$SU(N)$ Lattice Integrable Models and Modular Invariance

P. Di Francesco [$]

Joseph Henry Laboratories
Princeton University
Princeton, New Jersey 08544

and J.-B. Zuber*

Service de Physique Théorique de Saclay [1]
F-91191 Gif-sur-Yvette cedex, France

We first review some recent work on the construction of RSOS $SU(N)$ critical integrable models. The models may be regarded as associated with a graph, extending from $SU(2)$ to $SU(N)$ an idea of Pasquier, or alternatively, with a representation of the fusion algebra over non-negative integer valued matrices. Some consistency conditions that the Boltzmann weights of these models must satisfy are then pointed out. Finally, the algebraic connections between (a subclass of) the admissible graphs and (a subclass of) modular invariants are discussed, based on the theory of C-algebras. The case of G_2 is also treated.

[$] (pdf@pupthy.princeton.edu)
* (zuber@frsac11.bitnet)
[1] Laboratoire de l'Institut de Recherche Fondamentale du Commissariat à l'Energie Atomique

1. Introduction.

The two-dimensional unitary minimal conformal field theories with Virasoro central charge $c < 1$ are now fully classified and well understood. Apart from the constraint of unitarity which restricts the set of values of c to $c = 1 - \frac{6}{m(m+1)}$, $m = 2, 3, \cdots$, one-loop modular invariance has proved to be sufficiently powerful to determine entirely the operator contents of the different possible theories [1]. The general question of finding one-loop modular invariants for higher values of c is of great interest, but no classification has yet been proposed.

In this paper we are mostly interested in the unitary coset theories $SU(N)_{k-1} \times SU(N)_1/SU(N)_k$ of central charges $c = (N-1)\big(1 - N(N+1)/(k+N-1)(k+N)\big)$ and try to find a classifying principle of (part of) these theories other than one-loop modular invariance. We use a statistical mechanical approach, which consists of finding generalized height models subject to a set of constraints: integrability [2], explicit $SU(N)_q$ quantum group symmetry, restrictions on the degrees of freedom encoded in the topology of a graph. These are the "$SU(N)$ height models" defined in sect.2, already introduced in [3], which generalize the $SU(2)$ A, D, E models of Pasquier [4] and complement the RSOS models of reference [5]. We expect them to be critical and to be described in the continuum limit by the previous coset conformal field theories, with different height models leading in general to different modular invariant toroidal partition functions.

The construction of the $SU(N)$ height models consists of the two steps followed in sect.3: first find admissible graphs, the vertices of which become the heights of the models. From empirical remarks borrowed from the $N = 2, 3$ cases, we suggest the following possible restriction on these graphs: they should support a non negative integer matrix representation of the Kac-Moody $\widehat{SU(N)}_k$ fusion rules.

The second step is the determination of Boltzmann weights subject to the Yang-Baxter condition, and to other restrictions inherited from the $SU(N)$ quantum symmetry of the models. At this time, no general solution exists yet and this determination has been completed only in a few cases [3]: the existence of integrable models attached to these graphs remains therefore conjectural. We try to complement the methods used in [3] by exhibiting local constraints on these weights from the consideration of fixed boundary conditions as in [6] and [7].

In sect.4 we turn to the relations between these models and the coset modular invariants. A possible route would be to repeat the steps followed in the case of $SU(2)$: reformulation of the height model in the Coulomb gas language, computation of the toroidal partition function in the continuum limit and identification of the resulting combination of Gaussian partition functions with the modular invariant [4][8]. This program, however, looks extremely cumbersome. For $N \geq 3$, one deals with a $N-1$-component Gaussian field, whose action contains two parameters: the ordinary coupling constant g and the coefficient of the antisymmetric term. Combinations of the corresponding partition functions are difficult to identify and to reshuffle. This route will therefore not be explored in the present paper, and we rather try to establish shortcuts relating directly a given model with a modular invariant. In the case of $SU(2)$, the ADE classification had been based on the observation that the weights of $SU(2)$ labelling the diagonal terms in the modular invariants were nothing but the exponents of the Dynkin diagrams, i.e. characterize the eigenvalues of their adjacency matrix. The same appears in the case of $SU(3)$: diagonal terms of all known modular invariants match with the exponents characterizing the eigenvalues of some of the graphs. This empiric observation, however, does not say how to reconstruct the modular invariant from the graph. For all the modular invariants which are sums of square moduli of character combinations, ("type I" invariants, see sect. 4), we find that there is a well defined procedure to that effect, which relies on the theory of "C-algebras" [9]. A C-algebra, i.e. an associative and commutative algebra satifying some additional axioms (sect. 4), is attached to the graph; it contains a C-subalgebra which reproduces the extended fusion algebra of some Kac-Moody theory, and the dual C-algebra defines a partition of the set of exponents that yields the blocks of the invariant.

Though these steps remain empiric and there are still many loose ends in the program, it seems to us that it can be an interesting alternative to the difficult problem of finding modular invariants. Our considerations look quite general and apply as well to the case of G_2 cosets treated in an appendix.

2. General definitions : the SU(N) height models

We consider a statistical model defined on a square lattice with oriented bonds, say upward for vertical bonds and to the right for horizontal ones. To each node of the lattice associate a "height" i.e. a degree of freedom taking its values among the vertices of a given oriented finite connected "target" graph $\mathcal{G}$. The adjacency matrix of $\mathcal{G}$ is defined by:

$$G_{ab} = \text{number of links from vertex } a \text{ to vertex } b \tag{2.1}$$

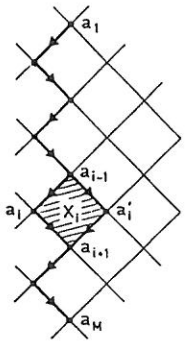
The following restriction is imposed on the possible configurations: we retain only the configurations where two nodes of the lattice connected by an oriented link have heights connected by a link of the same orientation on the target graph $\mathcal{G}$.

The possibility of having more than one link from vertex a to vertex b on $\mathcal{G}$ can be translated by a link variable taking G_{ab} possible values. (For simplicity, we omit to write them explicitly in the following.) We attach then to each height configuration (a, b, c, d) around a face of the square lattice a local Boltzmann weight $w(a, b, c, d|u)$ with spectral parameter u, satisfying the Yang-Baxter equation. This is expressed through the face transfer matrix formalism. After a rotation of the square lattice by 45 degrees, one considers the action of a local operator $\mathbf{X}_i(u)$ on the zig-zag line of fig. 1, affecting the i-th face only as:

$$\mathbf{X}_i(u) = \mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes X_i(u) \otimes \mathbf{1} \otimes \cdots \tag{2.2}$$

$$\langle a'_1, a'_2, \cdots, a'_M | \mathbf{X}_i(u) | a_1, a_2, \cdots, a_M \rangle = \prod_{j \neq i} \delta_{a_j, a'_j} w(a_i, a_{i-1}, a'_i, a_{i+1}|u) \tag{2.3}$$

Fig. 1. The zig-zag line to which height configurations are attached.



A sufficient condition for the model to be integrable is that $X_i(u)$ satisfies the celebrated Yang-Baxter equation:

$$X_i(u) X_{i+1}(u+v) X_i(v) = X_{i+1}(v) X_i(u+v) X_{i+1}(u) \tag{2.4}$$

One may look for solutions of this equation where the dependance on u is entirely contained in a function $\rho(u)$:

$$X_i(u) = \mathbf{1} + \rho(u) U_i \tag{2.5}$$

It is easy to show that a necessary and sufficient condition for (2.4) to be satisfied by the Ansatz (2.5) is:

$$\rho(u) = \left(\frac{\sin(\pi(\hat{\lambda} - u))}{\sin(\pi u)} \right)^{-1} \tag{2.6}$$

where $\hat{\lambda}$ is a parameter, (real in the critical region), and the U's satisfy the Hecke algebra:

$$U_i^2 = \beta U_i \tag{2.7a}$$

$$U_i U_j = U_j U_i \quad \text{if } |i - j| \geq 2 \tag{2.7b}$$

$$U_i U_{i+1} U_i - U_i = U_{i+1} U_i U_{i+1} - U_{i+1} \tag{2.7c}$$

where $\beta = 2\cos(\pi\hat{\lambda})$.

Representations of this algebra have been studied in [10] in a different context and turn out to coincide with those of [11] , obtained from the R-matrix of the quantum group $SU(N)_q$, with $\beta = q + q^{-1}$. The vertex as well as the height representations constructed in the latter both commute with the $SU(N)_q$ quantum group generators acting on a tensor product of its fundamental representation $V_f \otimes V_f \otimes \cdots \otimes V_f$, isomorphic to the height configuration space of the zig-zag line of fig. 1 . The U's found by this construction for given N consequently lie in the commutant of the quantum group $SU(N)_q$, described as a subalgebra of the Hecke algebra (2.7) obtained by imposing the vanishing of the $N+1$-th generalized antisymmetrizer S^-_{N+1} defined as follows. Let $X_i = q^{-1} - U_i$, $i = 1, 2, \cdots, N$. Due to (2.7) the X's satisfy the braid associativity condition:

$$X_i X_{i+1} X_i = X_{i+1} X_i X_{i+1} \tag{2.8}$$

Therefore they carry a representation of the symmetric group on $N+1$ objects, namely of its generators, the transpositions $\tau_{i,i+1}$ of neighbours. A generic permutation σ of the $N+1$ objects has a unique decomposition $\sigma = \prod_{i \in I_\sigma} \tau_{i,i+1}$ up to the braid equation for τ and the reduction $\tau^2 = 1$. If we define $X_\sigma = \prod_{i \in I_\sigma} X_i$, then the commutant of the $SU(N)_q$ quantum group is obtained from the Hecke algebra by setting:

$$S^-_{N+1} = \sum_{\sigma \in S_{N+1}} (-q)^{card(I_\sigma)} X_\sigma = 0 \tag{2.9}$$

We are naturally led to look for target graphs $\mathcal{G}$ which support a representation U of (2.7) and (2.9) for some N. This is precisely what we mean by "$SU(N)$ height models".

Let us now briefly describe the models based on the representations of [10] and [11] , studied in [12] and denoted in the following by $\mathbf{A}^{(n)}$, where n is an integer strictly greater than N, called the altitude of the model [1]. The heights of these models take their values among the level $k = n - N$ integrable representation labels of the Kac-Moody algebra $A^{(1)}_{N-1} = \widehat{SU(N)}$, i.e. they belong to the Weyl alcôve $P^{(n)}_{++}$:

$$P^{(n)}_{++} = \{\lambda = \sum_{i=1}^{N-1} \lambda_i \Lambda_i \mid \lambda_i \geq 1, \quad \sum_{i=1}^{N-1} \lambda_i \leq n-1\} \tag{2.10}$$

where Λ_i denote the fundamental weights of $SU(N)$. The corresponding graph $\mathcal{A}^{(n)}$ is built by drawing oriented links along the vectors (weights of the fundamental representation f):

$$e_1 = \Lambda_1 \tag{2.11a}$$

$$e_N = -\Lambda_{N-1} \tag{2.11b}$$

$$e_i = \Lambda_i - \Lambda_{i-1} \text{ for } 2 \leq i \leq N-1 \tag{2.11c}$$

between the points of $P^{(n)}_{++}$. The identity representation (1) of $\widehat{SU(N)}_k$, labelled by the apex θ of the alcôve:

$$\theta = \sum_{i=1}^{N-1} \Lambda_i \tag{2.12}$$

is only connected to the fundamental representation (f) of index $(\theta + e_1)$, whereas only the conjugate of the fundamental $(\bar{f}) = (\theta + e_N)$ is linked to it. The graph $\mathcal{A}^{(n)}$ indeed encodes the fusions of the representations of the Kac-Moody algebra $\widehat{SU(N)}_k$ by the fundamental (f):

$$(f) * (\lambda) = \sum_{\mu} A^{(n)}_{\lambda\mu}(\mu) \tag{2.13}$$

It is known [13] that the fusion matrices $(N^{(\lambda)})_{\mu\nu} = N_{\lambda\ \mu}^{\ \nu}$ (and in particular $N^{(f)} = A^{(n)}$) are diagonalized by the matrix S of the modular transformation $\tau \to -\frac{1}{\tau}$ acting linearly on the characters of the Kac-Moody integrable representations:

$$\chi_\lambda(-\frac{1}{\tau}) = \sum_{\mu \in P^{(n)}_{++}} S_\lambda^{\ \mu} \chi_\mu(\tau) \tag{2.14}$$

[1] In this paper, we adhere to the following conventions: boldface letters $\mathbf{A}^{(n)}$, $\mathbf{D}^{(n)}$, $\mathbf{E}^{(n)}$, $\mathbf{G}^{(n)}$, etc... refer to statistical mechanical models, calligraphic letters $\mathcal{A}^{(n)}, \cdots$ to graphs, roman capitals $A^{(n)}, \cdots$ to their adjacency matrix, and capitals in brackets $(A^{(n)}), \cdots$ to modular invariants.

in the sense that

$$N_{\lambda\ \mu}^{\ \nu} = \sum_{\rho} \frac{S_\lambda^{\ \rho} S_\mu^{\ \rho} S_\nu^{\ \rho *}}{S_1^{\ \rho}}. \tag{2.15}$$

The $\mathcal{A}^{(n)}$ graph enjoys therefore a self duality property : its eigenvalues read

$$\beta^{(\mu)} = \frac{S_f^{\ \mu}}{S_1^{\ \mu}} \tag{2.16}$$

for $\mu \in P^{(n)}_{++}$, which labels both the vertices and the eigenvalues of the graph. The representation of (2.7) and (2.9) attached to this graph reads:

$$U_i = \begin{array}{ccc} & \lambda & \\ \lambda+e_j & \diamond & \lambda+e_k \\ & \lambda+e_j+e_l & \end{array} = (1-\delta_{jl}) \frac{(s_{jl}(\lambda+e_j)s_{jl}(\lambda+e_k))^{\frac{1}{2}}}{s_{jl}(\lambda)} \tag{2.17}$$

where $s_{jl}(x) = \sin(\frac{\pi}{n} x.(e_j - e_l))$. The corresponding value of the parameter $\hat{\lambda}$ is merely $\frac{1}{n}$. That the representation (2.17) satisfies (2.9) is a highly non trivial identity which comes naturally out of its derivation in [11] .

More general Boltzmann weights w involving elliptic functions of nome $p \in \mathbf{C}$, which coincide with (2.17) in the limit $p \to 0$, have been derived in [12] . The "nome" p stands for a temperature in the model which drives it away from criticality. At $p = 0$ the models undergo a second order phase transition, and the elliptic functions reduce to the trigonometric functions entering ρ (2.6) . These critical models $\mathbf{A}^{(n)}$ have been shown [14] to be described in the continuum limit by the discrete series of unitary "$\mathcal{W}_N$-minimal" [2] conformal field theories described by the cosets:

$$SU(N)_{k-1} \times SU(N)_1 / SU(N)_k \tag{2.18}$$

with central charge

$$c = (N-1)\left(1 - \frac{N(N+1)}{n(n-1)}\right) \tag{2.19}$$

where n again denotes the altitude $n = N + k$. From the transfer matrix formalism on a cylinder [15], one finds that the continuum limit of the critical partition function with toroidal boundary conditions

$$Z = \sum_{\substack{\text{height} \\ \text{configurations}}} \prod_{\substack{\text{all} \\ \text{squares}}} w(\text{square}) \tag{2.20}$$

[2] that is, these coset theories are supposed to be minimal theories for some extended algebra $\mathcal{W}_N$, i.e. to involve only a finite number of representations of that algebra.

may be written as a sesquilinear combination of characters of the coset (2.18) :

$$Z(\tau) = \sum_{\alpha,\bar{\alpha}} \mathcal{N}_{\alpha,\bar{\alpha}} \chi_\alpha(\tau) \chi_{\bar{\alpha}}(\tau)^* \tag{2.21}$$

where τ stands for the modular ratio of the torus and $\mathcal{N}_{\alpha,\bar{\alpha}}$ are non negative integers. Eqn (2.21) expresses the Hilbert space of the conformal theory as a sum, weighted by the multiplicities $\mathcal{N}_{\alpha,\bar{\alpha}}$, of tensor products of left and right representation spaces of the symmetry $\mathcal{W}_N$-algebra of the coset (2.18). One imposes $\mathcal{N}_{1,1} = 1$, where 1 labels the identity representation of conformal weight 0, for the vacuum to be non degenerate. In the $\mathbf{A}^{(n)}$ case we have:

$$\mathcal{N}_{\alpha,\bar{\alpha}} = \delta_{\alpha,\bar{\alpha}} \tag{2.22}$$

But there are other partition functions of the form (2.21) with different operator contents.

A natural constraint arises from the statistical mechanical picture: the function (2.21) must be invariant under reparametrizations of the torus, i.e. under the action of the modular group on τ, generated by :

$$T : \tau \to \tau + 1 \quad \chi_\alpha(\tau+1) = \exp(2i\pi h_\alpha)\chi_\alpha(\tau) \tag{2.23a}$$

$$S : \tau \to -\frac{1}{\tau} \quad \chi_\alpha(-\frac{1}{\tau}) = \sum_{\beta \in P_{++}^{(N)}} S_\alpha^{\ \beta} \chi_\beta(\tau) \tag{2.23b}$$

where h_α is the conformal weight of the representation α. The solutions (2.21) to this constraint of modular invariance have been classified in [1] for the case $N = 2$. They are labelled by a pair of simply laced Lie algebras A, D, E of consecutive Coxeter numbers $n-1$ and n, if the central charge (2.19) is $c = 1 - 6/n(n-1)$. For $N \geq 3$ no complete classification is known. We believe that solutions to (2.7) and (2.9) other than $\mathbf{A}^{(n)}$ correspond to modular invariants other than the trivial diagonal solution (2.22). Among them, a family denoted $\mathbf{D}^{(n)}$ obtained from $\mathbf{A}^{(n)}$ by an "orbifold" procedure has already been identified and studied in [16]–[18] .

3. Graphs and Boltzmann weights

3.1. The graphs

Let us briefly describe the strategy followed in the $SU(2)$ case to find admissible target graphs. Given any non oriented graph $\mathcal{G}$, consider the highest eigenvalue $\beta = \beta^{(1)}$ of its adjacency matrix G and the corresponding Perron-Frobenius eigenvector with components $\{\psi_a^{(1)}\}$, $a \in v(\mathcal{G})$, the set of vertices or target heights of $\mathcal{G}$. These components are positive $\psi_a^{(1)} > 0$, $\forall a \in v(\mathcal{G})$ and their normalization is fixed by imposing that $\psi^{(1)}$ be of unit norm. Then the matrices :

$$U_i = \begin{array}{ccc} & a & \\ b & \diamond & c \\ & d & \end{array} = \delta_{a,d} \frac{(\psi_b^{(1)}\psi_c^{(1)})^{\frac{1}{2}}}{\psi_a^{(1)}} \tag{3.1}$$

solve equations (2.7) and (2.9) for $N = 2$ and the same β [4]. In fact any other eigenvector $\psi_a^{(\mu)}$ with no vanishing component, for an eigenvalue $\beta^{(\mu)}$ of G would be a solution as well, for $\beta = \beta^{(\mu)}$, but would not lead to real Boltzmann weights : such solutions are known [19] to be described by non unitary conformal theories in the continuum limit.

Returning to (3.1), a closer study of the model shows that it has the same critical behavior as the Q-state Potts model [2] with $Q = \beta^2$, i.e. it undergoes a second order phase transition iff $\beta \leq 2$. The case $\beta = 2$ corresponds to $c = 1$ non minimal theories [4] . For $\beta < 2$, due to the Cartan classification, the graph $\mathcal{G}$ has to lie among the A, D, E Dynkin diagrams corresponding to the simply laced Lie algebras (see [20] for a nice proof [3]). This reference to simply laced Lie algebras should not be misinterpreted as the appearance of some new symmetry: the quantum $SU(2)_q$ algebra seems to be the only symmetry concerning the whole set of $SU(2)$ height models. These models are described in the continuum limit by the (A_{n-2}, G) minimal series of [1] with central charge $c = 1 - \frac{6}{n(n-1)}$, where n is the Coxeter number of the Dynkin diagram $\mathcal{G}$ and $\beta = 2\cos(\frac{\pi}{n})$.

It has been noticed in [3] that the physical condition $\beta < 2$ can be replaced by the following. Let $N_{\lambda\mu}^{\ \ \nu}$ denote the fusion coefficients of the affine $\widehat{SU(2)}_k$ Kac-Moody theory at level $k = n-2$, with the integrable representations labels λ, μ, $\nu \in P_{++}^{(n)}$ defined in (2.10) . From Verlinde's formula (2.15) it follows that the matrices $N_{\mu\nu}^{(\lambda)} = N_{\lambda\mu}^{\ \ \nu}$ satisfy $N^{(\lambda)}N^{(\mu)} = \sum N_{\lambda\mu}^{\ \ \nu} N^{(\nu)}$, hence form the regular representation of this fusion algebra. One then looks for general matrix representations $V^{(\lambda)}$ of this fusion algebra:

$$V^{(\lambda)}V^{(\mu)} = \sum_\nu N_{\lambda\mu}^{\ \ \nu} V^{(\nu)} \tag{3.2}$$

[3] The quotients A_{2p}/Z_2 have also $\beta < 2$ but do not produce distinct statistical mechanical models [3].

The only solutions of (3.2) with non negative integer entries are such that $G = V^{(2)}$ has to be the adjacency matrix of an A, D, E Dynkin diagram. Here $(2) = (f)$ labels the fundamental representation of $\widehat{SU(2)}_k$. Indeed the V's are integer intertwiners between the matrices A and G at the same altitude n, due to (2.13) :

$$V^{(\mu)}G = \sum_{\nu} A_{\mu\nu} V^{(\nu)} \tag{3.3}$$

which implies that the eigenvalues of G lie among those of A with possible multiplicities. This provides an alternate though intriguing condition restricting the possible target graphs $\mathcal{G}$, which may be generalized to any N.

It is thus suggested that for arbitrary N, one should look for all representations of the $\widehat{SU(N)}_k$ fusion algebra over non negative integer matrices $V^{(\lambda)}_{ab}$, generalizing the regular representation provided by the fusion matrices $V^{(\lambda)}_{reg\ \mu\nu} = N_{\lambda\mu}{}^{\nu}$. These new representations are subject to the same additional conditions as the regular one:

i) the representation is consistent with the conservation of a $\mathbf{Z}_N$ charge, that is one can attach a $\mathbf{Z}_N$ variable $\tau(a)$ to each a, in such a way that

$$V^{(\lambda)}_{ab} \neq 0 \qquad \text{only if} \qquad \tau(a) + \tau(\lambda) = \tau(b) \tag{3.4}$$

where $\tau(\lambda)$ denotes the ordinary $\mathbf{Z}_N$ "N-ality" of points of $P^{(n)}_{++}$

$$\tau(\lambda) = \sum_{i=1}^{N-1} (i-1)\lambda_i \quad \mathrm{mod} N \tag{3.5}$$

ii) there is an involution $a \to \bar{a}$ under which τ is odd and

$$(V^{(\bar{\lambda})})_{\bar{a}\bar{b}} = (V^{(\lambda)})_{ab} = (V^{(\lambda)})_{\bar{b}\bar{a}} \tag{3.6}$$

This involution is a generalization of the conjugation of representations of $SU(N)$. Conditions i) and ii) imply that the matrices $V^{(\lambda)}$ commute among themselves, hence with their transpose $V^{(\lambda)\,t} = V^{(\bar{\lambda})}$, hence may be diagonalized simultaneously in an orthonormal basis $\psi^{(\rho)}_a$. Moreover, they may all be expressed polynomially in terms of the $N-1$ representatives of the fundamental representations of $SU(N)$: $V^{(\lambda)} = P^{(\lambda)}(V^{(f_1=f)}, V^{(f_2)}, \ldots, V^{(f_{N-1}=\bar{f})})$ with representation-independent polynomials $P^{(\lambda)}$.

As a consequence, it may be shown [3] that the eigenvalues of $V^{(\lambda)}$ are also eigenvalues of $V^{(\lambda)}_{reg} = N^{(\lambda)}$, given by Verlinde's formula as $S_\lambda{}^\rho / S_1{}^\rho$. Hence one may write

$$V^{(\lambda)}_{ab} = \sum_{\rho} \frac{\psi^{(\rho)}_a \psi^{(\rho)\,*}_b S_\lambda{}^\rho}{S_1{}^\rho} \tag{3.7}$$

Due to possible degeneracies of the eigenvalues of G, there might be some arbitrariness in the choice of the orthonormal basis of eigenvectors ψ, but it does not affect $V^{(\lambda)}$. If one defines $G = V^{(f)}$, its spectrum of eigenvalues is to be chosen among the values of (2.16). This G defines in this more abstract approach the adjacency matrix of a graph $\mathcal{G}$, and all the previous properties required on $V^{(\lambda)}$ may be rephrased in terms of the graph: the latter should be $N-$colourable (equ. (3.4)), be invariant under the involution $a \to \bar{a}$ (reversing all arrows), and most importantly, have the very restrictive spectral properties just mentioned. Notice that for $N \geq 4$, however, one needs more than just this matrix $G = V^{(f)}$ and $G^t = V^{(\bar{f})}$ to generate the whole set of matrices $V^{(\lambda)}$. Although the allowed configurations of neighbouring heights are fully specified by G, it seems that the representatives of the other fundamental weights are required to achieve a good description of the "path algebra".

To summarize, the first task seems to find all representations of the fusion algebra over matrices of non negative integers, endowed with some additional properties. In [3], a certain number of solutions to this problem have been given in the case of $SU(3)$. Some of them are displayed below.

3.2. The Boltzmann weights

The computation of Boltzmann weights for a given target graph $\mathcal{G}$ has been investigated in [3] , using the "cell calculus" (see also [21] and [22] for details), which consists of intertwining the $\mathbf{A}^{(n)}$ and $\mathbf{G}^{(n)}$ Boltzmann weights. The direct consequence of such a construction is that any algebra or identity satisfied by the $\mathbf{A}^{(n)}$ weights applies to $\mathbf{G}^{(n)}$ as well. Apart from (2.7) and (2.9) it is thus possible to derive other identities for the general Boltzmann weights, which could help to compute them directly.

First, the Markov trace condition [23] [24] states that:

$$\sum_c$$

$$\psi^{(1)}_c = f_N(\beta) G_{ab} \psi^{(1)}_b \tag{3.8}$$

where $f_N(\beta)$ is independant of the choice of representation U of (2.7) (2.9) . Using the $\mathbf{A}^{(n)}$ representation, one finds $f_N(\beta) = \mathcal{U}_{N-2}(\frac{\beta}{2})$, $\mathcal{U}_p$ being the p-th Chebishev polynomial of the second kind. Consider now a partition function (2.20) with cylindric boundary conditions, i.e. periodic along a vertical zig-zag line, and with a value of the height fixed to a (resp. to c) on the upper (resp. lower) boundary of the cylinder. Denote it by $Z_{ac}(\mathcal{A})$. Then by repeated use of the Hecke conditions (2.7) and the Markov property (3.8) it is easy to show that the modified partition function:

$$Z_{ac}^{mod}(\mathcal{G}) = \sum_{a,c\in v(\mathcal{G})} \psi_a^{(1)*} Z_{ac}(\mathcal{G}) \psi_c^{(1)} \tag{3.9}$$

is again independant of the representation U chosen. As this modification is irrelevant in the thermodynamic limit, this shows that at a given altitude n all the $\mathbf{G}^{(n)}$ models share the same critical properties as the $\mathbf{A}^{(n)}$ one, provided (3.8) holds.

The relation (3.8) clearly cannot hold for all the other eigenvectors of G: the relation could be inverted and would yield inconsistencies for the Boltzmann weights. This is not the case for (3.9). Suppose (3.9) holds for any eigenvector (normalized to unity) $\psi^{(\mu)}$ of G. The corresponding eigenvalues $\beta^{(\mu)}$ lie among those of A, except maybe for their multiplicities. They are labelled by a subset of (maybe repeated) weights in $P_{++}^{(n)}$, and coincide with (2.16). Let us call this set the exponents of $\mathcal{G}$, denoted $Exp(\mathcal{G})$. One has :

$$\begin{aligned}\sum_{a,c\in v(\mathcal{G})} \psi_a^{(\mu)*} Z_{ac}(\mathcal{G}) \psi_c^{(\mu)} &= \sum_{\lambda,\nu\in v(\mathcal{A})=P_{++}^{(n)}} S_\lambda{}^{\mu*} Z_{\lambda,\nu}(\mathcal{A}) S_\nu{}^{\mu} \\ &= \sum_{\nu\in P_{++}^{(n)}} Z_{1,\nu}(\mathcal{A}) \frac{S_\nu{}^{\mu}}{S_1{}^{\mu}}\end{aligned} \tag{3.10}$$

which can be inverted using the orthonormality of the ψ's :

$$Z_{ac}(\mathcal{G}) = \sum_{\lambda\in P_{++}^{(n)}} \left[\sum_{\mu\in Exp(\mathcal{G})} \psi_a^{(\mu)} \psi_c^{(\mu)*} \frac{S_\lambda{}^{\mu}}{S_1{}^{\mu}} \right] Z_{1,\lambda}(\mathcal{A}) \tag{3.11}$$

The expression in brackets with indices a, $c \in v(\mathcal{G})$ and $\lambda \in P_{++}(n)$ is readily recognized as the matrix element $V_{ac}^{(\lambda)}$ of the intertwiner defined in (3.7) such that $G = V^{(f)}$. We prefer to restate condition (3.11) in these terms:

$$Z_{ac}(\mathcal{G}) = \sum_{\lambda\in P_{++}^{(n)}} V_{ac}^{(\lambda)} Z_{1,\lambda}(\mathcal{A}) \tag{3.12}$$

This equation is indeed satisfied in the $SU(2)$ case for any finite size of the cylinder, where $Z_{1,\lambda}(\mathcal{A})$ is identified as the minimal Virasoro character $\chi_{1,\lambda}(\tau)$ [6]

$$Z_{ac}(\mathcal{G}) = \sum_{\lambda\in P_{++}^{(n)}} V_{ac}^{(\lambda)} \chi_{1,\lambda}(\tau) \tag{3.13}$$

In that form, it gives a natural interpretation of the property of the $V^{(\lambda)}$ to be non negative integers [25]; that they obey the fusion algebra has also been interpreted in this context in some cases in [7]. Equ. (3.12) still holds in the continuum limit for the off critical models, with $\tau = i\frac{\pi}{2} - \log 2\pi + \frac{4\pi^2}{n \log p}$ [6] and the nome p describing a temperature shift from the critical value. Eqn (3.12) gives a nice interpretation of the intertwiner entries as expressing a group theoretical decomposition of reducible $\mathbf{G}^{(n)}$ representations of the Hecke algebra (2.7) (2.9) onto $\mathbf{A}^{(n)}$ irreducible ones [26]. We expect (3.12) to generalize to any $N \geq 3$. What can we conversely learn from this identity? Considering the smallest possible cylinder of size 1×1, we get :

$$\sum_{b\in v(\mathcal{G})} \begin{array}{c} a \\ b \diamond b \\ c \end{array} = \sum_{\lambda\in P_{++}^{(n)}} V_{ac}^{(\lambda)} \begin{array}{c} \theta \\ \theta+e_1 \diamond \theta+e_1 \\ \theta+e_1+e_2 \end{array} \delta_{\lambda,\theta+e_1+e_2} = \beta V_{ac}^{(\theta+e_1+e_2)} \tag{3.14}$$

where θ is defined in (2.12) and labels the identity (1) representation of $\widehat{SU(N)}_k$, and e_i is as in (2.11) . It is easy to show that (3.14) and (3.8) are compatible. We learn that the projector $\frac{1}{\beta}U_i$ (2.7a) of size $G_{ac}^2 \times G_{ac}^2$ has rank $V_{ac}^{(\theta+e_1+e_2)}$.

The systematic solution of (3.8) and (3.12) might give clues on the general form of the Boltzmann weights w for a given graph $\mathcal{G}$ and simplify the general cell calculus. A number of $SU(3)$ solutions have been worked out completely though no simple expression seems to emerge [3] .

4. Modular invariance, graphs and fusion rules

4.1. Kac-Moody and coset modular invariants

We first recall a few general properties of modular invariant toroidal partition functions For a "rational" conformal field theory [27], this partition function Z may be recast into a *finite* sum of the form

$$Z(\tau) = \sum_{i,j} \mathcal{N}_{i,j}\chi_i(\tau)\chi_j(\tau)^* \tag{4.1}$$

where χ_i denote blocks, i.e. finite or infinite sums with integer coefficients of Virasoro characters, that are supposed to be the characters of some "extended algebra". Two situations may occur: either Z may be written as a sum of squares $Z = \sum |\chi_i|^2$ for some suitable choice of the blocks, or it cannot. This apparent tautology reflects two very different situations: in the first case, called "type I", Z appears as the trivial (diagonal) modular invariant of the extended algebra. The second case (type II) is obtained from the first by the action of some automorphism of the *extended* fusion algebra [28] [29].

These general considerations are illustrated on the case where the extended algebra contains a Kac-Moody algebra, that we take for definiteness as $\widehat{SU(N)}_k$ (see, however, some details about the case of G_2 in the Appendix). The set of characters of this Kac-Moody algebra, labelled by integrable weights belonging to $P_{++}^{(n)}$ (see (2.10)), forms a representation of the modular group. A study of the commutant of the modular group acting on this representation, i.e. of the general form of the modular invariants sesquilinear in the characters with arbitrary coefficients, has been recently completed [30]. It remains to find among these the physical invariants, with positive integer coefficients $\mathcal{N}_{\lambda\bar{\lambda}}$, (and moreover $\mathcal{N}_{11} = 1$). Beside the "trivial" diagonal modular invariant $Z = \sum_{\lambda\in P_{++}^{(n)}} |\chi_\lambda|^2$, that we denote as $(A^{(n)})$, other solutions are associated with the automorphism of the weight lattice, i.e. with the center Z_N [31] [32] and its subgroups [33]-[35]. When N is a prime number, as $N = 3$ for example, these solutions take only two distinct forms depending whether $n = k + N$ is or is not a multiple of N.

$$Z = \frac{1}{N} \sum_{\lambda\in Q\cap P_{++}^{(n)}} |\chi_\lambda + \chi_{\sigma\lambda} + \cdots + \chi_{\sigma^{N-1}\lambda}|^2 \qquad \text{if } N \text{ divides } n \tag{4.2}$$

and

$$Z = \sum_{\lambda\in Q\cap P_{++}^{(n)}} |\chi_\lambda|^2 + \sum_{\lambda\in P_{++}^{(n)}\setminus Q} \chi_\lambda\chi^*_{\sigma^\tau\lambda} \qquad \text{if } N \text{ does not divide } n \tag{4.3}$$

where σ denotes the Z_N automorphism

$$\sigma\lambda = \sum_{i=1}^{N-2} \lambda_{i+1}\Lambda_i + (n - \sum_{i=1}^{N-1} \lambda_i)\Lambda_{N-1} \tag{4.4}$$

and τ is the Z_N charge defined in (3.5). We refer to these modular invariants as $(D^{(n)})$.

The difficult task is to find the exceptional invariants $(E^{(n)})$. Various methods are known to produce invariants: conformal embeddings [36] , use of identities such as Mac-Donald's [37] [28]which give certain constant combinations of characters , or else [38].

All these invariants may be twisted by the conjugation of representations, i.e. the "right" characters $\chi_{\lambda'}(\bar{\tau})$ may be replaced by $\chi_{\bar{\lambda}'}(\bar{\tau})$. Although the partition functions are numerically the same (since $\chi_\lambda = \chi_{\bar{\lambda}}$), they may correspond to different theories.

Although these methods are known in the case of $\widehat{SU(2)}$ to exhaust all modular invariants, it is by no means clear whether they do for higher N. In Table I, is displayed the list of known modular invariants of $\widehat{SU(3)}$.

We now turn to the case of coset theories, $\widehat{SU(N)}_{k-1} \times \widehat{SU(N)}_1/\widehat{SU(N)}_k$, whose characters are obtained in the decomposition [39]

$$\chi_{(k-1)\,\lambda}\chi_{(1)\,\mu} = \sum \chi_{(k)\,\nu}\chi_{\text{coset}\,\lambda,\mu,\nu} \tag{4.5}$$

Suppose that we know a triplet of modular invariants pertaining to the levels $k-1$, 1 and k, described by matrices $\mathcal{N}_{(k-1)\,\lambda\lambda'}$, etc... This means that the matrix $\mathcal{N}_{(k-1)\,\lambda\lambda'}$ commutes with the generators $S_{(k-1)}$ and $T_{(k-1)}$ of modular transformations of the Kac-Moody characters (equ. (2.23)) of level $k-1$, and likewise for $\mathcal{N}_{(1)\,\mu\mu'}$ and $\mathcal{N}_{(k)\,\nu\nu'}$. From (4.5) it follows that the coset characters transform by the matrices $S_{(k-1)} \otimes S_{(1)} \otimes S^\dagger_{(k)}$ and $T_{(k-1)} \otimes T_{(1)} \otimes T^\dagger_{(k)}$, and that $\mathcal{N}_{(k-1)} \otimes \mathcal{N}_{(1)} \otimes \mathcal{N}_{(k)}$ commutes with these matrices. This produces an invariant provided all the triplets (λ, μ, ν) that appear in this tensor product correspond effectively to representations in the coset, i.e. have non vanishing branching functions in (4.5)[40]. In this particular class of coset $G\times G/G$, it has been proved [41] that the branching function $\chi_{\text{coset}\,\lambda,\mu,\nu}$ does not vanish if and only if $\lambda + \mu - \nu \in Q$, the root lattice. This imposes constraints on the possible invariants that one may construct in that way. (Conversely, it is not obvious that *all* modular invariants of the coset theory can be obtained in that way. To our knowledge, it has been proved only in the case $N = 2$.) For our present purpose, we focus our attention on the modular invariants which are diagonal

Table I. List of known $SU(3)_k$ Kac-Moody modular invariants; $n = k + 3$.

$$(A^{(n)}) \qquad Z = \sum_{\lambda \in P_{++}^{(n)}} |\chi_\lambda|^2$$

$$(D^{(n)}) \qquad Z = \begin{cases} \frac{1}{3} \sum_{\lambda \in Q \cap P_{++}^{(n)}} |\chi_\lambda + \chi_{\sigma\lambda} + \cdots + \chi_{\sigma^{N-1}\lambda}|^2 & \text{if 3 divides } n \\ \sum_{\lambda \in Q \cap P_{++}^{(n)}} |\chi_\lambda|^2 + \sum_{\lambda \in P_{++}^{(n)} \setminus Q} \chi_\lambda \chi^*_{\sigma^r \lambda} & \text{if 3 does not divide } n \end{cases}$$

$$(D^{(n)*}) \qquad Z = \begin{cases} \frac{1}{3} \sum_{\lambda \in Q \cap P_{++}^{(n)}} \left(\sum_{k=0}^{2} \chi_{\sigma^k \lambda} \right) \left(\sum_{k=0}^{2} \chi_{\sigma^k \bar{\lambda}} \right)^* & \text{if 3 divides } n \\ \sum_{\lambda \in Q \cap P_{++}^{(n)}} |\chi_\lambda \chi_{\bar{\lambda}} + \sum_{\lambda \in P_{++}^{(n)} \setminus Q} \chi_\lambda \chi^*_{\sigma^{r^2} \bar{\lambda}} & \text{if 3 does not divide } n \end{cases}$$

$$(E^{(8)}) \qquad Z = |\chi_{(1,1)} + \chi_{(3,3)}|^2 + |\chi_{(3,2)} + \chi_{(1,6)}|^2 + |\chi_{(2,3)} + \chi_{(6,1)}|^2 + |\chi_{(4,1)} + \chi_{(1,4)}|^2 + |\chi_{(1,3)} + \chi_{(4,3)}|^2 + |\chi_{(3,1)}\chi_{(3,4)}|^2$$

$$(E^{(8)*}) \qquad Z = |\chi_{(1,1)} + \chi_{(3,3)}|^2 + \left(\chi_{(3,2)} + \chi_{(1,6)}\right)\left(\chi_{(2,3)} + \chi_{(6,1)}\right)^* + \text{c.c.} + |\chi_{(4,1)} + \chi_{(1,4)}|^2 + \left(\chi_{(1,3)} + \chi_{(4,3)}\right)\left(\chi_{(3,1)}\chi_{(3,4)}\right) + \text{c.c.}$$

$$(E^{(12)}) \qquad Z = |\chi_{(1,1)} + \chi_{(10,1)} + \chi_{(1,10)} + \chi_{(5,5)} + \chi_{(5,2)} + \chi_{(2,5)}|^2 + 2|\chi_{(3,3)} + \chi_{(3,6)} + \chi_{(6,3)}|^2$$

$$(E_{MS}^{(12)}) \qquad Z = |\chi_{(1,1)} + \chi_{(10,1)} + \chi_{(1,10)}|^2 + |\chi_{(3,3)} + \chi_{(3,6)} + \chi_{(6,3)}|^2 + |\chi_{(5,5)} + \chi_{(5,2)} + \chi_{(2,5)}|^2 + |\chi_{(4,7)} + \chi_{(7,1)} + \chi_{(1,4)}|^2 + |\chi_{(7,4)} + \chi_{(1,7)} + \chi_{(4,1)}|^2 + 2|\chi_{(4,4)}|^2 + \left(\chi_{(2,2)} + \chi_{(8,2)} + \chi_{(2,8)}\right)\chi^*_{(4,4)} + \text{c.c.}$$

$$(E_{MS}^{(12)*}) \qquad Z = |\chi_{(1,1)} + \chi_{(10,1)} + \chi_{(1,10)}|^2 + |\chi_{(3,3)} + \chi_{(3,6)} + \chi_{(6,3)}|^2 + |\chi_{(5,5)} + \chi_{(5,2)} + \chi_{(2,5)}|^2 + 2|\chi_{(4,4)}|^2 + \left(\chi_{(4,7)} + \chi_{(7,1)} + \chi_{(1,4)}\right)\left(\chi_{(7,4)} + \chi_{(1,7)} + \chi_{(4,1)}\right)^* + \text{c.c.} + \left(\chi_{(2,2)} + \chi_{(8,2)} + \chi_{(2,8)}\right)\chi^*_{(4,4)} + \text{c.c.}$$

$$(E^{(24)}) \qquad Z = |\chi_{(1,1)} + \chi_{(22,1)} + \chi_{(1,22)} + \chi_{(5,5)} + \chi_{(5,14)} + \chi_{(14,5)} + \chi_{(11,11)} + \chi_{(11,2)} + \chi_{(2,11)} + \chi_{(7,7)} + \chi_{(7,10)} + \chi_{(10,7)}|^2 + |\chi_{(7,1)} + \chi_{(16,7)} + \chi_{(1,16)} + \chi_{(1,7)} + \chi_{(7,16)} + \chi_{(16,1)} + \chi_{(5,8)} + \chi_{(11,5)} + \chi_{(8,11)} + \chi_{(8,5)} + \chi_{(5,11)} + \chi_{(11,8)}|^2$$

Table II. List of known $SU(3)_{k-1} \times SU(3)_1 / SU(3)_k$ coset modular invariants, of the form $(A, A, G^{(n)})$, denoted here $(G^{(n)})$; $n = k + 3$.

$$(A^{(n)}) \qquad Z = \frac{1}{3} \sum_{\lambda \in P_{++}^{(n-1)}} \sum_{\mu \in P_{++}^{(n)}} |\chi_{\lambda,\mu}|^2$$

$$(D^{(n)}) \qquad Z = \frac{1}{9} \sum_{\lambda \in P_{++}^{(n-1)}} \sum_{\mu \in Q \cap P_{++}^{(n)}} |\sum_{k=0}^{2} \chi_{\lambda,\sigma^k \mu}|^2 \quad \text{if 3 divides n}$$

$$(D^{(n)*}) \qquad Z = \frac{1}{9} \sum_{\lambda \in Q \cap P_{++}^{(n)}} \left(\sum_{k=0}^{2} \chi_{\lambda,\sigma^k \mu} \right) \left(\sum_{k=0}^{2} \chi_{\lambda,\sigma^k \bar{\mu}} \right)^* \quad \text{if 3 divides n}$$

$$(E^{(8)}) \qquad Z = \frac{1}{3} \sum_{\lambda \in P_{++}^{(7)}} \Big[|\chi_{\lambda,(1,1)} + \chi_{\lambda,(3,3)}|^2 + |\chi_{\lambda,(3,2)} + \chi_{\lambda,(1,6)}|^2 + |\chi_{\lambda,(2,3)} + \chi_{\lambda,(6,1)}|^2 + |\chi_{\lambda,(4,1)} + \chi_{\lambda,(1,4)}|^2 + |\chi_{\lambda,(1,3)} + \chi_{\lambda,(4,3)}|^2 + |\chi_{\lambda,(3,1)}\chi_{\lambda,(3,4)}|^2 \Big]$$

$$(E^{(12)}) \qquad Z = \frac{1}{3} \sum_{\lambda \in P_{++}^{(11)}} \Big[|\chi_{\lambda,(1,1)} + \chi_{\lambda,(10,1)} + \chi_{\lambda,(1,10)} + \chi_{\lambda,(5,5)} + \chi_{\lambda,(5,2)} + \chi_{\lambda,(2,5)}|^2 + 2|\chi_{\lambda,(3,3)} + \chi_{\lambda,(3,6)} + \chi_{\lambda,(6,3)}|^2 \Big]$$

$$(E_{MS}^{(12)}) \qquad Z = \frac{1}{3} \sum_{\lambda \in P_{++}^{(11)}} \Big[|\chi_{\lambda,(1,1)} + \chi_{\lambda,(10,1)} + \chi_{\lambda,(1,10)}|^2 + |\chi_{\lambda,(3,3)} + \chi_{\lambda,(3,6)} + \chi_{\lambda,(6,3)}|^2 + |\chi_{\lambda,(5,5)} + \chi_{\lambda,(5,2)} + \chi_{\lambda,(2,5)}|^2 + |\chi_{\lambda,(4,7)} + \chi_{\lambda,(7,1)} + \chi_{\lambda,(1,4)}|^2 + |\chi_{\lambda,(7,4)} + \chi_{\lambda,(1,7)} + \chi_{\lambda,(4,1)}|^2 + 2|\chi_{\lambda,(4,4)}|^2 + \left(\chi_{\lambda,(2,2)} + \chi_{\lambda,(8,2)} + \chi_{\lambda,(2,8)}\right)\chi^*_{\lambda,(4,4)} + \text{c.c.} \Big]$$

$$(E_{MS}^{(12)*}) \qquad Z = \frac{1}{3} \sum_{\lambda \in P_{++}^{(11)}} \Big[|\chi_{\lambda,(1,1)} + \chi_{\lambda,(10,1)} + \chi_{\lambda,(1,10)}|^2 + |\chi_{\lambda,(3,3)} + \chi_{\lambda,(3,6)} + \chi_{\lambda,(6,3)}|^2 + |\chi_{\lambda,(5,5)} + \chi_{\lambda,(5,2)} + \chi_{\lambda,(2,5)}|^2 + 2|\chi_{\lambda,(4,4)}|^2 + \left(\chi_{\lambda,(4,7)} + \chi_{\lambda,(7,1)} + \chi_{\lambda,(1,4)}\right)\left(\chi_{\lambda,(7,4)} + \chi_{\lambda,(1,7)} + \chi_{\lambda,(4,1)}\right)^* + \text{c.c.} + \left(\chi_{\lambda,(2,2)} + \chi_{\lambda,(8,2)} + \chi_{\lambda,(2,8)}\right)\chi^*_{\lambda,(4,4)} + \text{c.c.} \Big]$$

$$(E^{(24)}) \qquad Z = \frac{1}{3} \sum_{\lambda \in P_{++}^{(11)}} \Big[|\chi_{\lambda,(1,1)} + \chi_{\lambda,(22,1)} + \chi_{\lambda,(1,22)} + \chi_{\lambda,(5,5)} + \chi_{\lambda,(5,14)} + \chi_{\lambda,(14,5)} + \chi_{\lambda,(11,11)} + \chi_{\lambda,(11,2)} + \chi_{\lambda,(2,11)} + \chi_{\lambda,(7,7)} + \chi_{\lambda,(7,10)} + \chi_{\lambda,(10,7)}|^2 + |\chi_{\lambda,(7,1)} + \chi_{\lambda,(16,7)} + \chi_{\lambda,(1,16)} + \chi_{\lambda,(1,7)} + \chi_{\lambda,(7,16)} + \chi_{\lambda,(16,1)} + \chi_{\lambda,(5,8)} + \chi_{\lambda,(11,5)} + \chi_{\lambda,(8,11)} + \chi_{\lambda,(8,5)} + \chi_{\lambda,(5,11)} + \chi_{\lambda,(11,8)}|^2 \Big]$$

in the weights of level $k-1$ and 1, i.e. of the form $\mathcal{N}_{(k-1)} = \mathbf{1}$, $\mathcal{N}_{(1)} = \mathbf{1}$ and only $\mathcal{N}_{(k)}$ may be non trivial. Even in that case, the previous constraint rules out the $(A, A, D^{(n)})$ and $(A, A, D^{(n)*})$ cases, whenever $n = k + N$ is not a multiple of N, as well as $(A, A, E^{(8)*})$. In the following, we drop the labels A, and denote this subclass of coset invariants $(G^{(n)})$. The resulting modular invariants are listed in Table II.

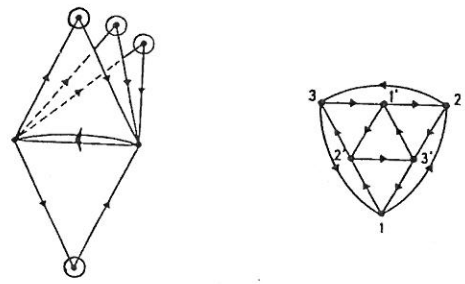
For the Kac-Moody and for this latter class of coset invariants, we may define the concept of exponents as the set $exp(G)$ of $SU(N)$ weights of level $k = n - N$ that label the *diagonal* terms of the modular invariant. In the case of $SU(2)$, this denomination is justified by the fact that these exponents are (quite remarkably and mysteriously) the exponents of the simply-laced Lie algebra that labels the invariant. Since we have also introduced the exponents of a graph $\mathcal{G}$ as specifying the labels of its eigenvalues, it is suggested that there may be a relation between the two notions. They do coincide in the one to one correspondence made in the $SU(2)$ case between the statistical models $\mathbf{G}^{(n)}$ and their toroidal partition functions in the continuum limit. We conjecture that such a correspondence holds for any N, namely that any $SU(N)$ unitary minimal coset modular invariant (G) is the continuum limit of a $SU(N)$ height model defined on a graph $\mathcal{G}$ with the same set of exponents $Exp(\mathcal{G}) = exp(G)$. For a given modular invariant this gives a set of eigenvalues (2.16) for the graph $\mathcal{G}$. Take for instance the $SU(3)$ $(D^{(6)})$ coset modular invariant:

$$Z_{(\text{coset},D^{(6)})} = \frac{1}{3} \sum_{r \in P_{++}^{(5)}} \left(|\chi_{r,\theta} + \chi_{r,\theta+3e_1} + \chi_{r,\theta-3e_3}|^2 + 3|\chi_{r,\theta+e_1+e_2}|^2 \right) \tag{4.6}$$

The corresponding set of eigenvalues is $\{2, 2\omega, 2\omega^2, 0, 0, 0\}$, where $\omega = \exp(\frac{2i\pi}{3})$. The only tricolourable graphs having those eigenvalues , $\mathcal{D}^{(6)}$ and $\mathcal{D}^{(6)*}$ are depicted in fig. 2. They belong to the set of solutions to (3.2) found in [3]. The two graphs certainly define different models, which are expected to yield the same toroidal partition function in the continuum limit. For all the coset $SU(3)$ modular invariants of Table II we found in the same way one or more graphs having the requisite eigenvalues. They are depicted in fig. 3. As a matter of fact they all yield representations of the Kac-Moody fusion rules as in (3.2). But the set of solutions to (3.2) is bigger and it is still unclear whether all the graphs yield modular invariants or whether they should verify additional properties. At any rate, the correspondence is at this stage uncomplete as one can neither write down a modular

invariant from the knowledge of a graph nor do the converse. The aim of the following sections is to attempt to clarify this point.

Fig. 2. The isospectral graphs $\mathcal{D}^{(6)}$ and $\mathcal{D}^{(6)*}$.



4.2. C-algebras defined by graphs

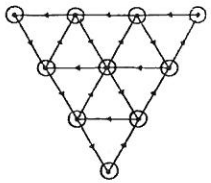
The so called C-algebras or character algebras [42] seem to provide a useful framework for the study of target graphs and the connection with modular invariants. Unfortunately, our considerations will remain empirical and limited to a subclass of graphs.

An associative algebra $\mathcal{U}$ over $\mathbf{C}$ with basis $\{x_a, a \in U = \{1, \cdots, d\}\}$ is called a C-algebra if the conditions $(i) - (vi)$ hold :

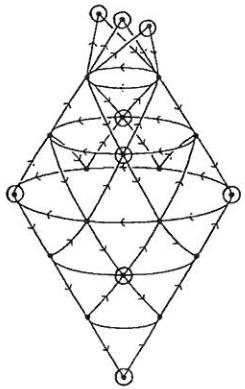
$$\begin{aligned}
&(i) \text{ The algebra is commutative } : x_a x_b = \sum_{c \in U} p_{ab}{}^c x_c, \text{ with } p_{ab}{}^c = p_{ba}{}^c. \\
&(ii) \ \mathcal{U} \text{ has an identity element } e = x_1 : p_{1b}{}^c = \delta_{bc}. \\
&(iii) \text{ All the } p_{ab}{}^c \text{ are real numbers.} \\
&(iv) \text{ There exists a permutation } a \to \sigma(a) \text{ such that : } p_{\sigma(a)\sigma(b)}^{\sigma(c)} = p_{ab}{}^c \ \forall a, b, c \in U. \\
&(v) \text{ There exist strictly positive numbers } k_a, \text{ such that } p_{ab}{}^1 = \delta_{a\sigma(b)} k_a. \\
&(vi) \text{ The } k_a, \text{ called the degrees of } \mathcal{U}, \text{ form a linear representation of } \mathcal{U}.
\end{aligned} \tag{4.7}$$

The simplest example of C-algebra is given by the Kronecker algebra of classes of a finite group. The permutation σ in (iv) maps a class on its inverse, and the linear representation

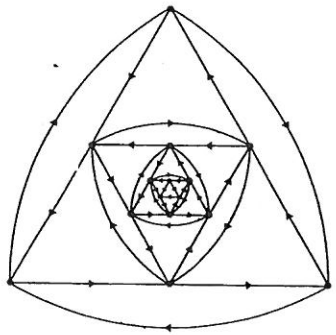
Fig. 3. List of some graphs for $SU(3)$. This is a subset of the graphs found in [3], to which we can associate a modular invariant in Table II. For type I graphs, the association is done following the method of sect 4.3, using the C-subalgebra denoted by circled vertices. For the other graphs, the association is conjectured.



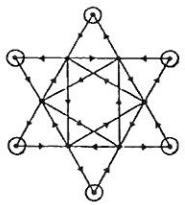
$\mathcal{A}^{(6)}$ I



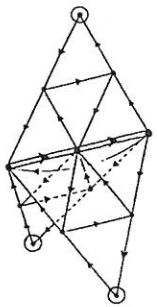
$\mathcal{D}^{(12)}$ I



$\mathcal{D}^{(12)*}$

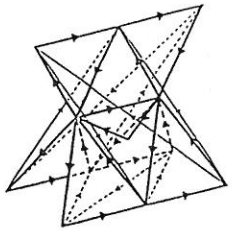


$\mathcal{E}^{(8)}$ I

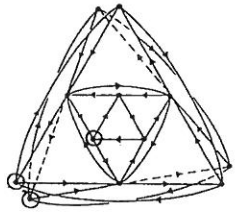


$\mathcal{E}_1^{(12)}$ I

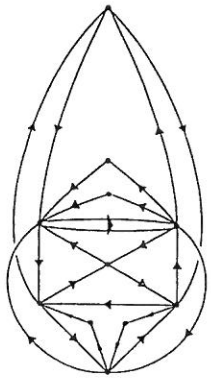
Fig. 3 (continued)



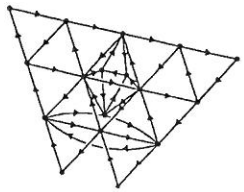
$\mathcal{E}_2^{(12)}$



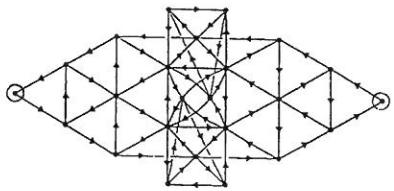
$\mathcal{E}_3^{(12)}$



$\mathcal{E}_4^{(12)}$ II



$\mathcal{E}_5^{(12)}$ II



$\mathcal{E}^{(24)}$ I

Table III
List of some graphs for $SU(3)$
with their exponents, their type and their associated coset modular invariant

Graphs	Altitude	Exponents	Type	Modular Invariant
$\mathcal{A}^{(n)}$	n	$P_{++}^{(n)}$	I	$(A^{(n)})$
$\mathcal{D}^{(n)}$	n $(\equiv 0 \bmod 3)$	$P_{++} \cap Q, (\frac{n}{3}, \frac{n}{3} triple)$	I	$(D^{(n)})$
$\mathcal{D}^{(n)*}$	n	$\{(j,j),(n-2j,j),(j,n-2j)\}$ $1 \leq j \leq E(\frac{n-1}{2})$		$(D^{(n)*})$
$\mathcal{E}^{(8)}$	8	$(1,1),(6,1),(1,6),(3,3),(3,2),(2,3),$ $(4,1),(3,4),(1,3),(1,4)(4,3),(3,1)$	I	$(E^{(8)})$
$\mathcal{E}_1^{(12)}$	12	$(1,1),(10,1),(1,10),(5,5),(5,2),(2,5),$ and twice $(3,3),(3,6),(6,3)$	I	$(E^{(12)})$
$\mathcal{E}_2^{(12)}$	12	$(1,1),(10,1),(1,10),(5,5),(5,2),(2,5),$ and twice $(3,3),(3,6),(6,3)$		$(E^{(12)})$
$\mathcal{E}_3^{(12)}$	12	$(1,1),(10,1),(1,10),(5,5),(5,2),(2,5),$ and twice $(3,3),(3,6),(6,3)$		$(E^{(12)})$
$\mathcal{E}_4^{(12)}$	12	$(1,1),(10,1),(1,10),(5,5),(5,2),(2,5),$ $(3,3),(3,6),(6,3)$ and twice $(4,4)$	II	$(E_{MS}^{(12)*})$
$\mathcal{E}_5^{(12)}$	12	$(1,1),(10,1),(1,10),(5,5),(5,2),(2,5),$ $(4,1),(7,4),(1,7),(1,4),(7,1),(4,7),$ $(3,3),(3,6),(6,3)$ and twice $(4,4)$	II	$(E_{MS}^{(12)})$
$\mathcal{E}^{(24)}$	24	$(1,1),(22,1),(1,22),(5,5),(14,5),(5,14),(7,7),(10,7),(7,10),$ $(11,11),(11,2),(2,11),(7,1),(16,7),(1,16),(1,7),(16,1),(7,16),$ $(5,8),(11,5),(8,11),(8,5),(11,8),(5,11)$	I	$(E^{(24)})$

(vi) is provided by the values of the identity representation character on the different classes : $\chi_1(C_a) = k_a = 1$. Notice that in this case the $p_{ab}{}^c$ are non negative integers.

A less trivial example is provided by the (extended) fusion rules of a conformal field theory. From the fusion numbers $N_{ij}{}^k$ which describe the number of couplings occuring between the primary fields i, j, and k, and are diagonalized by the unitary matrix $S_i{}^j$ we construct the C-algebra structure coefficients :

$$p_{ij}{}^k = \frac{\sqrt{k_i}\sqrt{k_j}}{\sqrt{k_k}} N_{ij}{}^k \tag{4.8}$$

with

$$\sqrt{k_i} = \left(\frac{S_i{}^1}{S_1{}^1}\right) \tag{4.9}$$

which are known to be non negative [29]. The permutation σ of (iv) is the conjugacy $i \to \bar{i}$ of the primary fields.

The notion of duality is of great importance in the C-algebra theory. Take the matrix representation $[P_a]_{bc} = \sqrt{k_a} p_{ab}{}^c$ of the C-algebra $\mathcal{U}$. The commuting matrices P_a are simultaneously diagonalizable. Let $\{E_\lambda, \lambda \in U^*\}$ denote the projectors on the corresponding common eigenspaces and ϑ stand for the sum of the degrees of $\mathcal{U}$:

$$\vartheta = \sum_{a \in U} k_a \tag{4.10}$$

One defines the matrices Π and Γ as follows :

$$P_a = \frac{1}{\vartheta} \sum_{\lambda \in U^*} \Pi_{a\lambda} E_\lambda \tag{4.11a}$$

$$E_\lambda = \sum_{a \in U} \Gamma_{\lambda a} P_a \tag{4.11b}$$

The idempotent operators E_λ, renormalized by a factor ϑ, generate another C-algebra, the dual C-algebra $\mathcal{U}^*$, for the Hadamard product

$$\vartheta E_\lambda \circ \vartheta E_\mu = \sum_{\nu \in D} q_{\lambda\mu}{}^\nu \vartheta E_\nu \tag{4.12}$$

where the Hadamard product of two matrices A and B is :

$$[A \circ B]_{\lambda\mu} = A_{\lambda\mu} B_{\lambda\mu}. \tag{4.13}$$

The dual structure coefficients $q_{\lambda\mu}{}^{\nu}$ are called the Krein parameters of the C-algebra $\mathcal{U}$. The linear representation of the dual algebra involved in (vi) of (4.7) is denoted by m_λ, these numbers are called the multiplicities of $\mathcal{U}$.

We now try to associate yet another associative and commutative algebra to the graphs $\mathcal{G}$, following an idea of Pasquier [24]. The generators N_a are attached to the vertices a of the graph and are represented by $|\mathcal{G}| \times |\mathcal{G}|$ matrices $N_{ab}{}^{c}$. Suppose first there exists some vertex a_1, denoted by abuse of notation 1, extremal in the sense that it is connected to only one pair of other vertices a_f and $a_{\bar{f}}$:

$$G_{1b} = \delta_{ba_f} \quad G_{b1} = \delta_{ba_f} \tag{4.14}$$

We then attach the identity element to a_1, $N_1 = \mathbf{1}$, the generators G and G^t to a_f and $a_{\bar{f}}$ respectively, $N_{a_f} = G$, $N_{a_f} = G^t$, and we try to solve the recursive relations for the N's:

$$GN_a = \sum_{b\in v(\mathcal{G})} G_{ab}N_b \tag{4.15}$$

Clearly this generalizes the observation made in (2.13) and this new algebra may be called the fusion algebra of the graph. The adjacency matrix G then appears as encoding some of the structure constants of this algebra, namely those pertaining to the fusions by the fundamental generator $N_{a_f} = G$. In the case of the graph $\mathcal{A}^{(n)}$, the matrices N reduce to the ordinary fusion matrices of $\widetilde{SU(N)}_k$: the fusion rules $N_{\lambda\mu}{}^{\nu}$ are entirely determined by the fundamental ones $A_{\lambda\mu} = N_{f\lambda}{}^{\mu}$. As the matrix A has only single eigenvalues, A defines up to phases the unitary diagonalization matrix S, which in turn yield unambiguously the coefficients $N_{\lambda\mu}{}^{\nu}$ by the Verlinde formula (2.15). For a general graph $\mathcal{G}$, two situations can occur. If G has only single eigenvalues, then the N's are again completely determined by the Verlinde formula (2.15) with S replaced by the diagonalization matrix ψ of G.

$$N_{ab}{}^{c} = \sum_{\lambda} \frac{\psi_a^{(\lambda)}\psi_b^{(\lambda)}\psi_c^{(\lambda)*}}{\psi_1^{(\lambda)}} \tag{4.16}$$

If some eigenvalues of G are degenerate, then the formula (4.16) is ambiguous as noticed before: there might be different choices of the eigenvectors $\psi^{(\lambda)}$ leading to different numbers N. But we require all the N's to be non negative integer numbers too[4]. The graph $\mathcal{G}$ is

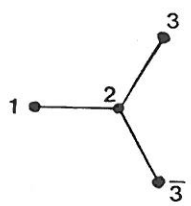
[4] This is done mainly here for mathematical reasons: the positivity condition plays a crucial role in the theory of c-algebras. This requirement is awaiting a good physical interpretation.

called type I if there exists a choice of eigenvectors for G yielding non negative integer N's. and of type II otherwise (i.e. N has some negative entries). Proceeding again empirically, we impose that the vertex a_1 be among the extremal vertices on $\mathcal{G}$ for which the component of the Perron-Frobenius eigenvector is minimal

$$\psi_1^{(1)} = \min_{a\in v(\mathcal{G})} \{\psi_a^{(1)}\} \tag{4.17}$$

The vertex 1 is chosen extremal so that the fundamental generator $N_{a_f} = G$ is associated with one and only one vertex of the graph $\mathcal{G}$. Notice that a given graph is required to have at least one extremal vertex to have a definite type. The graph $\mathcal{D}^{(6)*}$ of fig. 2 in which all vertices are equivalent and non extremal is not assigned any type. In some cases, one can however construct a C-algebra for such a graph. An identity vertex is chosen and G appears as the sum of generators associated with the vertices linked to the identity. The graph encodes the fusions by G, and one may try again to construct recursively the other generators associated with all the vertices of the graph. In the $\mathcal{D}^{(6)*}$ and $\mathcal{E}_3^{(12)}$ graphs of fig. 2 and fig. 3, the fusion coefficients (4.16) are still non negative integer numbers, and the graphs are respectively isospectral to the type I graphs $\mathcal{D}^{(6)}$ and $\mathcal{E}_1^{(12)}$. The forthcoming discussion will not however apply to them.

Fig. 4. The D_4 Dynkin diagram.



Given a graph $\mathcal{G}$, the easiest way of determining its type is to actually compute the N's. Consider for instance the case of $SU(2)$ associated with the D_4 Dynkin diagram depicted in fig. 4. Relations (4.15) read :

$$N_2N_2 = N_1 + N_3 + N_{\bar{3}} \tag{4.18a}$$

$$N_2N_3 = N_2 = N_2N_{\bar{3}} \tag{4.18b}$$

The unique 4-dimensional matrix representation of N with non negative integer entries reads up to the exchange $N_3 \leftrightarrow N_{\bar{3}}$:

$$N_1 = \mathbf{1} \tag{4.19a}$$

$$N_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{4.19b}$$

$$N_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{4.19c}$$

$$N_{\bar{3}} = N_3^t \tag{4.19d}$$

The D_4 diagram is therefore of type I. For $SU(2)$, the A, D_{even}, E_6 and E_8 Dynkin diagrams are type I, whereas the D_{odd} and E_7 are of type II. The symbols I, II in fig. 3 refer to the types of the graphs. It is striking to notice that the type I graphs correspond to type I modular invariants.

The question of uniqueness of the representation N_a in the type I cases remains open, though all the known examples give one and only one representation up to the action of the symmetry group of the graph on the vertices a. At any rate, we can now attach to any type I graph a C-algebra by :

$$p_{ab}{}^c = \frac{\psi_a^{(1)}\psi_b^{(1)}}{\psi_c^{(1)}\psi_1^{(1)}} N_{ab}{}^c, \qquad \text{for } a,b,c \in U = v(\mathcal{G}). \tag{4.20a}$$

$$q_{\lambda\mu}{}^\nu = \frac{\psi_1^{(\lambda)}\psi_1^{(\mu)}}{\psi_1^{(\nu)}\psi_1^{(1)}} M_{\lambda\mu}{}^\nu, \qquad \text{for } \lambda,\mu,\nu \in U^* = Exp(\mathcal{G}) \tag{4.20b}$$

where the Krein parameters are expressed in terms of the dual M :

$$M_{\lambda\mu}{}^\nu = \sum_{a\in U} \frac{\psi_a^{(\lambda)}\psi_a^{(\mu)}\psi_a^{(\nu)*}}{\psi_a^{(1)}} \tag{4.21}$$

These M are in general different from the N and take their values in the real numbers. However they turn out to be non negative in all the known type I cases [5].

[5] This too is awaiting a good interpretation; see however [24] and [43] for an interpretation in terms of an algebra of lattice local operators.

4.3. Modular invariants from C-algebra duality

The following development is based on the empirical observation that the fusion rules of the type I $\widehat{SU(N)}_k$ Kac-Moody theory labelled by G are given by a subset of the coefficients $N_{ab}{}^c$ attached to a type I graph $\mathcal{G}$ corresponding to the coset (G) theory, with $Exp(\mathcal{G}) = exp(G)$. Take for instance the D_4 ($\widehat{SU(2)}_4$) modular invariant :

$$Z_{D_4} = |\chi_1 + \chi_5|^2 + 2|\chi_3|^2 \tag{4.22}$$

The fusion rules with respect to the extended algebra are easily obtained by the $\widehat{SU(2)}_4 \to \widehat{SU(3)}_1$ conformal embedding. The $\widehat{SU(3)}_1$ characters are expressed in terms of the $\widehat{SU(2)}_4$ ones as:

$$\chi_\theta = \chi_1 + \chi_5 \tag{4.23a}$$

$$\chi_{\theta+e_1} = \chi_3 = \chi_{\theta-e_3} \tag{4.23b}$$

Thus the fusions are those $\widehat{SU(3)}_1$ encoded in the graph $\mathcal{A}^{(4)}$ of $SU(3)$ and turn out to be represented by the subset N_1, N_3, and $N_{\bar{3}}$ of (4.19).

We verified for all the known cases of type I graphs having the same exponents as a type I coset modular invariant, that the extended algebra of the corresponding non trivial Kac-Moody modular invariant theory is a subalgebra of the graph algebra N. We circled the vertices forming the subalgebra on the type I graphs of fig. 3. More precisely, there exists a subset T of $U = v(\mathcal{G})$ such that:

$$(i)\ 1 \in T \tag{4.24a}$$

$$(ii)\ \text{if } i \in T, \text{ then } \sigma(i) \in T. \tag{4.24b}$$

$$(iii)\ T \text{ is stable} : \forall i,j \in T, c \in U, \text{such that } N_{ij}{}^c \neq 0, \text{ then } c = k \in T. \tag{4.24c}$$

and the extended fusion algebra is given by $N_{ij}{}^k$ with i, j, $k \in T$. The conditions $(i)-(iii)$ define a C-subalgebra of $\mathcal{U}$ (4.20) , because $N_{ij}{}^c$ vanishes iff $p_{ij}{}^c$ does.

The positivity condition of the structure constants p and the Krein parameters q noticed in the previous section for a type I graph turns out to be a crucial hypothesis in the definition of the quotient of a C-algebra by a C-subalgebra [9]. Suppose a given C-algebra satisfies :

$$p_{ab}{}^c \geq 0 \qquad \forall a,b,c \in U \tag{4.25a}$$

$$q_{\lambda\mu}{}^\nu \geq 0 \qquad \forall \lambda,\mu,\nu \in U^* \tag{4.25b}$$

then for any proper subset T of U satisfying $(i)-(iii)$ of (4.24) , there exists a C-algebra, called the quotient $\mathcal{U}/\mathcal{T}$ of the C-algebra $\mathcal{U}$ with generators $\{x_a, a \in U\}$ by its C-subalgebra $\mathcal{T}$ with generators $\{x_i, i \in T\}$. Its generators read :

$$x_{T_\alpha} = \frac{1}{\varpi} \sum_{a \in T_\alpha} x_a \tag{4.26}$$

where $T_1 = T$ and T_α, $\alpha \in T^* \subset U^*$, are the equivalence classes in U for the relation $\sim$:

$$a, c \in U \ : \ a \sim c \iff \exists i \in T \text{ such that } \ p_{ai}{}^{c} \neq 0 \tag{4.27}$$

and :

$$\varpi = \sum_{i \in T} k_i. \tag{4.28}$$

Finally the C-algebra $\mathcal{T}$ has a dual $\mathcal{T}^*$, C-subalgebra of the dual $\mathcal{U}^*$ of $\mathcal{U}$, with generators $\{E_\alpha, \alpha \in T^* \subset U^*\}$ and there also exists a partition $\{T_i^*, i \in T\}$ of U^*, with $T_1^* = T^*$, formed by the equivalence classes of the relation $\approx$:

$$\lambda, \nu \in U^* \ : \ \lambda \approx \nu \iff \exists \alpha \in T^* \text{ such that } \ q_{\lambda\alpha}{}^{\nu} \neq 0 \tag{4.29}$$

and the quotient C-algebra $(\mathcal{U}/\mathcal{T})^* = \mathcal{U}^*/\mathcal{T}^*$ is generated by :

$$\vartheta E_{T_i^*} = \frac{1}{\kappa} \sum_{\lambda \in T_i^*} \vartheta E_\lambda \tag{4.30}$$

where ϑ has been defined in (4.10) and κ is the following sum of the multiplicities of $\mathcal{U}$:

$$\kappa = \sum_{\alpha \in T^*} m_\alpha \tag{4.31}$$

The matrices Π' and Γ' (4.11) of the quotient $\mathcal{U}/\mathcal{T}$ are expressed in terms of Π and Γ for $\mathcal{U}$:

$$\Pi'_{T_\alpha T_i^*} = \frac{1}{\varpi} \sum_{b \in T_\alpha} \Pi_{b\mu}, \quad \forall \mu \in T_i^* \tag{4.32a}$$

$$\Gamma'_{T_i^* T_\alpha} = \Gamma_{\mu b}, \quad \forall b \in T_\alpha, \forall \mu \in T_i^* \tag{4.32b}$$

whereas the degrees and multiplicities of $\mathcal{U}/\mathcal{T}$ are :

$$k'_{T_\alpha} = \frac{1}{\varpi} \sum_{a \in T_\alpha} k_a \tag{4.33a}$$

$$m'_{T_i^*} = m_\lambda, \qquad \forall \lambda \in T_i^* \tag{4.33b}$$

Returning to the C-algebra attached to a type I graph $\mathcal{G}$, this means that any choice of T satisfying (4.24a) - (4.24c) determines a partition $\{T_i^*, i \in T\}$ of $U^* = Exp(\mathcal{G})$ by duality. We found that for the choice of T yielding the extended fusion rules of the Kac-Moody theory with the same set of exponents, this partition determines precisely the form of the Kac-Moody modular invariant as well as the coset one:

$$\mathcal{N}_{\lambda\bar{\lambda}}^{K-M(G)} = \sum_{i \in T} \delta_{\lambda \in T_i^*} \delta_{\bar{\lambda} \in T_i^*} \tag{4.34a}$$

$$\mathcal{N}_{r,s;\bar{r},\bar{s}}^{coset(A,A,G)} = \frac{1}{N} \delta_{r,\bar{r}} \sum_{i \in T} \delta_{s \in T_i^*} \delta_{\bar{s} \in T_i^*} \tag{4.34b}$$

where the integrable $\widehat{SU(N)}_k$ weights $\lambda, \bar{\lambda}$ and $s, \bar{s}$ are considered as elements of $Exp(\mathcal{G})$ with a possible multiplicity, whereas $r, \bar{r} \in P_{++}^{(n-1)}$. For all the type I-irreducible graphs in fig. 3 we verified that the choice of C-subalgebra denoted by circled vertices yields the extended fusion algebra of a conformal field theory, the toroidal partition function of which is given by (4.34a) . The same remark holds when we replace $SU(N)$ with the compact Lie group G_2, see the appendix for more details.

Let us now describe the determination of the partition $\{T_i^*, i \in T\}$ of $\mathcal{G}$, giving the extended blocks of the resulting Kac-Moody conformal field theory. Using the expressions of the degrees and multiplicities of $\mathcal{U}$ as well as the matrices Π and Γ in terms of the basis of eigenvectors $\psi_a^{(\lambda)}$ of the adjacency matrix of the graph $\mathcal{G}$:

$$k_a = \left(\frac{\psi_a^{(1)}}{\psi_1^{(1)}}\right)^2 \tag{4.35a}$$

$$m_\lambda = \left(\frac{\psi_1^{(\lambda)}}{\psi_1^{(1)}}\right)^2 \tag{4.35b}$$

$$\Pi_{a\lambda} = \frac{\psi_a^{(1)} \psi_a^{(\lambda)}}{\psi_1^{(1)} \psi_1^{(\lambda)}} \tag{4.35c}$$

$$\Gamma_{\lambda a} = \frac{\psi_1^{(\lambda)}}{\psi_1^{(1)}} \left(\frac{\psi_a^{(\lambda)}}{\psi_a^{(1)}}\right)^*, \tag{4.35d}$$

we can eliminate the vectors ψ' of the quotient between (4.32a) and (4.32b) to obtain :

$$\sum_{b \in T_\alpha} \psi_b^{(1)} \psi_b^{(\lambda)} = \delta_{\lambda \in T^*} \frac{\psi_a^{(\lambda)}}{\psi_a^{(1)}} \sum_{b \in T_\alpha} (\psi_b^{(1)})^2, \qquad \forall a \in T_\alpha, \tag{4.36}$$

together with its dual version :

$$\sum_{\mu\in T_i^*} \psi_1^{(\mu)}\psi_a^{(\mu)} = \delta_{a\in T}\frac{\psi_a^{(\lambda)}}{\psi_1^{(\lambda)}}\sum_{\mu\in T_i^*}|\psi_1^{(\mu)}|^2, \qquad \forall\lambda\in T_i^*, \tag{4.37}$$

easily inverted to yield :

$$\sum_{j\in T}\psi_j^{(\lambda)}\psi_j^{(\mu)*} = \frac{\psi_1^{(\lambda)}\psi_1^{(\mu)}}{\sum_{\nu\in T_i^*}|\psi_1^{(\nu)}|^2}\delta_{\mu\in T_i^*}, \qquad \forall\lambda\in T_i^*. \tag{4.38}$$

Writing (4.38) for $\lambda=\mu\in T_i^*$, we get $(\sum_{j\in T}|\psi_j^{(\lambda)}|^2)(\sum_{\nu\in T_i^*}|\psi_1^{(\nu)}|^2) = (\psi_1^{(\lambda)})^2$, so that :

$$\sum_{i\in T}\psi_i^{(\lambda)}\psi_i^{(\mu)*} = \delta_{\mu\approx\lambda}\frac{\psi_1^{(\mu)}}{\psi_1^{(\lambda)}}\sum_{i\in T}|\psi_i^{(\lambda)}|^2 \tag{4.39}$$

where the equivalence relation $\approx$ between elements λ of $Exp(\mathcal{G})$ is defined in (4.29) . The r.h.s of (4.39) has to be symmetric, as $\psi_{\sigma(1)}^{(\lambda)} = \psi_1^{(\lambda)} = \psi_1^{(\lambda)*}$ are real, we finally get an intrinsic definition of the T_i^*'s as equivalence classes of $\approx$ up to a permutation of the indices $i\in T$:

$$\delta_{\lambda\approx\mu} = \sum_{i\in T}\sigma_i^{(\lambda)}\sigma_i^{(\mu)*} \tag{4.40}$$

where the matrix element :

$$\sigma_i^{(\lambda)} = \frac{\psi_i^{(\lambda)}}{(\sum_{k\in T}|\psi_k^{(\lambda)}|^2)^{\frac{1}{2}}} \tag{4.41}$$

is independant of $\lambda\in T_j^*, \forall j\in T$. It takes indeed the value $S_i^{\ j}$ of the S-modular transformation matrix (2.23b) acting on the extended characters $\chi_i(\tau) = \sum_{\lambda\in T_i^*}\chi_\lambda(\tau)$.

Let us return to the example D_4 (fig. 4) of $SU(2)$. Diagonalizing the solutions (4.19a) - (4.19d) simultaneously, we get unambiguous eigenvectors $\psi^{(\lambda)}$ which inserted into (4.40) with $T=\{1,3,\bar{3}\}$, lead to $T_1^*=\{1,5\}$, $T_3^*=\{3\}$ and $T_{\bar{3}}^*=\{3\}$. This partition of $Exp(D_4)$ coincides with the blocks in the modular invariant (4.22).

To summarize, given the type I graph $\mathcal{G}$ and a stable subset T of $v(\mathcal{G})$ satisfying (4.24a) - (4.24c) , we have obtained a dual partition T_i^* of the set $Exp(\mathcal{G})$ by (4.40) , yielding a type-I modular invariant (4.34a) , provided T gives the correct extended fusion rules for the extended theory. It must be clear that this connection between graphs and modular invariants remains at this stage both empiric and limited to the type-I graphs and modular invariants. Still we believe that there must be a deep reason beyond this connection, and that somehow it must extend to the type-II objects.

As a final remark let us point out that the connection between graphs and modular invariants seems to still hold for some of the graphs discussed before, which have no extremal vertices, thus no definite type, but are isospectral to type I ones. Take for instance the graph $\mathcal{E}_3^{(12)}$ of fig. 3. The circled vertices on fig. 3 define again a C-subalgebra of the graph C-algebra, for which we deduce a partition of $Exp(\mathcal{E}_3^{(12)})$ as before. This turns out to yield the extended blocks of the $SU(3)$ modular invariant $(E^{(12)})$ of table I which is also obtained from the type I isospectral graph $\mathcal{E}_1^{(12)}$ of fig. 3 and its circled vertices.

4.4. Modular invariants from matrix intertwiners

We finally mention another empiric connection between graphs and modular invariants. Once again it applies only to the type I objects. By inspection, it is easy to see that the blocks of the modular invariants labelled by G may be constructed as linear combinations of (Kac-Moody or $\mathcal{W}_N$) characters with coefficients given by certain matrix elements of the intertwining matrices $V^{(\lambda)}$ of (3.3), (3.7). For the Kac-Moody invariants, it reads

$$Z = \sum_i |\sum_\lambda V_{1i}^{(\lambda)}\chi_\lambda|^2 \tag{4.42}$$

where i runs over the subset of circled points of the graph $\mathcal{G}$ (i.e. over the sublagebra T). For illustration, we refer the reader to the appendix of [3] where he or she will find the coefficients of all type-I invariants (D_{even}, E_6 and E_8) of $SU(2)$ in some columns of the intertwiner denoted there $V^{(a=1)}$.

In fact the matrix elements of the intertwiner yielding the modular invariant blocks are labelled by the same vertices which form the subalgebra of previous section.

$$V_{1i}^{(\lambda)} = \delta_{\lambda\in T_i^*} = \sum_{j\in T} S_j^{\ i}\sigma_j^{(\lambda)*} \tag{4.43}$$

This relation between the ψ and S matrices turns out to be compatible with the modular invariance condition,which reads in the type I case:

$$\sum_{j\in T} S_i^{\ j}\delta_{\lambda\in T_j^*} = \sum_{\mu\in T_i^*} S_\lambda^{\ \mu} \tag{4.44}$$

5. Summary and Conclusions.

In this paper we have tried to advocate a new algebraic method of constructing modular invariants for $SU(N)$ Kac-Moody or coset theories.

We have introduced a class of statistical models, in which the height variables live on a graph. These models are expected to be critical and to be described by some coset theory in the continuum limit. By a well defined algebraic procedure, we have recovered *all* the known coset modular invariants of a certain type (type I, (A,A,G) see above) from type I graphs $\mathcal{G}$ and a given subset of their set of vertices. The missing coset models are first the (A, A, G) of type II, which are obtained from those of type I by an automorphism of their fusion algebra. Some of the graphs we have found seem to qualify to yield these theories, but we do not know what are the corresponding changes to bring to our algebraic procedure. Another class of coset theories are those obtained by combining different types of Kac-Moody invariants for each factor of the coset. It is not known how to reproduce such cosets by integrable lattice models.

There remain several unanswered questions in our construction. First what is a good classifying principle for the target graphs $\mathcal{G}$? The fact that the incidence matrix G of the graph should be the fundamental representative of a non negative integer matrix representation of the Kac-Moody fusion rules (3.2) enables one to define "intertwiners" $V^{(\lambda)}$ with the physical interpretation of sect.3: they express how the $\mathcal{G}$ cylindric partition function with fixed boundary conditions decomposes onto the $\mathcal{A}$ one. It is thus natural to expect non negative integers coefficients [6] [7]. But it is not clear if this principle is sufficient to select the "good" graphs supporting a representation of the Hecke algebra and leading to conformal field theories.

A second point which remains unclear is the actual link between these supposedly "good" graphs and modular invariants. The method presented above requires additional data, namely a subset of the set of vertices of $\mathcal{G}$ associated with an algebra which can be interpreted as a Kac-Moody fusion algebra. This means in particular that the S-matrix (4.41) has to satisfy: $S^4 = \mathbf{1}$. One could think, for a given type I graph, of inspecting all its possible subalgebras and checking whether this condition is realized. In principle one should test (4.44) for all possible C-subalgebras of type I graphs, which would yield all the type I modular invariants.

The last and most difficult part from the statistical mechanical point of view would be a thorough study of the models themselves, and an extension of their definition off criticality, which remains to be done even in the $SU(2)$ exceptional cases.

In spite of all these shortcomings, we believe to have pointed out a very striking algebraic connection between graphs, fusion algebras and modular invariants of coset theories, which seems to extend to any compact Lie group, as shown in the appendix for the G_2 case.

Acknowledgements.

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Appendix : the G_2 case

The Weyl alcôve $P_{++}^{(n)}$ of $\widehat{(G_2)}_k$ at the altitude $n = k + 4$ (the 4 added is the dual Coxeter number of the Lie algebra G_2), is generated by the two G_2 fundamental weights Λ_1, Λ_2 :

$$P_{++}^{(n)} = \{\lambda = \lambda_1\Lambda_1 + \lambda_2\Lambda_2 | \lambda_1, \lambda_2 \geq 1, \lambda_1 + 2\lambda_2 \leq n - 3\} \tag{5.1}$$

The fundamental fusion rules for $\widehat{(G_2)}_{k=3}$ are encoded in the $\mathcal{A}^{(n=7)}$ graph of fig. 5 . Like in the $SU(N)$ case we choose to perform the fusions with the fundamental representation $(f) = (2\Lambda_1 + \Lambda_2)$, corresponding to the 7-dimensional, self-conjugate representation of the ordinary G_2. The self-conjugacy implies that the graphs are unoriented like in the $SU(2)$ case. This fundamental representation enjoys the (defining) property that the Kac-Moody fusions at level k are simply given by the truncation of the infinite graph $\mathcal{A}^{(\infty)}$ of fig. 6 encoding the decomposition of tensor products of representations of the ordinary G_2 algebra by the fundamental one.

Fig. 5. The $\mathcal{A}^{(7)}$ graph of G_2.

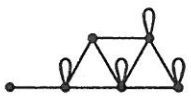
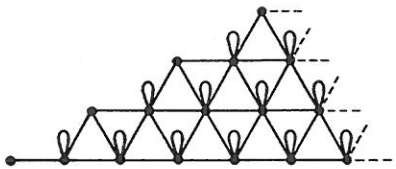


Fig. 6. The $\mathcal{A}^{(\infty)}$ graph of G_2.



The eigenvalues of the associated $A^{(n)}$ adjacency matrices read :

$$\beta^{(\lambda)} = \frac{S_f{}^{\lambda}}{S_1{}^{\lambda}} = 1 + 2\big(\cos(\frac{2\pi}{3n}\lambda_1) + \cos(\frac{2\pi}{3n}(\lambda_1 + 3\lambda_2)) + \cos(\frac{2\pi}{3n}(2\lambda_1 + 3\lambda_2))\big) \qquad (5.2)$$

for $\lambda \in P_{++}^{(n)}$ (5.1). Looking again for type I-irreducible representations of (3.2) where the $N_{\lambda\mu}{}^{\nu}$ now stand for the $(\widehat{G_2})_k$ fusion numbers, we found in addition to the trivial $\mathbf{A}^{(n)}$ solutions two exceptional graphs $\mathcal{E}^{(7)}$ and $\mathcal{E}^{(8)}$ with adjacency matrices $V^{(f)} = G$ at the levels 3 and 4, depicted in fig. 7 .

Fig. 7. The two exceptional graphs $\mathcal{E}^{(7)}$ and $\mathcal{E}^{(8)}$ of G_2.

Comparing with the known list of Kac-Moody modular invariants, we find a perfect agreement between the sets of exponents of the level 3 and 4 exceptional ones and our two

exceptional graphs. $Exp(\mathcal{G})$ is now the set of weights in $P_{++}^{(n)}$, possibly repeated, which parametrize the eigenvalues of the graph $\mathcal{G}$ through eqn (5.2). Apart from the (A) series:

$$Z^{K-M}_{(A^{(n)})} = \sum_{\lambda \in P_{++}^{(n)}} |\chi_\lambda(\tau)|^2 \qquad (5.3)$$

we have the two Kac-Moody modular invariants :

$$Z^{K-M}_{(E^7)} = |\chi_{\Lambda_1+\Lambda_2} + \chi_{2\Lambda_1+2\Lambda_2}|^2 + 2|\chi_{3\Lambda_1+\Lambda_2}|^2 \qquad (5.4a)$$

$$Z^{K-M}_{(E^8)} = |\chi_{\Lambda_1+\Lambda_2} + \chi_{4\Lambda_1+\Lambda_2}|^2 + |\chi_{\Lambda_1+2\Lambda_2} + \chi_{5\Lambda_1+\Lambda_2}|^2 + 2|\chi_{2\Lambda_1+2\Lambda_2}|^2 \qquad (5.4b)$$

The extended fusion rules of these theories are again given by the subalgebras of the $\mathcal{E}^{(7)}$ and $\mathcal{E}^{(8)}$ graphs associated with the circled vertices on fig. 7. The modular invariant combinations of characters (5.4a) - (5.4b) follow from the construction of sect. 4.3 , using C-algebra duality.

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