrm{U}_{-k}$ in general). Obviously $\phi_{j',j}(L)\bmod\mathrm{U}_{-2}$ is L_{-1}^{n}. In $\mathrm{U}_{-1}\bmod\mathrm{U}_{-3}$ let $\sigma_{j'j}(L_{-1},L_{-2})$ denote the projection of $\phi_{j',j}$ (with $[L_{-1},L_{-2}]\sim 0$). Then

$$\sigma_{j'j}^2(L_{-1},L_{-2})=\prod_{\substack{-j\leqslant M\leqslant j\\ -j'\leqslant M'\leqslant j'}}\left(L_{-1}^2+4(M\theta+M'\theta^{-1})^2L_{-2}\right). \tag{3.9}$$

Of course M and M' range respectively over $-j,-j+1,\dots,j$ and $-j',-j'+1,\dots,j'$. If the pair $(M,M')=(0,0)$ is allowed it corresponds to a square. Otherwise (M,M') and $(-M,-M')$ give identical factors. Hence the right-hand side is a perfect square. Rational theories correspond to values of θ such that there exist two coprime integers p and q satisfying

$$\theta q+\theta^{-1}p=0,\qquad c=1-6\frac{(p-q)^2}{pq}. \tag{3.10}$$

In the above formulas this may lead to the vanishing of some coefficients of L_{-2}.

(iii) The Virasoro algebra is a central extension of the algebra of vector fields on a circle ($c=0$), the Witt algebra, which admits representations on an infinite-dimensional vector space spanned by ψ_p, $p\in\mathbb{Z}$ of the form (λ,μ arbitrary)

$$l_{-k}\psi_p=[\mu+p-\lambda(k-1)]\psi_{p+k}. \tag{3.11}$$

For instance, on $\psi_p = 1/z^{p+\mu}$, l_{-k} is represented by $-z^{-k}(z\,\mathrm{d}/\mathrm{d}z + \lambda(k-1))$. Under $L_{-k} \to l_{-k}$ we have

$$\phi_{j',j}(l_{-1}, l_{-2}, \dots)\psi_0 = \varphi_{j',j}(\lambda, \mu)\psi_n . \tag{3.12}$$

Set

$$A(M, M') = \left[(j+M)\theta + (j'+M')\theta^{-1}\right]\left[(j+1-M)\theta + (j'+1-M')\theta^{-1}\right], \tag{3.13}$$

then

$$\varphi^2_{j',j}(\lambda, \mu) = \prod_{\substack{-j \leqslant M \leqslant j \\ -j' \leqslant M' \leqslant j'}} \left[(\mu + A(M, M'))(\mu + A(-M, -M')) - 4\lambda(M\theta + M'\theta^{-1})^2\right]. \tag{3.14}$$

Again the right-hand side is a perfect square.

These are, as far as we can say, all the known properties of singular vectors in the general case. Benoit and Saint-Aubin [8], however, have found an explicit form in the case where j' or j vanishes. This is the one we are about to exhibit here in a compact form which bears some striking resemblance to our formulation of the W-basis discussed in sect. 2.

Such a resemblance does not come as a complete surprise in view of the following observation. In the limit where in eqs. (3.7) $m \to 0$, hence $c \to \infty$, ("classical limit" [18]), the only values of h which remain finite correspond to $r = 2j' + 1 = 1$. Letting $n = s$, this limiting value is $(1-n)/2$, which is the value of the conformal weights of functions f discussed in sect. 2*. In this limit, it is suggested that the covariant operator ϕ mapping highest weight vectors f on their singular vector F should reduce to a covariant differential operator Q_n mapping $\mathscr{F}_{(1-n)/2}$ onto $\mathscr{F}_{(1+n)/2}$. The matching is provided by the following identification:

$$L_{-1} \to \mathrm{d},$$
$$-mL_{-k} \to \frac{a_2^{(k-2)}}{(k-2)!\sigma_1}, \qquad k \geqslant 2, \tag{3.15}$$

where $\sigma_1 = n(n^2-1)/6$ is the coefficient introduced in sect. 2. The form of the operator ϕ is thus known in this limit: $\phi = \Delta_2^{(n)}(a_2)$ and this suggests to recast the form of ϕ using the matrix formalism of sect. 2. Unexpectedly, but very fortunately, this matrix turns out to embody the whole form of ϕ_{0j} – beyond the limit $m \to 0$ – and enables one to reproduce the results of ref. [8].

* Letting $m \to -1$ would lead to similar conclusions, with the role of r and s interchanged.

Let us concentrate on the case $r = 1$, $n = s = 2j + 1$ as the other one is trivially related to it. So let once more

$$c = 13 + 6(t + t^{-1}),$$

$$h = -j - tj(j+1),$$

$$n = 2j + 1, \tag{3.16}$$

where $j = \frac{1}{2}, 1, \frac{3}{2}, \dots$. In the Verma module $V(c,h)$ we introduce a sequence of elements denoted

$$f = f_{-j}, f_{-j+1}, \dots, f_j, f_{j+1} = F, \tag{3.17}$$

where f_{-j} is the highest weight state f, f_{j+1} is the singular vector F and f_M ($j + M$ a non-negative integer) satisfies

$$L_0 f_M = (h + j + M) f_M. \tag{3.18}$$

We define the n-dimensional vectors

$$\boldsymbol{f} = (f_j, f_{j-1}, \dots, f_{-j})^{\mathrm{T}},$$

$$\boldsymbol{F} = (F, 0, \dots, 0)^{\mathrm{T}}. \tag{3.19}$$

We also use the notation $J_{\pm}$ introduced in sect. 2.

Proposition 3.2.
(i) *The equation*

$$\boldsymbol{F} = \left(-J_- + \sum_{k=0}^{n-1} L_{-k-1} (tJ_+)^k \right) \boldsymbol{f} \tag{3.20}$$

defines $F = f_{j+1}$ as a non-vanishing singular state at level n in $V(c,h)$, i.e.

$$p > 0, \qquad L_p f_{j+1} = 0, \qquad L_0 f_{j+1} = (h+n) f_{j+1}. \tag{3.21}$$

(ii) *Moreover,*

$$p > 0, \qquad L_p \boldsymbol{f} = \left[\left(J_0 - \frac{3p+1}{2} \right) - t^{-1} \frac{3p+1}{4} \right] (tJ_+)^p \boldsymbol{f}. \tag{3.22}$$

Before turning to the proof let us make a few remarks.

(a) The case $r = 2j' + 1$, $s = 1$ is simply obtained by changing $j \to j'$, $t \to t^{-1}$.

(b) Comparing with the classical case in sect. 2 we find a similar operator with

$$\mathrm{d} \to L_{-1}, \qquad W_{k+1} \to t^k L_{-k-1} = t^k \oint \frac{\mathrm{d}z}{2i\pi} z^{-k} T(z), \tag{3.23}$$

where

$$T(z) = \sum_{p \in \mathbb{Z}} \frac{L_p}{z^{p+2}} \tag{3.24}$$

is the energy–momentum tensor. Thus the operator $f \to F$ reads

$$-J_- + \sum_{k=0}^{n-1} \oint \frac{\mathrm{d}z}{2i\pi} T(z) \left(tz^{-1}J_+\right)^k = -J_- + \oint \frac{\mathrm{d}z}{2i\pi} \frac{zT(z)}{(z - tJ_+)}. \tag{3.25}$$

It also follows from the above that if we express the differential operator $Q_n(W_2, \ldots, W_n)$ of sect. 2, without ever commuting d and the W's (i.e. without derivatives of W's), at the price of having W's inserted between d's, nor commuting W's among themselves, an equivalent form of the relation between F and f is

$$F = Q_n\left(\mathrm{d} \to L_{-1}, W_k \to t^{k-1} L_{-k}\right) f. \tag{3.26}$$

(c) Eliminating all components f_j, $-j+1 \leqslant k \leqslant j$ one finds the explicit form of the operator ϕ: $f \equiv f_{-j} \to F$

$$\phi = \sum_{\substack{\text{partitions of } n \\ n = p_1 + \ldots + p_r,\ p_i \geqslant 1}} t^{n-r} \frac{(n-1)!^2}{\prod_{i=1}^{r-1}(p_1 + \ldots + p_i)(n - p_1 - \ldots - p_i)} L_{-p_1} \ldots L_{-p_r}, \tag{3.27}$$

which is the expression given by Benoit and Saint-Aubin [8].

We turn to the proof of the proposition and start with part (ii), since, as it will emerge, it entails part (i). We note that the action of L_p on f is in agreement with the commutation relations $(p, q > 0)$

$$\left[L_p, L_q\right] = (p - q) L_{p+q},$$

since

$$[L_p, L_q]f = \left[\left(J_0 - \frac{3q+1}{2}(1+\tfrac{1}{2}t^{-1})\right)t^q J_+^q, \left(J_0 - \frac{3p+1}{2}(1+\tfrac{1}{2}t^{-1})\right)t^p J_+^p\right]f$$

$$= (p-q)\left(J_0 - \frac{3p+3q+1}{2}(1+\tfrac{1}{2}t^{-1})\right)t^{p+q}J_+^{p+q}f$$

$$= (p-q)L_{p+q}f. \tag{3.28}$$

Hence it is enough to prove it for L_1 and L_2. For L_1 it reads component-wise

$$L_1 f_M = (t(M-2)-1)(j+M)(j+1-M)f_{M-1}. \tag{3.29}$$

This includes the case $M=-j$ where the coefficient on the r.h.s. vanishes and

$$L_1 f_{-j} \equiv L_1 f = 0 \tag{3.30}$$

by hypothesis, and if we extend it to $M=j+1$ it will also imply $L_1 f_{j+1} \equiv L_1 F = 0$. Since $f_{-j+1} = L_{-1}f_{-j}$,

$$L_1 f_{-j+1} = 2L_0 f_{-j} = -2j(1+t(j+1))f_{-j}, \tag{3.31}$$

which agrees with the above formula for $M=-j+1$. We can therefore assume the formula valid for $M' \leqslant M \leqslant j$ and prove it for $M+1$. By construction

$$f_{M+1} = \sum_{k=0}^{j+M} L_{-1-k} f_{M-k} t^k (J_+^k)_{M,M-k}, \tag{3.32}$$

while from the recursion hypothesis

$$M' \leqslant M, \qquad L_1 f_{M'} = (t(M'-2)-1)(J_+)_{M',M'-1} f_{M'-1}. \tag{3.33}$$

Using for $k \geqslant 0$ $L_1 L_{-1-k} = (2+k)L_{-k} + L_{-1-k}L_1$ we find

$$L_1 f_{M+1} = \sum_{k=0}^{j+M} (2+k)L_{-k} f_{M-k} t^k (J_+^k)_{M,M-k}$$

$$+ \sum_{k=0}^{j+M-1} (t(M-k-2)-1)L_{-1-k} f_{M-k} t^k (J_+^{k+1})_{M,M-k-1}, \tag{3.34}$$

where we use the structure of the matrix J_+ to identify

$$\left(J_+^k\right)_{M,M-k}\left(J_+\right)_{M-k,M-k-1}=\left(J_+^{k+1}\right)_{M,M-k-1}. \tag{3.35}$$

Hence splitting the first sum into the contribution $k=0$, $k>0$ and adding the second term we get

$$L_1 f_{M+1}=2L_0 f_M+\sum_{k=1}^{j+M}\left(t(M+1)-1\right)L_{-k}f_{M-k}t^{k-1}\left(J_+^k\right)_{M,M-k}. \tag{3.36}$$

Recognizing that

$$f_M=\sum_{k=1}^{j+M}L_{-k}f_{M-k}\left(J_+^{k-1}\right)_{M-1,M-k}, \tag{3.37}$$

and using the same property of J_+ we find

$$L_1 f_{M+1}=\left\{2L_0+\left(t(M+1)-1\right)\left(J_+\right)_{M,M-1}\right\}f_M. \tag{3.38}$$

Explicitly the bracket reads

$$\begin{aligned}&2M-2tj(j+1)+\left(t(M+1)-1\right)\left((j-M+1)(j+M)\right)\\&\quad=\left(t(M-1)-1\right)(j-M)(j+M+1),\end{aligned} \tag{3.39}$$

i.e. the formula is valid for $M+1$ if it is valid up to M.

A similar argument applies to the action of L_2 which is supposed to read component-wise:

$$L_2 f_M=\left[t^2\left(M-\tfrac{7}{2}\right)-\tfrac{7}{4}t\right]\left(j(j+1)-M(M-1)\right)\left(j(j+1)-(M-1)(M-2)\right)f_{M-2}. \tag{3.40}$$

This includes the cases $L_2 f_{-j}=L_2 f_{-j+1}=0$ valid by hypothesis. Again let us assume the formula true for $M'\leqslant M$ and let us establish it for $M+1$. We use for $k\geqslant 0$

$$L_2 L_{-1-k}=(3+k)L_{1-k}+L_{-1-k}L_2+\tfrac{1}{2}\delta_{k,1}\left(13+6\left(t+t^{-1}\right)\right), \tag{3.41}$$

and start from

$$f_{M+1}=L_{-1}f_M+L_2 f_{M-1}t\left(J_+\right)_{M,M-1}+\sum_{k=2}^{j+M}L_{-1-k}f_{M-k}t^k\left(J_+^k\right)_{M,M-k}. \tag{3.42}$$

From the recursion hypothesis,

$$L_2 f_{M+1} = 3L_1 f_M + t\left[4L_0 + \tfrac{1}{2}(13 + 6(t + t^{-1}))\right] f_{M-1}(J_+)_{M,M-1}$$
$$+ \sum_{k=2}^{j+M} (3+k) L_{1-k} f_{M-k} t^k (J_+)_{M,M-k}$$
$$+ \sum_{k=0}^{j+M-2} L_{-1-k} f_{M-k-2}\left(t^2\left(M - k - \tfrac{7}{2}\right) - \tfrac{7}{4}t\right) t^k \left(J_+^{k+2}\right)_{M,M-k-2}. \quad (3.43)$$

We take into account the expression for $L_1 f_M$ obtained above to rewrite

$$L_2 f_{M+1} = \left[3(t(M-2) - 1) + 4t(M-1) - 4t^2 j(j+1) + \tfrac{13}{2}t + 3t^2 + 3\right]$$
$$\times (j(j+1) - M(M-1)) f_{M-1}$$
$$+ \sum_{k=0}^{j+m-2} \left(t^2(5 + M - \tfrac{7}{2} - \tfrac{7}{4}t\right) L_{-1-k} f_{M-2-k} t^k \left(J_+^{k+2}\right)_{M,M-2-k}. \quad (3.44)$$

Comparing with

$$f_{M-1} = \sum_{k=0}^{j+M-2} L_{-1-k} f_{M-2-k} t^k (J_+)_{M-2,M-2-k}, \quad (3.45)$$

we get

$$L_{-2} f_{M+1} = A f_{M-1}, \quad (3.46)$$

where the factor A is

$$A = t(j(j+1) - M(M-1))\{3(M-2) + 4(M-1) - 4tj(j+1)$$
$$+ \tfrac{13}{2} + 3t + j(j+1) - (M-1)(M-2)\}\left(t\left(M + \tfrac{3}{2}\right) - \tfrac{7}{4}\right), \quad (3.47)$$

and after some rearrangement

$$A = \left(t^2\left(M - \tfrac{5}{2}\right) - \tfrac{7}{4}\right)(j(j+1) - M(M+1))(j(j+1) - M(M-1)). \quad (3.48)$$

This proves the formula for $L_2 f_{M+1}$. As above the induction holds up to $M = j$ in which case we obtain

$$L_2 f_{j+1} \equiv L_2 F = 0, \quad (3.49)$$

thus completing the proofs of parts (i) and (ii) of the proposition. Obviously $F \neq 0$ since the coefficient of L^n_{-1} is 1. Let us conclude by a few observations.

(a) If we rescale the components of f through

$$f_M = t^{j+M} \tilde{f}_M \left(J_+^{M+j}\right)_{M,-j}, \qquad -j \leqslant M \leqslant j, \tag{3.50}$$

the "descent" equation reads for $p \geqslant 1$ and $-j \leqslant M \leqslant j$

$$L_p \tilde{f}_M = \left(M - \frac{3p+1}{2}\left(1 + \tfrac{1}{2}t^{-1}\right)\right) \tilde{f}_{M-p}, \tag{3.51}$$

to be compared with the representation of the Witt algebra recalled above in eq. (3.11),

$$l_p \psi_r = (\mu + r + \lambda(p+1)) \psi_{r-p}, \qquad r, p \in \mathbb{Z}. \tag{3.52}$$

Here the grading is through integers shifted by μ, but an integral shift allows us to take μ mod 1. Thus the action of L_p on $\tilde{f}$'s, restricted to $p \geqslant 1$, can be identified with the one of l_p, provided

$$\lambda = -\tfrac{3}{2}\left(1 + \tfrac{1}{2}t^{-1}\right), \qquad \mu = \left(j + 1 + \tfrac{1}{2}t^{-1}\right) \bmod 1, \tag{3.53}$$

and of course these formulas hold with $t \to t^{-1}$ for the other series of singular vectors $r = 2j' + 1$, $s = 1$. A natural explanation for the origin of these descent equations will be provided in sect. 5.

(b) In the case $r = 1$, $s = 2j + 1$, we see that as $t \to 0$ or ∞ the singular vector

$$F = \phi(L) f \tag{3.54}$$

behaves as

$$t \to 0, \qquad \phi(L) \sim (L_{-1})^{2j+1},$$

$$t \to \infty, \qquad \phi(L) \sim t^{2j} L_{2j+1},$$

as expected (with $t \leftrightarrow t^{-1}$, $0 \leftrightarrow \infty$ and $j \leftrightarrow j'$ in the case $r = 2j' + 1$, $s = 1$). Moreover the conjecture of Feigin and Fuchs is verified: in both cases, $\phi(L)$ is invariant under $t \to t^{-1}$, $L_{-k} \to (-1)^{k-1} L_{-k}$.

(c) Let us look at $\phi(L)$ mod L_{-3}, again for $r = 1$, $s = 2j + 1$, assuming therefore

$$[L_{-1}, L_{-2}] \sim 0.$$

The projection of $\phi_{0,j}$ reads

$$\begin{aligned}\sigma_{0,j}(L) &= \det(-J_- + L_{-1} + tL_{-2}J_+)\\ &= \det\left(L_{-1} + \sqrt{tL_{-2}}(J_+ - J_-)\right)\\ &= \det\left(L_{-1} + \sqrt{-4tL_{-2}}J_2\right),\end{aligned} \tag{3.55}$$

where we wrote $J_\pm = J_1 \pm iJ_2$. But J_2 has the same spectrum as J_0, hence

$$\sigma_{0,j} = \prod_{-j\leqslant M\leqslant j}\left(L_{-1} + \sqrt{-4tL_{-2}}M\right), \tag{3.56}$$

or, grouping terms in M and $-M$

$$\sigma_{0,j}^2 = \sum_{-j\leqslant M\leqslant j}\left(L_{-1}^2 + 4L_{-2}(\theta M)^2\right), \tag{3.57}$$

while similarly

$$\sigma_{j',0}^2 = \prod_{-j'\leqslant M'\leqslant j'}\left(L_{-1}^2 + 4L_{-2}(\theta^{-1}M')^2\right), \tag{3.58}$$

again as expected.

(d) Finally let us check that the third general property also holds when we substitute for L_{-k} the operator

$$l_{-k}\psi_p = (\mu + p - \lambda(k-1))\psi_{p+k}, \tag{3.59}$$

and act with $\phi(L \to l)$ on ψ_0 obtaining $\varphi_{j',j}(\lambda,\mu)\psi_n$. Then (for $(0,j)$) in the matrix

$$-J_- + \sum_{k=0}^{2j} L_{-1-k}t^kJ_+^k$$

this amounts to replacing L_{-k-1} by $\mu + j + J_0 - \lambda k$ and to computing the determinant of the resulting matrix

$$\begin{aligned}\varphi_{0,j}(\lambda,\mu) &= \det\left(-J_- + \sum_{k=0}^{2j} t^kJ_+^k(\mu + j + J_0 - \lambda k)\right)\\ &= \det\left(-J_- + \frac{1}{1-tJ_+}(\mu + j + J_0) - \lambda\frac{tJ_+}{(1-tJ_+)^2}\right)\\ &= \det\left(-tJ_- + \frac{1}{1-J_+}(\mu + j + J_0) - \frac{\lambda J_+}{(1-J_+)^2}\right),\end{aligned} \tag{3.60}$$

where to obtain the last equality, we have used the automorphism $J_{\pm} \to t^{\mp 1} J_{\pm}$, $J_0 \to J_0$. Since $J_+^n = 0$, the matrix

$$U_\gamma = \frac{1}{(1-J_+)^\gamma} = \sum_{p \geqslant 0} \frac{\gamma(\gamma+1)\dots(\gamma+p-1)}{p!} J_+^p \tag{3.61}$$

is well defined and admits an inverse $U_\gamma^{-1} = U_{-\gamma} = (1-J_+)^\gamma$. From the commutation relations

$$\begin{aligned} \left[J_0, U_\gamma\right] &= \gamma J_+ U_{\gamma+1}, \\ \left[J_-, U_\gamma\right] &= -2\gamma U_{\gamma+1} J_0 - \gamma(\gamma+1) U_{\gamma+2} J_+, \\ \left[J_+, U_\gamma\right] &= 0, \end{aligned} \tag{3.62}$$

one derives that

$$\begin{aligned} &U_\gamma^{-1}\left(-tJ_- + \frac{1}{1-J_+}(\mu + j + J_0) - \frac{\lambda J_+}{(1-J_+)^2}\right)U_\gamma \\ &= -tJ_- + \frac{1}{1-J_+}(\mu + j + (1+2\gamma t)J_0) + \frac{\gamma + t\gamma(\gamma+1) - \lambda}{(1-J_+)^2} J_+, \end{aligned} \tag{3.63}$$

the determinant of which is again equal to $\varphi_{0,j}(\lambda, \mu)$. Choosing for γ any of the two roots of

$$\gamma + t\gamma(\gamma+1) = \lambda, \tag{3.64}$$

it follows that

$$\varphi_{0,j}(\lambda, \mu) = \det\left(-tJ_- + \frac{1}{1-J_+}(\mu + j + (1+2\gamma t))J_0\right). \tag{3.65}$$

Multiplying from the left by $1 = \det(1-J_+)$ and using the fact that $J_+J_- = (j+J_0)(j+1-J_0)$ from the form of the SU(2) Casimir operator, we get $\phi_{0,j}$ as the determinant of a lower triangular matrix

$$\begin{aligned} \varphi_{0,j}(\lambda, \mu) &= \det\left[-tJ_- + t(j(j+1) + J_0(1-J_0)) + \mu + j + (1+2\gamma t)J_0\right] \\ &= \prod_{-j \leqslant M \leqslant j} \left[t(j+M)(j+1-M) + \mu + j + M(1+2\gamma t)\right], \end{aligned} \tag{3.66}$$

which still involves γ. We therefore take the square and group terms pertaining to

M and $-M$ to get after some algebra using eq. (3.64) the desired result

$$\varphi_{0,j}^{2}(\lambda,\mu)=\prod_{-j\leqslant M\leqslant j}\{[\mu+(j+M)(1+(j+1-M)t)]\times[\mu+(j-M)(1+(j+1+M)t)]-4\lambda M^{2}t\} \quad (3.67)$$

independent of the choice of the root in (3.64).

We have thus verified the property (3.14) whenever j or j' vanishes. This is all we shall need in sect. 5 in the construction of the general singular vectors.

4. Fusion

To prepare for the computation of general singular vectors, we describe in this section the fusion of highest weight modules, introduced by Belavin et al. [3] as an appropriate adaptation of the notion of tensor product decomposition.

We fix throughout our discussion the central charge c and attach isomorphic Virasoro algebras to each point of the Riemann sphere and similarly heighest weight modules also parametrized by a point with coordinate x. To the latter we let correspond primary chiral fields $f_h(x)$ and their descendents. We refer to a finite non-decreasing sequence of positive integers $1\leqslant r_1\leqslant r_2\leqslant\ldots\leqslant r_k$ as a Young tableau Y, with $|\mathrm{Y}|=r_1+r_2+\ldots+r_k$ and write

$$L_{-r_1}\ldots L_{-r_k}\equiv L_{-\mathrm{Y}}. \quad (4.1)$$

Let $T(z)$ stand for the energy–momentum tensor. The descendents of $f_h(x)$,

$$L_{-\mathrm{Y}}f_h(x)\Leftrightarrow L_{-\mathrm{Y}}|c,h;x\rangle, \quad (4.2)$$

where it is understood that one operates with the Virasoro algebra at x, are obtained by applying repeatedly the short-distance expansion of T with the field f_h and its descendents. On the primary field, it reads

$$T(\xi)f_h(x)=\sum_{n=0}^{\infty}(\xi-x)^{n-2}L_{-n}f_h(x), \quad (4.3)$$

with

$$L_0f_h(x)=hf_h(x),\qquad L_{-1}f_h(x)=\frac{\mathrm{d}}{\mathrm{d}x}f_h(x), \quad (4.4)$$

and the above formula implies the highest weight property

$$p > 0, \qquad L_p f_h(x) = 0. \tag{4.5}$$

We can consider the fields as operators in a fixed vector space with a unique vacuum state invariant under global (Möbius) transformations, and a unique dual vacuum linear form (invariant under the same group). The second point of view, that we choose here, is to consider only correlation functions, i.e. vacuum expectation values of products of fields at distinct points. The latter are analytic functions of the argument of one of the fields on the universal covering of the Riemann sphere punctured at the arguments of the other fields. Henceforth fields will be understood as insertions into correlation functions, so will the corresponding relations. Only at the end of the process will we abstract the algebraic expressions.

The fusion of two highest weight modules attached respectively to the fields $f_0(x_0)$ and $f_1(x_1)$, where the subscripts imply the weights h_0 and h_1, to a third one relative to the field $f(x)$ (of weight h) will be possible if the corresponding three-point function

$$\langle f_0(x_0) f_1(x_1) f(x) \rangle = \frac{g(h_0, h_1, h)}{(x_0 - x_1)^{h_0 + h_1 - h}(x_0 - x)^{h_0 + h - h_1}(x_1 - x)^{h_1 + h - h_0}} \tag{4.6}$$

is non-vanishing. This implies selection or *fusion rules* on the weights h_0, h_1, h to be discussed below.

Among highest weight modules let us distinguish between Verma modules $\mathrm{V}(c,h)$, and irreducible ones $\mathrm{M}(c,h)$ where $\mathrm{M}(c,h)$ is possibly the quotient of $\mathrm{V}(c,h)$ by a maximal invariant submodule, the latter occurring when $\mathrm{V}(c,h)$ possesses singular vectors.

We define fusion among *irreducible* highest weight modules as a covariant linear map

$$\mathscr{F}\colon \quad \mathrm{M}(c, h_0; x_0) \otimes \mathrm{M}(c, h_1; x_1) \to \mathrm{M}(c, h; x). \tag{4.7}$$

The necessity of considering only irreducible modules will appear shortly to ensure unicity. As for the meaning of covariance, it also requires to be made explicit. Since we deal with three isomorphic but not identical copies of the Virasoro algebra it is not surprising that the product of highest weight states in $\mathrm{M}(c, h_0; x_0) \otimes \mathrm{M}(c, h_1; x_1)$ *does not* correspond to the highest weight state in $\mathrm{M}(c, h; x)$ but rather to an infinite linear combination

$$\mathscr{F}(|c, h_0; x_0\rangle \otimes |c, h_1; x_1\rangle) = \sum_{\mathrm{Y}} \beta_{\mathrm{Y}}(h_0, h_1, h | x_0, x_1, x) L_{-\mathrm{Y}} |c, h; x\rangle. \tag{4.8}$$

Due to the grading of the algebra and the modules, terms at a definite grade are well defined and convergence is understood in this weak sense. Part of the definition of fusion involves the determination of the coefficients β_Y. To obtain them and give a precise description of the correspondence $\mathscr{F}$, a disguised formulation of chiral vertex operators [19], we rely on Wilson's short-distance expansion of products of fields as a heuristic guide.

Let us choose the coordinates in such a way that

$$x_0 = x + \tfrac{1}{2}z, \qquad x_1 = x - \tfrac{1}{2}z. \tag{4.9}$$

This is not a serious restriction since global conformal transformations act transitively on triplets of points. As $z \to 0$ the product of fields $f_0(x_0)f_1(x_1)$ (inserted in correlations) is equivalent to a sum of expansions of the form

$$f_0(x_0)f_1(x_1) \sim \sum_h \left\{ \frac{g(h_0,h_1,h)}{z^{h_0+h_1-h}} \sum_Y z^{|Y|}\beta_Y(h_0,h_1,h) L_{-Y} f_h(x) \right\}. \tag{4.10}$$

As was mentioned above the choice of the mid-point $x = \frac{1}{2}(x_0 + x_1)$ is by no means mandatory and could well be modified using

$$f_h(x+x') = \exp(x' L_{-1}) f_h(x), \tag{4.11}$$

without changing the structure of the above expansion. It has however, the virtue that the coefficients satisfy the symmetry property

$$\beta_Y(h_0,h_1,h) = (-1)^{|Y|}\beta_Y(h_1,h_0,h), \tag{4.12}$$

disregarding a global phase arising from the prefactor $z^{h_0+h_1-h}$, conveniently absorbed in the normalization.

The covariance of $\mathscr{F}$ which will allow us to compute the β's consists in the following requirement. Consider a holomorphic univalent map $x \to \tilde{x} = g(x)$ in a common neighborhood U of x_0, x_1, x giving a one-to-one map $U \leftrightarrow g(U)$. We require that Wilson's expansion be such that both sides have identical transformation properties. Namely if

$$f_i(x_i)\,\mathrm{d}x_i^{h_i} = \tilde{f}_i(\tilde{x}_i)\,\mathrm{d}\tilde{x}_i^{h_i} \tag{4.13}$$

we impose that the coefficients β_Y appearing in the expansion of $f_0(x_0)/f_1(x_1)$ in terms of the descendents of $f((x_0+x_1)/2)$ *be the same* as those appearing in the corresponding quantities with tildes. The univalency of the map g is a weak requirement since we only need a pair of points $x_0, x_1 \to x$, so U can be shrunk at will and the condition reduces to $g' \neq 0$ at the common limiting point. We will see below how one applies this principle in practice to compute the coefficients β. We can thus define the map $\mathscr{F}$ on $\mathrm{M}(c, h_0; x_0) \otimes \mathrm{M}(c, h_1; x_1)$ by the rules

$$\mathscr{F}|c, h_0; x_0\rangle \otimes |c, h_1; x_1\rangle = \frac{1}{z^{h_0+h_1-h}} \sum_{\mathrm{Y}} z^{|\mathrm{Y}|} \beta_{\mathrm{Y}} L_{-\mathrm{Y}} |c, h; x\rangle, \tag{4.14}$$

$$\mathscr{F}(L_{-p} \otimes \mathbf{1}) = \frac{(-1)^p}{z^p}\left[h_1(p-1) + \frac{z}{2}L_{-1} - z\frac{\mathrm{d}}{\mathrm{d}z}\right] + \sum_{k \geqslant 0} \left(\frac{z}{2}\right)^k \binom{k+p-2}{k} L_{-p-k}, \tag{4.15a}$$

$$\mathscr{F}(\mathbf{1} \otimes L_{-p}) = \frac{1}{z^p}\left[h_0(p-1) - \frac{z}{2}L_{-1} - z\frac{\mathrm{d}}{\mathrm{d}z}\right] + \sum_{k \geqslant 0} \left(\frac{-z}{2}\right)^k \binom{k+p-2}{k} L_{-p-k}. \tag{4.15b}$$

Remarks

(a) For $p = 1$ in both (4.15a, b) the last sums reduce to L_{-1} (formally $\binom{-1}{0} = 1$) so that they read

$$\mathscr{F}(L_{-1} \otimes \mathbf{1}) = \frac{\mathrm{d}}{\mathrm{d}z} + \frac{L_{-1}}{2},$$

$$\mathscr{F}(\mathbf{1} \otimes L_{-1}) = -\frac{\mathrm{d}}{\mathrm{d}z} + \frac{L_{-1}}{2}. \tag{4.16}$$

(b) The expressions in (4.15) follow from the Ward identities [3]* for the insertions of energy-momentum tensors $T(\xi_1) \dots T(\xi_k)$ into correlation functions, in combinations with short-distance expansions for products of the form $T(\xi) f_i(x_i)$

$$\left\langle \left(L_{-p_1} \cdots L_{-p_r} f_0(x_0)\right) f_1(x_1) \dots \right\rangle = \mathscr{L}_{-p_1}(x_0) \dots \mathscr{L}_{-p_r}(x_0) \langle f_0(x_0) f_1(x_1) \dots \rangle, \tag{4.17}$$

where the differential operator $\mathscr{L}_{-p}$ reads

$$\mathscr{L}_{-p}(x_0) = \sum_{i \geqslant 1} \left(\frac{(p-1)h_i}{(x_i - x_0)^p} - \frac{1}{(x_i - x_0)^{p-1}} \frac{\partial}{\partial x_i} \right). \tag{4.18}$$

Once we have obtained the expressions for the singular vectors, eqs. (4.17) and (4.18) will enable us to translate them into partial linear differential equations satisfied by correlation functions involving the corresponding primary fields.

* This was also discussed in ref. [20]; the calculation of the coefficients β_{Y} presented in appendix A of that reference was, however, incorrect (but this did not affect the bulk of the paper). The following discussion gives the correct and systematic way to carry out these calculations.

Each $\mathscr{L}_{-p}$ is expanded for $x_0 \sim x$ in the form

$$\mathscr{L}_{-p}(x_0) = (-1/z)^p\left((p-1)h_1 + \tfrac{1}{2}z\partial_x - z\partial_z\right) + \sum_{k\geqslant 0}\left(\frac{z}{2}\right)^k\binom{p+k-2}{k}$$

$$\times \sum_{i\geqslant 2}(-1)^{p+k}\left[\frac{h_i(p+k-1)}{(x-x_i)^{p+k}} + \frac{1}{(x-x_i)^{p+k-1}}\partial_i\right], \qquad (4.19)$$

and the last sum is identified as L_{-p-k}. This is then iterated to get the general case. For the (locally nilpotent) subalgebra generated by the L_{-p}'s ($p>0$) the map $\mathscr{F}$ does not involve the central charge and $\mathscr{F}$ defines an isomorphism in the sense that

$$[\mathscr{F}(L_{-m}\otimes 1), \mathscr{F}(L_{-n}\otimes 1)] = (m-n)\mathscr{F}(L_{-m-n}\otimes 1) \qquad (4.20)$$

with $m,n>0$, and similarly with $x_0 \leftrightarrow x_1$.

(c) One passes from (4.15a) to (4.15b) by exchanging $h_0 \leftrightarrow h_1$, $z \leftrightarrow -z$, with x left invariant. This justifies our symmetric treatment in the general discussion.

(d) In the coïnciding point limit, i.e. as $z \to 0$, we have

$$\lim_{z\to 0} z^{-h}\mathscr{F}\left(z^{h_0}|c,h_0;x_0\rangle \otimes z^{h_1}|c,h_1;x_1\rangle\right) = |c,h;x\rangle, \qquad (4.21)$$

obtaining therefore the highest weight vector in $\mathrm{M}(c,h;x)$. This is compatible with the transformation properties of the fields since $(x_0-x_1)^h f((x_0+x_1)/2)$ behaves as $\mathrm{d}x^h f(x)$ as $x_0 - x_1 \to 0$. It shows moreover that the leading term in the short-distance expansion of two primary fields (corresponding to highest weight modules) is always a primary field (for the Virasoro algebra), a fact that we will recover and use later on.

A particular case of fusion occurs when two fields are identical. We normalize the two-point function $\langle f(x)f(y)\rangle$ to $(x-y)^{-2h}$. The identity or vacuum sector ($h=0$) occurs in the short-distance expansion of the product $f(x)f(y)$. Since on the Riemann sphere the value $\langle T(\xi)\rangle = 0$ is the only one compatible with the invariance under global conformal transformations, we find from the above, with

$$\mathscr{D}_r = \frac{(-1)^r}{x^r}\left[(r-1)h - x\frac{\mathrm{d}}{\mathrm{d}x}\right], \qquad (4.22)$$

$$\begin{aligned}\left\langle\left(L_{-r_1}\ldots L_{-r_k}f(x)\right)f(x_0)\right\rangle &= \mathscr{D}_{r_1}\ldots\mathscr{D}_{r_k}\frac{1}{x^{2h}}\\ &= \frac{(-1)^{\Sigma r_i}}{x^{2h+\Sigma r_i}}\left[(r_1+1)h + r_2 + \ldots + r_k\right]\\ &\quad\times\left[(r_2+1)h + r_3 + \ldots + r_k\right]\ldots\left[(r_k+1)h\right]. \qquad (4.23)\end{aligned}$$

On the other hand, normalizing the fields in such a way that $g(h_0, h_1, h) = 1$ we have from the three-point function

$$\langle f_0(x_0) f_1(x_1) f(0) \rangle = \frac{1}{z^{h_0+h_1-h}\left(x+\frac{1}{2}z\right)^{h_0+h-h_1}\left(x-\frac{1}{2}z\right)^{h_1+h-h_0}}$$

$$= \frac{1}{z^{h_0+h_1-h}x^{2h}} \sum_{n=0}^{\infty} \left(\frac{z}{2x}\right)^n \sum_{p+q=n} \frac{(-1)^q}{p!q!}$$

$$\times \frac{\Gamma(h_1-h_0-h+1)}{\Gamma(h_1-h_0-h+1-p)} \frac{\Gamma(h_0-h_1-h+1)}{\Gamma(h_0-h_1-h+1-q)}$$

$$= \frac{1}{z^{h_0+h_1-h}} \sum_{\mathrm{Y}} z^{|\mathrm{Y}|} \beta_{\mathrm{Y}} \langle (L_{-\mathrm{Y}} f(x)) f(0) \rangle . \tag{4.24}$$

By comparison we get the sum rules on the coefficients β_{Y},

$$\sum_{\substack{1 \leqslant r_1 \leqslant r_2 \dots \\ r_1+r_2+\dots=n}} \left[(r_1+1)h + \sum_{i\geqslant 2} r_i\right]\left[(r_2+1)h + \sum_{i \geqslant 3} r_i\right] \dots \beta_{r_1, r_2, \dots}$$

$$= \frac{1}{2^n} \sum_{p+q=n} (-1)^p \frac{1}{p!q!} \frac{\Gamma(h_1-h_0-h+1)}{\Gamma(h_1-h_0-h+1-p)} \frac{\Gamma(h_0-h_1-h+1)}{\Gamma(h_0-h_1-h+1-q)} . \tag{4.25}$$

In particular for $|\mathrm{Y}| = 1$ this fixes β_1 through

$$2h\beta_1 = h_0 - h_1 , \tag{4.26}$$

while for $|\mathrm{Y}| = 2$ we get the relation

$$2h(2h+1)\beta_{1,1} + 3h\beta_2 = \frac{h}{4} + \frac{(h_0-h_1)^2}{2} , \tag{4.27}$$

and so on. By examining further sum rules, starting more generally from

$$\langle f_0(x_0) f_1(x_1) (L_{-\mathrm{Y}} f(0)) \rangle ,$$

one would obtain an infinity of linear systems determining the coefficients β_{Y}.

The above method is equivalent to (but less effective than) the one following from the covariance principle. Let us show this explicitly by comparing the transformation properties of both sides of the short-distance expansion. With the notations introduced above we require that

$$\frac{1}{z^{h_0+h_1-h}}\sum_{Y} z^{|Y|}\beta_Y(L_{-Y}f)(x)$$

$$=\left(\frac{d\tilde{x}_0}{dx_0}\right)^{h_0}\left(\frac{d\tilde{x}_1}{dx_1}\right)^{h_1}\frac{1}{(\tilde{x}_0-\tilde{x}_1)^{h_0+h_1-h}}\sum_{Y}(\tilde{x}_0-\tilde{x}_1)^{|Y|}\beta_Y(L_{-Y}\tilde{f})\left(\frac{\tilde{x}_0+\tilde{x}_1}{2}\right), \tag{4.28}$$

where as before $x_0=x+\frac{1}{2}z$, $x_1=x-\frac{1}{2}z$ and

$$(L_{-Y}\tilde{f})\left(\frac{\tilde{x}_0+\tilde{x}_1}{2}\right)=L_{-Y}\exp\left[\left(\frac{\tilde{x}_0+\tilde{x}_1}{2}-\tilde{x}\right)L_{-1}\right]\tilde{f}(\tilde{x}). \tag{4.29}$$

We need here the transformation properties of the descendent fields $(L_{-Y}f)(x)$ with f primary. The easiest way is to apply the above formula in infinitesimal form. Set

$$\tilde{y}=y-\epsilon(y). \tag{4.30}$$

Eq. (4.28) reduces to

$$\left\{h_0\epsilon'(x_0)+\epsilon(x_0)\frac{\partial}{\partial x_0}+h_1\epsilon'(x_1)+\epsilon(x_1)\frac{\partial}{\partial x_1}\right\}\frac{1}{z^{h_0+h_1-h}}\sum_{Y} z^{|Y|}\beta_Y L_{-Y}f(x)$$

$$=\frac{1}{z^{h_0+h_1-h}}\sum_{Y} z^{|Y|}\beta_Y\delta_\epsilon[L_{-Y}f(x)]. \tag{4.31}$$

Choose

$$\epsilon(y)=\epsilon_k(y-x)^{k+1}, \qquad k\geqslant -1, \tag{4.32}$$

so that

$$\delta_\epsilon L_{-Y}f(x)=\epsilon_k L_k L_{-Y}f(x). \tag{4.33}$$

The covariance condition becomes, with $k\geqslant -1$,

$$\left[L_k-\left(\frac{z}{2}\right)^k(k+1)(h_0+(-1)^k h_1)-\left(\frac{z}{2}\right)^{k+1}\left\{\frac{1-(-1)^k}{2}\frac{\partial}{\partial x}+(1+(-1)^k)\frac{\partial}{\partial z}\right\}\right]$$

$$\times\frac{1}{z^{h_0+h_1-h}}\sum_{Y} z^{|Y|}\beta_Y L_{-Y}f(x)=0. \tag{4.34}$$

Since L_1 and L_2 generate by commutators the complete algebra of L_k's, $k \geqslant 1$ (recall that $L_{k+2} = ((-1)^k/k!)(\mathrm{ad}\, L_1)^k L_2)$ it is sufficient to impose the two relations pertaining to $k = 1$ and $k = 2$. For notational simplicity define

$$f^{(p)} = \sum_{|Y|=p} \beta_Y L_{-Y} f, \qquad p \geqslant 0, \tag{4.35}$$

where $f^{(0)}$ is equal to f. The above translates into the conditions

$$L_1 f^{(p)} = (h_0 - h_1) f^{(p-1)} + \tfrac{1}{4} L_{-1} f^{(p-2)}, \tag{4.36a}$$

$$L_2 f^{(p)} = \frac{h + p + 2(h_0 + h_1 - 1)}{4} f^{(p-2)}. \tag{4.36b}$$

It is understood that $f^{(p)}$ vanishes if $p < 0$. As a consequence, we recover the fact that $f^{(0)} \equiv f$ is a highest weight state (or primary field) as already claimed. Let us exemplify on the first few values of p how these equations determine recursively the coefficients β_Y assuming $\mathrm{M}(c, h; x)$ irreducible and the fusion rules (specified below) satisfied.

With $f^{(1)} = \beta_1 L_{-1} f$ the first equation gives as before

$$2h\beta_1 = h_0 - h_1 \Rightarrow \beta_1 = \frac{1}{2h}(h_0 - h_1) \quad \text{if } h \neq 0. \tag{4.37}$$

For $p = 2$,

$$f^{(2)} = \left(\beta_{1,1} L_{-1}^2 + \beta_2 L_{-2}\right) f, \tag{4.38}$$

we get from eq. (4.36a)

$$2h(2h+1)\beta_{1,1} + 3h\beta_2 = \frac{h}{4} + \frac{(h_0 - h_1)^2}{2}, \tag{4.39}$$

as before, while eq. (4.36b) yields

$$6h\beta_{1,1} + (4h + \tfrac{1}{2}c)\beta_2 = \frac{h}{4} + \frac{h_0 + h_1}{2}. \tag{4.40}$$

Provided the determinant

$$K_2 = h\left(16h^2 + 2h(c-5) + c\right) \tag{4.41}$$

is different from zero, we obtain, denoting $h_0 + h_1 = s$ and $h_0 - h_1 = d$,

$$\beta_{1,1} = \frac{2h^2 + h(c - 12s + 16d^2) + 2cd^2}{8h\left[16h^2 + 2h(c-5) + c\right]}, \tag{4.42a}$$

$$\beta_2 = \frac{h^2 + h(2s-1) + s - 3d^2}{\left[16h^2 + 2h(c-5) + c\right]}. \tag{4.42b}$$

For $p = 3$

$$f^{(3)} = (\beta_{1,1,1}L_{-1}^3 + \beta_{1,2}L_{-1}L_{-2} + \beta_3 L_{-3})f. \tag{4.43}$$

Again provided

$$K_3 = h(16h^2 + 2h(c-5) + c)(3h^2 + h(c-7) + c + 2) \tag{4.44}$$

is non-vanishing, we find, with the help of a computer,

$$\begin{aligned}\beta_{1,1,1} = \{&d(18h^3 + h^2(48d^2 - 108s + 15c + 18)\\ &+h(84s - 36cs + 22cd^2 - 52d^2 + 3c^2 - c - 20)\\ &+2c^2d^2 + 16cd^2 + c^2 - 12cs + 2c)\}\\ &\times\left[48h(16h^2 + 2h(c-5) + c)(3h^2 + h(c-7) + c + 2)\right]^{-1},\end{aligned} \tag{4.45a}$$

$$\begin{aligned}\beta_{1,2} = \{&d(3h^4 + h^3(6s + c - 16) + h^2(2cs + 9s - 9d^2 - 2c + 9)\\ &+h(3sc - 9s - 3cd^2 + 7d^2 - c) + c(s - d^2))\}\\ &\times\left[2h(16h^2 + 2h(c-5) + c)(3h^2 + h(c-7) + c + 2)\right]^{-1},\end{aligned} \tag{4.45b}$$

$$\beta_3 = \frac{d(h(s-1) + s - d^2)}{2h(3h^2 + h(c-7) + c + 2)}. \tag{4.45c}$$

The method determines recursively $f^{(p)}$, the component at level p, by solving at each stage a $P(p) \times P(p)$ linear system, with $P(p)$ the number of partitions of the integer p. The determinant of this linear system is nothing other than the Kac determinant at the corresponding level. If the complete determinant does not appear in some of the above expressions, this is due to cancellation of factors between numerator and denominator.

If $M(c,h) \equiv V(c,h)$, i.e. if $V(c,h)$ is irreducible, or equivalently does not possess singular vectors, no Kac determinant vanishes and the coefficients β_Y are determined from eqs. (4.36) for arbitrary h_0, h_1. On the other hand, assume that $V(c,h)$ is reducible, i.e. $M(c,h)$ is obtained by quotienting $V(c,h)$ by invariant submodules arising from singular vectors annihilated by L_k, $k>0$. When attempting to solve the system (4.36) at level p or higher, where p is the level of this singular vector, one can at best hope to determine $f^{(p)}$ up to this singular vector or its descendents. This is why we defined fusion among irreducible modules, so that in $M(c,h)$ we work modulo singular vectors and descendents. In other words, the latter can be set equal to zero. Still we have to make sure that the right-hand side of the corresponding linear system lies in the range of the linear operator on the left. This imposes conditions on the triplet h_0, h_1, h as necessary ones for fusion. We thus obtain *fusion rules*.

This is already apparent at level 1, where the Kac determinant reduces to h. From $2h\beta_1 = h_0 - h_1$ it follows that if h vanishes we must have $h_0 = h_1$. Thus two modules can have the vacuum ($h=0$) sector in their fusion rules only if they have equal weights, a well known property which entails that the only non-vanishing two-point functions of primary fields are those involving fields of equal weight.

Descendents of a singular vector at level p manifest themselves by the vanishing of the Kac determinant at level $p' > p$. Consistency requires that for this value of h, the right-hand side of the linear system continues to lie in the range of the linear operator on the left for the same values of h_0 and h_1. For instance, keeping $h=0$, the equations at level two read

$$0 = \frac{(h_0 - h_1)^2}{2}, \qquad c\beta_2 = h_0 + h_1. \tag{4.46}$$

We recover consistently the condition $h_0 = h_1$ and, assuming $c \neq 0$, we find $\beta_2 = (h_0 + h_1)/c$. This leaves $\beta_{1,1}$ arbitrary. But in the irreducible module $M(c,0)$ one quotients by descendents of the singular vector $L_{-1}|c,0\rangle \in V(c,0)$, and a term like $\beta_{1,1} L_{-1}^2 |c,0\rangle$ can be ignored.

Let us look at a further example, assuming $V(c,h)$ degenerate at level two, i.e.

$$16h^2 + 2h(c-5) + c = 0. \tag{4.47}$$

The fusion rules then require that

$$(h_0 + h_1)(2h+1) - 3(h_0 - h_1)^2 + h(h-1) = 0. \tag{4.48}$$

Assume for simplicity that $h \neq -\frac{1}{2}$ (i.e. $c \neq \infty$, the "classical" case). Solving for c and $s = h_0 + h_1$ (with $d = h_0 - h_1$) we get

$$s = \frac{3d^2 - h(h-1)}{2h+1}, \qquad c = \frac{-16h^2 + 10h}{2h+1}. \tag{4.49}$$

At level 2 we find

$$f^{(2)} = \frac{h + 2d^2}{8h(2h+1)} L_{-1}^2 f + \beta_2 \left[L_{-2} - \frac{3}{2(2h+1)} L_{-1}^2 \right] f, \tag{4.50}$$

where the factor of the arbitrary coefficient β_2 is the singular vector in V(c, h) set equal to zero in M(c, h). At level 3 we then find the consistent expression

$$\begin{aligned} f^{(3)} = d \Bigg\{ & L_{-1}^3 \frac{30h^3 + (20d^2 - 17)h^2 + (14d^2 - 11)h + 20d^2 - 2}{48h(h+2)(2h-1)(2h+1)(5h+1)} \\ & + L_{-1} L_{-2} \frac{h^2 - d^2}{h(h+2)(2h-1)(5h+1)} - L_{-3} \frac{h^2 - d^2}{2h(2h-1)(5h+1)} \Bigg\} f \\ & + \beta_2 \frac{d}{2(h+2)} L_{-1} \left[L_{-2} - \frac{3}{2(2h+1)} L_{-1}^2 \right] f \end{aligned} \tag{4.51}$$

which involves the first descendent of the singular vector at level 2 multiplied by the same arbitrary coefficiency β_2. The latter vanishes in M(c, h). Note that beyond the case $h = -\frac{1}{2}$ $(c = -\infty)$ one should also take care of the particular cases $h = -\frac{1}{5}$ $(c = -\frac{22}{5})$, $h = -2$ $(c = 28)$ and $h = \frac{1}{2}$ $(c = \frac{1}{2})$. The last one is the familiar case of the (chiral part of the) Ising model if we assume $h_0 = h_1 = h(1-h)/2(2h+1)$. Then the fusion rule of the spin operator (σ: $h_\sigma = \frac{1}{16}$) reads

$$\sigma\sigma \sim 1 + \epsilon \tag{4.52}$$

with ϵ the energy operator ($h_\epsilon = \frac{1}{2}$).

In order that fusion be uniquely defined (up to overall normalization) it was required that the target module M(c, h) be irreducible, but we did not yet use the irreducibility of the initial modules. In sect. 5 we will examine the consequences of quotienting by descendents of singular vectors in the initial spaces.

Remark. In applications it might be more convenient to use one of x_0 or x_1 as fusion point. For instance if $x \equiv x_1$ then with $x_0 = x + z$, $x_1 = x$ we write

$$\mathcal{F}|c, h_0; z+x\rangle \otimes |c, h_1; x\rangle = \frac{1}{z^{h_0+h_1-h}} \sum_{Y} z^{|Y|} \bar{\beta}_Y L_{-Y} |c, h; x\rangle, \quad (4.53)$$

$$\mathcal{F}(L_{-p} \otimes \mathbf{1}) = \frac{(-1)^p}{z^p}\left[h_1(p-1) + zL_{-1} - z\frac{\mathrm{d}}{\mathrm{d}z}\right] + \sum_{k \geq 0} z^k \binom{k+p-2}{k} L_{-p-k}, \quad (4.54a)$$

$$\mathcal{F}(\mathbf{1} \otimes L_{-p}) = \frac{1}{z^p}\left[h_0(p-1) - z\frac{\mathrm{d}}{\mathrm{d}z}\right] + L_{-p}, \quad (4.54b)$$

and $\mathcal{F}(L_{-1} \otimes \mathbf{1}) = \partial_z$, $\mathcal{F}(\mathbf{1} \otimes L_{-1}) = -\partial_z + L_{-1}$ while covariance reads

$$\left[L_k - \left(h_0(k+1)z^k + z^{k+1}\partial_z\right)\right] \frac{1}{z^{h_0+h_1-h}} \sum_{p \geq 0} z^p f^{(p)} = 0, \quad (4.55)$$

i.e.

$$L_1 f^{(p)} = (p - 1 + h + h_0 - h_1) f^{(p-1)}, \quad (4.56a)$$

$$L_2 f^{(p)} = (p - 2 + h + 2h_0 - h_1) f^{(p-2)}. \quad (4.56b)$$

5. General singular vectors

Our final goal is to obtain singular vectors in a Verma module $V(c, h_{j',j})$ from fusion, using the explicit expressions of sect. 2 for the particular cases $h_{j',0}$ and $h_{0,j}$. Before we proceed, it is good to have some examples obtained by direct computation. We give two of them.

Example 1. Singular vector at level 4 for $h_{\frac{1}{2},\frac{1}{2}} = -\frac{3}{4}(t + t^{-1} + 2)$.

Set

$$\tau_{\pm} = t + t^{-1} \pm 2, \quad (5.1)$$

so that $h_{\frac{1}{2},\frac{1}{2}} = -\frac{3}{4}\tau_+$. The singular vector is obtained by applying the following operator to the highest weight state:

$$\begin{aligned}\phi_{\frac{1}{2},\frac{1}{2}} = {} & L_{-1}^4 + \tau_+\tau_- L_{-2}^2 + \tfrac{3}{2}\tau_+ L_{-1}^2 L_{-2} + \tfrac{3}{2}\tau_- L_{-2} L_{-1}^2 \\ & - \tfrac{1}{2}(\tau_+ + \tau_-) L_{-1} L_{-2} L_{-1}.\end{aligned} \quad (5.2)$$

One can attempt to recast this in matrix form. One possibility is as follows. Write

$$F = \phi f, \quad (5.3)$$

and introduce the vector with components $(f_3, f_2, f_1, f_0 \equiv f)^{\mathrm{t}}$. Then, for instance

$$\begin{pmatrix} F \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} L_{-1} & \tau_- L_{-2} & -(\tau_-/2) L_{-3} & 0 \\ -1 & L_{-1} & 0 & (\tau_+/2) L_{-3} \\ 0 & -1 & L_{-1} & \tau_+ L_{-2} \\ 0 & 0 & -1 & L_{-1} \end{pmatrix} \begin{pmatrix} f_3 \\ f_2 \\ f_1 \\ f_0 \end{pmatrix}. \tag{5.4}$$

The f_k's satisfy the descent equations

$$L_1 f_0 = 0, \quad L_1 f_1 = 2hf_0, \quad L_1 f_2 = 2f_1, \quad L_1 f_3 = 2(h+3) f_2, \quad L_1 F = 0,$$
$$L_2 f_0 = 0, \quad L_2 f_1 = 0, \quad L_2 f_2 = \tfrac{16}{3} h f_0, \quad L_2 f_3 = 2(3+h) f_1, \quad L_2 F = 0. \tag{5.5}$$

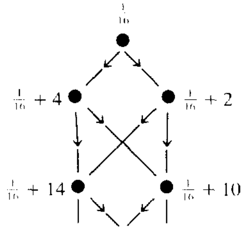
One verifies on this example the three general properties of singular vectors announced in sect. 3.

It is interesting to look at the value of the parameter t for which $h_{\frac{1}{2},\frac{1}{2}} = h_{0,\frac{3}{2}}$ say, namely the rational value $t = -\frac{3}{4}$, $c = \frac{1}{2}$ for which $h_{\frac{1}{2},\frac{1}{2}} = h_{0,\frac{3}{2}} = \frac{1}{16}$. This is a special case of the symmetry valid for $t = -m/(m+1)$, m integral, $h_{j',j} = h_{\bar{j}',\bar{j}}$, where

$$\bar{j} = \left(\frac{m-1}{2} - j\right), \qquad \bar{j}' = \left(\frac{m-2}{2} - j'\right) \qquad m \geqslant \sup(2j+1, 2j'+2).$$

Thus we deal with the chiral (or holomorphic) part of the spin operator of the Ising model. In this case there is also a degenerate vector at level 2,

$$\tilde{F} = \phi_{0,\frac{3}{2}} f = \left(L_{-1}^2 + tL_{-2}\right) f. \tag{5.6}$$



This does not contradict the general theory which says that some descendents of F and $\tilde{F}$ agree at a higher level (here 10 and 14). Indeed from the work of Feigin and Fuchs we have the diagram shown indexed by weights of Verma modules. An arrow indicates that the top module contains a submodule isomorphic to the bottom one. Moreover the intersection $\mathrm{V}(\frac{1}{2}, \frac{1}{16}+4) \cap \mathrm{V}(\frac{1}{2}, \frac{1}{16}+2)$ is generated by the linear span of submodules isomorphic to $\mathrm{V}(\frac{1}{2}, \frac{1}{16}+10)$ and $\mathrm{V}(\frac{1}{2}, \frac{1}{16}+14)$. In clear this means that at level 10 and 14 there exist singular vectors which are common descendents of F and $\tilde{F}$. Said otherwise, there exist pairs of (non-commuting) polynomials in the L_{-p}'s of respective grade 6 and 8 such that

$$\psi_6 F_4 = \tilde{\psi}_8 \tilde{F}_2$$

is a singular vector at level 10 and similarly

$$\psi'_{10} F_4 = \tilde{\psi}'_{12} \tilde{F}_2$$

is singular at level 14. Even in such a simple case the explicit construction of these pairs would be a major undertaking, unless we have a mean to factor (for $t = -\frac{3}{4}$) the expressions of the operators generating the corresponding singular vectors at level 10 and 14, in a non-unique way reflecting the non-commutativity of the ring.

Example 2. Case $j' = 1$, $j = \frac{1}{2}$, $h = -(2t^{-1} + \frac{5}{2} + \frac{3}{4}t)$.

$$F = \phi_{1,\frac{1}{2}} f,$$

with

$$\phi_{1,1/2} = L_{-1}^6 + AL_{-1}^4 L_{-2} + BL_{-1}^3 L_{-3} + CL_{-1}^2 L_{-2}^2 + DL_{-1}^2 L_{-4}$$
$$+ EL_{-1} L_{-2} L_{-3} + FL_{-1} L_{-5} + GL_{-2}^3 + IL_{-3}^2 + JL_{-6} + KL_{-2} L_{-4}, \quad (5.7)$$

and

$$A = \frac{8}{t} + 3t = \tau_1 + \tau_0 + \tau_{-1},$$

$$B = \frac{8}{t^2} - \frac{16}{t} + 24 - 6t,$$

$$C = \frac{16}{t^2} + 3t^2 = \tau_1 \tau_0 + \tau_0 \tau_{-1} + \tau_{-1} \tau_1,$$

$$D = -\frac{24}{t^2} + \frac{108}{t} - 72 + 18t,$$

$$E = \frac{32}{t^3} - \frac{32}{t^2} - \frac{24}{t} + 24t - 6t^2,$$

$$F = -\frac{16}{t^3} + \frac{136}{t^2} - \frac{264}{t} + 184 - 48t + 4t^2,$$

$$G = \frac{16}{t} - 8t + t^3 = \tau_1 + \tau_0 + \tau_{-1},$$

$$I = \frac{16}{t^4} - \frac{16}{t^3} - \frac{36}{t^2} + \frac{12}{t} + 20 - 12t + 2t^2,$$

$$J = \frac{80}{t^3} - \frac{272}{t^2} + \frac{316}{t} - 204 + 76t - 12t^2,$$

$$K = -\frac{32}{t^3} + \frac{80}{t^2} + \frac{24}{t} - 20 - 24t + 10t^2. \quad (5.8)$$

For comparison with the general properties we have introduced the parameters

$$\tau_{M',M} = 4(\theta^{-1}M' + \theta M)^2, \tag{5.9}$$

with

$$\tau_1 = \tau_{1,\frac{1}{2}} = t + 4 + 4t^{-1}, \quad \tau_0 = \tau_{0,\frac{1}{2}} = t, \quad \tau_{-1} = \tau_{-1,\frac{1}{2}} = t - 4 + 4t^{-1}. \tag{5.10}$$

It is not totally obvious how to cast the above expression into a matrix form. This justifies the present endeavour to use fusion as a device to obtain these matrix forms.

Let us look at the implications in the fusion process when $V(c, h_0)$ or $V(c, h_1)$ or both possess singular vectors. Suppose we started by "fusing" $f_0(x_0)$ and $f_1(x_1)$ and assume that $\phi_0 f_0(x_0)$ is again a primary field of weight $h_0 + n_0$. Starting from $(f^{(0)} \equiv f)$

$$f_0(x_0) f_1(x_1) \to \frac{1}{z^{h_0+h_1-h}} \sum_r z^r f^{(r)},$$

where the arrow denotes the fusion map of (4.7), we would derive

$$(\phi_0 f_0)(x_0) f_1(x_1) \to \frac{1}{z^{h_0+h_1-h+n_0}} \sum_r z^r \psi^{(r)},$$

with a leading term given by

$$\psi^{(0)} = \left\{ (-1)^{n_0} z^{h_0+h_1-h+n_0} \phi_0 \left(L_{-k} \to \frac{1}{z^k} \left[h_1(k-1) - z\frac{\mathrm{d}}{\mathrm{d}z} \right] \right) \frac{1}{z^{h_0+h_1-h}} \right\} f,$$

independent of the choice of the coordinate x (be it $(x_0 + x_1)/2$ or $x_1 \ldots$).

In sect. 3 we considered the representation of the Witt algebra

$$l_{-k} = -\frac{1}{z^k} \left[\lambda(k-1) + z\frac{\mathrm{d}}{\mathrm{d}z} \right] \tag{5.11}$$

acting on the basis $z^{-p-\mu}$ and described the effect of substituting l_{-k} for L_{-k} in $\phi_{j_0', j_0} \equiv \phi_0$ so that with the notations of this section

$$\psi^{(0)} = (-1)^{n_0} \varphi_{j_0', j_0}(\lambda, \mu) f \quad \lambda = -h_1, \quad \mu = h_0 + h_1 - h. \tag{5.12}$$

Consequently we obtain the following general result:

Proposition 5.1. *If* $\varphi_{j_0', j_0}(\lambda = -h_1, \mu = h_0 + h_1 - h) = 0$,

(i) *the irreducible module* $M(c, h)$ *does not occur in the fusion of* $M(c, h_0 + n_0) \otimes M(c, h_1)$;

(ii) *and as the first non-vanishing term* $\psi^{(r)}$, $r > 0$, *has to be a highest weight vector, it is necessarily a non-trivial singular vector in* $V(c, h)$.

A similar property holds of course with the roles of 0 and 1 interchanged.

Let us see how this works in particular interesting cases.

Case (a). Set $h_0 \equiv h_{\frac{1}{2},0} = -\frac{1}{2} - \frac{3}{4}t^{-1}$, $h = h_{0,j} = -j - tj(j+1)$. Then for $\lambda = -h_1$, $\mu = h_0 + h_1 - h$

$$\varphi_{\frac{1}{2},0}(\lambda,\mu) = \mu\left(\mu + 1 + \frac{1}{t}\right) - \frac{\lambda}{t},$$

quadratic in h_1, vanishes for

$$h_1 \equiv h_{\frac{1}{2},j} = -\frac{3}{4t} - \frac{1}{2} - 2j - j(j+1)t, \tag{5.13}$$

with a Verma module degenerate at level $2(2j+1)$ but also for

$$h_1 = \frac{1}{2} + \frac{1}{4t} - j(j+1)t, \tag{5.14}$$

which, using the general formula for singular vectors, one could interpret as $h_{-\frac{1}{2},j}$ (sic!) i.e. for generic t *it is not* the highest weight of a module with a singular vector. Let us choose this second value and look at the fusion symbolically written as

$$(\tfrac{1}{2},0) \otimes \text{“}(-\tfrac{1}{2},j)\text{”} \to (0,j).$$

We have

$$\phi_{\frac{1}{2},0} = L_{-1}^2 + \frac{1}{t}L_{-2}. \tag{5.15}$$

Taking for convenience the fusion point $x \equiv x_1$, we get from sect. 4

$$\mathcal{F}(L_{-1} \otimes 1) = \partial_z,$$

$$\mathcal{F}(L_{-2} \otimes 1) = \frac{h_1}{z^2} - \frac{1}{z}\partial_z + \sum_{k=1}^{\infty} z^{k-2}L_{-k}. \tag{5.16}$$

Transferring $\phi_{\frac{1}{2},0}$ on the h-module and requiring that $\mathrm{M}(c,h_0;x_0)$ be irreducible we find that

$$\left\{\partial_z^2 + \frac{1}{t}\left(\frac{h_1}{z^2} - \frac{1}{z}\partial_z + \sum_{k=1}^{\infty} z^{k-2}L_{-k}\right)\right\}\frac{1}{z^{h_0+h_1-h}}\sum_{p\geqslant 0} z^p f^{(p)} = 0, \tag{5.17}$$

which gives a recursive mean to compute the sequence $f^{(p)}$. Now $h_0 + h_1 - h = j - \frac{1}{2}t^{-1}$. Therefore the above reads

$$p((2j+1) - p)f^{(p)} = \frac{1}{t}\sum_{k\geqslant 1} L_{-k}f^{(p-k)}. \tag{5.18}$$

Of course one verifies that for $p=0$ this is an identity and that $f^{(p)}$ computed from this relation agrees with the value obtained from conditions (4.56a, b) of sect. 4 implementing covariance. The main importance of this exercise is to clarify the form we obtained for the singular vectors in the $(0, j)$ module starting from the second-order equation

$$\left\{\partial_z^2+\frac{1}{tz^2}(h_1-z\partial_z)+T^{(-)}(z)\right\}\left\{\frac{1}{z^{h_0+h_1-h}}\sum_{p\geqslant 0} z^p f^{(p)}\right\}=0, \tag{5.19}$$

where

$$T^{(-)}(z)=\sum_{k\geqslant 1} z^{k-2}L_{-k} \tag{5.20}$$

is the part of the energy–momentum tensor less singular than z^{-2}, expressing that the $h_0\equiv h_{1/2,0}$ Verma module is degenerate at level 2. Indeed the above recursion relation is nothing other than our previous matrix equation for the singular vector, giving a precise meaning to the intermediate stages if we relabel and rescale the components according to

$$f^{(j+M)}\equiv t^{-j-M}f_M, \qquad f_{-j}\equiv f,$$

using a representation where the generators of angular momentum read (with J_0 unchanged, $n=j+1$)

$$J_-\to\begin{pmatrix} 0 & & & & & \\ 1(n-1) & 0 & & & & \\ & 2(n-2) & 0 & & & \\ & & \ddots & \ddots & & \\ & & & & & \\ & & & & (n-1)1 & 0 \end{pmatrix},$$

$$J_+\to\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & & 1 \\ & & & & 0 \end{pmatrix}. \tag{5.21}$$

Then $f_{-j+1},\ldots,f_j$ are determined from the recursion relation and the condition

$f_{j+1} \equiv f^{(2j+1)} = 0$ gives an operator const. $\times\, \phi_{0,j} f_{-j} = 0$ (with a non-vanishing constant factor). Moreover the "descent" equations found among the components f_M agree with those derived from the covariance equations.

With this re-interpretation of the singular vectors of type $(0, j)$ we have completely fulfilled the expectations of the advocates of Liouville theory [18]. Of course with obvious changes we obtain the same for the singular vectors of type $(j, 0)$.

Case (b). We set $h_0 \equiv h_{0,j}$, $h_1 \equiv h_{j',0}$ and $h \equiv h_{j',j}$, so $\mu = h_0 + h_1 - h = 2jj'$ and $\lambda = -h_1 = j'(1 + (j'+1)/t)$. In the general expression

$$\varphi^2_{0,j} = \prod_{-j \leqslant M \leqslant j} \{[\mu + (j+M)(1 + t(j+1-M))] \times [\mu + (j-M)(1 + t(j+1+M))] - 4\lambda t M^2\} \tag{5.22}$$

the factor corresponding to $M = j$ vanishes, and this is of course expected since $\mathrm{V}(c, h)$ is degenerate at level $(2j+1)(2j'+1)$ but now we have to implement two distinct conditions, namely $(\phi_{0,j} f_0) f_1 = 0$ and $f_0(\phi_{j',0} f_1) = 0$. This is a rather ineffective manner to obtain the singular vector $h_{j',j}$ that we proposed in ref. [12] and that we will illustrate below.

For the same values of $h_0 \equiv h_{0,j}$ and $h \equiv h_{j',j}$ we have also the possibility to choose $h_1 \equiv h_{j',2j}$ as the second value for which the leading factor $(M = j)$ in $\phi_{0,j}$ above vanishes. This shows that we have several choices to obtain the (j', j) singular vector by fusion beyond the "natural" one $(0, j) \otimes (j', 0) \to (j', j)$.

Case (c). Inspired by case (a) we finally investigate the more promising fusion

$$(j' + \tfrac{1}{2}, 0) \otimes \text{``}(-\tfrac{1}{2}, j)\text{''} \to (j', j),$$

where

$$h_0 = -(j' + \tfrac{1}{2})\left[1 + \frac{j' + 1 + \frac{1}{2}}{t}\right], \tag{5.23}$$

and we have used the same abusive notation for

$$h_1 = \frac{1}{4t} + \frac{1}{2} - tj(j+1), \tag{5.24}$$

which again for generic t corresponds to an irreducible Verma module, while

$$h \equiv -\left[tj(j+1) + j + j' + 2jj' + \frac{j'(j'+1)}{t}\right]. \tag{5.25}$$

The singular vector operator is $\phi_0 \equiv \phi_{j'+\frac{1}{2},0}$ and applying proposition 5.1 we compute

$$\varphi^2_{j'+\frac{1}{2},0} \equiv \prod_{-j'-\frac{1}{2} \leq M' \leq j'+\frac{1}{2}} \left\{ \left[\mu + (j'+\tfrac{1}{2}+M')\left(1+\frac{j'+\frac{3}{2}-M'}{t}\right)\right] \right.$$

$$\left. \times \left[\mu + (j'+\tfrac{1}{2}-M')\left(1+\frac{j'+\frac{3}{2}-M'}{t}\right)\right] - \frac{4\lambda M'^2}{t} \right\}. \quad (5.26)$$

The factor corresponding to $M' = j'+\frac{1}{2}$ is $\mu[\mu+(2j'+1)(1+t^{-1})] - \lambda t^{-1}(2j'+1)^2$. Inserting the values

$$\mu = h_0 + h_1 - h = (2j'+1)\left(j - \frac{1}{2t}\right),$$

$$\mu + (2j'+1)\left(1+\frac{1}{t}\right) = (2j'+1)\left(j+1+\frac{1}{2t}\right),$$

$$-\frac{\lambda(2j'+1)^2}{t} = (2j'+1)^2\frac{h_1}{t} = (2j'+1)^2\left(\frac{1}{4t^2}+\frac{1}{2t} - j(j+1)\right), \quad (5.27)$$

we do indeed find that this factor vanishes. Consequently we obtain the singular vector in the module $V(c,h)$ by requiring that $\phi_0 f_0 \times f_1$ vanishes. It is once again simpler to take the fusion point at x_1. We have all data necessary since we know $\phi_0 \equiv \phi_{j'+\frac{1}{2},0}$,

$$\mathscr{F}(L_{-1} \otimes \mathbf{1}) = \partial_z,$$

$$\mathscr{F}(L_{-p} \otimes \mathbf{1}) = \frac{(-1)^p}{z^p}\left[h_1(p-1) + z(L_{-1} - \partial_z)\right] + \sum_{k=0}^{\infty} z^k \binom{k+p-2}{k} L_{-p-k}. \quad (5.28)$$

Very much as in case (a) this defines in the $V(c,h)$ module a set of intermediate stages between $f \equiv f^{(0)}$ and $f^{(n)}$, the singular vector set equal to zero in $M(c,h)$. It is readily seen that $(-1)^{2j'}\varphi_{j'+\frac{1}{2},0}$ $(\lambda = -h_1, \mu = h_0 + h_1 - h - n)$ is the coefficient of $f^{(n)}$ at level n in the recursion relation obtained from setting $\phi_0 f_0 \times f_1 = 0$. This vanishes again since indeed under $\mu \to \mu - (2j+1)(2j'+1)$ the above values get interchanged as $\mu \leftrightarrow -(\mu + 2j'+1)(1+t^{-1})$ in (5.27) and this is responsible for the fact that at this level we get directly the singular vector (for a generic value of c).

Let us first show how this reproduces the $(\frac{1}{2}, \frac{1}{2})$ singular vector. We need the operator

$$\phi_{1,0} \equiv L_{-1}^3 + \frac{2}{t}(L_{-1}L_{-2} + L_{-2}L_{-1}) + \frac{4}{t^2}L_{-3}, \tag{5.29}$$

with $h_1 = \frac{1}{4}t^{-1} + \frac{1}{2} - \frac{3}{4}t$. The substitution

$$\begin{aligned} &L_{-1} \to \partial_z, \\ &L_{-2} \to \frac{h_1}{z^2} - \frac{1}{z}\partial_z + \sum_{k=1}^{\infty} z^{k-2}L_{-k}, \\ &L_{-3} \to \frac{-2h_1}{z^3} + \frac{1}{z^2}\partial_z + \sum_{k=1}^{\infty} z^{k-3}(k-2)L_{-k}, \end{aligned} \tag{5.30}$$

yields

$$\begin{aligned} &\Bigg\{\partial_z^3 + \frac{4}{t}\left(h_1\frac{1}{z^2}\partial_z - \frac{1}{z}\partial_z^2 + \sum_{p\geq 1} z^{p-2}L_{-p}\partial_z\right) \\ &+\left(\frac{2}{t} + \frac{4}{t^2}\right)\left(\frac{-2h_1}{z^3} + \frac{1}{z^2}\partial_z + \sum_{p\geq 1} z^{p-3}(p-2)L_{-p}\right)\Bigg\}\sum_{k\geq 0} z^{k-1+t^{-1}}f^{(k)} = 0. \end{aligned} \tag{5.31}$$

This is recast in the recursion relation

$$\begin{aligned} 0 = {} & k(k-4)(k-2-t^{-1})f^{(k)} \\ & + 2t^{-1}\sum_{p\geq 1}\left[2(k-1-p+t^{-1}) + (p-2)(1+2t^{-1})\right]L_{-p}f^{(k-p)}. \end{aligned} \tag{5.32}$$

We see indeed that for $k = 0$ we have an identity while the condition for $k = 4$ will give the singular vector. Solving recursively

$$\begin{aligned} &f^{(1)} = \frac{2}{1+t}L_{-1}f, \\ &f^{(2)} = \left[\frac{1}{1+t}L_{-1}^2 + (1-t^{-1})L_{-2}\right]f, \\ &f^{(3)} = \frac{2}{3(t-1)}\left[\frac{1}{1+t}L_{-1}^3 + (1-t^{-1})L_{-1}L_{-2} + \frac{4}{t(t+1)}L_{-2}L_{-1} + \left(\frac{4}{t}-1\right)L_{-3}\right]f, \end{aligned} \tag{5.33}$$

and the fourth equation becomes

$$0 = 3L_{-1}f^{(3)} + 2(1+t^{-1})L_{-2}f^{(2)} + (4t^{-1}+1)L_{-3}f^{(1)} + 6t^{-1}L_{-4}f$$
$$\equiv \frac{2}{t^2-1}\phi_{\frac{1}{2},\frac{1}{2}}f, \tag{5.34}$$

where

$$\phi_{\frac{1}{2},\frac{1}{2}} = L_{-1}^4 + (t-t^{-1})^2 L_{-2}^2 + (t-t^{-1})\left(L_{-1}^2 L_{-2} + L_{-2}L_{-1}^2\right) + 4t^{-1}L_{-1}L_{-2}L_{-1}$$
$$+ \frac{(4-t)(t+1)}{t}L_{-1}L_{-3} + \frac{(4+t)(t-1)}{t}L_{-3}L_{-1} + 3(t-t^{-1})L_{-4}$$
$$= L_{-1}^4 + (t-t^{-1})^2 L_{-2}^2 + \frac{3(1+t)^2}{2t}L_{-1}^2 L_{-2} + \frac{3(1-t)^2}{2t}L_{-2}L_{-1}^2$$
$$-(t+t^{-1})L_{-1}L_{-2}L_{-1} \tag{5.35}$$

which agrees with the expression given at the beginning of this section where we used the notation $\tau_{\pm} = t^{-1} + t \pm 2 = (1 \pm t)^2/t$. The intermediate states $f \equiv f^{(0)}, f^{(1)}, f^{(2)}, f^{(3)}$ satisfy descent equations as did those we wrote at the beginning but do not coincide with those obtained there since our method does not treat t and t^{-1} symmetrically. Rather by rescaling

$$f \equiv f^{(0)} = \tilde{f}^{(0)}, \qquad f^{(1)} = \frac{2}{1+t}\tilde{f}^{(1)},$$
$$f^{(2)} = \frac{1}{1+t}\tilde{f}^{(2)}, \qquad f^{(3)} = \frac{2}{3(t^2-1)}\tilde{f}^{(3)}, \tag{5.36}$$

we obtain in the tilde basis

$$\begin{pmatrix} F_{\frac{1}{2},\frac{1}{2}} \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} L_{-1} & (t-t^{-1})L_{-2} & t^{-2}(4+t)(t-1)L_{-3} & 3(t-t^{-1})L_{-4} \\ -1 & L_{-1} & 4t^{-1}L_{-2} & t^{-2}(4-t)(t+1)L_{-3} \\ 0 & -1 & L_{-1} & (t-t^{-1})L_{-2} \\ 0 & 0 & -1 & L_{-1} \end{pmatrix} \begin{pmatrix} \tilde{f}^{(3)} \\ \tilde{f}^{(2)} \\ \tilde{f}^{(1)} \\ \tilde{f}^{(0)} \end{pmatrix}. \tag{5.37}$$

This calculation as we see is straightforward and leads directly to the (j',j) singular vector. Let us contrast it with the one resulting from the fusion $(0,j)\otimes(j',0)\to(j',j)$ using again $(0,\frac{1}{2})\otimes(\frac{1}{2},0)\to(\frac{1}{2},\frac{1}{2})$ as an example. Returning to the symmetric fusion point $x=(x_0+x_1)/2$, $z=x_0-x_1$, let $\phi\equiv\phi_{0,\frac{1}{2}}=L_{-1}^2+tL_{-2}$, $\phi_1\equiv\phi_{\frac{1}{2},0}=L_{-1}^2+t^{-1}L_{-2}$ and write

$$f_0(x_0)f_1(x_1)\to\frac{1}{z^{1/2}}\sum_{k\geqslant 0}z^k f^{(k)}(x)\,.$$

From $\phi_0 f_0=0$ and $\phi_1 f_1=0$ we get respectively for $k\geqslant 0$

$$0=\left[(k-\tfrac{3}{2})(k-\tfrac{1}{2})-(\tfrac{3}{4}+kt)\right]f^{(k)}+(k-\tfrac{3}{2}+\tfrac{1}{2}t)L_{-1}f^{(k-1)}+\tfrac{1}{4}L_{-1}^2 f^{(k-2)}$$
$$+t\sum_{q\geqslant 0}\frac{1}{2^{q-2}}L_{-q}f^{(k-q)}\,,\tag{5.38a}$$

$$0=\left[(k-\tfrac{3}{2})(k-\tfrac{1}{2})-(\tfrac{3}{4}+kt^{-1})\right]f^{(k)}-(k-\tfrac{3}{2}+\tfrac{1}{2}t^{-1})L_{-1}f^{(k-1)}+\tfrac{1}{4}L_{-1}^2 f^{(k-2)}$$
$$+t^{-1}\sum_{q\geqslant 0}\frac{(-1)^q}{2^{q-2}}L_{-q}f^{(k-q)}\,.\tag{5.38b}$$

For $k=0$ these two equations become identities as follows from the general discussion. We find that they agree for $k=1,2,3$ yielding

$$f^{(1)}=\frac{t-1}{2(t+1)}L_{-1}f\,,$$
$$f^{(2)}=\left(\tfrac{1}{8}L_{-1}^2+\tfrac{1}{2}L_{-2}\right)f\,,$$
$$f^{(3)}=\frac{1}{3(t^2-1)}\left(\frac{t^2+6t+1}{16}L_{-1}^3+(3t^2+2t+3)L_{-1}L_{-2}+tL_{-3}\right)f\,,\tag{5.39}$$

which can be compared with the values predicted in sect. 4 for the coefficients in the short-distance expansion. Now comes the equation for the singular vector in the module $h_{\frac{1}{2},\frac{1}{2}}$. Indeed when $k=4$ the equations read

$$0=4(2-t)f^{(4)}+A(t)f\,,$$
$$0=4(2-t^{-1})f^{(4)}+A(t^{-1})f\,.\tag{5.40}$$

where

$$A(t)=\frac{1}{32}\left[\frac{5+t}{3(t^2-1)}(t^2+6t+1)+1\right]L_{-1}^4$$
$$+\frac{1}{8}\left[\frac{(5+t)(3t^2+2t+3)}{3(t^2-1)}+1+t\right]L_{-1}^2L_{-2}$$
$$-\frac{t}{3(t+1)}L_{-1}L_{-3}+\frac{t}{2}L_{-2}^2+\frac{t}{t+1}L_{-4}. \tag{5.41}$$

Each equation separately involves the unknown contribution $f^{(4)}$, so it is insufficient to determine the singular vector. However, they should agree, so that the combination

$$\{(2-t^{-1})A(t)-(2-t)A(t^{-1})\}f$$

is proportional to the singular vector (set equal to zero in $\mathrm{M}(c,h)$). Normalizing to unity the coefficient of L_{-1}^4 we conclude that

$$\phi_{\frac{1}{2},\frac{1}{2}}=\frac{1}{t-t^{-1}}\left[(2-t^{-1})A(t)-(2-t)A(t^{-1})\right], \tag{5.42}$$

which equals the expression (5.2) as a little algebra readily shows.

This mechanism is general. In the fusion of $(0,j)\otimes(j',0)$ one verifies that in the (j',j) module the coefficients $f^{(k)}$, $0<k<(2j+1)(2j'+1)$ are determined recursively by the conditions $\phi_0 f_0=0$ or $\phi_1 f_1=0$ and that they agree. On the other hand, at level $n=(2j+1)(2j'+1)$ one finds

$$0=a_n f^{(n)}+Af, \qquad 0=b_n f^{(n)}+Bf, \tag{5.43}$$

with

$$a_n=D(j,j',n,t), \qquad b_n=D(j',j,n,t^{-1}),$$
$$D(j,j',k,t)=\prod_{M=-j}^{j}\left[(j-M)(2j'+1+t(j+M+1))-k\right], \tag{5.44}$$

so that up to normalization

$$\phi_{j',j}=\mathrm{const.}\times(b_nA-a_nB). \tag{5.45}$$

While this treatment is symmetric in j and j' it does not yield a matrix formulation for the singular vectors.

Hence we conclude that, among many possibilities, the fusions $(j'+\frac{1}{2},0)\otimes$ "$(-\frac{1}{2},j)$" $\to(j',j)$ or equivalently $(0,j+\frac{1}{2})\otimes$ "$(j',-\frac{1}{2})$" $\to(j',j)$ directly yield an expression for a singular vector. We choose the former to summarize the main result of this paper.

Proposition 5.2.

(*i*) *Verma modules with singular vectors have highest weights parametrized by a pair* (j', j) *with* j, j' *non-negative integers or half-integers* (*eq.* (3.6) *with* $\theta^2 = t$). *When* j *or* j' *vanishes the singular vector is given by the Benoit–Saint-Aubin formula* (3.20), $F = \phi_{0,j} f$ *or* $F = \phi_{j',0} f$.

(*ii*) *If none of* j *and* j' *vanishes, let* $f \equiv f^{(0)}$ *be the highest weight state in* $V(c, h)$, *set* $h_0 \equiv h_{j'+\frac{1}{2},0}$, $h_1 \equiv h_{-\frac{1}{2},j}$, $h \equiv h_{j',j}$ *and*

$$\bar{\phi} = \phi_{j'+\frac{1}{2},0}\left(L_{-p} \to \frac{(-1)^p}{z^p}\left[h_1(p-1) + z(L_{-1} - \partial_z) \right] + \sum_{k \geqslant 0} z^k \binom{k+p-2}{k} L_{-p-k} \right)$$

then the equation

$$\bar{\phi} \frac{1}{z^{h_0+h_1-h}} \sum_k z^k f^{(k)} = 0$$

determines recursively $f^{(k)}$ *for* $0 < k < n = (2j+1)(2j'+1)$ *in terms of* f *and yields at level* n

$$\phi_{j',j} f = 0$$

up to a non-vanishing factor. Moreover the intermediate coefficients $f^{(k)}$, $0 \leqslant k \leqslant n$ *satisfy "descent equations" given in* (4.56*a, b*).

To give a last non-trivial example, we recast, using this method, the $(1, \frac{1}{2})$ singular vector in the following unnormalized matrix form:

$$\left(\phi_{1,\frac{1}{2}} f^{(0)}, 0, \ldots, 0\right)^{\mathrm{T}} = A\left(f^{(5)}, \ldots, f^{(0)}\right)^{\mathrm{T}}, \tag{5.46}$$

where the matrix A reads:

$$\begin{pmatrix}
\frac{-5}{t-3} L_{-1} & \frac{-2t-4}{t-3} L_{-2} & \frac{-4t-3}{t-3} L_{-3} & \frac{-6t-2}{t-3} L_{-4} & \frac{-8t-1}{t-3} L_{-5} & \frac{-10t}{t-3} L_{-6} \\
\frac{-1}{2t} & \frac{-3}{5(t-2)} L_{-1} & \frac{-2t-2}{5(t-2)} L_{-2} & \frac{-4t-1}{5(t-2)} L_{-3} & \frac{-6t}{5(t-2)} L_{-4} & \frac{-8t+1}{5(t-2)} L_{-5} \\
0 & \frac{-1}{2t} & \frac{-1}{8(t-1)} L_{-1} & \frac{-2t}{8(t-1)} L_{-2} & \frac{-4t+1}{8(t-1)} L_{-3} & \frac{-6t+2}{8(t-1)} L_{-4} \\
0 & 0 & \frac{-1}{2t} & \frac{1}{9t} L_{-1} & \frac{-2t+2}{9t} L_{-2} & \frac{-4t+3}{9t} L_{-3} \\
0 & 0 & 0 & \frac{-1}{2t} & \frac{3}{8(t+1)} L_{-1} & \frac{-2t+4}{8(t+1)} L_{-2} \\
0 & 0 & 0 & 0 & \frac{-1}{2t} & \frac{5}{5(t+2)} L_{-1}
\end{pmatrix}.$$

We conclude by remarking that the key concept is fusion. It allows us to transfer equations for singular vectors in a reducible module V_0 to a module V, the best choice being by tensoring V_0 by an appropriate one chosen to be (generically) an irreducible Verma module. The possibility is dictated by the vanishing of the classical quantity $\varphi(-h_1, h_0 + h_1 - h)$, the expression of which plays a crucial role in the discussion.

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